Temporal logic control of nonlinear stochastic systems using a piecewise-affine abstraction

DOI:
10.1109/LCSYS.2022.3230765

Document status and date:
Published: 01/01/2023

Document Version:
Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

Temporal Logic Control of Nonlinear Stochastic Systems Using a Piecewise-Affine Abstraction

B. C. van Huijgevoort, S. Weiland, and S. Haesaert, *Member, IEEE*

***Abstract*—Automatically synthesizing controllers for continuous-state nonlinear stochastic systems, while giving guarantees on the probability of satisfying (infinite-horizon) temporal logic specifications crucially depends on abstractions with a quantified accuracy. For this similarity quantification, approximate stochastic simulation relations are often used. To handle the nonlinearity of the system effectively, we use finite-state abstractions based on piecewise-affine approximations together with tailored simulation relations that leverage the local affine structure. In the end, we synthesize a robust controller for a nonlinear stochastic Van der Pol oscillator.**

***Index Terms*—Formal specifications, stochastic systems, nonlinear dynamical systems, automatic control, cyber-physical systems.**

I. Introduction

THE DESIGN of controllers for safety-critical systems, such as airplanes, cars, and power systems, requires guarantees on their correct functioning. Although obtaining guarantees via analysis and verification is important, many of these systems are difficult to analyze and verify as they evolve over continuous spaces in a stochastic and generally nonlinear fashion. Therefore, we need methods that can handle simultaneously complex safety-critical requirements, large-scale continuous states, and stochastic and nonlinear state evolutions. Recent work [1], [2], has shown progress in the design of methods based on temporal logic specifications that can handle relevant safety specifications. Although these approaches scale to, respectively, more complex specifications [1] and larger stochastic systems with continuous states [2], they are still limited to linear or pseudo-linear stochastic systems.

For the nonlinear stochastic difference equations considered in this letter, less progress has been shown. Synthesizing a provably correct controller that guarantees the satisfaction of temporal logic specifications for nonlinear stochastic systems remains a very challenging problem and the number of methods that exist is very limited. More specifically, methods either focus on a specific type of specification [3], [4], use slope restrictions on the nonlinearity (pseudo-linearity) of the systems [2], [5] or consider a bounded stochastic disturbance [6], [7], [8]. When focusing on local behavior, many nonlinear systems behave almost linear. Therefore, a widely adopted approach in classical control [9], [10], [11] and for the verification of nonlinear *deterministic* systems [12] is performing a *piecewise-affine approximation* of the nonlinear system. In this letter, we leverage piecewise-affine approximations to synthesize a controller for temporal logic specifications.

To apply such a formal synthesis method with guarantees, the continuous-state behavior of systems is often approximated by a finite-state model [13], known as an abstraction. By quantifying the similarity between the continuous-state model and its finite-state abstraction, we obtain guarantees on the satisfaction of formal specifications. As in [1], [5], the similarity or deviations in probability and output of stochastic systems is expressed using approximate simulation relations [14]. Together with [15], this allows us to handle co-safe linear temporal logic specifications that are unbounded in time. In this letter, we develop tailored methods for the provably correct control design of nonlinear stochastic systems. More specifically, the conditions under which simulation relations can be established exist for nonlinear stochastic systems [14], [15]. However, the main challenge is to find such relations with efficient computation methods, where existing methods can only handle linear systems [16] or nonlinear systems with bounded slope [2], [5]. In this letter, we perform a piecewise-affine approximation step, such that we can use a method like [16]. By doing so, the computational cost can be managed and we enable an efficient implementation.

Literature. The existing methods for temporal logic control of nonlinear stochastic systems can be classified into abstraction-based and abstraction-free methods. As mentioned before, available results on abstraction-based methods for nonlinear stochastic systems [2], [5], [6], [7], [8] are either restricted in the type of systems or with respect to the specification. On the other hand, abstraction-free methods directly synthesize controllers for continuous-state systems and are generally based on Barrier certificates, which are either limited to finite-time horizon specifications [3], [4] or require supermartingale conditions [17].

To include piecewise-affine approximations into temporal logic control of nonlinear stochastic systems, we quantify the approximation error and construct a piecewise-affine abstraction (Section III). In Sections IV and V, we define a piecewise

Manuscript received 6 September 2022; revised 10 November 2022; accepted 4 December 2022. Date of publication 20 December 2022; date of current version 29 December 2022. This work was supported in part by the Project CODEC of the Research Program Veni which is (partly) financed by the Dutch Research Council (NWO) under Project 18244. Recommended by Senior Editor V. Ugrinovskii.
(Corresponding author: B. C. van Huijgevoort.)

The authors are with the Electrical Engineering Department, Eindhoven University of Technology, 5600 MB Eindhoven, The Netherlands (e-mail: b.c.v.huijgevoort@tue.nl).
Digital Object Identifier 10.1109/LCSYS.2022.3230765

simulation relation and describe the computation of the satisfaction probability of the specification. We end with the design of a controller for a stochastically perturbed Van der Pol Oscillator and discuss the results.

II. Problem Formulation and Approach

For a set $\mathbb{X}$ in Euclidean space,[1] the Borel measurable space is denoted as $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$ with $\mathcal{B}(\mathbb{X})$ the σ-algebra of the Borel sets [18]. A probability measure $\mathbb{P}$ over this space has realizations $x \sim \mathbb{P}$ with $x \in \mathbb{X}$. The set of probability measures on the measurable space $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$ is denoted by $\mathcal{P}(\mathbb{X})$. The weighted two-norm $||x||_D$ is defined as $||x||_D = \sqrt{x^\top Dx}$. Furthermore, I_N denotes the identity matrix of size $\mathbb{R}^N \times \mathbb{R}^N$. The Minkowski sum of two sets A and B is defined as $A \oplus B := \{a + b \mid a \in A, b \in B\}$.

A. Preliminaries

Model. Consider a system whose behavior can be modeled by a discrete-time nonlinear stochastic difference equation

$$M : \begin{cases} x_{t+1} = f(x_t) + Bu_t + w_t \\ y_t \quad = Cx_t, \quad \forall t \in \{0, 1, 2, \dots\}, \end{cases} \tag{1}$$

with state $x_t \in \mathbb{X} \subseteq \mathbb{R}^{n_x}$, input $u_t \in \mathbb{U} \subseteq \mathbb{R}^{n_u}$ disturbance $w_t \in \mathbb{W} \subseteq \mathbb{R}^{n_w}$ and output $y_t \in \mathbb{Y} \subseteq \mathbb{R}^{n_y}$. Furthermore, we have matrices $B \in \mathbb{R}^{n_x \times n_u}$, $C \in \mathbb{R}^{n_y \times n_x}$ and the nonlinear function $f : \mathbb{X} \to \mathbb{X}$ is assumed to be sufficiently smooth. The disturbance w_t is an independently and identically distributed (i.i.d.) noise signal with realizations $w \sim \mathbb{P}_w$ and the system is initialized at $x_0 \in \mathbb{X}$.

