Data-driven methods for distributed control of interconnected linear systems

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PROEFSCHRIFT

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Tom R. V. Steentjes

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Summary

Interconnected systems are omnipresent. The role of such systems in modern time becomes increasingly important, with examples including power networks, networks in systems biology, economic systems and chemical plant networks. From a control point-of-view, the ultimate goal is to influence an interconnected system such that it achieves a desired closed-loop performance or prescribed behavior. Models are typically not directly available for such systems, while data can be collected with increasing ease. Moreover, commonalities of interconnected systems, such as geographically distributed systems and a high dimension, impose restrictions on the controller synthesis and data collection. The challenge considered in this thesis is the use of data for the modeling and control of an interconnected system in a scalable manner with optimal or guaranteed control performance in some well-defined sense.

Non-centralized controllers, such as distributed controllers, yield the advantage of a non-classical communication pattern together with structured controller dynamics. We consider three main research problems in this thesis, connecting data of an interconnected system to distributed controller design. The first problem is to develop data-driven methods for modeling an interconnected system with the aim of distributed controller design. Then, the problem of synthesizing a distributed controller directly on the basis of data is considered, omitting the modeling of the underlying interconnected system. Finally, the problem of determining whether data are informative for distributed controller design with stability and performance guarantees is considered.

Provided the typical model-based nature of distributed controller synthesis methods, new modeling problems become apparent for identifying dynamical systems that are part of an interconnected system. These problems are related to local required model information, the distribution of identification with respect to an information pattern, and the orientation of identification towards a control-performance metric. In this thesis, it is shown how the data-driven modeling of linear interconnected systems in a closed-loop configuration can be performed by dynamic network identification. Further, results have been developed to perform

control-oriented identification of dynamic networks with respect to $\mathscr{H}_2$ performance criteria. Optimally, the control-oriented identification corresponds with a multi-input-multi-output prediction-error identification problem with external predictor inputs and internal predictor outputs. To allow for decentralized data collection and distributed estimation of model parameters, this thesis provides a solution to the distributed identification problem. For the corresponding distributed $\mathscr{H}_2$ controller synthesis problem, convex synthesis conditions have been developed, allowing for a scalable controller design.

Omitting the modeling step in the intertwined identification and control problem, leads to a data-driven philosophy for control, called direct data-driven control. This philosophy is particularly interesting for the control of interconnected systems, due to the complex model structures. In this thesis, we make the step from the well-established field of centralized direct data-driven control to distributed direct data-driven control. The development is based on the model-reference paradigm, utilizing a structured reference model that describes desired characteristics for the interconnected system. We introduce the notion of an ideal distributed controller that implements the structured reference model exactly. Two methods are provided for distributed direct data-driven controller synthesis via dynamic network identification, enabled by either tailor-made noise modeling or local controller identification in an auxiliary network.

Informativity is a fundamental property in data-driven control and identification of interconnected systems. The concept of informativity allows the use of data that is not necessarily persistently exciting of a sufficient order for identification, but is informative enough for, e.g., stabilization or performance-oriented controller synthesis. Conditions for informativity for linear systems with exact data and disturbed data with quadratic noise bounds have been studied in the literature. In this thesis we study informativity of data from interconnected systems for the synthesis of stabilizing distributed controllers and $\mathscr{H}_2/\mathscr{H}_\infty$ performance specifications. Alternative to data with quadratic noise bounds, we investigate informativity with a different prior knowledge on the noise, in the form of cross-covariance bounds. Comprehensive solutions are provided for determining informativity of data for (distributed) control and for the synthesis of distributed controllers from noisy data with guaranteed $\mathscr{H}_2$ or $\mathscr{H}_\infty$ performance.

Notation and symbols

$x^\top$	transpose of vector x
I	identity matrix
$\mathbf{1}$	column vector of all ones
0	zero, zero vector, zero matrix
A^{-1}	inverse of matrix A
$A^\top$	transpose of matrix A
$\operatorname{rank}(A)$	rank of matrix A
$\operatorname{trace}(A)$	trace of matrix A
$\operatorname{im} A$	image of matrix A
$\ker A$	kernel of matrix A
$A_\perp$	basis matrix of $\ker A$
$\operatorname{in}^- A$	number of negative eigenvalues of real symmetric matrix A
$\operatorname{in}^0 A$	number of zero eigenvalues of real symmetric matrix A
$\operatorname{in}^+ A$	number of positive eigenvalues of real symmetric matrix A
$\operatorname{in} A$	inertia of real symmetric matrix A, i.e., $(\operatorname{in}^- A, \operatorname{in}^0 A, \operatorname{in}^+ A)$
$\operatorname{diag}(A_1, \ldots, A_m)$	block-diagonal matrix with matrices $A_1, \ldots, A_m$ on its diagonal
$\operatorname{col}(A_1, \ldots, A_m)$	matrix that vertically stacks matrices $A_1, \ldots, A_m$
$\operatorname{row}(A_1, \ldots, A_m)$	matrix that horizontally stacks matrices $A_1, \ldots, A_m$
$\operatorname{diag}_{a \in \mathcal{A}} A_a$	block-diagonal matrix with matrices A_a, $a \in \mathcal{A}$, on its diagonal
$\operatorname{col}_{a \in \mathcal{A}} A_a$	matrix that vertically stacks matrices A_a, $a \in \mathcal{A}$
$\operatorname{row}_{a \in \mathcal{A}} A_a$	matrix that horizontally stacks matrices A_a, $a \in \mathcal{A}$

A_{ij}	the (i,j)-th entry of matrix A
$A \succ 0$	matrix A is positive definite
$A \succeq 0$	matrix A is positive semi-definite
$A \prec 0$	matrix $-A$ is positive definite
$A \preceq 0$	matrix $-A$ is positive semi-definite
$\mathbb{R}$	the set of real numbers
$\mathbb{R}_{\geq 0}$	the set of non-negative real numbers
$\mathbb{R}_{>0}$	the set of positive real numbers
$\mathbb{R}^n$	set of vectors with n real entries
$\mathbb{R}^{n\times m}$	set of $n \times m$ matrices with real entries
$\mathbb{S}^{n\times n}$	set of $n \times n$ symmetric matrices with real entries
$\mathbb{Z}$	the set of integer numbers
$\mathbb{N}$	the set of non-negative integer numbers
$\mathbb{N}_{>0}$	the set of positive integer numbers
$\mathbb{Z}_{[a:b]}$	the set $\mathbb{Z} \cap [a,b]$ for $a < b$, $a \in \mathbb{Z}$, $b \in \mathbb{Z}$
$B \setminus A$	the relative complement of set A with respect to set B
$\lvert A \rvert$	the cardinality of set A
$\lVert x \rVert$	Euclidean norm of a a vector $x \in \mathbb{R}^n$
ℓ_2^m	set of all Lebesgue measurable functions $d : \mathbb{N} \to \mathbb{R}^m$
$\mathcal{L}_2^m$	set of all Lebesgue measurable functions $d : \mathbb{R} \to \mathbb{R}^m$
$\lVert d \rVert_{\ell_2}$	ℓ_2 norm of $d \in \ell_2^m$
$\lVert d \rVert_{\mathcal{L}_2}$	$\mathcal{L}_2$ norm of $d \in \mathcal{L}_2^m$
$\lVert \Sigma \rVert_{\mathscr{H}_2}$	$\mathscr{H}_2$ norm of asymptotically stable system Σ
$\lVert \Sigma \rVert_{\mathscr{H}_\infty}$	$\mathscr{H}_\infty$ norm of asymptotically stable system Σ
$\deg(F)$	Degree of a polynomial function F
$\Delta \deg(F)$	Relative degree of a rational function F

Contents

Chapter 1

Introduction

1.1 Interconnected systems – taking control

Think about any device around you that is electrical or mechanical. Likely, this device consists of several interconnected components or is interconnected with one or multiple other devices. In the current era of technology, it is hard to imagine a world without complex technological systems that enhance our society. Be it the power generators feeding the electricity grid, irrigation systems that serve the demand for water in growing crops, or our mobile phone that allows us to connect to any other device connected to the internet. Even the room in our office is a system that is interconnected to other systems: walls of adjacent rooms transfer energy according to the second law of thermodynamics.

When we mention a *system*, we have to describe what we mean by this notion. Various (formal) definitions of a system can be given. On a high level, a system is an object to which variables are associated, room temperatures for example, that influence each other, i.e., variables that interact. Some of these variables can be observable, while other variables can be influenced. These variables may be called respectively outputs and inputs. When a variable, not necessarily an input or output, is associated with multiple systems, an interaction between the systems occurs, naturally. We say that the systems are *interconnected*. At this point, we do not formalize the notion of interconnected systems further. Instead, we discuss a couple of examples of such systems.

Example 1.1 (**Power networks**). *A power network consists of systems that consume and/or produce electrical energy and are interconnected through power transmission lines. In a classical power network, electrical energy is produced in central power stations, such as coal-fired power stations, nuclear power plants or hydro-electric power stations. Energy is consumed by the private and industrial* *sectors that have loads connected to the power grid. In this way the classical power grid can be seen as one-way pipeline, where the source (power station) has no real-time information about the termination points (consumers) (Farhangi, 2010). The classical power network is undergoing a rapid transformation. This transformation is partially induced by the integration of renewable energy sources, leading to distributed power consumption, storage,* and *generation by small-scale resources (Farhangi, 2010) (Figure 1.1).*

The transition from a classical power network to a power grid with distributed power generation, storage and consumption (smart grid), forms a technological challenge and offers opportunities at the same time. Indeed, the uni-directional nature, over-engineered properties and low efficiency of the existing power grid can be improved (Farhangi, 2010). These changes, however, can affect the synchronous stability of the power network, i.e., the ability to recover a synchronous frequency in the event of a disturbance (Tegling, 2018). The frequency is of main importance in the power network. It is a variable that is associated with each system in the power grid and has to be close to a nominal value, e.g. 50 *Hz, for safe and reliable operation of the grid. A change in frequency at one node propagates through a power network, as the nodes are (indirectly) coupled through their frequency and corresponding phase.*

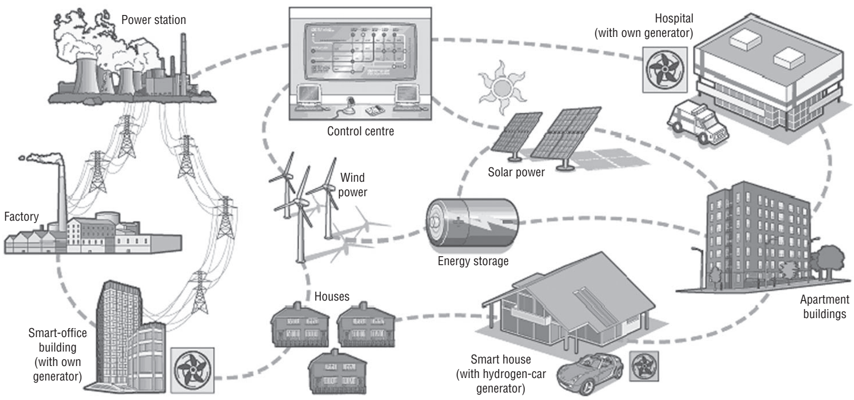

Figure 1.1: A smart grid is an electrical network that consists of renewable energy sources, traditional power plants, energy storage and users that consume and/or produce energy. Image adapted from (Kiesling, 2010).

Example 1.2 (Irrigation networks). *Withdrawal of water from surface and groundwater resources for irrigation currently accounts for* 70% *of all global water withdrawals (Nations, 2021). Efficient use of water in irrigation is therefore of* *paramount importance. An irrigation network serves the transportation of water from reservoirs to farms through an infrastructure of open-water channels, illustrated in Figure 1.2. In large-scale irrigation networks, the distribution of water in an irrigation network is typically performed under gravity (Cantoni et al., 2007). The flow of water in the network is controlled by gates, for example overshot gates (depicted in schematic form in Figure 1.3). An open-water channel in the network can be interpreted as a series of water pools that are linked through the control gates.*

Water losses in irrigation networks occur due to seepage and evaporation, but are mainly due to oversupply, resulting in spillage along, and at the end of, open water channels (Cantoni et al., 2007). Efficient operation of the network therefore requires that water levels along a channel remain close to reference set points, depending on the demand. The actuation of an upstream gate does not only influence the water level of a downstream pool. Indeed, also the water level of the upstream pool is influenced, leading to an interaction of the pools (systems) in the network.

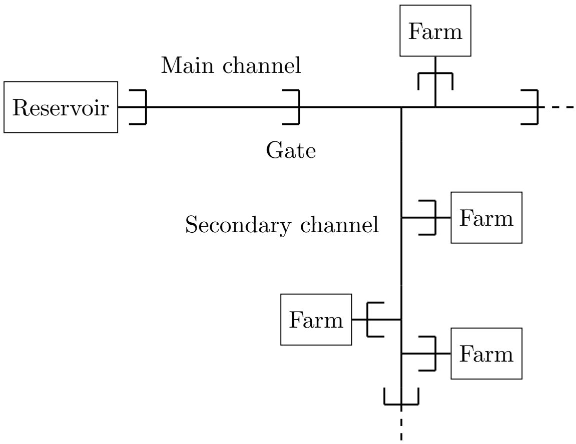

Figure 1.2: Top view of an irrigation network, illustrating the distribution of water through a network of channels and gates.

The urge to *understand* and *influence* systems that are around us, either natural or synthetic, has been part of human society for as long as we are aware of. This has brought laws of physics, understanding of the human body and the development of man-made systems currently running our society. The verb *understand* is intrinsically coupled with the verb *model*. In everyday life, we model the world around us in order to interact with it; we implicitly create a model of a conversation partner to predict his/her response, we implicitly create a model of our keyboard to hit the right key when we type and we model our surroundings when we cycle to avoid the collision with a tree.

Modeling systems explicitly requires a systematic approach. For example, modeling dynamical systems by a graphical model, e.g., a typical time trajectory or frequency function, is widely accepted for design purposes (Ljung, 1999). For systems that are too complex to be modeled graphically, a mathematical model may be necessary. Mathematical models describe relations among system variables by mathematical expressions. For models of dynamical systems, these mathematical models typically consist of a set of difference or differential equations. Paramount mathematical models in the systems and control field are polynomial models (Ljung, 1999), (Hannan and Deistler, 1987), transfer function models (Ljung, 1999), state-space models (Franklin et al., 2020) and behavioral models (Polderman and Willems, 1998), each of which has its own advantages and applications.

Modeling an interconnected system is a special and important case of mathematical modeling. Besides a set of mathematical expressions that model the dynamics, a graphical representation models the *structure* of the interconnected system (not to be confused with a graphical model, described in the previous paragraph). A graph consists of vertices and edges. Depending on the representation, vertices can represent variables and edges can represent dynamical relations, or vice versa. For linear interconnected systems, one such model is a *(module) dynamic network model*, defined as an interconnection of signals, represented by vertices, which are coupled through transfer functions (modules), represented by edges (Van den Hof et al., 2013). Another type of interconnected system model is given by the interconnection of (linear) state-space models, represented by vertices, and interconnection variables, represented by edges (Langbort and D'Andrea, 2003). Both representations will be used in the sequel. Since the latter can be equivalently represented by a dynamic network, and vice versa, let us now exemplify the modeling of an interconnected systems by a dynamic network model.

Example 1.3 (**Irrigation network (continued)**). *Let us consider an example of and irrigation network with three pools connected in series, as depicted in Figure 1.3. Each pool has a water level* w_i*,* $i = 1, 2, 3$*. The gate that is upstream for pool i yields a flow which, after a change of variables, results in a measure of the flow over gate i,* u_i *say. For the second pool, the flow over gate 2 influences both* w_2 *and the water level of the upstream pool,* w_1*. A first-order model for the continuous time dynamics for the water level in the first pool is (Cantoni et al.,* *2007)*

$$\alpha_1 \frac{\mathrm{d}}{\mathrm{d}t} w_1(t) = u_1(t - \tau_1) - u_2(t) + v_1(t), \quad t \in \mathbb{R},$$

where α_1 is a measure of the pool surface area, τ_1 is a delay associated with the time before the water reaches the point where the water level is measured and v_1 is a disturbance that represents, e.g., seepage and evaporation. A dynamic network model for the three pools is illustrated in Figure 1.4, where the interaction between w_1, u_1 and u_2 is represented by the blue modules (transfer functions) G_1 and G_{12}. Following a similar derivation for the other modules leads to all links represented by the blue modules.

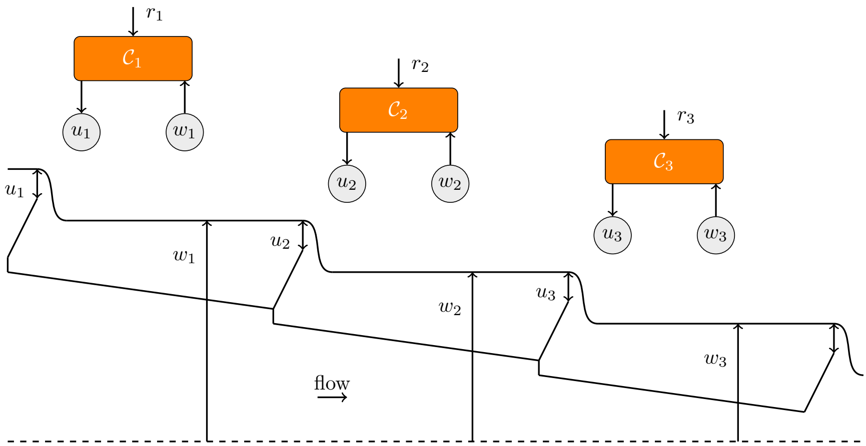

Figure 1.3: A controlled irrigation channel with decentralized controllers.

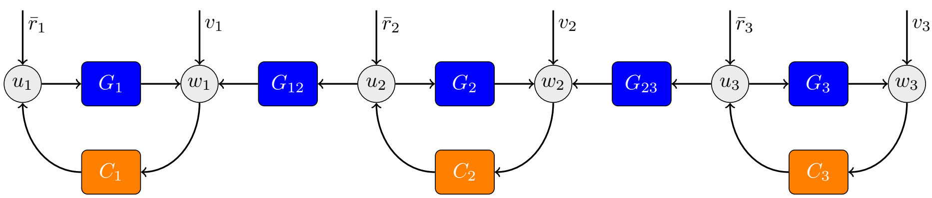

Figure 1.4: A controlled irrigation channel from a dynamic-network perspective.

Example 1.4 (**Power networks (continued)**). *The synchronization of frequencies in a power network is commonly studied through a model that consists* *of coupled swing equations, describing the behavior of the phase angle* θ_i *and frequency* $\frac{\mathrm{d}}{\mathrm{d}t}\theta_i = \omega_i$ *of* N *synchronous generators. Under simplifying assumptions, the linearized swing equation for generator* i *can be written as (Dörfler et al., 2013), cf. (Tegling and Sandberg, 2017)*

$$m_i \frac{\mathrm{d}^2}{\mathrm{d}t^2}\theta_i(t) + b_i \frac{\mathrm{d}}{\mathrm{d}t}\theta_i(t) = u_i - \sum_{j \in \mathcal{N}_i} k_{ij}(\theta_i(t) - \theta_j(t)), \quad t \in \mathbb{R},$$

where m_i *and* b_i *are inertia and damping coefficients, respectively. The coupling is defined by coefficients* k_{ij}*, denoting the susceptance of the transmission line between node* i *and a node* j *in* $\mathcal{N}_i$*, the set of neighbours, and* u_i *denotes the mechanical power supplied to/withdrawn from the network. This power network model has a mechanical analogue describing the motion of masses interconnected by springs that are moving along a circle (Dörfler et al., 2013), as depicted in Figure 1.5 on the left. A dynamic network model for the power network, where the interaction between* θ_1*,* θ_2 *and* θ_3 *is represented by the modules (transfer functions)* G_{ij}*, is shown in Figure 1.5 on the right (*$\bar{u}_i$ *is a filtered version of* u_i*).*

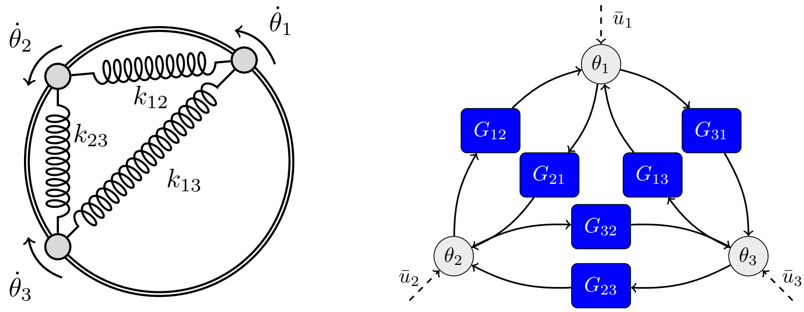

Figure 1.5: A mechanical analogue of the coupled swing equations (left) and a power network from a dynamic-network perspective (right).

By *influencing* a system, we change its behavior. To influence a system, additional dynamical laws are imposed on the variables; dynamics are added. From this point of view, adding a controller to a system is nothing but attaching another system, i.e., adding additional relationships between variables. A common and important type of controller is a *centralized* controller; a controller with unstructured/lumped dynamics. Such a control architecture can be perfectly suitable for systems that are ‘simple’, in the sense that there are, for example, a manageable number of system variables or that the systems are not geographically distributed. However, for more complex systems this architecture can lead to an impractical communication architecture, or even infeasibility. Structured controllers can overcome this culprit.

Consider for example the irrigation channel in Figure 1.3. We can add three systems $\mathcal{C}_i$, $i = 1, 2, 3$, to the interconnected system (cf. (Cantoni et al., 2007)), each of which opens or closes gate i, depending if the water level w_i is too low or too high, respectively. Albeit in rudimentary form, this controller is a structured controller with a structure that is *decentralized*. If gate i is not only actuated based on w_i, but also on w_{i+1}, additional (dynamical) relationships have to be imposed: the controllers are interconnected. This leads to a *distributed* controller. Centralized, distributed and decentralized controller architectures are exemplified in Figure 1.7. In the design of a distributed and decentralized controller, a controller is designed for meeting a given control objective while also considering the structure of the interconnected system. This is the essence in control of interconnected systems.

1.2 Challenges in control of interconnected systems

In order to design a controller for an interconnected system that meets desired control objectives, knowledge of the system is crucial. This knowledge can consist of the structure of the interconnected system, the dynamical relationships between variables, or both. With the increasing complexity of interconnected systems, such as power networks, the corresponding models become increasingly complex.

While one trend in the technological development of interconnected systems is the increase in complexity, another major trend is the increase in ease of access to information. Sensors are becoming less expensive and continuously improve in accuracy. With the classical power network having a few sensors, a smart grid contains sensors throughout (Farhangi, 2010). Clever use of the information provided by these sensors for obtaining the required knowledge of an interconnected system is a huge challenge that has lead to research problems in, for example, system identification, machine learning, systems biology, econometrics and control.

The main challenge in the control of interconnected systems is to deal with the complexity of the system and to determine how data can be used for the controller design. If a model of the interconnected system is not available, which data, e.g. sensor measurements, are required to obtain the required model information? If the structure of the interconnected system is not (completely) known, can it be deduced from the data? Is a complete model of the interconnected system actually necessary for the design of a decentralized/distributed controller?

To illustrate fundamental challenges in the use of data for control of interconnected systems, let us consider a simple version of the dynamic network discussed in Example 1.3 shown in Figure 1.6. Contrary to the ‘downstream’ irrigation network, an additional link may be present between u_1 and w_2. The decentralized control architecture in this scheme was originally considered in (Gudi and Rawlings, 2006) with $G_{12} = 0$, cf. (Van den Hof et al., 2018). Suppose the interaction dynamics G_{21} (depicted in green) have to be derived from data. Which signals in the scheme have to be measured? Do the controller dynamics need to be known? What conditions on the data have to be satisfied to obtain a meaningful model? These questions are all indirectly relevant to the controller design, since any modeling error can propagate in the design process and reduce the achieved performance of the synthesized controller.

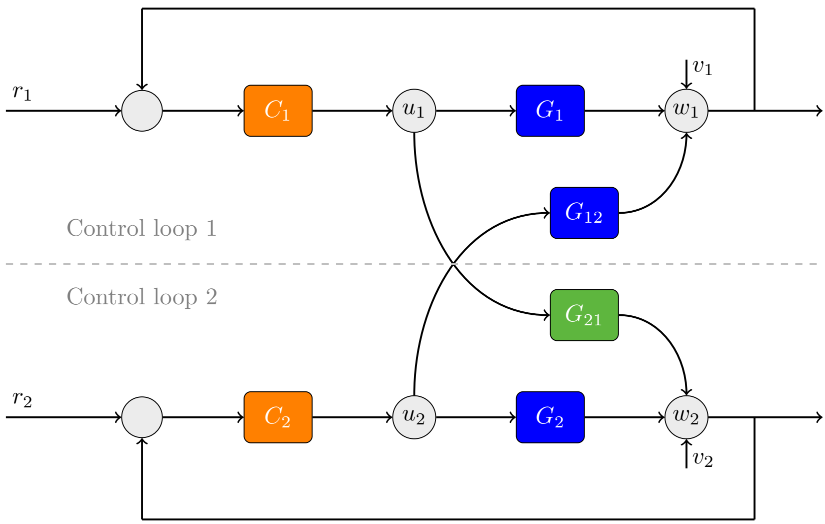

Figure 1.6: Two interconnected control loops.

