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Summary. We present an improvement on a previously proposed method for computing Gabor coefficients of characteristic functions with polygonal cross sections, based on a Taylor series expansion and Olver's algorithm. Several requirements are proposed to make the method more robust. Numerical evidence is given to show a convergent solution can be obtained based on a sufficiently high truncation number and working precision.

1 Introduction

In [1] a numerical method to compute Gabor coefficients for objects with polygonal cross sections was proposed. The key components of this method are: (1) a 1D-integral formulation derived from a double integral based on Gauss's theorem, (2) a Taylor series expansion of the complex error function, (3) derivation of a second-order inhomogeneous difference equation and (4) solution with Olver's algorithm. The main benefit of this method is that it transforms an integration problem into an evaluation problem, where the former can be computationally expensive when the complex error function is contained in the integrand. However, as observed in the second numerical experiment in [1], this method failed on some points.

We explain why this method failed on those points previously and we remedy the problem by introducing several requirements for this method. Furthermore, we show how this method can yield a convergent solution with these requirements.

2 Requirements of the Taylor-Olver method

Gabor coefficients for characteristic functions supported on a polygonal domain can be computed using the following fundamental integrals [1]:

$$I = \int_0^1 e^{-c_1x^2+c_2x}\mathrm{erf}(c_3x+c_4)dx, \tag{1}$$

where c_1, c_2, c_3, c_4 are given constants. Then we use a truncated Taylor series to approximate the complex error function:

$$\mathrm{erf}(z) \approx \frac{2}{\sqrt{\pi}}\sum_{n=0}^{N-1}\frac{(-1)^n z^{2n+1}}{(2n+1)n!}, \tag{2}$$

which thereafter yields an approximated integral $\tilde{I}$:

$$\tilde{I} = \frac{2e^{c_7}}{\sqrt{\pi}c_3}\sum_{m=0}^{2N-1} s_m I_m, \tag{3}$$

where

$$s_m = \begin{cases} \frac{(-1)^{\frac{m-1}{2}}}{m}, & m \text{ is odd},\\ 0, & m \text{ is even}, \end{cases}$$

$$I_m = \frac{1}{(\frac{m-1}{2})!}\int_{c_4}^{c_3+c_4} e^{-c_5y^2+c_6y}y^m dy,$$

and c_5, c_6, c_7 are given constants. Note that only the odd-indexed I_m contribute to the final result. By applying integration by parts, one obtains the following second-order difference equation:

$$I_{m-1} - b_m I_m - c_5 I_{m+1} = d_m, \forall m \geq 1 \tag{4}$$

where

$$b_m = -\frac{c_6(\frac{m-1}{2})!}{2(\frac{m}{2})!},$$

$$d_m = p(m, c_3+c_4) - p(m, c_4),$$

and $p(m,y) = \frac{1}{2(m/2)!}e^{-c_5y^2+c_6y}y^m$. The half-integer factorials in b_m are calculated with the Γ function. We use Olver's algorithm to solve this equation and assemble I_m together to get Gabor coefficients [1].

The following requirements emphasize three crucial points to obtain correct Gabor coefficients when using the proposed Taylor-Olver method.

Requirement 1: The truncation number N of the Taylor series in Eq. (2) must be sufficient. An accurate approximation of the partial sum in Eq. (2) to the complex error function $\mathrm{erf}(z)$ is a necessary condition to obtain an accurate approximated integral $\tilde{I}$ in Eq. (3). The truncation number N can be determined based on either desired accuracy [2] or numerical evidence.

Requirement 2: The working precision w, which indicates how many significant digits should be maintained in internal computations, must be high enough to guarantee the final accuracy. This is because:

- the integrals I_m can reach an extremely high value before vanishing eventually, e.g., the I_m in Fig. 3 of [1] reaches a level of 10^{360}, where the Gabor coefficient itself is a small number.
- the coefficient sequence s_m in (3) is alternating, therefore large cancellation errors occur if the working precision is not high enough.

- the truncated tridiagonal system is sensitive to d_m, which means the first term I_0 of the difference equation must be calculated with high accuracy.

Numerical evidence shows that a working precision $w = N$ yields stable results.