Remark 1: Without loss of generality, we assume that the output and input enter linearly. Systems with nonlinear terms $g(x_t)u_t$ and $h(x_t)$ instead of resp. Bu_t and Cx_t, can also be handled through piecewise-affine approximations.

A (finite) path $\boldsymbol{\omega}_{\to t} := x_0, u_0, x_1, u_1, \dots x_t$ of a system is built up from inputs u_t and realizations x_{t+1} based on (1) for a given state x_t, input u_t and disturbance w_t for each time step t. A control strategy $\boldsymbol{\mu} := \mu_0, \mu_1, \dots$ consists of maps $\mu_t(\boldsymbol{\omega}_{\to t})$ that determines an input u_t for each finite path of the model (1). In this letter, we focus on stationary control strategies $\boldsymbol{C} : u_t = \mu(\boldsymbol{\omega}_{\to t})$ that have a finite memory.

Specification. To express formal specifications, we use co-safe linear temporal logic (scLTL) [13], [19]. This language consists of atomic propositions $p_1, \dots, p_N$ that are either true or false. The set of atomic propositions is denoted as $AP = \{p_1, \dots, p_N\}$ and defines an alphabet 2^{AP}. The set of atomic propositions that are true form a letter in the alphabet, that is, $\pi \in 2^{AP}$. A word $\boldsymbol{\pi} = \pi_0\pi_1\pi_2 \dots$ is formed by a (possibly infinite) string of letters with associated suffix $\boldsymbol{\pi}_t = \pi_t\pi_{t+1}\pi_{t+2} \dots$ It is over these words that specifications are checked. A formal specification, written as a temporal logic formula ϕ, is formed by combining atomic propositions with logical and temporal operators as defined in the scLTL syntax: $\phi ::= p \mid \neg p \mid \phi_1 \wedge \phi_2 \mid \phi_1 \vee \phi_2 \mid \bigcirc \phi \mid \phi_1 \mathbin{\mathsf{U}} \phi_2$, with atomic proposition $p \in AP$. The semantics are given for suffixes $\boldsymbol{\pi}_t$. An atomic proposition $\boldsymbol{\pi}_t \models p$ holds if $p \in \pi_t$, while a negation $\boldsymbol{\pi}_t \models \neg p$ holds if $\boldsymbol{\pi}_t \not\models p$. Furthermore, a conjunction $\boldsymbol{\pi}_t \models \phi_1 \wedge \phi_2$ holds if both $\boldsymbol{\pi}_t \models \phi_1$ and $\boldsymbol{\pi}_t \models \phi_2$ are true, while a disjunction $\boldsymbol{\pi}_t \models \phi_1 \vee \phi_2$ holds if either $\boldsymbol{\pi}_t \models \phi_1$ or $\boldsymbol{\pi}_t \models \phi_2$ is true. Also, a next statement $\boldsymbol{\pi}_t \models \bigcirc\phi$ holds if $\boldsymbol{\pi}_{t+1} \models \phi$. Finally, an until statement $\boldsymbol{\pi}_t \models \phi_1 \mathbin{\mathsf{U}} \phi_2$ holds if there exists an $i \in \mathbb{N}$ such that $\boldsymbol{\pi}_{t+i} \models \phi_2$ and for all $j \in \mathbb{N}, 0 \leq j < i$ we have $\boldsymbol{\pi}_{t+j} \models \phi_1$. Via a labeling function $L : \mathbb{Y} \to 2^{AP}$, an output trajectory $\boldsymbol{y} = y_0y_1 \dots$ of a system (1) is translated to a word $\boldsymbol{\pi} = L(y_0)L(y_1) \dots$ A system satisfies a specification if the generated word $\boldsymbol{\pi} = L(\boldsymbol{y})$ satisfies the specification, i.e., $\boldsymbol{\pi}_0 \models \phi$.

B. Problem Statement

The goal of this letter is to automatically develop a controller $\boldsymbol{C}$, such that the controlled system $M \times \boldsymbol{C}$ satisfies a specification ϕ. Since we consider stochastic systems, we are interested in the satisfaction probability of a specification, denoted as $\mathbb{P}(M \times \boldsymbol{C} \models \phi)$.

Problem: Given model M as in (1), an scLTL specification ϕ and a probability $p \in [0, 1]$, design a controller $\boldsymbol{C}$, such that

$$\mathbb{P}(M \times \boldsymbol{C} \models \phi) \geq p. \tag{2}$$

We approach this problem by gridding the continuous-state space after approximating the nonlinearity of the model using a piecewise-affine function. This yields a finite-state abstraction of the original nonlinear model that is piecewise-affine. To compare the nonlinear model and the piecewise-affine abstract model, we locally couple the two models and define a piecewise approximate stochastic simulation relation similar to [14]. This computation is implicitly based on invariant set computations as in [16].

III. Piecewise-Affine Abstraction

In this section, we discuss the first step in designing a provably correct controller, namely constructing a piecewise-affine abstraction of the nonlinear system in (1).

Local affine approximation of $\boldsymbol{f(x_t)}$. In order to handle the non-linearity of $f(x_t)$ in (1), we use affine functions to locally approximate it in the bounded set $\mathbb{G} \subseteq \mathbb{X}$. To this end, we use Taylor's Theorem[2] [20, Sec. 4.10], [21, Sec. 2.4] to approximate the function $f(x_t)$ in (1) by its first-order Taylor polynomial $f_1(x_t)$. Using Taylor's inequality [22], an upper bound of the remainder $R_1(x_t)$ can be found. Define the bounded difference between $f(x_t)$ and its affine approximation $f_1(x_t)$ by $\kappa_t = f(x_t) - f_1(x_t) = R_1(x_t)$. Associate to this difference, bounded set $\mathcal{K} \subset \mathbb{R}^{n_x}$. Now, we get the following.

Theorem 1: Given a nonlinear function $f(x_t)$ that is sufficiently smooth, there exists a bounded vector $\kappa_t \in \mathcal{K}$, such that $f(x_t) = Ax_t + a + \kappa_t$ for $x \in \mathbb{G}$.