Practically speaking, modeling errors are unavoidable. Data can be corrupted due to noise in the measurements, a limited number of data is available or the model is not ‘rich’ enough to capture all the dynamics of the system. Such a practical situation poses additional challenges in the design of a performance-oriented decentralized/distributed controller. Indeed, if a consistent estimate cannot be obtained from data, then what is the ‘best’ model for controller design? This problem has been considered in the field of identification for control (Van den Hof, 1998) for ‘isolated’ systems, cf. (Van den Hof and Schrama, 1995), (Gevers, 2005), (Hjalmarsson, 2005). The identification of interaction dynamics, and, generally speaking, parts of interconnected systems for distributed control is a challenge that is yet to be addressed.

In safety-critical applications, an interconnected-system model that is 'oriented' towards the performance of a distributed controller may not suffice. Given uncertainty in the available data, how can one design a distributed controller that is guaranteed to meet the control objective? What kind of uncertainties can be dealt with and can additional/instrumental variables aid in the controller design?

So far, we have posed challenges on *how* to use data structured controller design for interconnected systems. A feature of interconnected systems is that they are typically spatially distributed and measurements are not performed in a central location. This leads to a non-classical information pattern for data-driven modeling and control. The non-classical information pattern can be illustrated by the example of the two interconnected control loops in Figure 1.6. Suppose there are two operators: the first operator has access to data $\{r_1, u_1, w_1\}$ and the second operator has access to data $\{r_2, u_2, w_2\}$. Given a control objective, how can the controllers C_1 and C_2 be designed on the basis of data given this information pattern? If the modeling of G_{21}, for example, is part of the control design, then what information must be shared between the operators to perform this modeling based on data? These questions are non-trivial. Hence, non-classical information patterns do not only lead to challenges for distributed controller design, but also for the underlying identification problem.

A model of an interconnected system captures the dynamics and structure that is required for the design of a controller. In this respect, the model can be regarded as a tool in the design of a distributed controller from data; an intermediate step in the design procedure from data. Are there situations where a structured controller can be constructed directly on the basis of data? This problem has been considered in the field of data-driven control for 'isolated' systems (Bazanella et al., 2012). Interconnected systems impose additional challenges to this problem, due to the structure of the interconnected system, controller and the available information pattern.

1.3 State of the art

In this section, we provide an overview of relevant literature and state-of-the-art methods for decentralized control, distributed control, distributed identification and estimation, data-driven control and informativity of data for control. Decentralized and distributed control can refer to the control of multi-agent systems or interconnected systems. In both cases, the control of multiple systems is considered. Interconnected systems are coupled through the principle of sharing variables, while multi-agent systems are typically decoupled, but interconnected by design to solve cooperative control problems. In what follows, we will mainly discuss literature related to interconnected systems.

1.3.1 Decentralized control

As opposed to centralized control of large-scale systems, *decentralized control* uses only locally available control variables. For feedback control, this means that only local measurements are performed for determining local corrective actions. Figure 1.7 exemplifies this concept. The feature of having non-identical measurement data available at controllers was first referred to as a *non-classical information pattern* in (Witsenhausen, 1968). This feature makes the decentralized controller easily implementable, but can lead to serious issues related to performance or even instability of the system (Aoki, 1972). Necessary and sufficient conditions for the existence of stabilizing local dynamic output feedback controllers were derived in (Wang and Davison, 1973). The conditions were stated in terms of the first-introduced notion of *fixed modes* of the decentralized control system. These fixed modes appeared to be a natural generalization of uncontrollable/unobservable modes of centralized control systems (Wang and Davison, 1973). Sandell et al. (1978) already concluded considerable progress in the early days of decentralized control, but observed that what lacked in the literature at the time was the solution to a desirable control structure and information distribution. Broad self-contained overviews of results and methods for decentralized control of large-scale systems are provided in (Sandell et al., 1978), (Siljak, 1991) and (Lunze, 1992). Noting that the design of decentralized controllers for the general case was still an open problem, the authors of (Scorletti and Duc, 2001) developed sufficient conditions for the existence of decentralized $\mathscr{H}_\infty$ controllers, introduced in (Scorletti and Duc, 1997). The conditions are derived for yielding dissipative properties of the subsystems and together with the performance objective, these yield a convex optimization problem with linear matrix inequality constraints. In (Rotkowitz and Lall, 2002), the notion of *quadratic invariance*, a property of information constraints with respect to the system, was introduced. This property can be interpreted as an algebraic condition that relates the controller structure and the system. Quadratic invariance was shown to be a sufficient condition for optimal decentralized control problems to be convex, for any norm of interest of the closed-loop system (Rotkowitz and Lall, 2006). It was shown in (Lessard and Lall, 2010) that convex optimization problems for decentralized control can be obtained for a broader class of information constraints with respect to the plant. This broader class is characterized by a property called *internal quadratic invariance*: a generalization of quadratic invariance. A generalization of decentralized control to the behavioral framework was considered in (Ishido and Takaba, 2007) and (Fiaz and Trentelman, 2010). Control in the behavioral framework is viewed as the restriction of a system's behavior via intersection with another behavior: the controller. A key problem in behavioral control is the characterization of all behaviors that can be achieved for the closed-loop system by interconnection with a controller. This problem is referred to as *implementability* and was extensively studied in (Belur and Trentelman, 2002). The natural analogue of implementability for decentralized control was considered in (Fiaz and Trentelman, 2010), called decentralized implementability. The corresponding problem is the characterization of all behaviors that can be achieved via decentralized control.

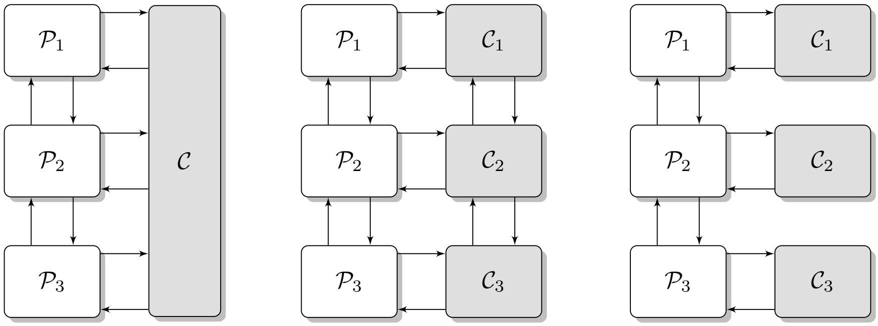

Figure 1.7: Example of a controlled system interconnection with centralized control (left), distributed control (middle) and decentralized control (right).

1.3.2 Distributed control

One can also think of a control scheme with non-classical information patterns, where local controllers can be interconnected. This concept is referred to in the literature as *distributed control*. While decentralized control was a natural way to address the non-classical information pattern at the time, distributed control can omit limitations induced by decentralized control (Langbort et al., 2004). These limitations include requirements on the information constraint, such as quadratic invariance (Rotkowitz and Lall, 2006), or the restriction to static feedback controllers as in e.g. (Scorletti and Duc, 2001). Sufficient conditions for the existence of controllers that are interconnected in the same way as the subsystems, i.e., plant and controller admit the same structure, while achieving $\mathscr{H}_\infty$ performance, are developed in (Langbort et al., 2004). The latter work originates from (D'Andrea and Dullerud, 2003), where a distributed control design for a class of interconnected identical systems was formulated. The control design of (Langbort et al., 2004) involves solving a linear matrix inequality. Even for a moderate number of subsystems and interconnection variables, this linear matrix inequality can be of large size (Langbort et al., 2004). Investigation of the structure of this inequality, however, led to ideas to distribute not only the controller itself, but also the *computation* of the controller (Langbort et al., 2004). A distributed algorithm for distributed control design via the method of alternating projections was proposed in (Langbort and D'Andrea, 2003), wherein design variables are shared via the same structure as the plant and controller have. The stabilization of non-linear discrete-time interconnected systems was considered in (Jokic and Lazar, 2009), for a given information pattern of the controller. The computation of optimal control actions therein is enabled by solving a convex optimization problem at each time step. In (Van Horssen and Weiland, 2016), a discrete-time analogue of the work in (Langbort et al., 2004) was presented. Additionally, synthesis of the distributed controller in (Van Horssen and Weiland, 2016) incorporates robust stability and robust $\mathscr{H}_\infty$ performance of the closed-loop system. A scalable $\mathscr{H}_\infty$ controller design for interconnected systems is presented in (Stürz et al., 2018). This design is shown to decompose into small-scale problems, given structural assumptions on the design variables. The synthesis problem is, however, not convex, but can be transformed into a bilinear matrix inequality (Stürz et al., 2018). Another distributed control strategy for interconnected systems is distributed model predictive control (MPC). Early work on this topic includes (Jia and Krogh, 2001) and (Camponogara et al., 2002), wherein a distributed MPC algorithm was proposed with an exchange of local state predictions between controllers. Stability constraints on one-step future predicted states guarantee asymptotic stability of the closed-loop interconnected system. A distributed MPC framework with stability and guaranteed feasibility is provided in (Venkat et al., 2007). In (Christofides et al., 2013), a review of results for distributed MPC is provided, including advantages and disadvantages of the various schemes. Since information exchange between controllers typically takes place over a digital network, the distributed MPC schemes are bound to phenomena as packet loss and delays. A classification of distributed MPC algorithms with respect to assumptions on the communication network is provided in (Grüne et al., 2014).

1.3.3 Distributed identification and estimation

Despite the extensive amount of developments in the field of control for interconnected systems, methods for obtaining the underlying system models are rare (Hansson and Verhaegen, 2014). As for data-driven estimation of individual dynamical systems, prediction-error identification methods provide well-established procedures for obtaining consistent system parameter estimates (Ljung, 1999). The focus on system identification of individual dynamical systems is, however, limited, for interconnected systems. Given a structure of a network of linear dynamical systems, various prediction error methods are readily operational for identifying these systems, while imposing certain conditions on the network (Van den Hof et al., 2013). The identification problem of such large-scale systems can typically be separated into multiple-input-single-output (MISO) identification problems (Rao et al., 1984), (Van den Hof et al., 2013). More precisely, consistent identification of a large-scale system can be performed via the identification of MISO building blocks, on the basis of measurements of multiple inputs and one, possibly disturbed, output, under the assumption that disturbances in the network are uncorrelated. Correlation of disturbances in a network can lead to confounding variables, which have to be taken into account in the identification problem. Confounding variables can be addressed by including additional inputs (Dankers et al., 2016), (Dankers et al., 2017) or outputs (Ramaswamy and Van den Hof, 2021) in the predictor model.

Although existing prediction error methods for dynamical networks can consistently identify local modules (single-input-single-output (SISO) systems), they require the output signal and multiple input signals for a MISO identification problem to be available centrally for global parameter estimation. Central data collection and computation of the module estimates may not always be desirable due to computational constraints or desired flexibility. A decomposition of the MISO identification problem into SISO identification problems to reduce computational complexity was suggested in (Rao et al., 1984). Therein, it was proposed to perform a decomposition of the parameter estimation via a Gauss-Seidel like algorithm, but a proof of convergence is absent.

Distributed estimation approaches can be divided into two distinct classes. The first class consists of consensus based methods, discerned by collaborative estimation of a *global* (common) parameter vector that is performed via a number of interconnected estimators. The estimation of a global parameter in (wireless) sensor networks was considered in (Schizas et al., 2008), (Cattivelli et al., 2008) and (Mateos et al., 2009). Therein, communication is employed between estimators to consent on a global estimate. More recent results include a performance analysis and reduced complexity of the algorithm from (Mateos et al., 2009) in (Mateos and Giannakis, 2012). In (Breschi, 2017), methods for cloud-aided estimation are presented for handling coupling constraints via the alternating direction method of multipliers (ADMM). The problem is discerned by a separable cost function, whereas the overlap of agent's parameters induces coupling constraints (consensus constraints). A similar (partial) consensus problem was considered in (Bäumelt, 2016) for a thermal modeling problem in buildings, where consensus constraints occur due to parameters such as a mutual wall conductance between building zones. Parameter consensus was utilized to relax persistence of excitation conditions in system identification via an ensemble of identical systems in (Papusha et al., 2014).

The second class of distributed estimation is also enabled by collaborative estimation via interconnected estimators. Therein each estimator is, however, concerned with the estimation of a *local* parameter vector. Results for parameter estimation in static interconnected systems were derived in (Marelli and Fu, 2015). For distributed state estimation, we single out moving-horizon methods (Farina et al., 2010), local plug-and-play estimators (Riverso et al., 2013) and Kalman filtering (Marelli et al., 2018) for networks of linear systems. In (Hansson and Verhaegen, 2014), the problem of distributed system identification for interconnected systems was considered. A distributed implementation solution via the use of distributed ADMM optimization was proposed, which may lead to local optimal solutions, however.

1.3.4 Data-driven control and informativity for control

Methods for the design of controllers on the basis of data can be divided into two classes: (i) indirect data-driven control and (ii) direct data-driven control. Indirect data-driven control is model based: first a plant model is estimated on the basis of data and consecutively a controller design is performed on the basis of the plant model. The problem of identying a model for control that leads to the best control performance is considered in the field of identification for control (Van den Hof and Schrama, 1995), cf. (Van den Hof, 1998), (Gevers, 2005), (Hjalmarsson, 2005). In direct data-driven control, the plant modeling step is omitted; a controller is synthesized directly from data. Typical advantages of direct-data driven controller design are that no loss of data can occur due to under-modeling of the plant and the order of the controller can be fixed. An exhaustive survey on data-driven control methods, both direct and indirect, is provided in (Hou and Wang, 2013). Virtual reference feedback tuning (VRFT) (Campi et al., 2002), (Bazanella et al., 2012) is a 'one-shot' method for designing a controller directly on the basis of data. In this method, a model-reference control problem is essentially reformulated into a system identification problem, through the generation of a virtual closed-loop system that is compatible with the data. Iterative feedback tuning (IFT) (Hjalmarsson et al., 1998), (Hjalmarsson, 2002) shares with VRFT the property of being direct, i.e., no model is identified in the procedure. A distinguishing feature of the IFT algorithm is, however, that it is iterative by design, where a gradient estimation of the control criterion is performed at each iteration to tune the controller (Hjalmarsson et al., 1998). Optimal controller identification (OCI) (Campestrini et al., 2017) also solves a model-reference control problem, by embedding the control design problem in the prediction-error identification of an optimal controller. This method was extended in (Huff et al., 2019), for the identification of controllers for multi-input-multi-output systems. Other state-of-the-art methods for direct data-driven control are correlation-based tuning (CbT) (van Heusden et al., 2011), asymptotically exact (Formentin et al., 2015) and moment-matching (Breschi et al., 2019) controller tuning.

A recent trend in data-based system analysis and control originates from Willems' fundamental lemma (Willems et al., 2005). Applications include data-based predictive control (Coulson et al., 2019), (Allibhoy and Cortés, 2021), the data-based parametrization of stabilizing controllers (De Persis and Tesi, 2020) and robust data-based state-feedback design with noisy data (Berberich et al., 2020). The data-based verification of dissipativity properties was considered in (Koch et al., 2020a), (Koch et al., 2020b), which allows to determine system measures such as the $\mathscr{H}_\infty$ norm or passivity properties from data corrupted by a noise signal satisfying quadratic bounds. A similar noise description was considered in (De Persis and Tesi, 2020) and (van Waarde et al., 2022), of which the latter extends the data-based controller design results in (van Waarde et al., 2020) to the noisy case via a matrix S-lemma. The data-based conditions in (van Waarde et al., 2022) are necessary and sufficient for stabilizing state feedback synthesis, including $\mathscr{H}_2$ or $\mathscr{H}_\infty$ performance specifications. Trade-offs in using quadratic bounds, such as energy bounds, or instantaneous noise bounds for determining controllers from data are discussed in (Bisoffi et al., 2021a). Data-driven results for analysis and controller synthesis via Petersen's lemma and Finsler's lemma have been developed in (Bisoffi et al., 2021b) and (van Waarde and Camlibel, 2021), respectively. These results bring alternative data-driven conditions with respect to the matrix S-lemma, where the Finsler's lemma in (van Waarde and Camlibel, 2021) leads to conditions for both exact and noisy data.

1.4 Problem statement

We have observed that design of controllers from data has been extensively studied in the literature. However, most of the theory that has been developed, applies only to isolated systems or small-scale interconnected systems where the interconnection structure is not taken into account. From a control point-of-view, methods are available for the synthesis of a structured controller, provided that a model of the target interconnected system is available, which is not realistic in general. For interconnected systems, the synergy between control and identification is a field that is relatively unexplored. With data that is obtained according to a non-classical information pattern, a controller that is an interconnected system itself, and a global control objective that is represented by stability or a performance criterion such as an $\mathscr{H}_2/\mathscr{H}_\infty$ norm, we formulate the following research question.

Under which conditions can data from an interconnected system be used for the design of a distributed controller, with the objective of achieving optimal and/or guaranteed control performance?

This research question contains several aspects. Let us consider these aspects separately now by taking the state-of-the-art into account, in order to formulate sub-questions from the research question.

- *Use data from an interconnected system for the design of a distributed controller:* As discussed in Section 1.3.4, control design based on data can be separated into two categories on a high level: indirect data-driven control and direct data-driven control. We will consider indirect and direct distributed data-driven control for interconnected systems in the research. For indirect design, we therefore consider the data-driven modeling of dynamic networks and the model-based synthesis of distributed controllers. For direct design, we consider the design of distributed controllers based directly on data, which has been only considered for centralized controller design in the state-of-the-art, as discussed in Section 1.3.4.

- *Achieving optimal performance:* The performance in the research will be measured by control performance criteria, such as $\mathscr{H}_2$ and $\mathscr{H}_\infty$ control criteria. These criteria will be considered for the synthesis of distributed controllers. In an indirect design, a distributed controller is model-based and, hence, the performance will be limited by the quality of the model. Quality of the model can be measured by asymptotic statistical properties, such as consistency, or the ability to capture information about the dynamics that lead to a controller with the best performance. In a direct design, distributed model-reference control problems will be considered; structured versions of model-reference control problems in state-of-the-art data-driven control methods such as VRFT and OCI. The performance is measured with respect to an ($\mathscr{H}_2$) model-reference control criterion for distributed control.

- *Achieving guaranteed performance:* In the research we distinguish optimal performance from guaranteed performance. Guaranteed performance refers to the achieved performance of a distributed controller that is designed based on a finite number of data samples, compared to data-driven controller design with a performance that is asymptotically optimal in the number of data points. Because state-of-the-art methods for guaranteed performance only allow 'unstructured' analysis and synthesis, guaranteed performance analysis and synthesis for interconnected systems will be considered in the research.

- *Under which conditions:* The conditions refer to the assumptions on the experimental setup under which the data-driven controller design is performed. This includes conditions on the measurement data, sensing and actuating locations, unmeasured exogenous disturbances and the considered controller class.

The discussed aspects lead to a number of sub-questions related to three main topics that will be discussed in Part I to Part III of this thesis:

- Part I: Network identification and distributed control
- Part II: Distributed data-driven model-reference control
- Part III: Distributed data-driven control with guarantees

Next, we will formulate sub-questions for these topics in the following three subsections corresponding to the three Parts of this thesis, including the considered approach to solve the problems.

1.4.1 Design of a distributed controller through data-driven modeling

Given the model-based nature of the majority of methods for distributed controller design, we will first consider the problem of distributed control on the basis of data according to a model-based philosophy. In the model-based philosophy, two steps can be distinguished. The first step is to derive a suitable model for controller design. Suitable in this context means a model that can be used in the synthesis process and that is most relevant for control, e.g., a model that leads to the best control performance.

For an unknown interconnected system, how to obtain the most relevant model from data for control?

Models that are used in model-based distributed control are typically structured with respect to subsystems, cf. (Langbort et al., 2004), (Van Horssen and Weiland, 2016), (Chen et al., 2019). The approach taken in *Chapter 2* is to take this structure into account in the data-driven modeling of subsystems, via a dynamic-network approach to system identification (Van den Hof and Ramaswamy, 2021), (Gevers et al., 2018). Both indirect and direct identification prediction-error identification methods are considered for obtaining consistent estimates of an interconnected system, which may be interconnected with a (preliminary) distributed controller. In the case that the subsystems cannot be represented by the considered model, bias and variance aspects play an important role in the performance resulting from the identified model. We will approach this problem by 'shaping' the *bias* through a modified identification criterion, to minimize the resulting performance degradation for an $\mathscr{H}_2$ control criterion (for reference tracking).

The second step is to design a distributed controller on the basis of the model that leads to an optimal performance.

How to obtain an optimal distributed controller for a model of the interconnected system?

The approach taken to distributed controller design in *Chapter 4*, is to consider dynamical subcontrollers which are interconnected with each other according to the interconnection structure of the plant, according to the framework introduced by (Langbort et al., 2004), or decentralized. We consider the objective of finding a distributed controller such that the controlled network has an $\mathscr{H}_2$ or $\mathscr{H}_\infty$ norm that is upper-bounded by a given value. Instead of viewing a linear interconnected system as an interconnection of transfer functions or state-space systems, an interconnection of linear systems can also be considered of which the dynamics are defined by their *behavior*; a subset of all possible outcomes/trajectories. Interconnections of systems in the behavioral setting do not assume any directions of interconnection signal, i.e., no input or output is assigned to a signal. We will consider distributed control in a behavioral setting in *Chapter 5*. This approach widens the previously considered class of linear interconnected systems considered for distributed controller design and is representation-free. We will consider the application of a canonical distributed controller developed in Chapter 5 to module dynamic networks in *Chapter 6*.

An important aspect in the data-driven modeling of interconnected systems, is the information pattern induced by distributed sensing. The distribution of information poses additional challenges in the data-driven modeling and control design for interconnected systems. Similar to the non-classical information patterns in decentralized and distributed control schemes, one can think of limited information availability for local "identification modules", which are destined to estimate a part of the interconnected system (or controller) dynamics. We refer to the concept of interconnected identification modules with non-classical information patterns as distributed identification. In this line of thought, a natural question that arises is as follows:

How can required model or controller dynamics be obtained from measurement data, given a non-classical information pattern for identification?

In *Chapter 3*, we step away from the classical (central) information pattern deployed in classical identification. We approach this step by considering the direct method for network identification in Chapter 2. The direct method yields a network identification problem that is separable into distinct multiple-input-single-output identification problems if the external noise signals are uncorrelated with respect to each other. The idea is then to assign an estimator (identification module) to each module in the network to estimate the corresponding parameters. In our approach to distributed identification, we consider the development of a distributed version of the well-established recursive least squares estimator (Kay, 1993), (Mendel, 1973), to distribute the estimation both spatially as well as temporally.

1.4.2 Design of a distributed controller directly on the basis of data

In the design of a distributed controller for an interconnected system for which measurement data is available, the ultimate goal is to determine a controller that achieves a given control objective, rather than determining a model of the underlying system. From this point-of-view, the main research question does not involve a modeling problem, but concerns directly the design of a distributed controller on the basis of data. This reasoning leads to the following research question:

Under which conditions can an optimal distributed controller be designed directly on the basis of data?

To answer this question, we consider the development of two different distributed data-driven controller design methods, extending VRFT (Campi et al., 2002) and OCI (Campestrini et al., 2016) to construct distributed controllers based on interconnected system data. Direct data-driven methods, such as VRFT and OCI, are typically based on a model-reference framework. We therefore consider the development of a distributed model-reference framework for dynamic networks in *Chapter 6*, based on the theory for distributed control in a behavioral setting from Chapter 5. The approach continues by solving the distributed model-reference control problem directly from data. This problem is considered in *Chapter 7* and *Chapter 8* with a philosophy coming from, respectively, VRFT and OCI.

1.4.3 Data-based conditions for the existence of a distributed controller with guaranteed performance

With safety-critical applications in mind, a controller that is optimal based on an infinite number of data points, i.e., an optimal controller that can be estimated consistently, may not suffice. In such applications, the challenge is to design a distributed controller on the basis of a finite number of data points, while *guaranteeing* a desired stability or performance objective. This leads to the following research question:

How to find a distributed controller on the basis of a (finite) number of data that is guaranteed to yield a prescribed performance level and under which conditions on the data does such a controller exist?