Requirement 3: Truncation number N' in Olver's algorithm should be large enough. Olver's algorithm transforms an semi-infinite matrix system, which is corresponding to the difference equation, into a truncated tridiagonal system. Olver provided a way to automatically determine the truncation number based on the desired accuracy [3]. Numerical evidence shows that a truncation number of the tridiagonal system $N' = 1.5N$ yields a stable result.

3 Numerical Results

To demonstrate the importance of above requirements, we recalculated one of the failed integrals in the second experiment in [1] for $c_1 = 3.14$, $c_2 = 49.3 - 5.1i$, $c_3 = -1.8$ and $c_4 = 15.4 - 14.5i$ in Eq. (1). One can observe the range of $\mathrm{erf}(c_3 x + c_4)$ for $x \in [0, 1]$ in Fig. 1. A large truncation number N is needed for the Taylor series to converge due to a relatively large distance from the origin, which therefore makes this I one of the most difficult ones in the triangle example to compute with the Taylor-Olver method.

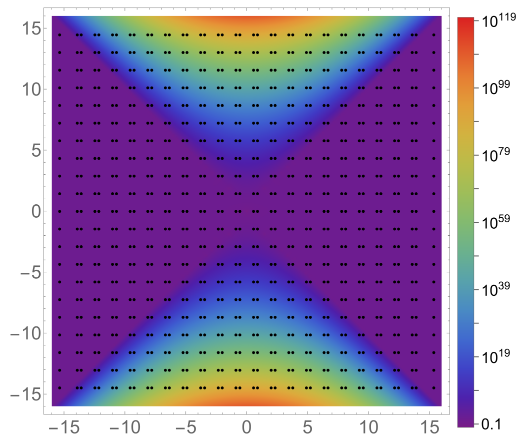

Fig. 1: Magnitude of $\mathrm{erf}(z)$ on a log scale. The black dots represent all c_4 occurring in Simulation 2 in [1]

Following above three requirements, we obtained the result $\tilde{I} = -1.87 \times 10^{27} + 1.72 \times 10^{26} i$. Compared with a high accuracy numerical reference, this solution has absolute error 1.52×10^{-95} and relative error 8.09×10^{-123}. The truncation number of the Taylor series is $N = 2800$, the working precision used is $w = 2800$, the dimension of the truncated tridiagonal system in Olver's algorithm is $N' = 4200$. Fig. 2 shows the computed integral sequence I_m from Eq. (3), compared with the numerical reference.

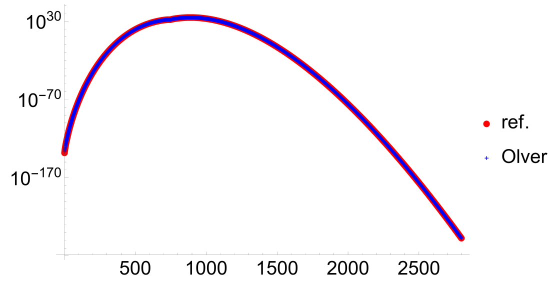

Fig. 2: Solution of Eq. (4) based on Olver's algorithm.

Fig. 3 shows a convergent solution obtained by increasing the truncation number N of the Taylor series, as along as the proposed requirements are satisfied. This result also implies that an insufficient truncation number of the Taylor series can be catastrophic.

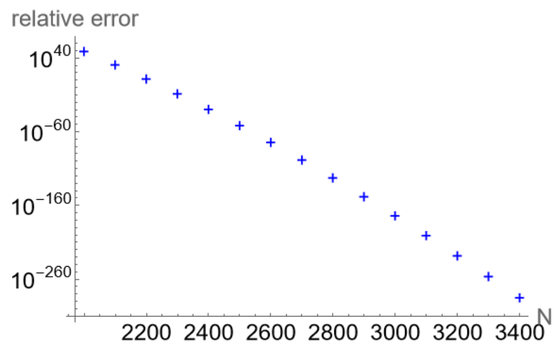

Fig. 3: Convergence obtained by increasing the truncation number of the Taylor series in Eq. (2).

Overall, we proposed three requirements to make the previous Taylor-Olver method more robust. In the future, optimization of the working precision should be considered to reduce the computation time.

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