The proof follows from the extension of Taylor's Theorem to higher-dimensional functions as in [21, Sec. 2.4].

Piecewise-affine finite-state abstraction. To construct such an abstraction, we need two different partitionings of the state space. A coarse partitioning to construct the piecewise-affine approximation of the nonlinear dynamics and a fine grid to get a finite-state approximation of the affine dynamics. To obtain the coarse partitions, we partition the state space $\mathbb{X}$ with polytopic cells $\hat{P}_i$ with $i \in \{1, \dots, N_P\}$, such that it covers the complete state space, that is $\bigcup_i \hat{P}_i = \mathbb{X}$ and such that the partitions do not overlap $\hat{P}_i \cap \hat{P}_j = \emptyset$ for $i \neq j$. Similarly, to

[1] We limit our results to sets in Euclidean spaces, which are measurable and separable.

[2] The first reference discusses the most common 1D case, while an extension to multivariate functions is given in the second reference.

obtain the fine grid we grid the state space $\mathbb{X}$ in a finite number of regions $\mathbb{A}_j \subset \mathbb{X}$, such that $\bigcup_j \mathbb{A}_j = \mathbb{X}$ and $\mathbb{A}_j \cap \mathbb{A}_l = \emptyset$ for $j \neq l$ hold. In each region, a representative point $\hat{x}_j \in \mathbb{A}_j$ is chosen. Together, these points make up the set of abstract states, $\hat{x} \in \hat{\mathbb{X}} = \{\hat{x}_1, \hat{x}_2, \dots, \hat{x}_{N_A}\}$.

After performing the affine approximation for each of the partitions $\hat{P}_i \subseteq \mathbb{G}$, we select a finite number of inputs from $\mathbb{U}$ to form the abstract input space $\hat{\mathbb{U}}$. Now, we can approximate the behavior of the affine approximation of the nonlinear dynamics by a finite-state abstract system. To this end, consider the operator $\Pi : \mathbb{X} \to \hat{\mathbb{X}}$ that maps states from the original state space to the abstract state space. Then in region $\hat{P}_i$ the dynamics of the abstract system equal

$$\hat{x}_{t+1} = \Pi(A\hat{x}_t + B\hat{u}_t + a + \hat{w}_t), \tag{3}$$

with states $\hat{x} \in \hat{P}_i \subset \hat{\mathbb{X}}$, initial state $\hat{x}_0 = \Pi(x_0)$, inputs $\hat{u} \in \hat{\mathbb{U}}$, and disturbances $\hat{w} \in \mathbb{W}$ with realizations $\hat{w} \sim \mathbb{P}_{\hat{w}}$. Next, we introduce a bounded vector $\beta \in \mathcal{B} \subset \mathbb{X}$, that pushes the state to its representative point. Then, with a slight abuse of notation[3] the state dynamics of the local abstract system (3) for $\hat{x} \in \hat{P}_i$ satisfy $\hat{x}_{t+1} \in A\hat{x}_t + B\hat{u}_t + a + \hat{w}_t + \mathcal{B}$. More precisely, there exist $\beta \in \mathcal{B}$, such that $\hat{x}_{t+1} = A\hat{x}_t + B\hat{u}_t + a + \hat{w}_t + \beta_t$. Now, we can write the state dynamics of the local abstract system as an affine system described by

$$\hat{x}_{t+1} = A\hat{x}_t + B\hat{u}_t + a + \hat{w}_t + \beta_t \;\text{ for }\; \hat{x} \in \hat{P}_i. \tag{4}$$

To define a piecewise-affine finite-state abstraction, we translate the abstract system (4) that locally approximates the nonlinear system (1) to a piecewise-affine system that approximates the complete nonlinear system as

$$\hat{M} : \begin{cases} \hat{x}_{t+1} = A_i\hat{x}_t + B\hat{u}_t + a_i + \hat{w}_t + \beta_t \;\text{ for }\; \hat{x} \in \hat{P}_i \\ \hat{y} = C\hat{x}_t, \end{cases} \tag{5}$$

with states $\hat{x} \in \hat{\mathbb{X}} \subset \mathbb{X}$, initial state $\hat{x}_0$, inputs $\hat{u} \in \hat{\mathbb{U}}$ and disturbance $\hat{w} \in \mathbb{W}$.

Concluding, we have constructed a piecewise-affine finite-state system which approximates the behavior of a nonlinear continuous-state system (1). This piecewise-affine system consists of local affine dynamics defined over partitions $\hat{P}_i$.

IV. Piecewise Stochastic Simulation Relation

In this section, we discuss how to quantify the difference between the original nonlinear stochastic model and the abstract model obtained via piecewise affine approximations.

A. Similarity Quantification

To quantify the similarity between the models (1) and (5), we start by defining a local metric for the error dynamics based on coupling the models through their inputs and stochastic disturbances. First, we couple the inputs u and $\hat{u}$ by using an interface function [23] denoted as

$$\mathcal{U}_v : \hat{\mathbb{U}} \times \hat{\mathbb{X}} \times \mathbb{X} \to \mathbb{U}. \tag{6}$$

This function assigns input u to the abstract input $\hat{u}$ given the states $\hat{x}$ and x of the abstract and original model, respectively. Next, we couple disturbances $w \sim \mathbb{P}_w$ and $\hat{w} \sim \mathbb{P}_{\hat{w}}$ as in [16] using the following definition based on [24].

[3] Here, the Minkowski sum of the two sets is neglected.

Definition 1 (Coupling Probability Measures): A coupling of probability measures $\mathbb{P}_w$ and $\mathbb{P}_{\hat{w}}$ on the same measurable space $(\mathbb{W}, \mathcal{B}(\mathbb{W}))$ is any probability measure $\mathcal{W}$ on the product measurable space $(\mathbb{W} \times \mathbb{W}, \mathcal{B}(\mathbb{W} \times \mathbb{W}))$ whose marginals are $\mathbb{P}_w$ and $\mathbb{P}_{\hat{w}}$, that is,

$$\mathcal{W}(\hat{A} \times \mathbb{W}) = \mathbb{P}_{\hat{w}}(\hat{A}) \textit{ for all } \hat{A} \in \mathcal{B}(\mathbb{W})$$
$$\mathcal{W}(\mathbb{W} \times A) = \mathbb{P}_{w}(A) \textit{ for all } A \in \mathcal{B}(\mathbb{W}).$$

We can trivially extend this to Borel measurable stochastic coupling kernels

$$\mathcal{W} : \hat{\mathbb{U}} \times \hat{\mathbb{X}} \times \mathbb{X} \to \mathcal{P}(\mathbb{W}^2). \tag{7}$$

Now, we can quantify the similarity between the stochastic models M (1) and $\hat{M}$ (5) using a simulation relation [14].