To answer this question, we start with considering the design of a distributed controller based on a finite number of data samples in *Chapter 9.* Each noise signal affecting a system in the network is assumed to satisfy a quadratic matrix inequality in this chapter, which can represent, for example, bounds on the energy or magnitude of the noise signal. This characterization of the noise allows us to derive a parametrization of the set of system matrices for each subsystem that are compatible with the data. Such parametrization of system matrices have been considered in the literature for single/unstructured systems for analysis and control, cf. (Koch et al., 2020b), (van Waarde et al., 2022), (Berberich et al., 2021). Through a dual parametrization, we can employ a linear fractional transformation (LFT) representation for each subsystem that enables the use of a robust variant of the analysis conditions in Chapter 4 and distributed controller existence conditions.

To determine under which conditions on the data a controller exists that yields a guaranteed performance, we consider the concept of informativity in *Chapter 10* and *Chapter 11.* Data being not informative enough for system identification does not imply data are not informative for controller design per se. In Chapter 10, we start by developing conditions on the data for the existence of stabilizing, $\mathscr{H}_2$ or $\mathscr{H}_\infty$ controllers that are unstructured, i.e., centralized. The noise in this analysis is assumed to satisfy bounds on the squared sample cross-covariance. These bounds represent prior knowledge about the system in a relative form, i.e., with respect to an known signal that is instrumental. The bounds generalize quadratic noise bounds as considered in the literature and Chapter 9. Because of the quadratic parametrization of the corresponding set of feasible systems, a matrix version of the S-lemma (van Waarde et al., 2022) is considered to determine non-conservative conditions on the data for $\mathscr{H}_2$ or $\mathscr{H}_\infty$ controller design. Prior knowledge on the noise is considered in the form of (component-wise) linear sample cross-covariance bounds, which is fundamentally different from the prior knowledge assumed in Chapter 10, where bounds on the squared sample-cross covariance matrix are specified with respect to the partial order on positive-semi-definite matrices. The approach in Chapter 11 is to use the convexity of the set of feasible systems (polyhedra) to derive informativity conditions on the data for $\mathscr{H}_2$ or $\mathscr{H}_\infty$ centralized and distributed controller design.

1.5 Overview of contents

In *Chapter 2*, we consider the problem of deriving models of linear interconnected systems with the purpose of designing a decentralized or distributed controller. For a controlled network that is structured in a distributed control structure, with a clear distinction between the structured plant and distributed controller, it is shown that identification methods developed for dynamics networks can be specified to provide consistent and approximate models of the structured plant. We will show how both direct and indirect methods for network identification can be specified/generalized for the considered network setup. Furthermore, both subsystem identification (local) as well as full-network identification (global) are considered. The choice for indirect/direct or local/global identification can be made by considering assumptions on the network, depending, e.g., on the knowledge of the (preliminary) distributed controller or correlation of external noise signals that affect subsystems in the network. In practical situations, a mismatch between the chosen model class and true system is inevitable. For this situation, we develop a procedure for determining a performance-oriented model of the interconnected system. In this procedure, the performance degradation (difference between designed and achieved controlled network) is minimized in the data-driven modeling of the network for an $\mathscr{H}_2$ reference tracking criterion.

In *Chapter 3*, a distributed identification method is developed for estimating modules in a dynamic network. The identification problem in dynamic networks is separable into distinct multiple-input-single-output identification problems if the external noise signals are uncorrelated with respect to each other. We develop a distributed recursive estimation scheme for multi-input-single-output models that are linear in the parameters, linking an estimator (identification module) to each module in the network. Connecting the identification modules through a mutual fusion center for communication, leads to a distributed identification scheme for each multi-input-single-output system. By Lyapunovs second method, sufficient conditions are derived for asymptotic convergence of the estimators to the true parameter values, which lead to asymptotic unbiasedness in the presence of additive output noise.

In *Chapter 4*, we consider the problem of synthesizing a distributed or decentralized controller for linear interconnected systems. The considered objective is to find a linear and dynamic distributed controller such that the controller network has an $\mathscr{H}_2$ or $\mathscr{H}_\infty$ norm that is upper-bounded by a given value. We recall results for the $\mathscr{H}_\infty$-case from the literature and treat the $\mathscr{H}_2$-case in detail. Specifically, we consider the $\mathscr{H}_2$-analysis of interconnected systems, then move to the analysis of controlled interconnected systems, and finally consider the synthesis of structured controllers that achieve $\mathscr{H}_2$ performance. For the latter, we develop convex and structured conditions for the existence of a distributed or decentralized $\mathscr{H}_2$ controller for interconnected systems with an arbitrary interconnection structure. The existence conditions serve as a preliminary step in the controller synthesis and its solutions yield a distributed controller achieving the specified performance upper-bound. The effectiveness and scalability of the developed distributed $\mathscr{H}_2$ controller synthesis method is demonstrated for small- to large-scale oscillator networks on a cycle graph.

In *Chapter 5*, distributed control is considered from a more general perspective. We introduce distributed control in a behavioral setting, where a distributed controller is an interconnection of controllers with the same interconnection structure as the interconnected system to be controlled. Given a desired (structured) behavior for the interconnected system, we provide conditions under which this desired behavior can be implemented, i.e., there exists a distributed controller that achieves the desired system behavior. The proof is constructive and yields a canonical distributed controller described by the behavior of the to-be-controlled and desired interconnected system. Furthermore, regularity of the canonical distributed controller will be analyzed; with respect to the interconnection between the controller and the plant, as well as the interconnections between subsystems of the distributed controller.

Chapter 6 is enabled by the canonical distributed controller in Chapter 5 and considers the specific case of interconnected systems defined by transfer functions. We develop a distributed controller that implements a desired behavior described by transfer functions; a structured reference model. This controller, an *ideal distributed controller*, consists of local controllers that depend explicitly on the transfer functions of the corresponding subsystems of the to-be-controlled interconnected system and reference model. We first analyze the properties of this distributed controller and provide conditions on the reference model and interconnected system for which proper or stable distributed controllers are obtained. Since the structured reference model is a key preliminary ingredient in the data-driven controller design introduced in Chapter 7 and Chapter 8, the synthesis of a structured reference model is also considered, based on the analysis conditions for $\mathscr{H}_2$ and $\mathscr{H}_\infty$ performance.

In *Chapter 7*, a method is developed for determining a distributed controller directly on the basis of data, without an intermediate step of modeling the underlying interconnected system. We first consider a case without unmeasured exogenous disturbances. The method is based on the structured reference model and ideal-distributed-controller concept introduced in Chapter 6. Given the structure of the interconnected system and reference model, a *virtual reference network* is introduced, analogous to a virtual reference setup in VRFT. The node variables in this network are comprised of measured node signals and virtual reference signals that correspond to the measured signal. Consequently, a collection of unknown modules in the network form an ideal distributed controller that is to be determined. We show how the ideal-distributed-controller synthesis problem is transformed into a network identification problem. We then show what the effect of noise is on distributed controller estimates if the identification procedure is not adapted. This motivates the sensible modeling of the noise. A method is provided to obtain consistent distributed controller estimates, by considering tailor-made noise models in the network identification.

The problem of determining a distributed controller directly on the basis of data is revisited in *Chapter 8*. Here, we show that a to-be-controlled interconnected system is equivalent to an interconnected system with the modules related to the reference model and the (inverse of) dynamics of an ideal distributed controller. This framework allows the data-driven control problem to be analyzed by dynamic network identification results. We call this method distributed optimal controller identification (OCI). This approach to distributed data-driven control, is directly applicable to interconnected systems subject to unmeasured exogenous disturbances and we show that consistent controller estimates are obtained if the noise is modeled correctly, i.e., if the considered class of noise models is sufficiently rich. Via an example, we show how the achieved performance of the controlled network improves by the application of the developed method, compared to the direct data-driven controller design where noise is not modeled or an instrumental variable method is used.

Chapter 9 considers the problem of determining a distributed controller from data with *guaranteed* performance, for a finite number of data samples. Each noise signal affecting a system in the network is assumed to satisfy a quadratic matrix inequality in this chapter, which can represent, for example, bounds on the energy or magnitude of the noise signal. With this characterization of the noise, we derive a parametrization of the system matrices for each subsystem and introduce a dual parametrization that allows the application of performance and stability conditions in primal form. Via the parametrizations, we employ a linear fractional transformation (LFT) representation for each subsystem that enables us to use a robust variant of the convex conditions in Chapter 4 to develop sufficient convex conditions based on noisy data for the performance analysis of interconnected systems. Then, data-based conditions are presented that are sufficient for the existence of a distributed controller that achieves a given $\mathscr{H}_\infty$ performance level.

In *Chapter 10*, the informativity of data for controller design is analyzed. In this chapter, we start by developing conditions on the data for the existence of stabilizing, $\mathscr{H}_2$ or $\mathscr{H}_\infty$ controllers that are unstructured, i.e., centralized. The noise in this analysis is assumed to satisfy bounds on the squared sample cross-covariance. These bounds represent prior knowledge about the system in a relative form, i.e., with respect to a known signal that is instrumental. The bounds generalize quadratic noise bounds as considered in the literature and Chapter 9.

We recall a matrix type S-procedure from the literature that allows us to derive necessary and sufficient conditions for informativity of data for controller design. These conditions are also constructive, in the sense that a controller that solves the control problem can be derived from its solution set.

The informativity of noisy data is revisited in *Chapter 11.* Prior knowledge on the noise is considered in the form of (component-wise) linear sample cross-covariance bounds, which is fundamentally different from the prior knowledge assumed in Chapter 10, where bounds on the squared sample-cross covariance matrix are specified with respect to the partial order on positive-semi-definite matrices. We show how finite sets of input-state data together with the linear cross-covariance bounds lead to a set of feasible system matrices that is either an unbounded or bounded polyhedron, depending on the input, state, and instrumental variable data. We show how convexity of the set of feasible systems lead to conditions under which the data are informative for quadratic stabilization, which are also necessary in case the polyhedron is bounded. The results are then naturally extended for informativity for $\mathscr{H}_2$ and $\mathscr{H}_\infty$ control. The construction of ellipsoidal supersets of the polytope is considered subsequently, to reduce the computational complexity, while adding conservatism to the resulting informativity conditions. Finally, we show how the cross-covariance bounds apply to interconnected systems and develop sufficient conditions on informativity of data collected from interconnected systems for the design of a distributed controller.

A graph representing the relations between Chapter 2 to Chapter 11 from this thesis is shown in Figure 1.8. An arrow pointing from one node to an other node indicates that the results from the source chapter extend to the sink chapter, or that the theory in the source node is applicable to the sink node. Bidirectional arrows indicate a general relation, e.g., a mutual control objective is considered. Whether a relation is strong or weak is indicated by the link being solid or dashed, respectively.

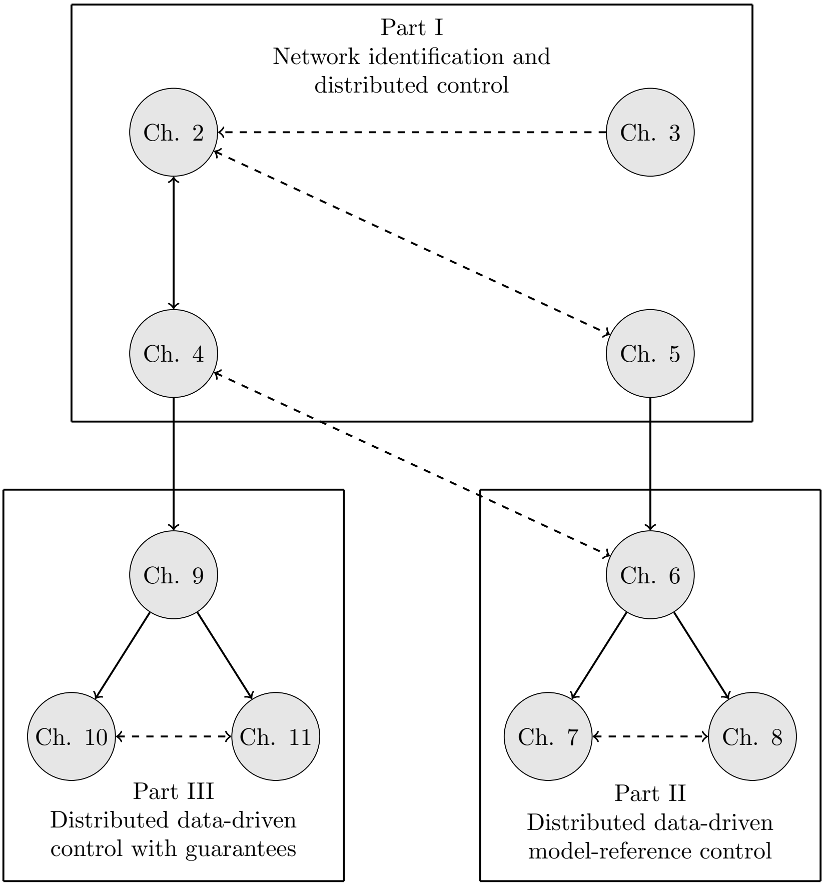

Figure 1.8: Relations between chapters in this thesis.

1.6 Summary of publications

A part of the results in this thesis has been published elsewhere in a different form.

Chapter 3 contains results that have been presented in:

- T. R. V. Steentjes, M. Lazar, and P. M. J. Van den Hof. A recursive estimation approach to distributed identification of large-scale multi-input-single-output FIR systems. *IFAC-PapersOnLine*, 51(23):236 – 241, 2018. 7^{th} IFAC Workshop on Distributed Estimation and Control in Networked Systems

Chapter 4 contains results that have been presented in:

- T. R. V. Steentjes, M. Lazar, and P. M. J. Van den Hof. Scalable distributed $\mathscr{H}_2$ controller synthesis for interconnected linear discrete-time systems. *IFAC-PapersOnLine*, 54(9):66–71, 2021c. 24^{th} International Symposium on Mathematical Theory of Networks and Systems

Chapter 6 contains results that have been presented in:

- T. R. V. Steentjes, M. Lazar, and P. M. J. Van den Hof. Data-driven distributed control: Virtual reference feedback tuning in dynamic networks. In *Proc. 59th IEEE Conference on Decision and Control (CDC)*, pages 1804–1809, 2020

Chapter 7 contains results that have been presented in:

- T. R. V. Steentjes, M. Lazar, and P. M. J. Van den Hof. Data-driven distributed control: Virtual reference feedback tuning in dynamic networks. In *Proc. 59th IEEE Conference on Decision and Control (CDC)*, pages 1804–1809, 2020
- T. R. V. Steentjes, P. M. J. Van den Hof, and M. Lazar. Handling unmeasured disturbances in data-driven distributed control with virtual reference feedback tuning. *IFAC-PapersOnLine*, 54(7):204–209, 2021d. 19^{th} IFAC Symposium on System Identification SYSID 2021

Chapter 8 contains results that have been presented in:

- T. R. V. Steentjes, M. Lazar, and P. M. J. Van den Hof. Controller identification for data-driven model-reference distributed control. In *Proc. 2021 European Control Conference (ECC)*, pages 2358–2363, Rotterdam, The Netherlands, 2021b

Chapter 9 contains results that have been presented in:

- T. R. V. Steentjes, M. Lazar, and P. M. J. Van den Hof. $\mathcal{H}_\infty$ performance analysis and distributed controller synthesis for interconnected linear systems from noisy input-state data. In *Proc. 60th IEEE Conference on Decision and Control (CDC)*, pages 3717–3722, Austin, Texas, USA, 2021a

Chapter 10 contains results that have been presented in:

- T. R. V. Steentjes, M. Lazar, and P. M. J. Van den Hof. On data-driven control: Informativity of noisy input-output data with cross-covariance bounds. *IEEE Control Systems Letters*, 6:2192–2197, 2022b

Chapter 11 contains results that have been presented in:

- T. R. V. Steentjes, M. Lazar, and P. M. J. Van den Hof. Informativity conditions for data-driven control based on input-state data and polyhedral cross-covariance noise bounds. 2022a. Submitted to the *25th International Symposium on Mathematical Theory of Networks and Systems (MTNS 2022)*

1.7 Other publications

Other publications that are not included in this thesis are:

- T. R. V. Steentjes, A. I. Doban, and M. Lazar. Construction of continuous and piecewise affine Lyapunov functions via a finite-time converse. *IFAC-PapersOnLine*, 49(18):13–18, 2016. 10$^{\text{th}}$ IFAC Symposium on Nonlinear Control Systems

- T. R. V. Steentjes, A. I. Doban, and M. Lazar. Feedback stabilization of positive nonlinear systems with applications to biological systems. In *Proc. 2018 European Control Conference (ECC)*, pages 1619–1624, 2018

- T. R. V. Steentjes, M. Lazar, and A. I. Doban. Construction of continuous and piecewise affine feedback stabilizers for nonlinear systems. *IEEE Transactions on Automatic Control*, 66(9):4059–4068, 2021e

Part I

Network identification and distributed control

Chapter 2

Control-oriented identification of dynamic networks

Distributed controller design methods typically rely on a model of the underlying interconnected system that has to be controlled. Due to the complex interconnection structure or dynamics, such models are typically not directly available. In this chapter we consider the problem of deriving models of interconnected systems for distributed controller design. This problem is first approached by considering the identification of subsystems of an interconnected system that is in closed-loop with a distributed controller. The identification can be performed by considering direct and indirect network identification methods for the specific closed-loop network structure. The network structure and controller dynamics can be used explicitly in the identification objective by considering a tailor-made parametrization of the closed-loop transfer matrices. In the case of approximate modeling, an $\mathscr{H}_2$ control performance criterion can be taken into account, leading to full-network identification problem with a tailor-made parametrization.

2.1 Introduction

State-of-the-art distributed controller design methods are typically based on a model of the to-be-controlled interconnected system. Several examples of such methods for the design of distributed and decentralized controllers have been discussed in Section 1.3. A convex scalable method for the design of distributed $\mathscr{H}_2$ controllers is presented in Chapter 4 of this thesis. Interconnected system models are, however, typically not directly available for controller design. When data from the interconnected system is available, two approaches to controller design can be followed: (i) indirect data-driven control and (ii) direct data-driven control. Indirect data-driven control is model based: first a plant model is estimated on the basis of data and consecutively a controller design is performed based on the model. The modeling step is omitted in direct-data driven control; a controller is synthesized directly from data. Direct data-driven distributed control is extensively treated in Part II and III of this thesis.

Indirect data-driven distributed control is concerned with the estimation of an interconnected-system model from data and distributed controller design based on the model. The identification of interconnected systems, also called dynamic network identification in the literature (Van den Hof et al., 2013), is a topic that has received increased interest over the past years. Specific network identification methods have been developed for various applications, such as the identification of plant dynamics in decentralized control loops (Gudi and Rawlings, 2006), (Van den Hof et al., 2018) and the reconstruction of biological networks (Yuan et al., 2011). Stemming from open-loop and closed-loop identification, prediction-error identification methods for generic dynamic networks have been developed in the literature (Van den Hof et al., 2013), (Gevers and Bazanella, 2015), (Dankers et al., 2016), (Linder, 2017), (Weerts et al., 2018), (Gevers et al., 2018), (Ramaswamy et al., 2019), (Bazanella et al., 2019). Other methods for identification of interconnected systems have been developed in an alternative setting, e.g., through Wiener filters (Materassi and Salapaka, 2012) and subspace identification (Haber and Verhaegen, 2014). Despite the fact that distributed controller design is one of the motivations for dynamic network identification (Van den Hof et al., 2013), links between dynamic network identification and distributed controller design are sparse in the literature. The identification of interconnection dynamics in two interconnected control loops was motivated by model-predictive controller design in (Gudi and Rawlings, 2006). However, for general distributed control schemes, indirect data-driven distributed control results have yet to be developed, to the best of the author's knowledge.

Identification for controller design of single-input-single-output/unstructured multi-input-multi-output systems has been investigated in the field of 'identification for control' (Van den Hof and Schrama, 1995), (Gevers, 2005). In this field, the performance criterion for control is taken into account, yielding a control-oriented identification procedure that minimizes a performance degradation criterion. Iteratively performing control-oriented identification and controller design aims at minimizing the achieved performance, see e.g., (Zang et al., 1991), (Anderson and Kosut, 1991), (Lee et al., 1995) or (Van den Hof and Schrama, 1995) for an overview.

In this chapter, we consider two problems related to the identification of network dynamics for distributed controller design. First, we consider the problem of identifying a model of an interconnected system that operates in a closed-loop setting with a distributed controller. This yields a dynamic network with a particular structure. We will show how identification methods that have been developed for dynamic network identification, can be adapted to this specific situation to deal with this particular structure and the distributed controller dynamics. Furthermore, we will consider a network identification problem with a tailor-made parametrization of network transfer matrices. In this problem, the structure of the closed-loop network and knowledge of the distributed controller dynamics are explicitly used for identification of the subsystems of the interconnected system.

Next, we consider the control-oriented identification of interconnected systems with respect to an $\mathscr{H}_2$ reference tracking criterion. The identification problem is control-oriented in the sense that the aim is to minimize the performance degradation, by shaping the bias of the subsystem models in an approximate modeling setting. We will show that the identification criterion of the closed-loop network identification method with a tailor-made parametrization, can be interpreted as a performance degradation term for the $\mathscr{H}_2$ reference tracking criterion.

2.2 Preliminaries

2.2.1 Dynamic network and distributed controller

Consider a network of L linear systems on a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, with vertex set $\mathcal{V} := \{1, \ldots, L\}$ and edge set $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$, described by

$$y_i(t) = G_i^0(q)u_i(t) + \sum_{j \in \mathcal{N}_i} G_{ij}^0(q)w_j(t) + v_i(t), \quad i \in \mathcal{V},\ t \in \mathbb{Z}, \tag{2.1}$$

where y_i is the output of system i, u_i is the control input, v_i is an unmeasured exogenous disturbance signal (process noise), q^{-1} is the shift operator, i.e., $q^{-1}x(t) = x(t-1)$. The rational transfer functions G_i^0 and G_{ij}^0 describe the local and coupling dynamics, respectively. The vector process $v := \mathrm{col}(v_1, \ldots, v_L)$ is modeled as a stationary stochastic process with a rational spectral density matrix Φ_v, such that there exists a white noise process $e := \mathrm{col}(e_1, \ldots, e_L)$ and a rational transfer matrix H_0 that is monic, stable and minimum phase, satisfying $v(t) = H_0(q)e(t)$. Subsystems are interconnected through the outputs y_j or inputs u_j, $j \in \mathcal{V}$, corresponding to $w_j = y_j$ or $w_j = u_j$, respectively. Specifically, the output of system $i \in \mathcal{V}$ is influenced by the outputs or inputs of systems $j \in \mathcal{N}_i$, where $\mathcal{N}_i \subseteq \mathcal{V}$ denotes the set of incoming neighbours such that $(j, i) \in \mathcal{E}$ $(\mathcal{N}_i = \{j \in \mathcal{V} \,|\, G_{ij}^0 \neq 0\})$.

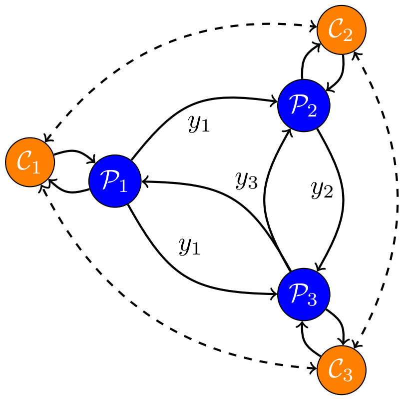

Figure 2.1: Three systems $\mathcal{P}_i$ interconnected through outputs y_i, $i \in \{1, 2, 3\}$, and controlled by controllers $\mathcal{C}_i$.

To each system $i \in \mathcal{V}$, a reference signal r_i is assigned. This induces a tracking error $z_i := r_i - y_i$; the difference between the reference and observed output signal. We define the data-generating interconnected system (or true interconnected system) as an interconnected system with subsystems $\mathcal{P}_i$, defined by the dynamics (2.1) and performance output $z_i = r_i - y_i$:

$$\mathcal{P}_i : \begin{cases} y_i = G_i^0(q)u_i + \sum_{j \in \mathcal{N}_i} G_{ij}^0(q)w_j + v_i, \\ z_i = r_i - y_i, \end{cases} \qquad i \in \mathcal{V}. \tag{2.2}$$

An example network ($|\mathcal{V}| = 3$) with output coupling is shown in Figure 2.1.

A dynamical distributed controller is interconnected with the interconnected system through the tracking errors and control inputs. More specifically, the distributed controller consists of subcontrollers $\mathcal{C}_i$, described by

$$\mathcal{C}_i : \begin{cases} u_i = C_i(q)z_i + \sum_{j \in \mathcal{N}_i} C_{ij}(q)\eta_{ij}, \\ \zeta_{ij} = K_{ij}(q)z_i + \sum_{h \in \mathcal{N}_i} K_{ijh}(q)\eta_{ih}, \ j \in \mathcal{N}_i, \end{cases} \qquad i \in \mathcal{V}, \tag{2.3}$$

which takes z_i as an input and generates u_i as an output. Two subcontrollers $\mathcal{C}_i$ and $\mathcal{C}_j$ are interconnected through their interconnection variables $\eta_{ij} = \zeta_{ji}$ only if $j \in \mathcal{N}_i$. To be consistent with the distributed control setting in Chapter 4, the controller interconnections are considered to be bi-directional. We note, however, that the reasoning in this chapter is analogous for a setting with uni-directional communication constraints in the distributed controller. Decentralized control schemes form a special case of distributed control schemes where the interconnection signal dimension is zero, i.e., a decentralized controller module $\mathcal{C}_i$ is described by $u_i = C_i(q)z_i$.

2.2.2 Closed-loop network dynamics

The controller interconnection equations $\eta_{ij} = \zeta_{ji}$ for each pair of controllers $\mathcal{C}_i$ and $\mathcal{C}_j$ that are interconnected can be compactly written as $\zeta = \Delta_c \eta$, where Δ_c is a binary matrix and $\zeta := \text{col}(\zeta_1, \ldots, \zeta_L)$, $\eta := \text{col}(\eta_1, \ldots, \eta_L)$. With $z := \text{col}(z_1, \ldots, z_L)$, $u := \text{col}(u_1, \ldots, u_L)$, the distributed controller dynamics (2.3) are compactly described by

$$\begin{bmatrix} u \\ \zeta \end{bmatrix} = \begin{bmatrix} C_{\mathcal{D}} & C_{\mathcal{DD}} \\ K_{\mathcal{DD}} & K_{\mathcal{D}} \end{bmatrix} \begin{bmatrix} z \\ \eta \end{bmatrix}, \quad \zeta = \Delta_c \eta. \tag{2.4}$$

Elimination of the controller interconnection variables ζ and η in (2.4) leads to a transfer matrix for the distributed controller from z to u:

$$u = C_{\mathcal{I}}(q) z, \quad \text{where } C_{\mathcal{I}} = C_{\mathcal{D}} + C_{\mathcal{DD}}(\Delta_c - K_{\mathcal{D}})^{-1} K_{\mathcal{DD}}.$$

Since the plant interconnection signals w_j are equal to either u_j or y_j, we can write $w = \begin{bmatrix} \Delta_y & \Delta_u \end{bmatrix} \text{col}(y, u)$, where Δ_y and Δ_u are binary diagonal matrices such that each row of $\begin{bmatrix} \Delta_y & \Delta_u \end{bmatrix}$ sums up to one, i.e., $\begin{bmatrix} \Delta_y & \Delta_u \end{bmatrix} \mathbf{1} = \mathbf{1}$ where $\mathbf{1}$ denotes the column vector of all ones. Define $G^0_{\mathcal{D}} := \text{diag}(G^0_1, \ldots, G^0_L)$ and let G_0 be a matrix whose (i,j)-th entry is G^0_{ij}, $(i,j) \in \mathcal{V}^2$. We arrive at a compact representation for (2.1):

$$y = G_0 \Delta_y y + (G_0 \Delta_u + G^0_{\mathcal{D}}) u + H_0 e.$$

The closed-loop network (with tracking errors $(z_1, \ldots, z_L)$ eliminated) is therefore described by

$$\begin{bmatrix} y \\ u \end{bmatrix} = \underbrace{\begin{bmatrix} G_0 \Delta_y & G_0 \Delta_u + G^0_{\mathcal{D}} \\ -C_{\mathcal{I}} & 0 \end{bmatrix}}_{G^0_{\mathcal{I}}} \begin{bmatrix} y \\ u \end{bmatrix} + \begin{bmatrix} 0 \\ C_{\mathcal{I}} \end{bmatrix} r + \begin{bmatrix} H_0 \\ 0 \end{bmatrix} e, \tag{2.5}$$

where $G_0 \Delta_y$ and $G_0 \Delta_u$ are hollow matrices and $G^0_{\mathcal{D}}$ is a diagonal matrix.

Remark 2.2.1. *In the closed-loop data-generating system* (2.5)*, the control inputs u_i are assumed to be disturbance free, which is reflected by the (exact) equality $u = C_{\mathcal{I}}(q)(r - y)$. This an assumption of a principal nature, which is also a common assumption in the literature on classical closed-loop identification, cf. (Van den Hof, 1998), (Forssell and Ljung, 1999), (Gevers, 2005), (Hjalmarsson, 2005). The assumption of disturbance-free inputs in the experimental setup is evaluated in more detail in (Van den Hof, 1998, Section 7).*

2.2.3 State-space representation

A transfer function representation of the interconnected system and distributed controller is considered in this chapter, following the convention of dynamic network representations (Van den Hof et al., 2013). State-space representations of subsystems $\mathcal{P}_i$ and subcontrollers $\mathcal{C}_i$ will be considered for distributed controller design in Chapter 4. A state-space representation of $\mathcal{P}_i$ is of the form

$$x_i(t+1) = A_i x_i(t) + \sum_{j\in\mathcal{N}_i} A_{ij} w_j(t) + B_i u_i(t), \tag{2.6}$$
$$y_i(t) = C_i x_i(t) + \sum_{j\in\mathcal{N}_i} C_{ij} w_j(t) + D_i u_i(t) + v_i(t), \quad z_i(t) = r_i(t) - y_i(t),$$

and satisfies $G_i^0(q) = C_i(qI - A_i)^{-1}B_i + D_i$ and

$$G_{ij}^0(q) = C_i(qI - A_i)^{-1}A_{ij} + C_{ij}, \quad j \in \mathcal{N}_i.$$

A representation (2.6) can always be constructed, but is in general not unique (Kailath, 1980), (Hannan and Deistler, 1987). One state-space representation for $\mathcal{P}_i$ is derived in Appendix 2.A. A state-space representation for $\mathcal{C}_i$ can be obtained *mutatis mutandis*.

2.3 Network identification in the presence of distributed control

Consider the closed-loop interconnected system described by (2.2)-(2.3). The problem of identifying a transfer function G_{ij}^0 or collection of transfer functions $(G_i^0, (G_{ij}^0)_{j\in\mathcal{N}_i})$ is commonly referred to as a local identification problem in dynamic network identification (Ramaswamy et al., 2019), (Van den Hof et al., 2019), (Van den Hof and Ramaswamy, 2021). Methods for the local module identification problem can be categorized into two main approaches: *direct* methods and *indirect* methods. A direct method is characterized by the use of 'node' signals w_j (u_j or y_j in the considered setting) as predictor inputs, i.e., as inputs for predicting y_i through a parametrized model. In an indirect method, external signals r_j are chosen as predictor inputs to predict y_i, and a post-processing step is typically required to extract the local estimated dynamics. The synergy of the two methods led to a generalized method for local module identification, developed by Ramaswamy et al. (2019).

2.3.1 Direct method

The main idea behind the direct method for identification in a closed loop, is to use the input-output data of the plant for identifying a model without taking the closed-loop situation directly into account, i.e., by disregarding the presence of the controller (Van den Hof, 1998). Similarly, in the direct method to local module identification in dynamic networks, a module or MISO system is identified based on its input-output data, without considering the rest of the network in the identification procedure (directly), cf. (Van den Hof et al., 2013). Although not considered direcly, the network as a whole is still of importance for the direct method for the analysis of informativity of the input-output data for identification, which will be discussed later in this subsection.

The direct method typically requires the inclusion of a model for the noise filters in the estimation problem. When the process noise signals in $\{v_1, v_2, \ldots, v_L\}$ are uncorrelated (Φ_v is a diagonal matrix) such that $v_i = H_i^0(q)e_i$, the identification problem reduces to a MISO identification problem (Van den Hof et al., 2013). Consider the parametrized transfer function models $G_i(\theta_i)$, $(G_{ij}(\theta_i))_{j\in\mathcal{N}_i}$ and $H_i(\theta_i)$ for, respectively, G_i^0, $(G_{ij}^0)_{j\in\mathcal{N}_i}$ and H_i^0, with a one-step-ahead predictor for y_i given by

$$\hat{y}_i(t|t-1;\theta_i) := (1 - H_i^{-1}(\theta_i))y_i(t) + H_i^{-1}(\theta_i)\left(G_i(\theta_i)u_i(t) + \sum_{j\in\mathcal{N}_i} G_{ij}(\theta_i)w_j(t)\right).$$

The corresponding prediction error $\varepsilon_i(t,\theta_i) := y_i(t) - \hat{y}_i(t|t-1;\theta_i)$ is minimized in the direct method in a least-squares sense (Van den Hof et al., 2013):

$$\hat{\theta}_i^N := \arg\min_{\theta} \frac{1}{N}\sum_{t=0}^{N-1} \varepsilon_i(t,\theta_i)^2. \tag{2.7}$$

Under weak assumptions[1], the parameter estimate converges asymptotically in the number of data as $\hat{\theta}_i^N \to \theta_i^*$ with probability 1 as $N \to \infty$, where $\theta_i^* := \arg\min_\theta \bar{E}\varepsilon_i^2(t,\theta)$, $\bar{E} := \lim_{N\to\infty} \frac{1}{N}\sum_{t=0}^{N-1} E$ and E is the expectation. When the transfer function models are evaluated at θ_i^*, consistent estimates are obtained.

Definition 2.3.1. *The module transfer functions G_i^0, G_{ij}^0, $j \in \mathcal{N}_i$, and H_i^0 are said to be estimated consistently if, respectively, $G_i(q,\theta_i^*) = G_i^0(q)$, $G_{ij}(q,\theta_i^*) = G_{ij}^0(q)$, $j \in \mathcal{N}_i$, and $H_i(q,\theta_i^*) = H_i^0(q)$.*

Consistent identification through the direct method in dynamic network identification is dependent on the chosen parametrization, the information present in the predictor input and output signals and correlation of the external signals.

[1] These assumptions are standard in the identification literature and are related to the stability of the predictor model and bounded moments of the noise (for convergence), see e.g., (Ljung, 1999) for details.

Proposition 2.3.1 (Van den Hof et al. (2013)). *Consider the closed-loop network described by* (2.2)-(2.3)*, the direct method estimator* (2.7) *and let the following conditions hold:*

(1) the noise v_i is uncorrelated to the reference r_j, for all $j \in \mathbb{Z}_{[1:L]}$,

(2) the noise v_i is uncorrelated to the noise v_j, for all $j \in \mathbb{Z}_{[1:L]} \setminus \{i\}$,

(3) every path in the network graph from vertex i to itself contains at least one transfer function with a delay,

(4) the spectral density of $\mathrm{col}(y_i, u_i, w_{h1}, \ldots, w_{hL})$, $h_\bullet \in \mathcal{N}_i$ is positive definite for almost all $\omega \in [-\pi, \pi]$,

(5) there exists θ_i^0 so that $G_i(\theta_i^0) = G_i^0$, $G_{ij}(\theta_i^0) = G_{ij}^0$ and $H_i(\theta_i^0) = H_i^0$.

Then G_i^0, G_{ij}^0, $j \in \mathcal{N}_i$, and H_i^0 are estimated consistently.

Data-informativity for the direct method is formulated in terms of a positive-definite power spectrum of $\mathrm{col}(y_i, u_i, w_{h1}, \ldots, w_{hL})$, $h_\bullet \in \mathcal{N}_i$, requiring sufficient excitation from external signals. Excitation can be both provided by reference signals r_j, $j \in \mathcal{V}$, as well as disturbance signals e_j, $j \in \mathcal{V}$, because both types of signals contribute to the excitation of the predictor inputs.

While this informativity condition is rather implicit, it can be translated to path-based conditions on the closed-loop network graph for informativity that holds generically, i.e., independent of the transfer function coefficients. The next lemma follows *mutatis mutandis* from Proposition 1 by Van den Hof and Ramaswamy (2020).

Lemma 2.3.1. *Consider the vector $w_\mathcal{D} := \mathrm{col}(y_i, u_i, w_{h1}, \ldots, w_{hL}) \in \mathbb{R}^p$, $h_\bullet \in \mathcal{N}_i$ and let the stacked vector of external signals $\mathrm{col}(r, e)$ have a power spectrum that is positive definite almost everywhere. Then $\Phi_{w_\mathcal{D}}$ is positive definite almost everywhere if there are p vertex-disjoint*[2] *paths from $\mathrm{col}(r, e)$ to $w_\mathcal{D}$.*

By (2.2) and (2.3), we observe that there exists a path from r_i to u_i (through C_i) and that there exists a path from e_i to y_i (through H_i^0). These two paths are clearly disjoint. Hence, if there exist $|\mathcal{N}_j|$ vertex-disjoint paths (of which each path is vertex disjoint with $r_i \to u_i$ and $e_i \to y_i$), then the condition in Lemma 2.3.1 is satisfied.

Corollary 2.3.1. *Let $\Phi_{(r,e)}$ be positive definite almost everywhere and let $r_\mathcal{V}$ and $e_\mathcal{V}$ be stacked vectors of r_j and e_j, with $j \in \mathcal{V} \setminus \{i\}$. Then $\Phi_{w_\mathcal{D}}$ is positive definite almost everywhere if there are $|\mathcal{N}_i|$ vertex-disjoint paths from $\mathrm{col}(r_\mathcal{V}, e_\mathcal{V})$ to $w_{\mathcal{N}_i}$.*

[2] A set of paths is vertex disjoint if no two of them have one or more vertices in common.

For the condition on the vertex-disjoint paths to hold, it is clearly necessary that there are at least $|\mathcal{N}_i|$ external signals in $(r_\mathcal{V}, e_\mathcal{V})$. Let us illustrate the direct method and its sufficient informativity conditions via a simple example network.

Example 2.1. *Consider the two interconnecting control loops introduced in Chapter 1, depicted in Figure 2.2. The identification of the interconnection dynamics* G^0_{21} *has been treated in detail by Gudi and Rawlings (2006). In view of the direct method, the dynamics of* $\mathcal{P}_1$ *and* $\mathcal{P}_2$ *can be identified through two identification experiments.*

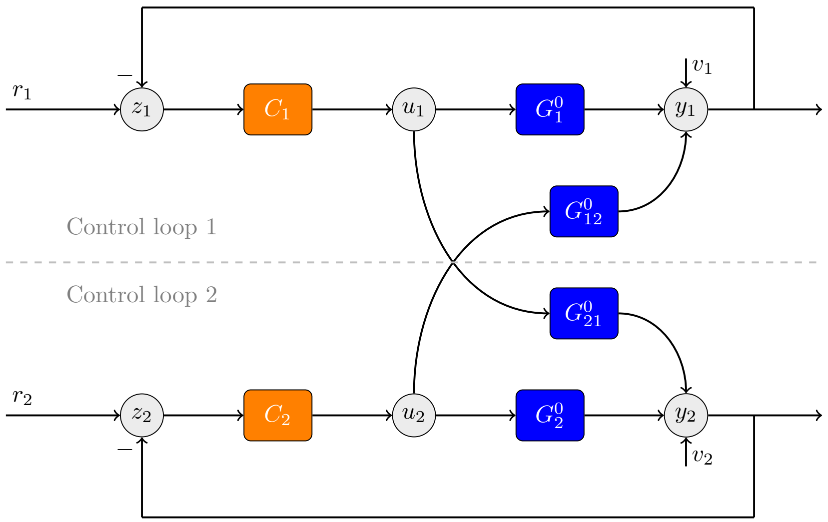

Figure 2.2: Two interconnected control loops considered by Gudi and Rawlings (2006), cf. (Van den Hof et al., 2018).

The identification of $\mathcal{P}_2$ *for example, consists of modeling* G^0_2, G^0_{21} *and, possibly,* H^0_2. *Finding the parameter estimates in* (2.7), *requires a prediction for* y_2 *with predictor inputs* u_2 *and* u_1. *By choosing a suitable model class according to condition 5 in Proposition 2.3.1, consistent estimates can be obtained if the data informativity condition (condition 4, Proposition 2.3.1) holds, i.e., if the power spectrum of* $\mathrm{col}(u_1, u_2, y_2)$ *is positive definite. By Corollary 2.3.1, only one path from* r_1 *and* e_1 *to* u_1 *is required. It is clear that, due to the feedback loop from* y_1 *to* z_1, *sufficient excitation can be provided by choosing either* r_1 *or* e_1 *to have a positive definite power spectrum. Notice that it is not required to measure these external signals, nor to know the controller dynamics.*

Instead of the identification of one subsystem $\mathcal{P}_i$, identification of the full plant $(\mathcal{P}_i)_{i\in\mathcal{V}}$ can be performed by the separate identification of $\mathcal{P}_i$, $i \in \mathcal{V}$ (which may be performed in parallel). When Φ_v is a non-diagonal matrix, confounding variables are present in the experimental setup (Van den Hof and Ramaswamy, 2021). The implication is that consistent estimates cannot be obtained through the direct method with a MISO predictor model, in general. Confounding variables due to noise correlation can be handled by adapting the predictor model, e.g., through adding additional variables to the predictor output, as described in (Ramaswamy and Van den Hof, 2021). Alternatively, correlated noise processes can be handled by the use of an indirect identification method, as discussed in the next subsection.

2.3.2 Indirect method

Based on the dynamics of the closed-loop network in (2.5), the closed-loop network can be written as

$$\begin{bmatrix} y \\ u \end{bmatrix} = (I - G^0_{\mathcal{I}})^{-1} \left(\begin{bmatrix} 0 \\ C_{\mathcal{I}} \end{bmatrix} r + \begin{bmatrix} H_0 \\ 0 \end{bmatrix} e \right) = \begin{bmatrix} T^0_y \\ T^0_u \end{bmatrix} r + \bar{v},$$

where

$$\begin{bmatrix} T^0_y \\ T^0_u \end{bmatrix} := (I - G^0_{\mathcal{I}})^{-1} \begin{bmatrix} 0 \\ C_{\mathcal{I}} \end{bmatrix} \quad \text{and} \quad \bar{v} := (I - G^0_{\mathcal{I}})^{-1} \begin{bmatrix} H_0 \\ 0 \end{bmatrix} e$$

If T^0_y and T^0_u would be known/given and the transfer matrices G_0 and $G^0_{\mathcal{D}}$ are unknown, one can use the definition of $\mathrm{col}(T^0_y, T^0_u)$ above in order to retrieve G_0 and $G^0_{\mathcal{D}}$. It is well known (Ljung, 1999), cf. (Gevers et al., 2018), (Van den Hof and Ramaswamy, 2021), that a consistent estimate $\mathrm{col}(\hat{T}_y, \hat{T}_u)$ of $\mathrm{col}(T^0_y, T^0_u)$ can be obtained through open-loop MIMO identification, provided that the power spectrum of r is positive definite for a sufficiently high number of frequencies. Given the definition of $\mathrm{col}(T^0_y, T^0_u)$ and (2.5), estimates of G_0 and $G^0_{\mathcal{D}}$ can be obtained from $\mathrm{col}(\hat{T}_y, \hat{T}_u)$, by solving the equation

$$\begin{bmatrix} I - \hat{G}\Delta_y & -(\hat{G}\Delta_u + \hat{G}_{\mathcal{D}}) \\ C_{\mathcal{I}} & I \end{bmatrix} \begin{bmatrix} \hat{T}_y \\ \hat{T}_u \end{bmatrix} = \begin{bmatrix} 0 \\ C_{\mathcal{I}} \end{bmatrix} \tag{2.8}$$

in $\hat{G}$ and $\hat{G}_{\mathcal{D}}$. Consistent estimates of G_0 and $G^0_{\mathcal{D}}$ are obtained through (2.8) if the estimates $\hat{T}_y$ and $\hat{T}_u$ are consistent.

Remark 2.3.1. *Notice that the first block row in* (2.8) *is sufficient for solving the equation in $\hat{G}$ and $\hat{G}_{\mathcal{D}}$, given $\hat{T}_y$ and $\hat{T}_u$. Moreover, if the controller transfer matrix $C_{\mathcal{I}}$ is known, then knowledge of either $\hat{T}_y$ or $\hat{T}_u$ is sufficient. Indeed, given $C_{\mathcal{I}}$, $\hat{T}_u$ can be obtained by solving the second block row in* (2.8) *given $\hat{T}_y$.* Vice versa*, the transfer matrix $\hat{T}_y$ can be obtained from $\hat{T}_u$.*

Not all the transfer functions in $G_{\mathcal{I}}^0$ have to be identified for controller design. The transfer matrix $C_{\mathcal{I}}$ (which captures dynamics of the distributed controller in the loop) is either known or its dynamics are not of interest in the considered identification problem. The dynamics of one subsystem $\mathcal{P}_i$, $i \in \mathcal{V}$, (or all subsystems) can be identified by solving a subset of the equations in (2.8), which also requires only the identification of a submatrix of $\mathrm{col}(T_y^0, T_u^0)$ (Gevers et al., 2018).

Let us elaborate on the indirect identification of G_i^0, $(G_{ij}^0)_{j\in\mathcal{N}_i}$. Consider a vector of reference signals $r_{\mathcal{D}} = \mathrm{col}_{j\in\mathcal{D}}\, r_j$, with $\mathcal{D} \subseteq \mathcal{V}$. Now, consider two transfer matrices of interest: the transfer matrix $T_{i,r_{\mathcal{D}}}^0$ from external reference signals $r_{\mathcal{D}}$ to y_i and the transfer matrix $T_{w_{\mathcal{N}},r_{\mathcal{D}}}^0$ from $r_{\mathcal{D}}$ to $w_{\mathcal{N}} := \mathrm{col}(u_i, w_{h_1}, \ldots, w_{h_L})$, $h_\bullet \in \mathcal{N}_i$. The main principle of the local indirect method is that estimates of G_i^0, $(G_{ij}^0)_{j\in\mathcal{N}_i}$ can be obtained, by post-processing estimates of $T_{i,r_{\mathcal{D}}}^0$ and $T_{w_{\mathcal{N}},r_{\mathcal{D}}}^0$. To estimate the transfer matrices from $r_{\mathcal{D}}$ to y_i and $w_{\mathcal{N}}$, we collect the output y_i, control input u_i, and the in-neighbour nodes in the predictor output $w_{\mathcal{Y}} = \mathrm{col}(y_i, u_i, w_{h_1}, \ldots, w_{h_L}) = \mathrm{col}(y_i, w_{\mathcal{N}})$. By minimizing the prediction error

$$\varepsilon_i(t,\theta) = \bar{H}_i(q,\theta)^{-1}\left(w_{\mathcal{Y}} - \begin{bmatrix} T_{i,r_{\mathcal{D}}} \\ T_{w_{\mathcal{D}},r_{\mathcal{D}}} \end{bmatrix}(q,\theta) r_{\mathcal{D}}\right), \tag{2.9}$$

in a least-squares sense, an estimate $(\hat{T}_{i,r_{\mathcal{D}}},\ \hat{T}_{w_{\mathcal{N}},r_{\mathcal{D}}})$ of $(T_{i,r_{\mathcal{D}}}^0, T_{w_{\mathcal{N}},r_{\mathcal{D}}}^0)$ is obtained. Since this is an open-loop identification problem, the noise model $\bar{H}_i$ in (2.9) does not have to be identified consistently for the consistent estimation of the tranfer functions of interest (hence, e.g. $\bar{H}_i = I$ can be chosen). The following result follows from Theorem 5.2, by Gevers et al. (2018).

Theorem 2.3.1. *Take $r_{\mathcal{D}} = \mathrm{col}(r_i, r_{h_1}, \ldots, r_{h_L})$, $h_\bullet \in \mathcal{N}_i$. If the estimate $(\hat{T}_{i,r_{\mathcal{D}}}, \hat{T}_{w_{\mathcal{N}},r_{\mathcal{D}}})$ of $(T_{i,r_{\mathcal{D}}}^0, T_{w_{\mathcal{N}},r_{\mathcal{D}}}^0)$ is consistent, then consistent estimates of G_i^0, $(G_{ij}^0)_{j\in\mathcal{N}_i}$ are given by*

$$\begin{bmatrix} \hat{G}_i & \hat{G}_{\mathcal{N}_i} \end{bmatrix} = \hat{T}_{i,r_{\mathcal{D}}} \hat{T}_{w_{\mathcal{N}},r_{\mathcal{D}}}^{-1}. \tag{2.10}$$

Consistent estimates of $(T_{i,r_{\mathcal{D}}}^0, T_{w_{\mathcal{N}},r_{\mathcal{D}}}^0)$ can be obtained via standard open-loop identification, which requires a full-order model for the elements of $T_{i,r_{\mathcal{D}}}^0$, $T_{w_{\mathcal{N}},r_{\mathcal{D}}}^0$ and $r_{\mathcal{D}}$ is required to be persistently exciting of a sufficiently high order (Ljung, 1999).