Definition 2 [$(\epsilon, \boldsymbol{\delta})$-stochastic simulation relation]: Let stochastic models M and $\hat{M}$ with metric output space $(\mathbb{Y}, \mathbf{d}_{\mathbb{Y}})$, interface function $\mathcal{U}_v$ (6), and stochastic kernel $\mathcal{W}$ (7) be given. If there exists a measurable relation $\mathcal{R} \subseteq \hat{\mathbb{X}} \times \mathbb{X}$, with $(\hat{x}_0, x_0) \in \mathcal{R}$, and such that

1) $\forall(\hat{x}, x) \in \mathcal{R} : \boldsymbol{d}_{\mathbb{Y}}(\hat{y}, y) \leq \epsilon$, and
2) $\forall(\hat{x}, x) \in \mathcal{R}, \forall \hat{u} \in \hat{\mathbb{U}} : (\hat{x}^+, x^+) \in \mathcal{R}$ holds with probability at least $1 - \boldsymbol{\delta}(\hat{x})$, with $\boldsymbol{\delta} : \hat{\mathbb{X}} \to [0, 1]$.

then $\hat{M}$ is $(\epsilon, \boldsymbol{\delta})$-stochastically simulated by M, and this simulation relation is denoted as $\hat{M} \preceq_{\epsilon}^{\boldsymbol{\delta}} M$.

We refer to ϵ as the (metric) output deviation and to $\boldsymbol{\delta}$ as the probabilistic or stochastic deviation function. Note that unlike [14] the stochastic deviation is not uniform for the whole state space. Instead it is introduced as a function $\boldsymbol{\delta} : \hat{\mathbb{X}} \to [0, 1]$ that depends on the abstract state $\hat{x}$. If $\boldsymbol{\delta}(\hat{x})$ is a piecewise constant function, then we refer to the simulation relation as a *piecewise stochastic simulation relation*. We have defined a measure to quantify the difference between two models on a global level, that is, over the full state space. The question is now how we can compute it based on the given piecewise-affine structure of the abstractions.

B. Piecewise Similarity Quantification

Consider a simulation relation given as

$$\mathcal{R} := \left\{(\hat{x}, x) \in \hat{\mathbb{X}} \times \mathbb{X} \mid ||x - \hat{x}||_D \leq \epsilon\right\}, \tag{8}$$

with a suitable weighting matrix D. It can be seen that

$$C^\top C \preceq D \tag{9}$$

implies that the first condition of Def. 2 is satisfied. Next, we use relation (8) to show that a global $(\epsilon,\boldsymbol{\delta})$-stochastic simulation relation can be computed with a piecewise constant probability deviation function $\boldsymbol{\delta} : \hat{\mathbb{X}} \to [0, 1]$ defined on state partition $\hat{P}_i$ as $\boldsymbol{\delta}(\hat{x}) = \delta_i$ if $\hat{x} \in \hat{P}_i$. Function $\boldsymbol{\delta}$ assigns a constant local probability deviation to each partition in the abstract state space $\hat{\mathbb{X}}$ based on a local similarity quantification derived using the local error dynamics.

Local stochastic error dynamics. Consider a local interface function $u_t = \mathcal{U}_{vi}(\hat{u}_t, \hat{x}_t, x_t)$ as

$$u_t = \hat{u}_t + K_{f,i}(x_t - \hat{x}_t), \tag{10}$$

with feedback matrix $K_{f,i} \in \mathbb{R}^{n_u \times n_x}$ and a local stochastic kernel $\mathcal{W}_i$, assigning to each $(\hat{u}, \hat{x}, x)$ a probability measure

$$\mathcal{W}_i : \hat{\mathbb{U}} \times \hat{P}_i \times \mathbb{X} \to \mathcal{P}(\mathbb{W}^2). \tag{11}$$

For $\hat{x} \in \hat{P}_i$, we have $\hat{x}_{t+1} = A_i\hat{x}_t + B\hat{u}_t + a_i + \hat{w}_t + \beta_t$ and if $\|x - \hat{x}\|_D \le \epsilon$ holds, then there exists a κ_t such that

$$x_{t+1} = A_i x_t + Bu_t + a_i + w_t + \kappa_t \text{ with } \kappa_t \in \mathcal{K}_i.$$

Given that x_t belongs to P_i defined as

$$P_i := \{x \in \mathbb{X} \mid \exists \hat{x} \in \hat{P}_i : \|x - \hat{x}\|_D \le \epsilon\}. \tag{12}$$

Following the previous section, the set $\mathcal{K}_i$ is defined as

$$\mathcal{K}_i := \sup_{x \in P_i}(R_1(x)), \tag{13}$$

with $R_1(x)$ the remainder of the first-order Taylor polynomial [21, Sec. 2.4]. For $\hat{x}_t \in \hat{P}_i$ and $x_t \in P_i$, we get the error dynamics $x_{\Delta t} := x_t - \hat{x}_t$ equal to

$$x_{\Delta t+1} = (A_i + BK_{f,i})x_{\Delta t} + (w_t - \hat{w}_t) + \kappa_t - \beta_t \tag{14}$$

with $\kappa \in \mathcal{K}_i$ and $\beta \in \mathcal{B}$ and with $(\hat{w}_t, w_t) \sim \mathcal{W}_i$.

Local coupling and interface functions with $\boldsymbol{\delta} = \delta_i$. Following [16], we make sure that the second condition of Def. 2 is satisfied by finding a global invariant set $\{x_\Delta \mid \|x_\Delta\|_D \le \epsilon\}$ parameterized with a global D for the error dynamics (14). Together with D, we have to compute an optimal local interface function (10) and coupling (11) for all partitions $\hat{P}_i$, with $i \in \{1, \dots N_p\}$. More precisely, we design $\mathcal{W}_i$ and $K_{f,i}$ such that the probability $1 - \delta_i$ with which $\|x_{\Delta t+1}\|_D \le \epsilon$ holds is maximized. To this end, consider local coupling $\hat{w} = w + F_i(x - \hat{x})$ that holds with probability $1 - \delta_i$. The coupling term F_i introduced in [16] reduces the complexity of the design of $\mathcal{W}_i$ as it allows to write the design problem as a set of implications or matrix inequalities. That is, a relation between this term and the probability deviation δ_i can been derived as upper bound

$$||F_i(x - \hat{x})|| \le r_i := \left|2\,\mathrm{idf}\left(\frac{1-\delta_i}{2}\right)\right|. \tag{15}$$

Here, idf denotes the inverse distribution function of a Gaussian distribution $\mathcal{N}(0, I)$.