Example 2.2. *Let us return to the two interacting control loops in Example 2.1. For the identification of G_2^0 and G_{21}^0 through the indirect method, we can estimate the transfers $\mathrm{col}(r_1, r_2) \to \mathrm{col}(u_1, u_2)$ and $\mathrm{col}(r_1, r_2) \to y_2$. Consistent estimates $\hat{T}_{w_{\mathcal{N}},r_{\mathcal{D}}}$ and $\hat{T}_{2,r_{\mathcal{D}}}$ can be obtained with the prediction error (2.9), provided that (r_1, r_2) is persistently exciting of sufficiently high order. Consistent estimates of G_2^0 and G_{21}^0 are then obtained as $\begin{bmatrix} \hat{G}_2 & \hat{G}_{21} \end{bmatrix} = \hat{T}_{2,r_{\mathcal{D}}} \hat{T}_{w_{\mathcal{N}},r_{\mathcal{D}}}^{-1}$.*

The external signal vector $r_{\mathcal{D}}$ can be chosen in other ways than stated in Theorem 2.3.1, with $\mathcal{D}$ any subset of $\mathcal{V}$, cf. (Van den Hof and Ramaswamy, 2021), (Shi et al., 2022). This may yield a non-square matrix $\hat{T}_{w_{\mathcal{N}},r_{\mathcal{D}}}$, such that a consistent estimate is

$$\begin{bmatrix}\hat{G}_i & \hat{G}_{\mathcal{N}_i}\end{bmatrix} = \hat{T}_{i,r_{\mathcal{D}}}\hat{T}^{\dagger}_{w_{\mathcal{D}},r_{\mathcal{D}}},$$

where $\hat{T}^{\dagger}_{w_{\mathcal{N}},r_{\mathcal{D}}}$ is a right-inverse of $\hat{T}_{w_{\mathcal{N}},r_{\mathcal{D}}}$. It should be noted that the existence of the right inverse of $\hat{T}_{w_{\mathcal{N}},r_{\mathcal{D}}}$ requires a sufficient number of (measured) external excitation signals in $r_{\mathcal{D}}$ ($|\mathcal{D}| \geq |\mathcal{N}_i| + 1$) (Van den Hof and Ramaswamy, 2021).

2.3.3 Tailor-made parametrization

From the previous subsection, we have observed that the indirect identification of a plant in a closed-loop network requires two steps: (i) the identification of a transfer matrix from (a subset of) external signals to (a subset of) node signals (u, y) and (ii) the computation of the plant transfer functions through post-processing. In a classical closed-loop situation, the two steps of the indirect identification procedure can be combined into a single step, by employing a tailor-made parametrization where knowledge of the controller is used (Van den Hof, 1998). In this subsection, we will consider the identification in a closed-loop network with a tailor-made parametrization, where knowledge of the controller and the network structure is used in the parametrization.

Let us consider a predictor model where the outputs $(y_1, \ldots, y_L)$ are predicted on the basis of the external reference signals $(r_1, \ldots, r_L)$. A one-step-ahead predictor for the output y_i with predictor inputs $(r_1, \ldots, r_L)$ will be of the form

$$\hat{y}_i(t|t-1;\theta) := \sum_{j=1}^{L} T_{ij}(q,\theta) r_j(t), \tag{2.11}$$

with $T_{ij}(q,\theta)$ the parametrized transfer between r_j and y_i. Now, since the structure of the transfer r_j and y_i is known (due to (2.5)), this means that $T_{ij}(q,\theta)$ can be parametrized in terms of the parameters of $(G_i(\theta))_{i\in\mathcal{V}}$ and $(G_{ij}(\theta))_{(i,j)\in\mathcal{E}}$.

We will make this parametrization explicit. From (2.5), it follows that

$$y = G_0\Delta_y y + (G_0\Delta_u + G^0_{\mathcal{D}})C_{\mathcal{I}}(r - y) + H_0 e$$

and, hence, with $\bar{H}_0 := \left(I - G_0\Delta_y + (G_0\Delta_u + G^0_{\mathcal{D}})C_{\mathcal{I}}\right)^{-1} H_0$,

$$y = \underbrace{\left(I - G_0\Delta_y + (G_0\Delta_u + G^0_{\mathcal{D}})C_{\mathcal{I}}\right)^{-1}(G_0\Delta_u + G^0_{\mathcal{D}})C_{\mathcal{I}}}_{=:T_0} r + \bar{H}_0 e. \tag{2.12}$$

Therefore, given a parametrization $G(\theta)$ and $G_{\mathcal{D}}(\theta)$ of G_0 and $G_{\mathcal{D}}^0$, the parametrization $T(\theta)$ of T_0 in (2.12) can be taken as

$$T(\theta) = (I - G(\theta)\Delta_y + (G(\theta)\Delta_u + G_{\mathcal{D}}(\theta))C_{\mathcal{I}})^{-1} (G(\theta)\Delta_u + G_{\mathcal{D}}(\theta))C_{\mathcal{I}}, \quad (2.13)$$

such that the parametrized transfers $T_{ij}(\theta)$ in (2.11) are taken as the (i,j)-th entry of $T(\theta)$.

This leads to an identification problem $\arg\min_\theta V_{TM}(\theta)$, with

$$V_{TM}(\theta) := \sum_{i=1}^{L} \frac{1}{N} \sum_{t=0}^{N-1} \varepsilon_i^2(t,\theta), \;\; \varepsilon_i(t,\theta) = y_i(t) - \sum_{j=1}^{L} T_{ij}(q,\theta) r_j(t), \quad i \in \mathcal{V}. \quad (2.14)$$

Remark 2.3.2. *Notice that explicit knowledge of the experimental setup is used in the identification. Indeed, from* (2.13)*, we observe that in the tailor-made parametrization knowledge is used of the network structure, the fact that the plant operates in closed loop, and the dynamics of the distributed controller are used. This also implies that the distributed controller (at least the transfer matrix $C_{\mathcal{I}}$) should be known in order to compute the prediction error in* (2.14)*. In contrast, in the indirect method described in* (2.3.2)*, an unstructured estimate of T_0 is made in the first step, using no knowledge of the network structure or controller.*

Remark 2.3.3. *The noise filter $\bar{H}_0$ is not identified through the presented approach, but is implicitly modeled with a fixed noise model $\bar{H} = I$. Notice that there is no assumption on the correlation between noise signals e_i and e_j, $(i,j) \in \mathcal{V}^2$, $i \neq j$.*

Consistent estimates $G(\hat{\theta}_N)$ and $G_{\mathcal{D}}(\hat{\theta}_N)$ of G_0 and $G_{\mathcal{D}}^0$ are obtained if the power spectrum of r is positive definite for a sufficiently high number of frequencies, provided that r and e are uncorrelated and that the model set parametrized by (2.13) is uniformly stable[3]. We will not go into further detail on consistency of the tailor-made identification method here, since the tailor-made parametrization is particularly of interest for identification for controller design in case the system is *not* in the model set.

2.4 Network identification for distributed control

2.4.1 Control-oriented identification problem

In the previous section, the problem of identifying subsystems in a closed-loop setting with a distributed control architecture was considered. In this section,

[3]Uniform stability here refers to the stability of the (derivatives) of predictor filters and connectedness of the parameter set, cf. (Van den Hof, 1998) for more details.

we will consider the identification of subsystems $\mathcal{P}_i$, $i \in \mathcal{V}$, with the objective of designing a distributed controller. The control problem that is considered here is a distributed $\mathscr{H}_2$ control problem, which will be considered in Chapter 4 for the problem of synthesizing a distributed controller in a general setting. In the setting of this chapter this is a reference tracking problem, where the tracking errors of all subsystems should be minimal for any choice of reference signals. More specifically, for the interconnected system (2.2) with subsystems $\mathcal{P} = (\mathcal{P}_1, \dots, \mathcal{P}_L)$, the control problem is to find a distributed controller (2.3) with controllers $\mathcal{C} = (\mathcal{C}_1, \dots, C_L)$ such that $\|F(\mathcal{P}, \mathcal{C})\|_{\mathscr{H}_2}$ is minimal, where $F(\mathcal{P}, \mathcal{C})$ is the transfer $\mathrm{col}(r_1, \dots, r_L) \to \mathrm{col}(z_1, \dots, z_L)$ for the closed-loop network described by (2.2)-(2.3).

In Chapter 4, the dynamics of the interconnected system (2.2) are assumed to be known and a solution is provided to the distributed $\mathscr{H}_2$ control problem. In reality, these dynamics are typically not known, i.e., for each $i \in \mathcal{V}$, the transfer functions describing $\mathcal{P}_i$, G_i^0 and G_{ij}^0 with $j \in \mathcal{N}_i$, are not known. The model-based controller design in Chapter 4 therefore requires a model of the interconnected system, $\hat{\mathcal{P}} = (\hat{\mathcal{P}}_1, \dots, \hat{\mathcal{P}}_L)$.

Loosely speaking, the problem of concern is to identify a model $\hat{\mathcal{P}}$ of the interconnected system and to synthesize a distributed controller $\mathcal{C} = (\mathcal{C}_1, \dots, C_L)$ based on $\hat{\mathcal{P}}$ such that $\|F(\mathcal{P}, \mathcal{C})\|_{\mathscr{H}_2}$ is minimal, i.e., such that the closed-loop data-generating system achieves a high performance in terms of reference tracking. Of course, without further considerations, only $\|F(\hat{\mathcal{P}}, \mathcal{C})\|_{\mathscr{H}_2}$ is optimized in a model-based controller design. That is, the closed-loop interconnected system model achieves a high performance. Evaluation of the performance with a model-based design is possible via the performance inequality (Van den Hof and Schrama, 1995), cf. (Schrama, 1992):

$$\begin{aligned} \big| \|F(\hat{\mathcal{P}}, \mathcal{C})\|_{\mathscr{H}_2} - \|F(\mathcal{P}, \mathcal{C}) - F(\hat{\mathcal{P}}, \mathcal{C})\|_{\mathscr{H}_2} \big| & \\ \le \|F(\mathcal{P}, \mathcal{C})\|_{\mathscr{H}_2} & \\ \le \|F(\hat{\mathcal{P}}, \mathcal{C})\|_{\mathscr{H}_2} + \|F(\mathcal{P}, \mathcal{C}) - F(\hat{\mathcal{P}}, \mathcal{C})\|_{\mathscr{H}_2}, & \end{aligned} \tag{2.15}$$

where we can distinguish three main terms: the achieved performance $\|F(\mathcal{P}, \mathcal{C})\|_{\mathscr{H}_2}$, the designed performance $\|F(\hat{\mathcal{P}}, \mathcal{C})\|_{\mathscr{H}_2}$, and the performance degradation term $\|F(\mathcal{P}, \mathcal{C}) - F(\hat{\mathcal{P}}, \mathcal{C})\|_{\mathscr{H}_2}$.

The achieved performance is 'squeezed' in between the upper and lower bounds dictated by the triangular inequality, if the performance degradation is kept (relatively) small. The optimization of $\|F(\mathcal{P}, \mathcal{C})\|_{\mathscr{H}_2}$ through model-based design could therefore be performed by minimizing the upper-bound in (2.15), while taking into account that the performance degradation should be relatively small (with respect to the designed performance) (Van den Hof and Schrama, 1995). State-of-the-art methods for distributed controller design and network identification can either optimize the distributed controller or interconnected system model, but are not fit to optimize both simultaneously in the upperbound in (2.15)[4]. Iterative schemes for identification and control aim at minimizing the upperbound by alternately solving an identification problem and a control problem, cf. (Van den Hof and Schrama, 1995).

Given a model $(\hat{\mathcal{P}}_1, \ldots, \hat{\mathcal{P}}_L)$ of the interconnected system, $\|F(\hat{\mathcal{P}}, \mathcal{C})\|_{\mathscr{H}_2}$ is a distributed $\mathscr{H}_2$ controller criterion which should be optimized for $\mathcal{C}_1, \ldots, \mathcal{C}_L$. Given a distributed controller $\mathcal{C}_1, \ldots, \mathcal{C}_L$, the performance degradation $\|F(\mathcal{P}, \mathcal{C}) - F(\hat{\mathcal{P}}, \mathcal{C})\|_{\mathscr{H}_2}$ is a model mismatch criterion. Now, consider the problem of determining $\hat{\mathcal{P}}_1, \ldots, \hat{\mathcal{P}}_L$ from data such that the performance degradation is optimized, i.e., the problem

$$\min_{\hat{\mathcal{P}}_1, \ldots, \hat{\mathcal{P}}_L} \|F(\mathcal{P}, \mathcal{C}) - F(\hat{\mathcal{P}}, \mathcal{C})\|_{\mathscr{H}_2}. \tag{2.16}$$

Notice that this is a control-oriented identification problem with a criterion that is determined by the transfer matrix of interest F and the $\mathscr{H}_2$ performance metric. Problem (2.16) is an identification problem with a non-trivial identification criterion. This problem has been thoroughly analyzed in the field of identification for control for single-input-single-output systems in a standard control loop (Van den Hof and Schrama, 1995).

For the case of interconnected systems, (2.16) is a network identification problem. We distinguish two cases: (i) consistent identification of $\mathcal{P}_i$, $i \in \mathcal{V}$ (exact modeling) and (ii) identification of $\mathcal{P}_i$, $i \in \mathcal{V}$, with a model set that is not rich enough to capture all dynamics of the network (approximate modeling).

2.4.2 Exact modeling

Consider the direct method described in Section 2.3.1. As a consequence of Proposition 2.3.1, the sum of the asymptotic identification criteria $\bar{E}\varepsilon_i^2(t, \theta)$ leads to an identification criterion with a minimizing argument that also minimizes the control-oriented identification criterion in (2.16). Indeed, if the consistency conditions in Proposition 2.3.1 are satisfied for each i, then a consistent estimate $\hat{\mathcal{P}}_i$ equals $\mathcal{P}_i$, for all $i \in \mathcal{V}$. Hence, $(\hat{\mathcal{P}}_1, \ldots, \hat{\mathcal{P}}_L)$ clearly is a minimizing argument of $\|F(\mathcal{P}, \mathcal{C}) - F(\hat{\mathcal{P}}, \mathcal{C})\|_{\mathscr{H}_2}$. Similarly, it is clear that consistent estimates of $\hat{G}_i^0$, $(\hat{G}_{ij}^0)_{j \in \mathcal{N}_i}$ obtained through the indirect method form a model $(\hat{\mathcal{P}}_1, \ldots, \hat{\mathcal{P}}_L)$ that is a solution to (2.16).

[4]The optimization of $\|F(\mathcal{P}, \mathcal{C})\|_{\mathscr{H}_2}$ directly on the basis of data, instead of the upperbound, is a challenging problem on its own. This problem will be discussed in detail in Part II in a model-reference control setting.

2.4.3 Approximate modeling

The estimates that follow from the direct method are consistent and therefore solve the control-oriented identification problem. However, when the 'system in the model set' assumption is violated[5], then consistency is not guaranteed, which implies that an estimated model $\hat{P}$ is in general not a minimizing argument of $\|F(\mathcal{P}, \mathcal{C}) - F(\hat{\mathcal{P}}, \mathcal{C})\|_{\mathscr{H}_2}$. The question is then: how to identify $(\mathcal{P}_1, \ldots, \mathcal{P}_L)$ via approximate modeling, such the performance degradation is minimized? We will show that the identification criterion of the indirect identification method with tailor-made parametrization, described in Section 2.3.3, can be interpreted as the performance degradation term in (2.15) under conditions on the experimental setup.

Let us have a closer look at the performance degradation for an approximated model $\hat{\mathcal{P}}$. Let $\Delta F_{ij} := F_{ij}^0 - \hat{F}_{ij}$, where F_{ij}^0 is the transfer $r_j \to z_i$ of the closed-loop network (2.2)-(2.3)[6]. By the definition of the $\mathscr{H}_2$ norm, we can rewrite the performance degradation as

$$\begin{aligned}\|F(\mathcal{P}, \mathcal{C}) - F(\hat{\mathcal{P}}, \mathcal{C})\|_{\mathscr{H}_2} &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \operatorname{trace} \Delta F(e^{i\omega})^* \Delta F(e^{i\omega}) \, \mathrm{d}\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \Delta F_{11}^* \Delta F_{11} + \cdots + \Delta F_{L1}^* \Delta F_{L1} \\ &\quad + \Delta F_{12}^* \Delta F_{12} + \cdots + \Delta F_{L2}^* \Delta F_{L2} \\ &\qquad \vdots \\ &\quad + \Delta F_{1L}^* \Delta F_{1L} + \cdots + \Delta F_{LL}^* \Delta F_{LL} \, \mathrm{d}\omega \\ &= \sum_{j=1}^{L}\sum_{i=1}^{L} \frac{1}{2\pi}\int_{-\pi}^{\pi} \Delta F_{ij}^* \Delta F_{ij} \, \mathrm{d}\omega. \end{aligned} \tag{2.17}$$

In order to minimize the performance degradation, prediction-error identification with predictor inputs as reference signals $r_1, \ldots, r_L$ and predictor outputs as tracking errors $z_1, \ldots, z_L$ therefore appears to be a natural choice. Indeed, it can be shown that the prediction-error identification criterion[7] with a tailor-made parametrization from Section 2.3.3, can be interpreted as a performance degradation term.

[5]That is, there does not exist a θ_i^0 such that $G_i(\theta_i^0) = G_i^0$, $G_{ij}(\theta_i^0) = G_{ij}^0$ and $H_i(\theta_i^0) = H_i^0$. This is referred to as approximate modeling in the identification literature (system is not in the model set).

[6]Since $z_i = r_i - y_i$, F_{ij}^0 is directly related to the transfer T_{ij}^0 from r_j to y_i as $F_{ij}^0 = -T_{ij}^0$ for $i \neq j$ and $F_{ii}^0 = 1 - T_{ii}^0$.

[7]Notice that the output y_i is predicted in Section 2.3.3 instead of the tracking error z_i. This is equivalent, however, since the prediction error in (2.14) is $\varepsilon_i(t, \theta) = y_i - \hat{y}(t|t-1; \theta) = (r_i - \hat{y}(t|t-1; \theta)) - z_i$, where the first term is a prediction of z_i with a tailor-made parametrization.

Given the tailor-made parametrization $T_{ij}(\theta)$ of T_{ij}^0 as defined in (2.13), consider the parametrization $F_{ij}(\theta)$ of F_{ij}^0 as $F_{ij}(\theta) = -T_{ij}(\theta)$, $i \neq j$ and $F_{ii}(\theta) = 1 - T_{ii}(\theta)$. The following result provides an expression for the asymptotic identification criterion corresponding to the network identification problem (2.14).

Lemma 2.4.1. *Consider the asymptotic prediction-error identification criterion* $\bar{V}_{TM}(\theta) := \sum_{i=1}^{L} \bar{E}\varepsilon_i^2(t,\theta)$, *with the prediction errors defined in* (2.14). *If the conditions*

1. *for all* $(i,j) \in \mathbb{Z}^2_{[1:L]}$, r_i *is uncorrelated to* e_j,
2. *for all* $(i,j) \in \mathbb{Z}^2_{[1:L]}$, $i \neq j$, r_i *is uncorrelated to* r_j,

are satisfied, then

$$\bar{V}_{TM}(\theta) = \sum_{j=1}^{L}\sum_{i=1}^{L} \frac{1}{2\pi}\int_{-\pi}^{\pi} \Delta F_{ij}(\theta)^* \Delta F_{ij}(\theta)\Phi_{r_j} + E^*(\omega)(\bar{H}_i^0)^*\bar{H}_i^0 E(\omega)\,\mathrm{d}\omega,$$

where E denotes the Fourier transform of e.

Proof. Parseval's theorem provides an expression for $\bar{V}_{TM}$ in the frequency domain:

$$\bar{V}_{TM}(\theta) = \sum_{i=1}^{L} \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_{\varepsilon_i}(\omega)\,\mathrm{d}\omega.$$

By condition 1. and 2., it follows that

$$\Phi_{\varepsilon_i} = \sum_{j=1}^{L} |F_{ij}^0 - F_{ij}(\theta)|^2 \Phi_{r_j} + E^*(\omega)(\bar{H}_i^0)^*\bar{H}_i^0 E(\omega), \quad i \in \mathcal{V}.$$

Hence, we find that

$$\begin{aligned}\bar{V}_{TM}(\theta) &= \sum_{i=1}^{L} \frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_{j=1}^{L} |F_{ij}^0 - F_{ij}(\theta)|^2 \Phi_{r_j} + E^*(\omega)(\bar{H}_i^0)^*\bar{H}_i^0 E(\omega)\,\mathrm{d}\omega \\ &= \sum_{i=1}^{L}\sum_{j=1}^{L} \frac{1}{2\pi}\int_{-\pi}^{\pi} \Delta F_{ij}^*(\theta)\Delta F_{ij}(\theta)\Phi_{r_j} + E^*(\omega)(\bar{H}_i^0)^*\bar{H}_i^0 E(\omega)\,\mathrm{d}\omega,\end{aligned}$$

which yields the assertion. □

Two observations can be made: (i) the identification criterion $\bar{V}_{TM}$ consists of a part that depends on the parameter vector θ and a part that is independent of θ, and (ii) the identification criterion is tunable through Φ_{r_j}, i.e., through the choice of external reference signals. Under the conditions in Lemma 2.4.1, we find that the minimizing argument of $\bar{V}_{TM}$ is equal to

$$\begin{aligned}\arg\min_{\theta} \bar{V}_{TM}(\theta) &= \arg\min_{\theta} \sum_{i=1}^{L}\sum_{j=1}^{L} \frac{1}{2\pi}\int_{-\pi}^{\pi} \Delta F_{ij}^*(\theta)\Delta F_{ij}(\theta)\, \mathrm{d}\omega \\ &\quad + \sum_{i=1}^{L} \frac{1}{2\pi}\int_{-\pi}^{\pi} E^*(\omega)(\bar{H}_i^0)^*\bar{H}_i^0 E(\omega)\, \mathrm{d}\omega \\ &= \arg\min_{\theta} \sum_{i=1}^{L}\sum_{j=1}^{L} \frac{1}{2\pi}\int_{-\pi}^{\pi} \Delta F_{ij}^*(\theta)\Delta F_{ij}(\theta)\, \mathrm{d}\omega\end{aligned}$$

for constant reference spectra $\Phi_{r_i} = 1$, $i \in \mathcal{V}$. Hence, we find that a plant model $\hat{\mathcal{P}}$ obtained through the minimization of $\bar{V}_{TM}$ will be a minimizing argument of $\|F(\mathcal{P},\mathcal{C}) - F(\hat{\mathcal{P}},\mathcal{C})\|_{\mathscr{H}_2}$ by (2.17). That is, the parameter vector that minimizes the asymptotic identification criterion $\bar{V}_{TM}$ is a minimizing argument of the performance degradation.

Remark 2.4.1. *The rationale behind the prediction-error* (2.14) *is that the estimator bias in the case of approximate modeling is 'shaped' such that models G_i and G_{ij} give a small $\mathscr{H}_2$ norm for the transfer functions of interest for control. This idea has been explored in (Ljung, 1999, Section 13.5) for 'pulling' the bias towards models that give a small sensitivity in SISO control loops. It should be noted that if the model class is chosen such that condition 5. of Proposition 2.3.1 is satisfied (the system is in the model set), then there is no bias (asymptotically) and the identification approach in this subsection still yields consistent estimates of G_i^0 and G_{ij}^0, $j \in \mathcal{N}_i$, under sufficient excitation through r.*

2.4.4 Iterative identification and controller design

As mentioned in Section 2.4, the designed performance and performance degradation in (2.15) cannot be minimized simultaneously. Iterative schemes have been developed in the field of identification for control to alternately minimize the designed performance and performance degradation via controller design and control-oriented identification, respectively (Van den Hof and Schrama, 1995). Considering these iterative schemes, an scheme in which the interconnected system identification and distributed controller design are performed iteratively, has the form

$$(\hat{\mathcal{P}}_1, \ldots, \hat{\mathcal{P}}_L)(k+1) = \arg \min_{\tilde{\mathcal{P}}_1,\ldots,\tilde{\mathcal{P}}_L} \|F(\mathcal{P}, \mathcal{C}(k)) - F(\tilde{\mathcal{P}}, \mathcal{C}(k))\|_{\mathscr{H}_2}$$
$$(\mathcal{C}_1, \ldots, \mathcal{C}_L)(k+1) = \arg \min_{\tilde{\mathcal{C}}_1,\ldots,\tilde{\mathcal{C}}_L} \|F(\hat{\mathcal{P}}(k+1), \tilde{\mathcal{C}})\|_{\mathscr{H}_2}.$$

Here k denotes the iteration number. This iterative scheme can be interpreted as a 'windsurfer approach'[8] for the distributed controller design of interconnected systems.

The iterative control scheme can be initiated by performing experiments on the closed-loop network (2.2) with a preliminary (distributed) controller $\mathcal{C}(k)$. The plant models $(\hat{\mathcal{P}}_1, \ldots, \hat{\mathcal{P}}_L)(k+1)$ are obtained through identification with a control-oriented identification criterion as discussed in Section 2.4.3 (or Section 2.4.2 in the case of exact modeling). Based on the model $(\hat{\mathcal{P}}_1, \ldots, \hat{\mathcal{P}}_L)(k+1)$, a distributed or decentralized $\mathscr{H}_2$ controller $(\mathcal{C}_1, \ldots, \mathcal{C}_L)(k+1)$ can be designed such that $\|F(\hat{\mathcal{P}}(k+1), \tilde{\mathcal{C}})\|_{\mathscr{H}_2}$ is minimized, via the method described in Chapter 4. Iterations are continued by setting $k \to k+1$ and repeating the identification and controller design.

2.5 Conclusions

In this chapter we have considered the problem of identifying subsystems of an interconnected system for the design of a decentralized or distributed controller. We have first considered the identification of subsystems of a network in closed-loop with a (distributed) controller, yielding a network with a special structure. It has been shown that consistent estimates of the subsystems can be obtained through both direct and indirect network identification methods. When a tailor-made parametrization is used in the network identification, the structure of the closed-loop network and the controller dynamics can be explicitly taken into account in the predictor model. The identification criterion corresponding to the network identification method with a tailor-made parametrization can be interpreted as the performance degradation for an $\mathscr{H}_2$ performance criterion for reference tracking.

[8] The windsurfer approach is an adaptive robust control design method introduced by Anderson and Kosut (1991). It is motivated by the way humans learn windsurfing.

Appendix

2.A Derivation of a state-space representation

The transfer matrix $[G_i^0 \; G_{\mathcal{N}_i}]$ has a (left) matrix fraction description[9] (Hannan and Deistler, 1987)

$$[G_i^0 \; G_{\mathcal{N}_i}^0] = P_i^{-1}[Q_i \; Q_{\mathcal{N}}],$$

where P, Q and $Q_{\mathcal{N}}$ are polynomial matrices given by $P(q^{-1}) = I + P_1 q^{-1} + \cdots + P_l q^{-l}$, $Q(q^{-1}) = Q_0 + Q_1 q^{-1} + \cdots + Q_l q^{-l}$ and $Q_{\mathcal{N}}(q^{-1}) = Q_0^{\mathcal{N}} + Q_1^{\mathcal{N}} q^{-1} + \cdots + Q_l^{\mathcal{N}} q^{-l}$. The dynamics of $\mathcal{P}_i$ are therefore described by $z_i = r_i - y_i$ and

$$P(q^{-1})y_i(t) = \begin{bmatrix} Q(q^{-1}) & Q_{\mathcal{N}}(q^{-1}) \end{bmatrix} \begin{bmatrix} u_i(t) \\ w_{\mathcal{N}_i}(t) \end{bmatrix} + P_i(q^{-1})v_i(t)$$

or, equivalently, defining $\bar{y}_i := y_i - v_i$,

$$\begin{aligned} \bar{y}_i(t) + P_1\bar{y}_i(t-1) + \cdots + P_l\bar{y}_i(t-l) &= Q_0 u(t) + Q_1 u(t-1) + \cdots + Q_l u(t-l) \\ &+ Q_0^{\mathcal{N}} w_{\mathcal{N}_i}(t) + Q_1^{\mathcal{N}} w_{\mathcal{N}_i}(t-1) + \cdots + Q_l^{\mathcal{N}} w_{\mathcal{N}_i}(t-l). \end{aligned}$$

Define the vector signal $x_i(t) := \mathrm{col}(\bar{y}_i(t)), \bar{y}_i(t-1), \ldots, \bar{y}_i(t-l), \bar{u}_i(t), \ldots, \bar{u}_i(t-1), \ldots, \bar{u}_i(t-l))$, where $\bar{u}_i = \mathrm{col}(u_i, w_{\mathcal{N}_i})$, and define $\bar{P} := \mathrm{row}(-P_1, \ldots P_l)$ and $\bar{Q} := \mathrm{row}(Q_1, Q_1^{\mathcal{N}}, \ldots, Q_l, Q_l^{\mathcal{N}})$, such that

$$\bar{y}_i(t) = \begin{bmatrix} Q_0 & Q_0^{\mathcal{N}} \end{bmatrix} \bar{u}_i(t) + \begin{bmatrix} \bar{P} & \bar{Q} \end{bmatrix} x_i(t). \tag{2.18}$$

[9] A simple but rudimentary way of obtaining a matrix fraction description of a rational matrix G, is to write $G = p^{-1}Q$, where p is the least common multiple of the denominator polynomials of the entries of G.

From (2.18) it follows that a state-space representation of $\mathcal{P}_i$ is

$$\begin{aligned} x_i(t+1) &= \underbrace{\left[\begin{array}{cc|cc} \multicolumn{2}{c|}{\bar{P}} & \multicolumn{2}{c}{\bar{Q}} \\ \hline I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 \end{array}\right]}_{=:A_i} x_i(t) + \underbrace{\left[\begin{array}{c} Q_0 \\ \hline 0 \\ I \\ 0 \\ 0 \end{array}\right]}_{=:B_i} u_i(t) + \underbrace{\left[\begin{array}{c} Q_0^{\mathcal{N}} \\ \hline 0 \\ 0 \\ I \\ 0 \end{array}\right]}_{=:A_{\mathcal{N}_i}} w_{\mathcal{N}_i}(t), \\ y_i(t) &= \underbrace{\begin{bmatrix} \bar{P} & \bar{Q} \end{bmatrix}}_{=:C_i} x_i(t) + \underbrace{Q_0}_{=:D_i} u_i(t) + \underbrace{Q_0^{\mathcal{N}}}_{=:C_{\mathcal{N}_i}} w_{\mathcal{N}_i}(t) + v_i(t), \end{aligned} \tag{2.19}$$

with $z_i = r_i - y_i$.

Chapter 3

Distributed identification in dynamic networks

The identification of dynamic networks can typically be separated into the identification of several multi-input-single-output (MISO) systems. In this chapter, we separate the identification in dynamic networks further, by developing a novel approach to distributed identification of MISO systems. The distributed identification is discerned by the local estimation of local parameters, which correspond to a module in the MISO system. The local estimators are derived from the standard recursive least squares estimator and require limited information exchange. By Lyapunov's second method, sufficient conditions are derived for asymptotic convergence of the estimators to the true parameters in the absence of disturbances, which lead to asymptotic unbiasedness in the presence of additive output disturbances.

3.1 Introduction

Prediction-error identification methods provide a powerful tool for obtaining consistent system parameter estimates (Ljung, 1999). However, when dealing with large scale interconnected systems, such as the ones arising from biology or power grids, the identification problem becomes more challenging. Given a network of linear dynamical systems, various prediction error methods are readily operational for identifying these systems (Rao et al., 1984), (Van den Hof et al., 2013).

This chapter is based on the publication: T. R. V. Steentjes, M. Lazar, and P. M. J. Van den Hof. A recursive estimation approach to distributed identification of large-scale multi-input-single-output FIR systems. *IFAC-PapersOnLine*, 51(23):236 – 241, 2018. 7^{th} IFAC Workshop on Distributed Estimation and Control in Networked Systems

The identification problem of such large-scale systems can typically be separated into multiple-input-single-output (MISO) identification problems, in the case that the process disturbances are uncorrelated over the channels (Rao et al., 1984), (Van den Hof et al., 2013). More precisely, identification of a large-scale system can be performed via the identification of MISO building blocks, on the basis of measurements of multiple inputs and one, possibly disturbed, output. Figure 3.1 shows such a MISO building block. In the case that the process noise is correlated over the channels, confounding variables can require the addition of inputs or outputs to the predictor model, leading to extended MISO (Dankers et al., 2016) or MIMO (Ramaswamy and Van den Hof, 2021) identification problems.

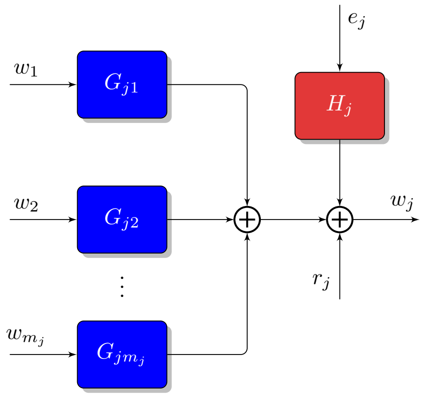

Figure 3.1: MISO system interconnection with $m_j \in \mathbb{N}$ subsystems.

Although existing prediction error methods for dynamical networks can consistently identify local modules (single-input-single-output (SISO) systems), the identification problem that has to be solved is typically a MISO identification problem, with measurement data required to be available centrally for parameter estimation. Central data collection and computation of the module estimates may not always be desirable due to computational constraints or desired flexibility. A further decomposition of the MISO identification problem into SISO identification problems to reduce computational complexity was also suggested in (Rao et al., 1984). Therein, it was proposed to perform a decomposition of the parameter estimation via a Gauss-Seidel like algorithm, but a proof of convergence is absent.

Distributed estimation has caught a vast amount of attention in the literature. Existing approaches can be divided into two distinct classes. The first class consists of consensus based methods, discerned by collaborative estimation of a global (common) parameter vector that is performed via a number of interconnected estimators (Mateos and Giannakis, 2012), (Papusha et al., 2014). The second class is also enabled by collaborative estimation via interconnected estimators. Therein each estimator is, however, concerned with the estimation of a local parameter vector. We refer to the results derived for parameter estimation in static large-scale systems (Marelli and Fu, 2015), distributed state estimation via moving-horizon methods (Farina et al., 2010) and distributed identification via ADMM (Hansson and Verhaegen, 2014).

In this work, we consider the distributed identification of transfer function modules in dynamic networks. We consider that the identification problem can be separated into MISO identification problems, which holds true under the assumption that noise processes are uncorrelated over the channels. To the best of the author's knowledge, the distribution of the identification of modules in interconnected systems has only been considered by Hansson and Verhaegen (2014) in the optimization layer through ADMM; a method that performs alternating optimization steps and can often be successfully applied, but has no guarantee for convergence (Hansson and Verhaegen, 2014). The distribution of the identification of modules in dynamic networks is therefore a largely unsolved problem in the literature.

We develop a distributed solution for the MISO prediction error identification problem (Van den Hof et al., 2013). Predictor models are employed that are linear with respect to the parameters and can be specified by orthogonal basis functions (Heuberger et al., 2005). These models are chosen to be linear because of the explicit solution to the corresponding least squares problem for identification, which allows recursive parameter estimation via recursive least squares methods. The use of orthogonal basis functions allows the specification of model structures such as FIR or ARX, while more advanced basis functions can improve the accuracy of the model (Heuberger et al., 2005). These predictor models serve as a basis for the developed distributed identification method. The distributed identification scheme is composed of local recursive estimators that are coupled with local SISO modules. Intercommunication of the local estimators is accomplished through the transmission of scalar signals between recursions via a mutual fusion center.

3.2 Preliminaries

The sets of non-negative integers and non-negative reals are denoted by $\mathbb{N}$ and $\mathbb{R}_{\geq 0}$, respectively. Given $a \in \mathbb{Z}$, $b \in \mathbb{Z}$ such that $a < b$, we denote $\mathbb{Z}_{[a:b]} := \{a, a+1, \dots, b-1, b\}$. Let $I_n \in \mathbb{R}^{n \times n}$ denote the identity matrix. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to belong to class $\mathcal{K}$ ($\alpha \in \mathcal{K}$), if it is continuous, strictly increasing and $\alpha(0) = 0$. It is said to belong to class $\mathcal{K}_\infty$ ($\alpha \in \mathcal{K}_\infty$), if additionally $\alpha(r) \to \infty$ as $r \to \infty$. For an $x \in \mathbb{R}^n$, let $\|x\|_2$, or simply $\|x\|$, denote the 2-norm of x.

3.2.1 Concepts from Lyapunov theory

Consider the discrete-time, time-varying system

$$x(t+1) = f(x(t), t), \quad x_0 := x(t_0),\ t_0 \in \mathbb{N}, \tag{3.1}$$

with $f : \mathbb{R}^n \times \mathbb{N} \to \mathbb{R}^n$, $f(0, \cdot) = 0$. Let the solution of (3.1) initialized in $x_0 \in \mathbb{R}^n$ at time $t_0 \in \mathbb{N}$ be denoted by $s(t, t_0, x_0)$.

Definition 3.2.1. *The origin equilibrium of* (3.1) *is called stable if for each* $\varepsilon > 0$ *and each* $t_0 \in \mathbb{N}$*, there exists* $\delta = \delta(\varepsilon, t_0)$ *so that*

$$\|x_0\| < \delta \;\Rightarrow\; \|s(t, t_0, x_0)\| < \varepsilon, \quad \forall t \geq t_0.$$

Definition 3.2.2. *The origin equilibrium of* (3.1) *is called attractive if there is a* $\delta > 0$ *such that*

$$\begin{aligned} &\textit{For each } \varepsilon > 0 \textit{ there exists } T = T(\varepsilon, t_0) \textit{ such that} \\ &\|x_0\| < \delta \;\Rightarrow\; \|s(t, t_0, x_0)\| < \varepsilon, \quad \forall t \geq t_0 + T. \end{aligned} \tag{3.2}$$

By the definition of a function limit at infinity, (3.2) is equivalent with:

$$\|x_0\| < \delta \;\Rightarrow\; \|s(t, t_0, x_0)\| \to 0 \text{ as } t \to \infty.$$

Definition 3.2.3. *The origin equilibrium of* (3.1) *is called globally attractive if*

$$x_0 \in \mathbb{R}^n \;\Rightarrow \|s(t, t_0, x_0)\| \to 0 \textit{ as } t \to \infty.$$

Definition 3.2.4. *The origin equilibrium of* (3.1) *is called asymptotically stable if it is stable and attractive.*

Definition 3.2.5. *The origin equilibrium of* (3.1) *is called globally asymptotically stable if it is stable and globally attractive.*

Theorem 3.2.1. *The origin is a stable equilibrium of* (3.1) *if there is a function* $W : \mathbb{R}^n \times \mathbb{N} \to \mathbb{R}$*, so that*

$$k_1(\|\xi\|) \leq W(\xi, \tau) \leq k_2(\|\xi\|, \tau), \quad \forall (\xi, \tau) \in \mathbb{R}^n \times \mathbb{N}, \tag{3.3}$$

$$\Delta W(\xi, \tau) \leq 0, \quad \forall (\xi, \tau) \in \mathbb{R}^n \times \mathbb{N}, \tag{3.4}$$

with $k_1 \in \mathcal{K}_\infty$*,* $k_2(\cdot, \tau) \in \mathcal{K}_\infty$ *for each* $\tau \in \mathbb{N}$ *and* $\Delta W(\xi, \tau) := W(f(\xi, \tau), \tau + 1) - W(\xi, \tau)$*.*

Proof. See Appendix 3.A. □

Theorem 3.2.2. *The origin is a globally asymptotically stable equilibrium of* (3.1) *if there is a function* $W : \mathbb{R}^n \times \mathbb{N} \to \mathbb{R}$ *so that*

$$k_1(\|\xi\|) \leq W(\xi, \tau) \leq k_2(\|\xi\|, \tau), \quad \forall(\xi, \tau) \in \mathbb{R}^n \times \mathbb{N}, \tag{3.5}$$

$$\Delta W(\xi, \tau) \leq -k_3(\|\xi\|), \quad \forall(\xi, \tau) \in \mathbb{R}^n \times \mathbb{N}, \tag{3.6}$$

with $k_1 \in \mathcal{K}_\infty$*,* $k_2(\cdot, \tau) \in \mathcal{K}_\infty$ *for each* $\tau \in \mathbb{N}$ *and* $k_3 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ *a continuous and positive definite function with* $k_3(0) = 0$*.*

Proof. See Appendix 3.B. □

Note the absence of a uniform upperbound on W in Theorem 3.2.2. This avoids the need for a uniform lower bound on, or termination of gain/covariance matrix recursions as in (Mendel, 1973), (Udink ten Cate, 1979), for proving convergence of the recursive estimation scheme in Section 3.6.

Definition 3.2.6. *A function* $W : \mathbb{R}^n \times \mathbb{N} \to \mathbb{R}$ *that satisfies* (3.5) *and* (3.6) *is called a Lyapunov function for* (3.1)*.*

3.3 Identification in dynamic networks

We consider the system setup of Van den Hof et al. (2013): a dynamic network that describes the relations among L internal variables w_i, $i \in \{1, \dots, L\} =: \mathcal{V}$ on a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, with edges $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$:

$$w_j(t) = \sum_{i \in \mathcal{N}_j} G^0_{ji}(q) w_i(t) + r_j(t) + v_j(t), \quad j \in \mathcal{V}, \tag{3.7}$$

where

- q^{-1} is the delay operator, i.e., $q^{-1}x(t) = x(t-1)$,
- $\mathcal{N}_j \subseteq \mathcal{V}$ is the set of nodes $i \in \mathcal{V}$ for which $G^0_{ji} \neq 0$, i.e., for which $(i, j) \in \mathcal{E}$, where G^0_{ji} is a rational and strictly proper transfer function,
- v_j is an unmeasured process noise signal. The process $v := \mathrm{col}(v_1, \dots, v_L)$ is modeled as a stationary stochastic process with a rational and diagonal spectral density matrix Φ_v (uncorrelated over channels), such that there exist stationary zero-mean white-noise processes e_j with variance σ_j^2 and transfer functions H^0_j that are monic, stable and minimum phase, satisfying $v_j(t) = H^0_j(q) e_j(t)$.

Collecting all network variables w_j, r_j and v_j in vectors yields an expression

$$\begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_L \end{bmatrix} = \begin{bmatrix} 0 & G_{12}^0 & \cdots & G_{1L}^0 \\ G_{21}^0 & 0 & \cdots & G_{2L}^0 \\ \vdots & \vdots & \ddots & \vdots \\ G_{L1}^0 & G_{L2}^0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_L \end{bmatrix} + \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_L \end{bmatrix} + \begin{bmatrix} H_1 e_1 \\ H_2 e_2 \\ \vdots \\ H_L e_L \end{bmatrix}, \tag{3.8}$$

which describes the dynamics of the network. With straightforward vector and matrix notations, (3.8) is equivalently written as

$$w = G^0 w + r + H^0 e.$$

We will assume throughout this chapter that the network is well-posed, i.e., $(I - G^0)^{-1}$ exists, and that $(I - G^0)^{-1}$ is stable.

3.3.1 Direct method in dynamic network identification

Given the description of the dynamic network in (3.7), each scalar internal variable w_j, is described by a multi-input-single-output (MISO) system

$$w_j(t) = \sum_{i \in \mathcal{N}_j} G_{ji}^0(q) w_i(t) + r_j(t) + H_j^0(q) e_j(t).$$

This MISO structure is the starting point for the distributed identification method that is developed in this chapter. In the direct method (Van den Hof et al., 2013), a module G_{ji}^0 (for some $i \in \mathcal{N}_j$ is identified by identifying all G_{jl}^0, $l \in \mathcal{N}_j$ simultaneously. That is, by solving a single optimization problem

$$\min_\theta V_j(\theta), \quad V_j(\theta) := \sum_{t=0}^{N-1} \varepsilon_j^2(t, \theta), \tag{3.9}$$

where ε_j is the prediction error, $\varepsilon(t,\theta) := w_j(t) - \hat{w}_j(t|t-1;\theta)$, based on the one-step ahead predictor $\hat{w}_j$, defined by (Van den Hof et al., 2013)

$$\begin{aligned} \hat{w}_j(t|t-1;\theta) = \big(1 - H_j^{-1}(q,\theta)\big)\, w_j(t) \\ + H_j^{-1}(q,\theta)\Big(\sum_{i \in \mathcal{N}_j} G_{ji}(q,\theta) w_i(t) + r_j(t)\Big), \end{aligned} \tag{3.10}$$

where $G_{ji}(q,\theta)$ and $H_j(q,\theta)$ are models of G_{ji}^0 and H_j^0, with $H_j(q,\theta)$ a monic transfer function and θ the parameter vector.

In this chapter, we consider predictors $\hat{w}_j$ which are linear with respect to the parameters. This requires that $H_j^{-1}(q,\theta)G_{ji}(q,\theta)$ and $H_j^{-1}(q,\theta)$ are linear in the parameters, such that the predictor (3.10) becomes

$$\begin{aligned}\hat{w}_j(t|t-1;\theta) &= \sum_{k=1}^{n_a} a_k F_k^j(q)(r_j(t)-w_j(t)) + \sum_{i\in\mathcal{N}_j}\sum_{k=0}^{n_b^i} b_k^i L_k^j(q) w_i(t) + r_j(t)\\ &= \varphi_j^\top(t)\theta_j + \sum_{i\in\mathcal{N}_j}\varphi_i^\top(t)\theta_i + r_j(t), \qquad (3.11)\end{aligned}$$

where the monicity of $H_j(q,\theta)$ is used. In the predictor expression (3.11), the parameter vector is $\theta = \mathrm{col}_{l\in\mathcal{N}_j\cup\{j\}}\,\theta_l$, where $\theta_j := \mathrm{col}(a_1,\ldots,a_{n_a})$ and $\theta_i := \mathrm{col}(b_1^i,\ldots,b_{n_b^i}^i)$, $i\in\mathcal{N}_j$, and F_k^j, L_k^j are (orthogonal) basis functions, cf. (Van den Hof et al., 1995), (Heuberger et al., 2005). In prediction-error identification, a set of basis functions $(F_k)_{k\in\{1,\ldots,n\}}$ is typically chosen as a set of orthonormal basis functions, where orthonormality is reflected by the property

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} F_k(e^{i\omega})F_k(e^{-i\omega})\,\mathrm{d}\omega = 1 \quad\text{and}\quad \frac{1}{2\pi}\int_{-\pi}^{\pi} F_k(e^{i\omega})F_l(e^{-i\omega})\,\mathrm{d}\omega = 0,\ k\neq l.$$

A special choice of basis functions is $F_k(q) = q^{-k}$, for which the model structure becomes an ARX or FIR model structure (which will be shown in Example 3.1 and 3.2 for the predictor (3.11)). Since the accuracy of the model is limited by the choice of basis functions (Van den Hof et al., 1995), more advanced basis functions have been developed in the literature, such as Laguerre functions, Kautz functions and generalizations of the aforementioned functions (Heuberger et al., 2005). Let us briefly exemplify the predictor via two model structures for G_{ji} and H_j that lead to a linear prediction expression of the form (3.11).

Example 3.1 (ARX model structure)**.** *In an ARX model structure, parametrized models are transfer functions that are rational with respect to the delay operator, such that*

$$\begin{aligned}G_{ji}(q,\theta) &= \frac{B_i(q,\theta)}{A(q,\theta)}, \quad i\in\mathcal{N}_j,\\ H_j(q,\theta) &= \frac{1}{A(q,\theta)},\end{aligned}$$

where

$$\begin{aligned}A(q,\theta) &= 1 + a_1q^{-1} + a_2q^{-2} + \cdots + a_{n_a}q^{-n_a},\\ B_i(q,\theta) &= b_0^i + b_1^iq^{-1} + b_2^iq^{-2} + \cdots + b_{n_b^i}q^{-n_b^i},\end{aligned}$$

with $\theta = \mathrm{col}_{l\in\mathcal{N}_j\cup\{j\}}\,\theta_l$ *and* θ_j, θ_i *as defined in the paragraph following* (3.11)*. It follows by* (3.10) *that*

$$\hat{w}_j(t|t-1;\theta) = \sum_{k=1}^{n_a} a_k q^{-k}(r_j(t) - w_j(t)) + \sum_{i\in\mathcal{N}_j}\sum_{k=0}^{n_b^i} b_k^i q^{-k} w_i(t) + r_j(t),$$

which can be written as (3.11) *with* $F_k^j(q) = L_k^j(q) = q^{-k}$*. The second equality in* (3.11) *follows with*

$$\begin{aligned}\varphi_j^\top(t) &= \begin{bmatrix} r_j(t-1) - w_j(t-1) & \cdots & r_j(t-n_a) - w_j(t-n_a)\end{bmatrix},\\ \varphi_i^\top(t) &= \begin{bmatrix} w_i(t) & w_i(t-1) & \cdots & w_i(t-n_b^i)\end{bmatrix}, \quad i\in\mathcal{N}_j.\end{aligned}$$

Example 3.2 (FIR model structure)**.** *An FIR model structure is a special, but important, case of an ARX model structure where* $A(q,\theta) = 1$*. It follows that the predictor* (3.10) *is*

$$\hat{w}_j(t|t-1;\theta) = \sum_{i\in\mathcal{N}_j}\sum_{k=0}^{n_b^i} b_k^i q^{-k} w_i(t) + r_j(t) = \sum_{i\in\mathcal{N}_j} \varphi_i^\top(t)\theta_i + r_j(t),$$

with θ_i *and* φ_i *as defined in Example 3.1.*

Because the predictor (3.11) has the property of being linear-in-the-parameters, the least-squares identification problem (3.9) has a closed-form solution:[1]

$$\hat{\theta} := \arg\min_{\theta} V_j(\theta) = (\Phi^\top\Phi)^{-1}\Phi^\top \mathbf{w}_j, \tag{3.12}$$

with $\mathbf{w}_j$ a vectorized form of w_j, i.e., $\mathbf{w}_j := \mathrm{col}(w_j(0),\dots,w_j(N-1))$, and $\Phi^\top := (\varphi(0),\dots,\varphi(N-1))$ with $\varphi(t) := \mathrm{col}_{l\in\mathbb{N}_j\cup\{j\}}$. The parameter estimate $\hat{\theta}$ is referred to as the *least squares estimator* (LSE). The covariance matrix of the LSE is $\Sigma = \sigma_j^2(\Phi^\top\Phi)^{-1}$ (Kay, 1993).