As concluded from the error dynamics in (14), together with the coupling, the interface function can be used to further compensate for the error in the state by computing a feedback-term $K_{f,i}$. To satisfy the bound $u \in \mathbb{U}$ we shrink $\hat{\mathbb{U}}$ by α, such that $\hat{\mathbb{U}} \subset \alpha\mathbb{U}$ and we compute u_u, such that

$$||K_{f,i}(x - \hat{x})|| \le u_u \tag{16}$$

implies $K_{f,i}(x - \hat{x}) \in (1 - \alpha)\mathbb{U}$. Taken together, we conclude the following.

Lemma 1 (Piecewise Requirements): Consider stochastic models M (1) and $\hat{M}$ (5) for which a simulation relation $\mathcal{R}$ (8) with weighting matrix D satisfying (9) is given. If there exist matrices F_i, and $K_{f,i}$ such that the following implications are satisfied for a given $\boldsymbol{\delta}(\hat{x})$

$$x_\Delta^\top D x_\Delta \le \epsilon^2 \implies \begin{cases} x_\Delta^\top F_i^\top F_i x_\Delta & \le r_i^2 \\ x_\Delta^\top K_{f,i}^\top K_{f,i} x_\Delta & \le u_u^2 \\ x_{\Delta t+1}^\top D x_{\Delta t+1} & \le \epsilon^2, \end{cases} \tag{17}$$

with $x_{\Delta t+1}$ in (14) and r_i in (15), then there exists coupling kernels $\mathcal{W}_i$ and interfaces $\mathcal{U}_{vi}$ such that

$$\forall(\hat{x}, x) \in \hat{P}_i \times \mathbb{X}, \forall \hat{u} \in \hat{\mathbb{U}} : (\hat{x}^+, x^+) \in \mathcal{R} \tag{18}$$

holds with probability $1 - \delta_i$ for all $\hat{P}_i$, with $i \in \{1, \dots N_p\}$ and with $\bigcup_i \hat{P}_i = \hat{\mathbb{X}}$.

Proof: It can readily be seen that the first and second implication in (17) are sufficient conditions for the bounds on resp. the *coupling compensator term* (15) and the feedback-term (16). Assume that bounded sets $\mathcal{K}_i$ are given and define the sets $S_i := \{(\hat{x}, x) \in \hat{P}_i \times \mathbb{X} \mid ||x - \hat{x}||_D \le \epsilon\}$. The last implication in (17) is a sufficient condition for sets S_i to be controlled invariant sets according to [16, Definition 7] with disturbance $\beta + \kappa \in \mathcal{B} \bigoplus \mathcal{K}_i$. If the implications in (17) hold, then the bounds in [16, Th. 8] are satisfied and S_i are controlled-invariant sets. Following the proofs of [16, Th. 8 and Lemma 6] we can conclude that this implies the existence of coupling kernels $\mathcal{W}_i$ and interfaces $\mathcal{U}_{vi}$ such that Lemma 1 holds. An algorithm to obtain matrix D and bounded sets $\mathcal{K}_i$ is explained in the Appendix. ∎

From local to piecewise similarity quantification. To obtain a global similarity quantification, we define a piecewise stochastic kernel $\mathcal{W}$ and a piecewise interface function $\mathcal{U}_v$. Since $\hat{\mathbb{U}} \times \hat{P}_i \times \mathbb{X}$ for $i \in \{1, \dots, N_P\}$ is a partitioning of $\hat{\mathbb{U}} \times \hat{\mathbb{X}} \times \mathbb{X}$ we use the local stochastic coupling kernel $\mathcal{W}_i : \hat{\mathbb{U}} \times \hat{P}_i \times \mathbb{X} \to \mathcal{P}(\mathbb{W}^2)$ to compute the piecewise stochastic coupling kernel $\mathcal{W} : \hat{\mathbb{U}} \times \hat{\mathbb{X}} \times \mathbb{X} \to \mathcal{P}(\mathbb{W}^2)$ as

$$\mathcal{W}(\cdot \mid \hat{u}, \hat{x}, x) = \mathcal{W}_i(\cdot \mid \hat{u}, \hat{x}, x) \text{ if } \hat{x} \in \hat{P}_i. \tag{19}$$

Similarly, the interface function can be composed as

$$\mathcal{U}_v(\hat{u}_t, \hat{x}_t, x_t) = \mathcal{U}_{vi}(\hat{u}_t, \hat{x}_t, x_t) \text{ if } \hat{x} \in \hat{P}_i. \tag{20}$$

We can now show that these functions constitute to a $(\epsilon, \boldsymbol{\delta})$-stochastic simulation relation for simulation relation (8).

Theorem 2 (Piecewise Stochastic Similarity): Let stochastic models M (1) and $\hat{M}$ (5) be given. Then the interface function $\mathcal{U}_v$ (20) and the global Borel measurable stochastic kernel $\mathcal{W}$ (19) computed for the simulation relation (8) based on (17) define an $(\epsilon,\boldsymbol{\delta})$-stochastic simulation relation in a piecewise manner as given in Def. 2 if

- it holds that $(\hat{x}_0, x_0) \in \mathcal{R}$, and if
- the simulation relation satisfies matrix inequality (9).

Proof: The proof builds on Lemma 1, and can be sketched as follows. The first condition of Def. 2 holds by choosing matrix D, such that (9) holds. This is proven in the proof of [16, Th. 9]. Lemma 1 shows that if (17) is satisfied then a local stochastic kernel $\mathcal{W}_i$ as in (11) exists, such that (18) holds with probability $1 - \delta_i$. By choosing the interface function $\mathcal{U}_v$ as (20), and the global stochastic kernel as in (19) the second condition in Def. 2 is satisfied. ∎

V. Temporal Logic Control

In this section, we discuss how to compute the satisfaction probability of (infinite-horizon) temporal logic specifications based on the dynamic programming mappings from [15]. Next, we apply the method from this letter to a nonlinear stochastic case study and discuss the results.