3.3.2 Recursive least squares

In practice, computing the LSE can be undesirable when all the data $\mathbf{w}_j$ and Φ are not available at once or when (3.12) is computationally intractable, for example. Instead, one can use a recursive LSE (Kay, 1993), which updates the LSE each time new data is available.

Let $\hat{\theta}(k)$ denote the LSE of θ based on $k+1$ data samples of the output node $\mathbf{w}_j(k) = (w_j(0) \ \cdots \ w_j(k))^\top$ and regressor $\Phi(k) = (\varphi(0) \ \cdots \varphi(k))^\top$. The

[1] We omit r_j here and in the sequel for clarity of exposition. The reasoning throughout this chapter is identical for the case where r_j is non-zero.

recursive LSE reads as follows (Kay, 1993). First, compute the "batch" estimator $\hat{\theta}(k)$ for $k \in \mathbb{N}$:

$$\begin{aligned}\hat{\theta}(k) &= (\Phi(k)^\top \Phi(k))^{-1}\Phi(k)^\top \mathbf{w}_j(k), \\ \Sigma(k) &= \sigma_j^2(\Phi(k)^\top \Phi(k))^{-1}.\end{aligned} \tag{3.13}$$

When new data is available, update the estimator according to

$$\begin{aligned}\hat{\theta}(k+1) &= \hat{\theta}(k) + \alpha(k)\Sigma(k)\varphi(k+1)(w_j(k+1) - \varphi^\top(k+1)\hat{\theta}(k)), \\ \alpha(k) &:= \frac{1}{\sigma_j^2 + \varphi^\top(k+1)\Sigma(k)\varphi(k+1)}.\end{aligned} \tag{3.14}$$

The covariance matrix of the updated LSE is

$$\Sigma(k+1) = (I - \alpha(k)\Sigma(k)\varphi(k+1)\varphi^\top(k+1))\Sigma(k). \tag{3.15}$$

Remark 3.3.1. *The recursive LSE and covariance matrix can be written in a more compact form, using the prediction error definition in* (3.14) *and applying the matrix inversion lemma (Guttman, 1946) to* $\Sigma(k+1)$ *in* (3.15), *as*

$$\hat{\theta}(k+1) = \hat{\theta}(k) + \alpha(k)\Sigma(k)\varphi(k+1)\varepsilon_j(k+1, \hat{\theta}(k)), \tag{3.16}$$

$$\Sigma^{-1}(k+1) = \Sigma^{-1}(k) + \frac{1}{\sigma_j^2}\varphi(k+1)\varphi^\top(k+1), \tag{3.17}$$

respectively, cf. (Kay, 1993).

Remark 3.3.2. *One can avoid the computation of a batch LSE* (3.13) *completely, by initialization of the recursive LSE* (3.16) *from "scratch" with* $\hat{\theta}(-1) = 0$ *and* $\Sigma(-1) = cI$, *with* $c \in \mathbb{R}_{\geq 0}$ *(Kay, 1993).*

3.4 Problem formulation

Given the MISO prediction error identification problem described in Section 3.3.1, central collection of $m_j = |\mathcal{N}_j|$ node signals w_i and one node signal w_j is required[2] for the central computation of $\hat{\theta}$, using either the LSE (3.12) or the recursive LSE (3.16). From a distributed point of view, however, local module parameter estimators $\hat{\theta}_i$ for θ_i, may be preferred, due to computational or communication constraints. We will refer to the concept of distributed identification, as the local parameter estimation for G_{ji} via a local identification module, with intercommunication between local identification modules. The distributed identification

[2]We remark that if all w_i's are uncorrelated, then SISO identification (without modeling other subsystems) provides consistent estimates. This will lead to increased variance, however.

concept is illustrated in Figure 3.2: Each subsystem G_{ji}, $i \in \mathcal{N}_j$ is coupled with an identification module $\mathcal{I}_i$, which measures node signal w_i and is connected to some module $\mathcal{B}$. Module I_j is an identification module that aims at modeling the noise filter H_j (if parametrized). Module $\mathcal{B}$ describes the relation between sent and received signals of all modules $\mathcal{I}_l$, $l \in \mathcal{N}_j \cup \{j\}$. Given this distribution, two problems arise, related to the local identification and communication. Firstly, is there an $\mathcal{I}_l$ that arrives at unbiased estimates of the true parameter θ_l^0? Consequently, if the answer is affirmative, what signals have to be shared between the identification modules $\mathcal{I}_l$, $l \in \mathcal{N}_j \cup \{j\}$, i.e., what should $\mathcal{B}$ describe?

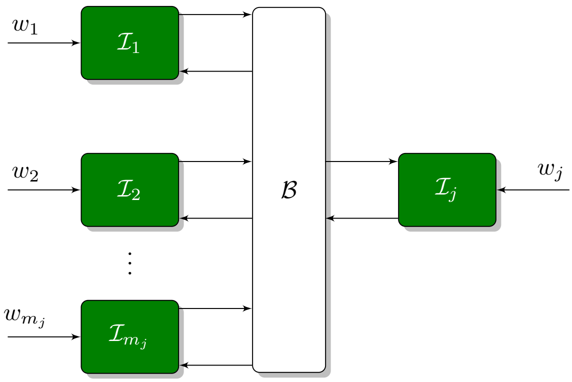

Figure 3.2: Distributed identification scheme with identification modules $\mathcal{I}_i$, $i \in \mathcal{N}_j \cup \{j\}$ and communication module $\mathcal{B}$.

Since the data matrix Φ is in general non-sparse, the identification problem $\min_\theta V_j(\theta)$ is in general non-separable. Therefore, it is not clear how the LSE (3.12) can be adopted in a distributed identification scheme. The recursive LSE, however, can be advantageous for the distribution of the parameter estimation. Indeed, one can exploit structures for the parameter covariance matrix $\Sigma(k)$, such as diagonal or block-diagonal structures, in order to "separate" the estimation problem w.r.t. θ_l, $l \in \mathcal{N}_j \cup \{j\}$. Finally, asymptotic unbiasedness of the developed distributed identification procedure should be assessed, i.e., we need to verify whether $\lim_{k\to\infty} \mathrm{E}\,\hat{\theta}_l(k) \to \theta_l^0$, where $\hat{\theta}_l(k)$ denotes the proposed estimator for θ_l^0 based on $k+1$ data samples.

3.5 Distributed estimation algorithm

Inspired by the recursive LSE (3.16), we develop a distributed recursive estimator: for each $i \in \mathbb{N}_{[1:m_j]}$, let the local parameter estimate $\hat{\theta}_i : \mathbb{N} \to \mathbb{R}^{n_i}$ of θ_i^0 be defined recursively by

$$\hat{\theta}_i(k+1) = \hat{\theta}_i(k) + \alpha_i(k)\Sigma_i(k)\varphi_i(k+1)(w_j(k+1) - \sum_{l\in\mathcal{N}_j\cup\{j\}} \varphi_l(k+1)\hat{\theta}_l(k)), \tag{3.18}$$

with $\alpha_i : \mathbb{N} \to \mathbb{R}$ and $\Sigma_i : \mathbb{N} \to \mathbb{R}^{n_i\times n_i}$. Comparing the local estimator update equation (3.18) with the recursive LSE (3.16), the matrix Σ_i has the interpretation of a local covariance matrix of $\hat{\theta}_i$. Let $\Sigma_i : \mathbb{N} \to \mathbb{R}^{n_i\times n_i}$ be defined recursively by

$$\Sigma_i^{-1}(k+1) = \Sigma_i^{-1}(k) + \frac{1}{\gamma_i^2(k)}\varphi_i(k+1)\varphi_i^\top(k+1), \tag{3.19}$$

with $\gamma_i : \mathbb{N} \to \mathbb{R}$. The scalars $\alpha_i(k)$ and $\gamma_i(k)$ are related to sufficient conditions for consistency of estimator (3.18), which will be provided in Section 3.6.

Consider the stacked vector $\hat{\theta}_B(k) := \mathrm{col}_{l\in\mathcal{N}_j\cup\{j\}}\,\hat{\theta}_l(k)$. Define $A_B(k) := \mathrm{diag}_{l\in\mathcal{N}_j\cup\{j\}}\,\alpha_l(k)I_{n_l}$ and let $\Gamma_B(k) := \mathrm{diag}_{l\in\mathcal{N}_j\cup\{j\}}\,\gamma_l(k)I_{n_l}$. Accordingly, let

$$\begin{aligned}\Sigma_B(k) &:= \mathrm{diag}_{l\in\mathcal{N}_j\cup\{j\}}\,\Sigma_l(k),\\ \varphi_B(k) &:= \mathrm{diag}_{l\in\mathcal{N}_j\cup\{j\}}\,\varphi_l(k)\varphi_l^\top(k).\end{aligned}$$

For the estimator update we can then write

$$\hat{\theta}_B(k+1) = \hat{\theta}_B(k) + A_B(k)\Sigma_B(k)\varphi(k+1)\varepsilon(k+1,\hat{\theta}_B(k))$$

with

$$\Sigma_B^{-1}(k+1) = \Sigma_B^{-1}(k) + \Gamma_B^{-2}(k)\varphi_B(k+1).$$

The latter equations seem to resemble (3.16) and (3.17), which describe the recursive LSE. Note, however, that the matrix Σ_B is block diagonal, while the covariance matrix Σ is dense, in general.

Now, let identification module $\mathcal{I}_i$ be described by (3.18) and (3.19) so that

$$\mathcal{I}_i : \begin{cases} \hat{\theta}_i(k+1) = \hat{\theta}_i(k) + \alpha_i(k)\Sigma_i(k)\varphi(k+1)\varepsilon_j(k+1,\hat{\theta}_B(k)) \\ \Sigma_i^{-1}(k+1) = \Sigma_i^{-1}(k) + \frac{1}{\gamma_i^2(k)}\varphi_i(k+1)\varphi_i^\top(k+1). \end{cases}$$

Writing the distributed estimator (3.18) as

$$\begin{aligned}&\hat{\theta}_i(k+1) = \hat{\theta}_i(k) + \alpha_i(k)\Sigma_i(k)\varphi_i(k+1)\\ &\cdot(w_j(k+1) - \varphi_i^\top(k+1)\hat{\theta}_i(k) - \sum_{l\in(\mathcal{N}_j\cup\{j\})\setminus\{i\}} \varphi_l^\top(k+1)\hat{\theta}_l(k)),\end{aligned}$$

it becomes apparent what information exchange is required between identification modules, assuming that each module $\mathcal{I}_i$ can measure node $w_i(t)$ and (indirectly) receive node $w_j(t)$. The local recursive estimator $\hat{\theta}_i(k+1)$ depends on an "autonomous" part plus a contribution from other identification modules $\mathcal{I}_l$, $l \in (\mathcal{N}_j \cup \{j\}) \setminus \{i\}$. The inputs from other subsystems and parameter vectors $\hat{\theta}_l$ are not required to be known. Indeed, only the scalar products $\varphi_l^\top(k+1)\hat{\theta}_l(k) \in \mathbb{R}$ need to be known, for all $l \in (\mathcal{N}_j \cup \{j\}) \setminus \{i\}$, which we will refer to as the local predictions. The appropriate communication can be achieved if, for example, at every time step, each $\mathcal{I}_i$ sends the local prediction $\varphi_i^\top(k+1)\hat{\theta}_i(k) \in \mathbb{R}$ to all other identification modules $\mathcal{I}_l$, $l \in (\mathcal{N}_j \cup \{j\}) \setminus \{i\}$ with a corresponding definition for $\mathcal{B}$. The latter corresponds to an all-to-all communication, however, and can be inefficient for a large number of identification modules, i.e., for large m_j. One can instead consider $\mathcal{B}$ to be described by the static relation

$$\mathcal{B}: \quad \varepsilon_j(k+1, \hat{\theta}_B(k)) = w_j(k+1) - \varphi_j^\top(k+1)\hat{\theta}_j(k) - \sum_{i \in \mathcal{N}_i} \varphi_i^\top(k+1)\hat{\theta}_i(k)$$

and consider the following distributed identification procedure to improve efficiency in the communication:

For all $l \in \mathcal{N}_j \cup \{j\}$, initialize $\mathcal{I}_l$ at $k = 0$ with $\hat{\theta}_i(0) \in \mathbb{R}^{n_l}$ and $0 \prec \Sigma_l(0) \in \mathbb{R}^{n_l \times n_l}$. For each time $k \in \mathbb{N}$ perform

(i) For each $i \in \mathcal{N}_j$, $\mathcal{I}_i$ measures $w_i(k+1)$ and sends the local prediction $\varphi_i^\top(k+1)\hat{\theta}_i(k)$ to $\mathcal{B}$. Identification module I_j measures $w_j(k+1)$ and sends $w_j(k+1) - \varphi_j^\top(k+1)\hat{\theta}_j(k)$ to $\mathcal{B}$.

(ii) $\mathcal{B}$ returns the prediction error $\varepsilon(k+1, \hat{\theta}_B(k))$ to $\mathcal{I}_l$, $l \in \mathcal{N}_j \cup \{j\}$.

(iii) For each $l \in \mathcal{N}_j \cup \{j\}$, $\mathcal{I}_l$ computes $\hat{\theta}_l(k+1)$ and $\Sigma_l(k+1)$ by (3.18) and (3.19), respectively.

Remark 3.5.1. *The distributed identification procedure can be viewed as a central fusion, distributed computation scheme: local estimations are obtained by modules $\mathcal{I}_l$, $l \in \mathcal{N}_j \cup \{j\}$, which all connect to $\mathcal{B}$. This scheme reflects the interconnection of the MISO system in Figure 3.1, where all modules G_{ji}, $i \in \mathcal{N}_j$, and noise filter H_j connect to a single summation point.*

3.6 Convergence analysis

Now that the central and distributed estimators are updated according to (3.16) and (3.18), respectively, let us analyze the asymptotic properties of the estimators. In this section, we will first assume that no noise affects $w_j(t)$, i.e., the noise signal $v_j(t) = 0$ for $t \in \mathbb{N}$, and that the model structure defining the predictor (3.11) is rich enough to capture the dynamics of the MISO system i.e., the node signal w_j can be described by $w_j(t) = \varphi^\top(t)\theta^0$ for some θ_0. We will analyze the desired convergence $\hat{\theta} \to \theta^0$ via Lyapunov's second method, as was done in the analysis of gradient algorithms for deterministic parameter estimation in (Udink ten Cate and Verbruggen, 1978) and (Mendel, 1973).

3.6.1 Central recursive LSE

We will briefly pay attention to a convergence result for the recursive LSE, to show the analogy with the convergence result for the distributed recursive estimator in Section 3.6.2.

Consider the estimator error $\tilde{\theta}(k) := \hat{\theta}(k) - \theta^0 \in \mathbb{R}^n$. In the absence of noise ($v_j(t) = 0$), it follows from (3.16) that the recursive LSE error dynamics are described by

$$\tilde{\theta}(k+1) = \tilde{\theta}(k) - \alpha(k)\Sigma(k)\varphi(k+1)\varphi^\top(k+1)\tilde{\theta}(k), \tag{3.20}$$

$$\Sigma^{-1}(k+1) = \Sigma^{-1}(k) + \frac{1}{\sigma_j^2}\varphi(k+1)\varphi^\top(k+1). \tag{3.21}$$

Observe that the origin is clearly an equilibrium of difference equation (3.20).

Convergence

The following result demonstrates that the estimation error converges to zero for the recursive LSE in the deterministic case, i.e., when the noise $v(t) = 0$ for all $t \in \mathbb{N}$. A similar result was proven in (Udink ten Cate and Verbruggen, 1978, Appendix B), for a least-squares like gradient algorithm.

Proposition 3.6.1. *Let $W_C : \mathbb{R}^n \times \mathbb{N} \to \mathbb{R}$ be defined by*

$$W_C(\xi, \tau) := \xi^\top \Sigma^{-1}(\tau)\xi$$

and let $\Sigma(k)$ satisfy (3.21), *$\Sigma(0) \succ 0$. Assume that $\tilde{\theta}(k)$ and $\varphi(k+1)$ are not orthogonal for all $k \in \mathbb{N}$. Then $W_C : \mathbb{R}^n \times \mathbb{N} \to \mathbb{R}$ is a Lyapunov function for* (3.20).

Proof. See Appendix 3.C. □

Remark 3.6.1. *When $\tilde{\theta}(k)$ and $\varphi(k+1)$ are orthogonal, the error system* (3.20) *is stable, but not guaranteed to be asymptotically stable, and convergence cannot be concluded. Orthogonality can, however, always be avoided by utilizing input signals with sufficient independent frequencies (Mendel, 1973).*

3.6.2 Distributed recursive estimator

Estimator error dynamics

Consider the distributed recursive estimator (3.18). When no noise is present in the measured node $w_j(t)$ ($v_j(t) = 0$), the distributed estimator update (3.18) can be written as

$$\hat{\theta}_i(k+1) = \hat{\theta}_i(k) + \alpha_i(k)\Sigma_i(k)\varphi_i(k+1)\Big(\sum_{l\in\breve{\mathcal{N}}_j}\varphi_l^\top(k+1)\theta_l^0 - \sum_{l\in\breve{\mathcal{N}}_j}\varphi_l^\top(k+1)\hat{\theta}_l(k)\Big)$$

where we used $w_j(k) = \sum_{l\in\breve{\mathcal{N}}_j}\varphi_l^\top(k)\theta_l^0$, with $\breve{\mathcal{N}}_j := \mathcal{N}_j \cup \{j\}$. Now, define the error vector $\tilde{\theta}_B(k) := \hat{\theta}_B(k) - \theta^0 \in \mathbb{R}^n$. We then have

$$\begin{aligned}
\tilde{\theta}_B(k+1) &= \hat{\theta}_B(k+1) - \theta^0 \\
&= \hat{\theta}_B(k) - \theta^0 + A_B(k)\Sigma_B(k)\varphi(k+1)\sum_{l\in\breve{\mathcal{N}}_j}\varphi_l^\top(k+1)(\theta_l^0 - \hat{\theta}_l(k)) \\
&= \tilde{\theta}_B(k) - A_B(k)\Sigma_B(k)\varphi(k+1)\varphi^\top(k+1)\tilde{\theta}_B(k) \\
&= F(k)\tilde{\theta}_B(k)
\end{aligned}$$

with $F(k) := I_n - A_B(k)\Sigma_B(k)\varphi(k+1)\varphi^\top(k+1)$.

Recalling the difference equation for the gain matrix $\Sigma_B(k)$, we conclude that the error behavior of the distributed recursive estimator (3.18) is described by

$$\tilde{\theta}_B(k+1) = F(k)\tilde{\theta}_B(k), \tag{3.22}$$

$$\Sigma_B^{-1}(k+1) = \Sigma_B^{-1}(k) + \Gamma_B^{-2}(k)\varphi_B(k+1), \tag{3.23}$$

where $\Gamma_B^{-2}(k) = \mathrm{diag}_{l\in\breve{\mathcal{N}}_j}\,\gamma_l^{-2}(k)I_{n_l}$.

Convergence

The following result proves the existence of the scalar functions $\gamma_i(k)$ for each estimator, such that the distributed estimation error vector converges to zero in the deterministic case.

Theorem 3.6.1. *Let $W_B : \mathbb{R}^n \times \mathbb{N} \to \mathbb{R}$ be defined by*

$$W_B(\xi,\tau) := \xi^\top\Sigma_B^{-1}(\tau)\xi$$

and let $\Sigma_B(\cdot)$ satisfy (3.23), $\Sigma_B(0) \succ 0$. For all $i \in \mathbb{N}_{[1:m]}$, let $\alpha_i = \alpha_B$, with $\alpha_B(k) := (\sigma_j^2 + \sum_{l\in\breve{\mathcal{N}}_j}\varphi_l^\top(k+1)\Sigma_l(k)\varphi_l(k+1))^{-1}$. Assume that $\tilde{\theta}_B(k)$ and $\varphi(k+1)$ are not orthogonal for all $k \in \mathbb{N}$. Then there exist γ_i, $i \in \breve{\mathcal{N}}_j$, such that $W_B : \mathbb{R}^n \times \mathbb{N} \to \mathbb{R}$ is a Lyapunov function for (3.22).

Proof. We will first prove that there exists $k_1 \in \mathcal{K}_\infty$ s.t. $W_B(\xi, \tau) \geq k_1(\|\xi\|)$ for all $(\xi, \tau) \in \mathbb{R}^n \times \mathbb{N}$, by induction. Let $\xi \in \mathbb{R}^n$ be arbitrary and let $k_1(r) := \lambda_{\min}(\Sigma_B^{-1}(0))r^2$. We claim that $W_B(\xi, \tau) \geq k_1(\|\xi\|)$ for all $\tau \in \mathbb{N}$. For the base case $\tau = 0$ the statement is true, since we have $W_B(\xi, 0) \geq \lambda_{\min}(\Sigma_B^{-1}(0))\|\xi\|^2 = k_1(\|\xi\|)$. Now, let $W_B(\xi, k) \geq k_1(\|\xi\|)$ be true for some $k \in \mathbb{N}$. Then

$$\begin{aligned}
W_B(\xi, k+1) &= \xi^\top \Sigma_B^{-1}(k+1)\xi \\
&= \xi^\top \Sigma_B^{-1}(k)\xi + \xi^\top \underbrace{\Gamma_B^{-2}(k)\varphi_B(k+1)}_{\geq 0}\xi \\
&\geq \xi^\top \Sigma_B^{-1}(k)\xi \geq k_1(\|\xi\|),
\end{aligned}$$

thus the statement is also true for $k+1$. We conclude that $W_B(\xi, \tau) \geq k_1(\|\xi\|)$ for all $(\xi, \tau) \in \mathbb{R}^n \times \mathbb{N}$. For the upperbound, let $k_2(r, k) := \lambda_{\max}(\Sigma_B^{-1}(k))r^2$. Then $W_B(\xi, \tau) \leq \lambda_{\max}(\Sigma_B^{-1}(\tau))\|\xi\|^2 = k_2(\|\xi\|, \tau)$ for all $\xi \in \mathbb{R}^n$.

Let us now analyze the one-step-difference $\Delta W_B(k) := W_B(\tilde{\theta}_B(k+1), k+1) - W_B(\tilde{\theta}_B(k), k)$. Using the distributed estimator error dynamics (3.22), we find

$$\begin{aligned}
\Delta W_B(k) &= \tilde{\theta}_B^\top(k+1)\Sigma_B^{-1}(k+1)\tilde{\theta}_B(k+1) - \tilde{\theta}_B^\top(k)\Sigma_B^{-1}(k)\tilde{\theta}_B(k) \\
&= \tilde{\theta}_B^\top(k+1)\left(\Sigma_B^{-1}(k+1) - \Sigma_B^{-1}(k)\right)\tilde{\theta}_B(k+1) \\
&\quad + \tilde{\theta}_B^\top(k+1)\Sigma_B^{-1}(k)\tilde{\theta}_B(k+1) - \tilde{\theta}_B^\top(k)\Sigma_B^{-1}(k)\tilde{\theta}_B(k) \\
&= \overline{\Delta W}_B(k) + \tilde{\theta}_B^\top(k+1)\left(\Sigma_B^{-1}(k+1) - \Sigma_B^{-1}(k)\right)\tilde{\theta}_B(k+1),
\end{aligned}$$

where

$$\begin{aligned}
\overline{\Delta W}_B &:= \tilde{\theta}_B^\top(k+1)\Sigma_B^{-1}(k)\tilde{\theta}_B(k+1) - \tilde{\theta}_B^\top(k)\Sigma_B^{-1}(k)\tilde{\theta}_B(k) \\
&= \theta_B^\top\Sigma_B^{-1}\tilde{\theta}_B - 2\tilde{\theta}_B^\top\varphi\varphi^\top\Sigma_B A_B \Sigma_B^{-1}\tilde{\theta}_B \\
&\quad + \tilde{\theta}_B^\top\varphi\varphi^\top\Sigma_B A_B\Sigma_B^{-1}A_B\Sigma_B\varphi\varphi^\top\tilde{\theta}_B - \theta_B^\top\Sigma_B^{-1}\tilde{\theta}_B \\
&= \tilde{\theta}_B^\top\varphi\varphi^\top\Sigma_B A_B\Sigma_B^{-1}A_B\Sigma_B\varphi\varphi^\top\tilde{\theta}_B - 2\tilde{\theta}_B^\top\varphi\varphi^\top\Sigma_B A_B\Sigma_B^{-1}\tilde{\theta}_B.
\end{aligned}$$

Now, since $\alpha_i = \alpha_B$ for all $i \in \mathbb{N}_{[1:m]}$, we have that $\overline{\Delta W}_B$ simplifies to

$$\begin{aligned}
\overline{\Delta W}_B &= \alpha_B^2\tilde{\theta}_B^\top\varphi\varphi^\top\Sigma_B\varphi\varphi^\top\tilde{\theta}_B - 2\alpha_B\tilde{\theta}_B^\top\varphi\varphi^\top\tilde{\theta}_B \\
&= -\alpha_B(\tilde{\theta}_B^\top\varphi)^2\left(2 - \alpha_B\varphi^\top\Sigma_B\varphi\right),
\end{aligned}$$

so that $\overline{\Delta W}_B$ is negative when

$$0 < \alpha_B < \frac{2}{\varphi^\top\Sigma_B\varphi}.$$

Since $\alpha_B = (\sigma_j^2 + \sum_{l\in\breve{\mathcal{N}}_j} \varphi_l^\top \Sigma_l \varphi_l)^{-1}$, the latter condition is satisfied, such that $\overline{\Delta W}_B < 0$.