A. Dynamic Programming

In correct-by-design control synthesis, an scLTL specification ϕ can be written as a *deterministic finite-state automaton* (DFA), characterized by the tuple $\mathcal{A}_\phi = \{Q, q_0, \Sigma, \tau_\mathcal{A}, F\}$ [13]. Here, the set of states is denoted by Q with initial state q_0. The input alphabet and transition function are respectively denoted by $\Sigma = 2^{AP}$ and $\tau_\mathcal{A} : Q \times \Sigma \to Q$. Finally, F denotes the set of accepting states. The word $\boldsymbol{\pi}$ satisfies the specification ϕ, that is $\boldsymbol{\pi} \models \phi$, if the word $\boldsymbol{\pi}$ is *accepted* by the DFA $\mathcal{A}_\phi$. This means that there exists a trajectory $q_0, q_1, \ldots, q_f$ with $q_f \in F$ starting at q_0 and evolving according to $q_{t+1} = \tau_{\mathcal{A}}(q_t, \pi_t)$. By analyzing the product composition between the system M and the specification DFA $\mathcal{A}_\phi$, denoted as $M \otimes \mathcal{A}_\phi$, we can compute the satisfaction probability. The composition $M \otimes \mathcal{A}_\phi$ consists of states $(x_t, q_t) \in \mathbb{X} \times Q$. For a given input u_t, it evolves from (x_t, q_t) to (x_{t+1}, q_{t+1}) by following the stochastic transition from x_t to x_{t+1} in (1) and from q_t to $q_{t+1} = \tau_{\mathcal{A}}(q_t, L(Cx_t))$. Hence, computing the satisfaction probability is equivalent to solving a reachability problem of the composition $M \otimes \mathcal{A}_\phi$, which can be written as a dynamic program. With a slight abuse of notation we refer to the stationary policy of this composed system as $\boldsymbol{\mu} : \hat{\mathbb{X}} \times Q \to \hat{\mathbb{U}}$.

We use the abstract model to compute the satisfaction probability, since this is not possible for the original model due to its continuous states. The satisfaction probability with policy $\boldsymbol{\mu}$ in time horizon $[1, \ldots, N]$ is expressed by the value function $V_N^{\boldsymbol{\mu}}(\hat{x}, q) : \hat{\mathbb{X}} \times Q \to [0, 1]$, which is equivalent to the probability that the trajectory starting at (x, q) and generated by applying $\boldsymbol{\mu}$ to $M \otimes \mathcal{A}_\phi$ reaches the target set F within this time horizon. The value function is defined as $V_N^{\boldsymbol{\mu}}(\hat{x}, q) := \mathbb{E}_{\boldsymbol{\mu}}(\max_{0 \le t \le N} 1_F(q_t) | (\hat{x}_0, q_0))$, with indicator function 1_F equal to 1 if $q \in F$ and 0 otherwise. The value function can also be computed recursively for a policy $\boldsymbol{\mu}_i = (\mu_{i+1}, \ldots, \mu_N)$ with horizon $N - i$ as $V_{N-k+1}^{\boldsymbol{\mu}_{k-1}}(\hat{x}, q) = \mathbf{T}^{\mu_k}(V_{N-k}^{\boldsymbol{\mu}_k})(\hat{x}, q)$, initialized with $V_0 \equiv 0$. Here, operator $\mathbf{T}^{\mu_k}(\cdot)$ is defined as $\mathbf{T}^{\mu_k}(V)(\hat{x}, q) := \mathbb{E}_{\mu_k}(\max\{1_F(q^+), V(\hat{x}^+, q^+)\})$, with DFA transitions $q^+ = \tau_{\mathcal{A}_\phi}(q, L(Cx^+))$. For a stationary policy $\boldsymbol{\mu}$, the infinite-horizon value function is computed as $V_\infty^{\boldsymbol{\mu}} = \lim_{N \to \infty} (\mathbf{T}^{\boldsymbol{\mu}})^N V_0$ initialized with $V_0 \equiv 0$. The policy-optimal converged value function V_∞^* is computed with the operator $\mathbf{T}^*(\cdot) := \sup_\mu \mathbf{T}^\mu(\cdot)$. The corresponding satisfaction probability can now be computed as $\mathbb{P}^{\boldsymbol{\mu}} := \max(\mathbf{1}_F(\bar{q}_0, V_\infty^*(x_0, \bar{q}_0)))$ with $\bar{q}_0 = \tau(q_0, L(Cx_0))$.

To cope with the output deviation ϵ and with probability deviations described by the function $\boldsymbol{\delta}(\hat{x})$, we define a robust dynamic programming mapping similar to [14], as

$$\mathbf{T}_{\epsilon,\boldsymbol{\delta}}^{\mu_k}(V)(\hat{x}, q) := \mathbf{L}\Big(\mathbb{E}_\mu\big(\min_{q^+ \in Q^+} \max\{1_F(q^+), V(\hat{x}^+, q^+)\}\big) - \boldsymbol{\delta}(\hat{x})\Big),$$

with $\mathbf{L} : \mathbb{R} \to [0, 1]$ a truncation function $\mathbf{L}(\cdot) := \min(1, \max(0, \cdot))$ and with $Q^+(q, \hat{y}^+) := \{\tau_{\mathcal{A}}(q, L(y^+)) \mid \|y^+ - \hat{y}^+\| \le \epsilon\}$. We can now compute the robust satisfaction probability by considering the first time instance based on x_0, that is, $\mathbb{R}^{\boldsymbol{\mu}} := \max(\mathbf{1}_F(\bar{q}_0, V_\infty^{\boldsymbol{\mu}}(x_0, \bar{q}_0)))$ with $\bar{q}_0 = \tau_{\mathcal{A}}(q_0, L(Cx_0))$. This probability is robust since it gives a lower bound on the probability in (2), i.e., $\mathbb{R}_{\epsilon,\boldsymbol{\delta}}(\hat{M} \times \hat{\boldsymbol{C}} \models \phi) \le \mathbb{P}(M \times \boldsymbol{C} \models \phi)$.

B. Case Study

We have applied this method to a forced, stochastically perturbed Van der Pol oscillator with state dynamics

$$\begin{aligned} x_{1t+1} &= x_{1t} + x_{2t}\tau + w_{1t} \\ x_{2t+1} &= f_2(x_t) + u_t + w_{2t}, \end{aligned}$$

with nonlinear function $f_2(x_t) = x_{2t} + (-x_{1t} + (1 - x_{1t}^2)x_{2t})\tau)$. Here, $\tau = 0.1$ is the sampling time and $w \sim \mathcal{N}(0, 0.2I_2)$ is a Gaussian disturbance. The output equals the state, that is $y_t = x_t$ and we have states $x \in \mathbb{X} = [-3, 3]^2$, input

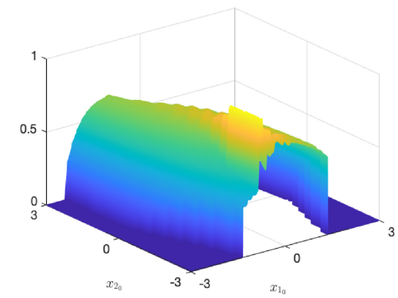

Fig. 1. Robust satisfaction probability $\mathbb{R}_{\epsilon,\boldsymbol{\delta}}(\hat{M} \times \hat{\boldsymbol{C}} \models \phi)$.

$u \in [-1, 1]$ outputs $y \in \mathbb{Y} = \mathbb{X}$, safe region $P_1 = \mathbb{X}$ and goal region $P_2 = [-1.2, -0.9] \times [-2.9, -2]$. The specification $\phi = P_1 \cup P_2$ means stay in the safe region, while reaching the goal region.

We obtained an abstract model with state dynamics as in (3) by partitioning the state space with square regions of width 0.01 leading to[4] $\beta \in \mathcal{B} = [-0.01, 0.01]^2$ and with $\hat{u} \in \hat{\mathbb{U}} = \{-0.6, 0, 0.6\}$ leaving some input action for the feedback part, namely $-0.4 \le K_{f,i}(x - \hat{x}) \le 0.4$. Next, we used 1600 equally sized square partitions to obtain a piecewise-affine abstraction as in (5). We then selected $\epsilon = 0.08$ and computed a corresponding probability deviation function $\boldsymbol{\delta}(\hat{x})$ such that the implications in (17) are satisfied. We computed a global stochastic kernel $\mathcal{W}$ (19) and interface function $\mathcal{U}_v$ (20) and used Theorem 2 to obtain an $(\epsilon, \boldsymbol{\delta})$-stochastic simulation relation. Finally, we obtained a robust controller $\boldsymbol{C}$ and the robust satisfaction probability shown in Fig. 1. The MATLAB implementation takes 55 minutes on a computer with a 2, 3 GHz Quad-Core Intel Core i5 processor and 16 GB MHz memory, while using 225Mb memory to store the variables in the workspace. 44.5% of the computation time is spent on gridding and 55% on computing the matrices in Lemma 1.

Comparison to available software tools. Similar case studies have been presented in [7], [25], where [7] considers an autonomous Van der Pol oscillator and [25] combines the input with a multiplicative noise term. However, the results presented in [7], [25] are limited to verification or a reachability analysis instead of the control synthesis performed in this letter. Furthermore, we have chosen a more stochastic variant with a Gaussian disturbance instead of a uniform distribution with bounded support as used in [7], [25]. The unbounded nature of the Gaussian disturbance contributes significantly to the difficulty of this case study.

Reduced output dimension. Consider the Van der Pol oscillator with state dynamics as before, but with output $y_t = [1, 0]\ x_t$ and with $\mathbb{Y} = [-3, 3]$. Since the words $\boldsymbol{\pi}$ are defined over the outputs of the system, we adjust the regions of the specification accordingly, that is $P_1 = [-3, 3]$ and $P_2 = [-1.2, -0.9]$. Next, we perform the same steps (with the same parameters) as before to compute the robust satisfaction probability in Fig. 2.

VI. Conclusion

Concluding, to the best of our knowledge this letter is the first to describe a temporal logic control method for

[4]Normally, you get $\mathcal{B} = [-0.005, 0.005]^2$, however, our implementation uses an efficient tensor-based computation that leads to a bigger set for β.

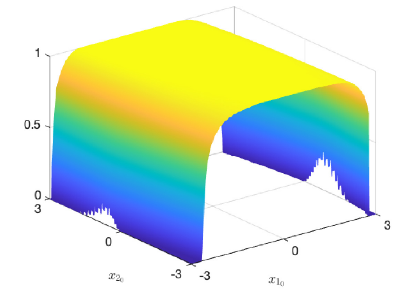

Fig. 2. Robust satisfaction probability $\mathbb{R}_{\epsilon,\delta}(\hat{M} \times \hat{C} \models \phi)$ for the van der Pol oscillator with $C = [1, 0]$.

Algorithm 1 Get Weighting Matrix D and Bounded Sets $\mathcal{K}_i$

1: **Input:** $M, \hat{M}, \epsilon$
2: Set $D = C^\top C$
3: Compute P_i and $\mathcal{K}_i$ using (12) and (13)
4: Choose $N_i \leq N_P$ to compute a suitable value for D
5: $D \leftarrow$ solve optimization problem (21)
6: Update P_i and $\mathcal{K}_i$ using (12) and (13)

nonlinear *stochastic* models that uses piecewise-affine approximations. By using a state-dependent probability deviation, a lower bound on the satisfaction probability is computed. The method described in this letter can handle (unbounded) scLTL specifications and is applicable to nonlinear systems with an unbounded disturbance. For future work, we aim to reduce the computation time for the similarity quantification by using alternatives to obtain the matrices in Lemma 1.

Appendix
Implementation Details

Here we detail how to obtain matrices F_i and $K_{f,i}$ such that the implications in Lemma 1 are satisfied. To this end, we introduce Algorithm 1 to obtain bounded sets $\mathcal{K}_i$ and global matrix D that satisfies (9) and likely implies the existence of matrices F_i and $K_{f,i}$. The algorithm is based on using an optimistic preliminary estimate of matrix D, and sets P_i and $\mathcal{K}_i$ (steps 2, 3). Next, we update these estimates by solving the following optimization problem for a small number of partitions $N_i \leq N_P$ (steps 4, 5).

$$\min_{D_{inv},L_i,Q_i,r_i} \; r_i^2 \;\; \text{s.t.} \;\; D_{inv} \succ 0,$$

$$\begin{bmatrix} D_{inv} & D_{inv}C^T \\ CD_{inv} & I \end{bmatrix} \succeq 0, \forall i \in \{1, \ldots, N_i\}: \tag{21a}$$

$$\begin{bmatrix} \frac{1}{\epsilon^2} D_{inv} & L_i^T \\ L_i & r_i^2 I \end{bmatrix} \succeq 0, \begin{bmatrix} \frac{1}{\epsilon^2} D_{inv} & Q_i^T \\ Q_i & u_u^2 I \end{bmatrix} \succeq 0, \tag{21b}$$

$$\begin{bmatrix} \lambda D_{inv} & * & * \\ 0 & (1-\lambda)\epsilon^2 & * \\ A_i D_{inv} + B_i Q_i + B_{w,i} L_i & \psi_l & D_{inv} \end{bmatrix} \succeq 0 \tag{21c}$$

where $D_{inv} = D^{-1}$, $L_i = F_i D_{inv}$, $Q_i = K_{f,i} D_{inv}$ and $\psi_l \in vert(\mathcal{B} \bigoplus \mathcal{K}_i)$. This optimization problem is parameterized in $\lambda \in [0, 1]$ and constructed by following [16, Sec. 4]. Together with the given value of ϵ, we use matrix D to update sets P_i and $\mathcal{K}_i$ (step 6). Since we already obtained matrix D, we can compute matrices F_i and $K_{f,i}$ from Lemma 1 for all $i \in \{1, \ldots, N_P\}$ in parallel by formulating an optimization problem similar to (21) with constraints (21b)-(21c).

References

[1] B. C. van Huijgevoort and S. Haesaert, "Multi-layered simulation relations for linear stochastic systems," in *Proc. Eur. Control Conf. (ECC)*, 2021, pp. 728–733.
[2] A. Lavaei, S. Soudjani, and M. Zamani, "Compositional construction of infinite abstractions for networks of stochastic control systems," *Automatica*, vol. 107, pp. 125–137, Sep. 2019.
[3] A. Nejati, S. Soudjani, and M. Zamani, "Compositional construction of control barrier certificates for large-scale stochastic switched systems," *IEEE Control Syst. Lett.*, vol. 4, pp. 845–850, 2020.
[4] P. Jagtap, S. Soudjani, and M. Zamani, "Formal synthesis of stochastic systems via control barrier certificates," *IEEE Trans. Autom. Control*, vol. 66, no. 7, pp. 3097–3110, Jul. 2021.
[5] B. Zhong, A. Lavaei, M. Zamani, and M. Caccamo, "Automata-based controller synthesis for stochastic systems: A game framework via approximate probabilistic relations," 2021, *arXiv:2104.11803*.
[6] M. Dutreix, J. Huh, and S. Coogan, "Abstraction-based synthesis for stochastic systems with omega-regular objectives," *Nonlinear Anal. Hybrid Syst.*, vol. 45, Aug. 2022, Art. no. 101204.
[7] R. Majumdar, K. Mallik, and S. Soudjani, "Symbolic controller synthesis for Büchi specifications on stochastic systems," in *Proc. 23rd Conf. Hybrid Syst. Comput. Control*, 2020, pp. 1–11.
[8] R. Majumdar, K. Mallik, A.-K. Schmuck, and S. Soudjani, "Symbolic qualitative control for stochastic systems via finite parity games," *IFAC-PapersOnLine*, vol. 54, no. 5, pp. 127–132, 2021.
[9] M. Johansson, "Piecewise linear control systems," Ph.D. dissertation, Dept. Autom. Control, Lund Inst. Technol., Lund, Sweden, 1999.
[10] L. Rodrigues and J. P. How, "Synthesis of piecewise-affine controllers for stabilization of nonlinear systems," in *Proc. 42nd IEEE Conf. Decis. Control*, vol. 3, 2003, pp. 2071–2076.
[11] L. Rodrigues and S. Boyd, "Piecewise-affine state feedback for piecewise-affine slab systems using convex optimization," *Syst. Control Lett.*, vol. 54, no. 9, pp. 835–853, 2005.
[12] E. Asarin, T. Dang, and A. Girard, "Hybridization methods for the analysis of nonlinear systems," *Acta Informatica*, vol. 43, no. 7, pp. 451–476, 2007.
[13] C. Belta, B. Yordanov, and E. A. Gol, *Formal Methods for Discrete-Time Dynamical Systems*, vol. 89. Cham, Switzerland: Springer, 2017.
[14] S. Haesaert, S. Soudjani, and A. Abate, "Verification of general Markov decision processes by approximate similarity relations and policy refinement," *SIAM J. Control Optim.*, vol. 55, no. 4, pp. 2333–2367, 2017.
[15] S. Haesaert and S. Soudjani, "Robust dynamic programming for temporal logic control of stochastic systems," *IEEE Trans. Autom. Control*, vol. 66, no. 6, pp. 2496–2511, Jun. 2021.
[16] B. C. Van Huijgevoort and S. Haesaert, "Similarity quantification for linear stochastic systems: A coupling compensator approach," *Automatica*, vol. 144, Oct. 2022, Art. no. 110476.
[17] M. Anand, A. Lavaei, and M. Zamani, "Compositional synthesis of control barrier certificates for networks of stochastic systems against ω-regular specifications," 2021, *arXiv:2103.02226*.
[18] A. W. Knapp, "Chapter VI. Measure theory for euclidean space," in *Basic Real Analysis*. New York, NY, USA: Birkhäuser, 2016, pp. 334–394.
[19] O. Kupferman and M. Y. Vardi, "Model checking of safety properties," *Formal Methods Syst. Design*, vol. 19, no. 3, pp. 291–314, 2001.
[20] R. A. Adams and C. Essex, *Calculus: A Complete Course*, vol. 4, 7th ed. Toronto, ON, Canada: Pearson Addison Wesley, 2009.
[21] D. Lou, "Parameterized model order reduction with applications to thermal systems," Ph.D. dissertation, Dept. Electr. Eng., Technische Universiteit Eindhoven, Eindhoven, The Netherlands, Nov. 2021.
[22] E. W. Weisstein, "Taylor's inequality." 2014. [Online]. Available: https://mathworld.wolfram.com/
[23] A. Girard and G. J. Pappas, "Hierarchical control system design using approximate simulation," *Automatica*, vol. 45, no. 2, pp. 566–571, 2009.
[24] F. den Hollander, "Probability theory: The coupling method," Lecture Notes, Math. Inst., Leiden Univ., Leiden, The Netherlands, 2012.
[25] A. Abate et al., "ARCH-COMP21 category report: Stochastic models," in *Proc. 8th Int. Workshop Appl. Verification of Continuous Hybrid Syst. (ARCH21)*, 2021, pp. 55–89. [Online]. Available: https://easychair.org/publications/paper/GjCj