By equation (3.23), the one-step-difference is equal to

$$\begin{aligned}\Delta W_B(k) &= \overline{\Delta W}_B + \tilde{\theta}_B^\top(k+1)\Gamma_B^{-2}(k)\varphi_B(k+1)\tilde{\theta}_B(k+1)\\ &= \overline{\Delta W}_B + \sum_{i\in\breve{\mathcal{N}}_j} \frac{1}{\gamma_i^2(k)}\tilde{\theta}_i^\top(k+1)\varphi_i\varphi_i^\top\tilde{\theta}_i(k+1)\\ &\leq \overline{\Delta W}_B + \sum_{i\in\breve{\mathcal{N}}_j} \frac{1}{\gamma_i^2(k)} \sum_{l\in\breve{\mathcal{N}}_j} (\tilde{\theta}_l^\top(k+1)\varphi_l)^2,\end{aligned}$$

where we used the Cauchy-Schwarz inequality. The decrease condition $\Delta W_B(k) < 0$ is therefore satisfied whenever $\gamma_i(k)$, $i \in \mathbb{N}_{[1:m]}$, are so large that

$$\sum_{i=1}^{m} \frac{1}{\gamma_i^2(k)} < \frac{|\overline{\Delta W}_B|}{\tilde{\theta}_B^\top(k)F^\top(k)\varphi_B F(k)\tilde{\theta}_B(k)}$$

for all $k \in \mathbb{N}$, which is equivalent to the existence of $k_3 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that (3.6) holds (Malisoff and Mazenc, 2009). This concludes the proof. □

Remark 3.6.2. *The difference in the stability analysis of the distributed estimator w.r.t. the recursive LSE is induced by* (3.23)*. The block-diagonality of φ_B requires conditions on γ_i for stability, whereas a multiplication of $\varphi\varphi^\top$ with σ^{-2} in* (3.21) *suffices for stability of the recursive LSE.*

Remark 3.6.3. *The proof of Theorem 3.6.1 gives exact conditions on the scalar functions $\gamma_i(k)$. When $\gamma_i(k) = \gamma_i$ is chosen to be a constant, it suffices to assume that $\gamma_i \in \mathbb{R}$ is large enough, such that $\Delta W_B(k) < 0$.*

In the presence of noise, the error dynamics for the distributed estimator are described by

$$\tilde{\theta}_B(k+1) = F(k)\tilde{\theta}_B(k) + G(k)v(k+1),$$

with $G(k) := A_B(k)\Sigma_B(k)\varphi(k+1)$. The following result provides sufficient conditions for asymptotic unbiasedness of the distributed estimator $\hat{\theta}_B$.

Proposition 3.6.2. *Let $\prod_{\tau=t+1}^{k} F(\tau)G(t)$ and $v_j(t)$ be statistically independent for all $t \in \mathbb{N}$. If there exists a Lyapunov function for* (3.22)*, then $\lim_{k\to\infty} \mathrm{E}\,\hat{\theta}_B(k) = \theta^0$.*

Proof. We refer the reader to the proof of (Mendel, 1973, Theorem 2-5). □

3.6.3 Network conditions

We will now translate the conditions in Theorem 3.6.1 and Proposition 3.6.2 for asymptotic convergence and unbiasedness to conditions on the dynamic network described by (3.8).

One of the sufficient conditions in Theorem 3.6.1 for asymptotic convergence of the distributed estimates $\hat{\theta}_l(k)$ to θ_l^0 as $k \to \infty$, is that for all $k \in \mathbb{N}$ it holds that $\tilde{\theta}_B(k)$ and $\varphi(k+1)$ are not orthogonal. This is naturally satisfied if for all $k \in \mathbb{N}$ it holds that $\hat{\theta}_i(k) - \theta_i^0$ and $\varphi_i(k+1)$ are not orthogonal. As noted in Remark 3.6.1, orthogonality of $\hat{\theta}_B(k) - \theta_0$ and $\varphi(k+1)$ can always be avoided by choosing 'input' signals that are excited at a sufficient number of independent frequencies (Mendel, 1973). These conditions therefore translate to excitation conditions on w_i, $i \in \mathcal{N}_j$, and w_j. The following corollary therefore follows from Theorem 3.6.1.

Corollary 3.6.1. *Let $v_j = 0$ and let the spectral density of $\operatorname{col}_{l\in\breve{\mathcal{N}}_j} w_l$, denoted $\Phi_{j,\mathcal{N}_j}(\omega)$, be positive definite for almost all $\omega \in [-\pi, \pi]$. Further, let γ_l be chosen as in the constructive proof of Theorem 3.6.1. Then each estimate $\hat{\theta}_l(k) \to \theta_l^0$ for $k \to \infty$.*

In the presence of noise ($v_j \neq 0$), the statistical independence of the terms $\prod_{\tau=t+1}^{k} F(\tau)G(t)$ and the noise $v_j(t)$ yields asymptotically unbiased estimates (Proposition 3.6.2). The dependence of $\prod_{\tau=t+1}^{k} F(\tau)G(t)$ on φ_j and thus w_j in the case that $A(q,\theta) \neq 1$, can yield the independence condition to fail to hold. If $A(q,\theta) = 1$ (as in the FIR case, for example), then the independence condition can be satisfied. An additional sufficient condition is that the node variables w_i, $i \in \mathcal{N}_j$ do not depend on v_j, i.e., there is no path from v_j to w_i for each $i \in \mathcal{N}_j$:

Corollary 3.6.2. *Suppose there does not exist a path from node j to node i for each $i \in \mathcal{N}_j$. Let the spectral density of $\operatorname{col}_{l\in\breve{\mathcal{N}}_j} w_l$, denoted $\Phi_{j,\mathcal{N}_j}(\omega)$, be positive definite for almost all $\omega \in [-\pi, \pi]$ and let γ_l be chosen as in the constructive proof of Theorem 3.6.1. Then $\lim_{k\to\infty} \mathrm{E}\,\hat{\theta}_B(k) = \theta^0$.*

3.7 Numerical example

Consider the data generating system (3.7) with $m_j = 20$ subsystems (depicted in Figure 3.1), so that $w_j(t) = \sum_{i\in\mathcal{N}_j} G_{ji}^0(q)w_i(t) + v_j(t)$, with $G_{ji}^0 = B_i^0(q) = b_0^i + b_1^i q^{-1} + \cdots + b_{n_b^i}^i q^{-n_b^i}$ and $v(t)$ zero-mean white Gaussian noise with standard deviation $\sigma_j = 0.1$. For this illustrative example, the subsystems $G_{ji}^0(q)$ of the data generating system are constructed in a random fashion as follows: each subsystem has $n_b^i + 1 \in \mathbb{N}$ unknown parameters, which is an integer drawn from a discrete uniform distribution $\mathcal{U}\{1,10\}$ using the MATLAB function `randi`. The unknown parameters $b_k^i \in \mathbb{R}$, $k \in \mathbb{N}_{[0:n_b^i]}$, $i \in \mathcal{N}_j$, are drawn from a normal distribution $\mathcal{N}(0,1)$ in MATLAB using `randn`. The total number of to-be-estimated parameters is $n = \sum_{i \in \mathcal{N}_j}(n_b^i + 1) = 102$. The rest of the network, i.e., the rows in (3.8) except row j, is chosen such that the nodes w_i are persistently exciting of sufficiently high order. More specifically, the rest of the network is described by $w_i = v_i$, $i \in \{1,\ldots,20\} = \mathcal{V}$, where v_i is a Gaussian white-noise process with unit variance.

We apply the distributed recursive estimation procedure from Section 3.5. The local estimators $\hat{\theta}_i : \mathbb{N} \to \mathbb{R}_i^n$ are described by (3.18) with $\alpha_i = \alpha_B$, $i \in \mathbb{N}_{[1:20]}$, as defined in Theorem 3.6.1. The matrices $\Sigma_i : \mathbb{N} \to \mathbb{R}^{n_i \times n_i}$ are described by (3.19), with $\gamma_i(k) = \gamma = 100$. For comparison, we apply a corresponding central recursive estimator, i.e., the recursive LSE (3.16) with the update for the matrix $\Sigma : \mathbb{N} \to \mathbb{R}^{n \times n}$ described by $\Sigma^{-1}(k+1) = \Sigma^{-1}(k) + \gamma^{-2}\varphi(k+1)\varphi^\top(k+1)$ instead of (3.17).

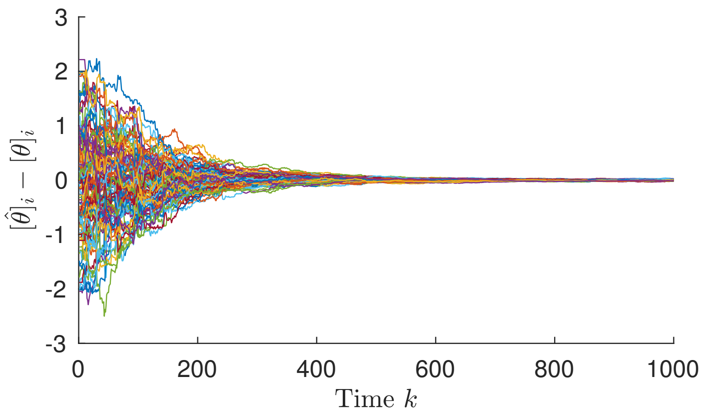

Figure 3.3: Evolution of the estimation error for all parameters $[\hat{\theta}]_j - [\theta^0]_j$, $j \in \mathbb{N}_{[1:102]}$, for the central identification of a MISO system with $m_j = 20$ subsystems.

Figure 3.3 and 3.4 show the evolution of the estimation error over time for the central and distributed estimator, respectively, initialized in $\hat{\theta}(1) = \hat{\theta}_B(1) = 0$ and $\Sigma(1) = \Sigma_B(1) = 100 I_{102}$. The overall estimation errors $\|\hat{\theta}(k) - \theta^0\|^2$ and $\|\hat{\theta}_B(k) - \theta^0\|^2$ are shown in Figure 3.5 in blue and red, respectively. We observe a lower decrease rate for the estimation errors in the distributed identification scheme w.r.t. the central scheme, in general, while convergence is observed for both schemes.

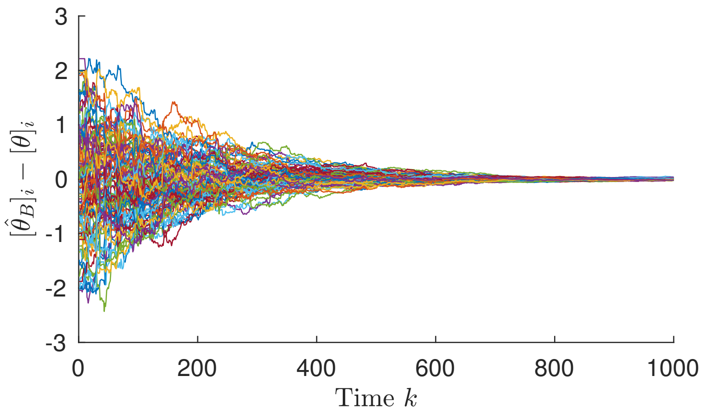

Figure 3.4: Evolution of the estimation error for all parameters $[\hat{\theta}_B]_j - [\theta^0]_j$, $j \in \mathbb{N}_{[1:102]}$, for the distributed identification of a MISO system with $m_j = 20$ subsystems.

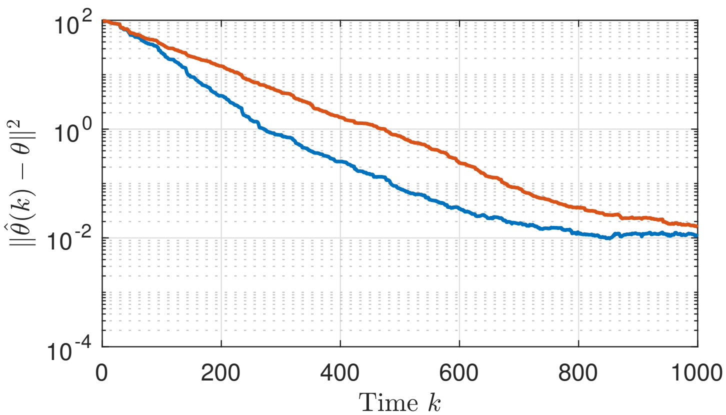

Figure 3.5: Estimation error for the central estimator $\|\tilde{\theta}(k)\|^2$ (blue) and distributed estimator $\|\tilde{\theta}_B(k)\|^2$ (red) for $\gamma = 100$.

3.8 Conclusions

We have stated a recursive estimation algorithm for the distributed identification of MISO systems in dynamic networks, enabled by a prediction-error identification problem with a predictor that is linear in the parameters. The distributed identification scheme consists of local identification modules, which estimate a subvector of the total parameter vector corresponding to subsystems and, possibly, a noise filter. Via Lyapunov's second method, we have obtained sufficient conditions for asymptotic convergence of the estimators to the true parameters in the absence of noise, which leads to asymptotic unbiasedness in the presence of noise at the system's output.

Regarding communication, information exchange between identification modules is required to be performed through a mutual fusion center in the developed method. The communication protocol only requires regressors to be communicated, which should not yield concerns regarding privacy of parameter estimates or measurement data. The present communication features a single point of failure, however. The application of alternative communication architectures for distributed identification of dynamic networks is a problem that could be explored in future research.

Appendix

3.A Proof of Theorem 3.2.1

Proof. The proof follows the same line of reasoning as the proof for the *continuous-time* version of the theorem (Vidyasagar, 1993, Section 5.3.1, Theorem 1). We give the proof for completeness.

Let $\varepsilon > 0$ and $t_0 \in \mathbb{N}$ be given. We will show that there exists $\delta(\varepsilon, t_0) > 0$ so that

$$\|x_0\| < \delta(\varepsilon, t_0) \;\Rightarrow\; \|s(t, t_0, x_0)\| < \varepsilon \quad \forall t \geq t_0.$$

Take $\delta = \delta(\varepsilon, t_0) > 0$ so that

$$k_2(\delta, t_0) < k_1(\varepsilon).$$

Such a $\delta > 0$ always exists, since $k_1(\varepsilon) > 0$ and $k_2(\delta, t_0) \to 0$ as $\delta \to 0$. Now, let $\|x_0\| < \delta$. Then

$$W(x_0, t_0) \leq k_2(\delta, t_0) < k_1(\varepsilon).$$

From (3.4), it follows that for all $t \geq t_0$ we have

$$W(s(t, t_0, x_0), t) \leq W(x_0, t_0).$$

Since $W(s(t, t_0, x_0), t) \geq k_1(\|s(t, t_0, x_0)\|)$ by (3.3), we have

$$\begin{aligned} k_1(\|s(t, t_{,0}, x_0\|) &\leq W(s(t, t_0, x_0), t) \\ &\leq W(x_0, t_0) \leq k_2(\delta, t_0) < k_1(\varepsilon), \end{aligned}$$

which implies

$$\|s(t, t_0, x_0)\| < \varepsilon, \quad \forall t \geq t_0.$$

Therefore, the origin equilibrium of (3.1) is stable, which concludes the proof. $\square$

3.B Proof of Theorem 3.2.2

Proof. Let $t_0 \in \mathbb{N}$ and $x_0 \in \mathbb{R}^n$. Since (3.5) and (3.6) imply conditions (3.3) and (3.4), the origin is stable by Theorem 3.2.1. It remains to be proven that the origin is globally attractive, i.e., $\lim_{t\to\infty} s(t, t_0, x_0) = 0$.

Suppose that the origin is not attractive, i.e., there exists an $x_0 \in \mathbb{R}^n$ such that

$$\neg \left[\lim_{t\to\infty} \|s(t, t_0, x_0)\| = 0\right]$$

is true ($\neg a$ denotes the negation of assertion a). Then there is a sufficiently small positive number $c \in \mathbb{R}_{>0}$ such that for all $T \in \mathbb{N}$, $T \geq t_0$, there exists some $t \geq T$ with $\|s(t, t_0, x_0)\| \geq c$. Hence, there exists a sequence $\{t_1, t_2, \ldots, t_k\}$ with $t_0 < t_1 < t_2 < \cdots < t_k$ and $t_k \to \infty$ as $k \to \infty$, which satisfies

$$\|s(t_i, t_0, x_0)\| \geq c, \quad i = 1, \ldots, k. \tag{3.24}$$

Indeed, let $t_1 \geq t_0$ be such that $\|s(t_1, t_0, x_0)\| \geq c$ and let $T_1 := t_1 + 1$. Then there exists some $t_2 \geq T_1$ such that $\|s(t_2, t_0, x_0)\| \geq c$. Now, assume that $\|s(t_i, t_0, x_0)\| \geq c$ for some $i \in \{1, \ldots, k-1\}$ and define $T_i := t_i + 1$. Then there exists some $t_{i+1} \geq T_i$ such that $\|s(t_{i+1}, t_0, x_0)\| \geq c$. Therefore, (3.24) holds true by the principle of natural induction.

Since $\|s(t_i, t_0, x_0)\| \geq c$, we have that $k_1(\|s(t_i, t_0, x_0)\|) \geq k_1(c)$ and hence

$$W(s(t_i, t_0, x_0), t) \geq k_1(c) > 0$$

for all $i \in \{1, \ldots, k\}$, by (3.5). Define a non-decreasing and positive-definite function $\rho : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ by

$$\rho(s) = \inf_{z \geq s} k_3(z),$$

so that $\rho(s) \leq k_3(s)$ for all $s \in \mathbb{R}_{\geq 0}$. Then $\rho(\|s(t_i, t_0, x_0)\|) \geq \rho(c)$, since $\rho : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is non-decreasing, so that

$$\Delta W(s(t_i, t_0, x_0), t_i) \leq -\rho(c), \tag{3.25}$$

for all $i \in \{1, \ldots, k\}$, by (3.6). Define the set $\bar{T}_k := \mathbb{Z}_{[t_0:t_k]} \setminus \{t_1, t_2, \ldots, t_k\}$. The upperbound in (3.25) implies

$$\begin{aligned}
W(s(t_k,t_0,x_0),t) &= W(x_0,t_0) + \sum_{\tau=t_0}^{t_k} \Delta W(s(\tau,t_0,x_0),\tau) \\
&= W(x_0,t_0) + \sum_{i=1}^{k} \Delta W(s(t_i,t_0,x_0),t_i) + \sum_{\tau\in\bar{T}_k} \Delta W(s(\tau,t_0,x_0),\tau) \\
&\leq W(x_0,t_0) + \sum_{i=1}^{k} \Delta W(s(t_i,t_0,x_0),t_i) \\
&\leq W(x_0,t_0) + \sum_{i=1}^{k} -\rho(c) = W(x_0,t_0) - \rho(c)(k-1)
\end{aligned}$$

where the first inequality follows from the non-positiveness of $\Delta W(s(\tau,t_0,x_0),\tau)$, $\tau \in \bar{T}_k$, by (3.6), and the second inequality follows from (3.25). Hence, we have

$$0 < k_1(c) \leq W(s(t_k,t_0,x_0),t_k) \leq W(x_0,t_0) - \rho(c)(k-1).$$

For sufficiently large values of k, the right-hand side of the latter inequality becomes negative, which cannot be true. Therefore, we conclude that the origin is attractive, i.e., $\lim_{t\to\infty} s(t,t_0,x_0) = 0$, which concludes the proof. □

3.C Proof of Proposition 3.6.1

Proof. We will first prove condition (3.5). Let $\xi \in \mathbb{R}^n$ be arbitrary and let $k_1(r) := \lambda_{\min}(\Sigma^{-1}(0))r^2$. We claim that $W_C(\xi,\tau) \geq k_1(\|\xi\|)$ for all $\tau \in \mathbb{N}$. For the base case $\tau = 0$ the statement is true, since we have $W_C(\xi,0) \geq \lambda_{\min}(\Sigma^{-1}(0))\|\xi\|^2 = k_1(\|\xi\|)$. Now, let $W_C(\xi,k) \geq k_1(\|\xi\|)$ be true for some $k \in \mathbb{N}$. Then

$$\begin{aligned}
W_C(\xi,k+1) &= \xi^\top \Sigma^{-1}(k+1)\xi \\
&= \xi^\top \Sigma^{-1}(k)\xi + \left(\frac{\xi^\top \varphi(k+1)}{\sigma_j}\right)^2 \geq k_1(\xi),
\end{aligned}$$

thus the statement is also true for $k+1$. We conclude that $W_C(\xi,\tau) \geq k_1(\|\xi\|)$ for all $(\xi,\tau) \in \mathbb{R}^n \times \mathbb{N}$. For the upperbound, let $k_2(\xi)(r,k) := \lambda_{\max}(\Sigma^{-1}(k))r^2$. Then $W_C(\xi,\tau) \leq \lambda_{\max}(\Sigma^{-1}(\tau))\|\xi\|^2 = k_2(\|\xi\|,\tau)$ for all $\xi \in \mathbb{R}^n$.

We investigate the one-step difference $\Delta W_C(k) := W_C(\tilde{\theta}(k+1),k+1) -$

$W_C(\tilde{\theta}(k), k)$. Using the estimator error dynamics (3.20), we find that

$$\begin{aligned}\Delta W_C(k) &= \tilde{\theta}^\top(k+1)\Sigma^{-1}(k+1)\tilde{\theta}(k+1) - \tilde{\theta}^\top(k)\Sigma^{-1}(k)\tilde{\theta}(k) \\ &= \tilde{\theta}^\top(k)\Sigma^{-1}(k+1)\tilde{\theta}(k) - \tilde{\theta}^\top(k)\Sigma^{-1}(k)\tilde{\theta}(k) \\ &\quad + \alpha^2(k)\tilde{\theta}^\top(k)\varphi(k+1)\varphi^\top(k+1)\Sigma(k)\Sigma^{-1}(k+1) \\ &\quad\quad \cdot \Sigma(k)\varphi(k+1)\varphi^\top(k+1)\tilde{\theta}(k) \\ &\quad - 2\tilde{\theta}^\top(k)\Sigma^{-1}(k+1)\alpha(k)\Sigma(k)\varphi(k+1)\varphi^\top(k+1)\tilde{\theta}(k).\end{aligned}$$

Substituting the covariance matrix update equation (3.21) into the latter equation, we determine that

$$\Delta W_C(k) = (\tilde{\theta}\varphi)^2\left(\frac{1}{\sigma_j^2} + \alpha^2\varphi^\top\Sigma\varphi + \frac{\alpha^2}{\sigma_j^2}\left(\varphi^\top\Sigma\varphi\right)^2 - 2\alpha - 2\frac{\alpha}{\sigma_j^2}\varphi^\top\Sigma\varphi\right),$$

where we omitted the time dependence of the variables on the RHS for brevity. Recalling the definition of $\alpha(k)$, we can further rewrite $\Delta W(k)$ as

$$\Delta W_C(k) = -\frac{\tilde{\theta}^\top\varphi\varphi^\top\tilde{\theta}}{\sigma_j^2 + \varphi^\top\Sigma\varphi}.$$

It is now easily seen that

$$\Delta W_C(k) = -\frac{\tilde{\theta}^\top(k)\varphi(k+1)\varphi^\top(k+1)\tilde{\theta}(k)}{\sigma_j^2 + \varphi^\top(k+1)\Sigma(k)\varphi(k+1)} < 0$$

if $\tilde{\theta}(k)^\top\varphi(k+1) \neq 0$, which implies the existence of $k_3 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ so that (3.6) holds (Malisoff and Mazenc, 2009), which concludes the proof. $\square$

Chapter 4

Scalable distributed $\mathscr{H}_2$ and $\mathscr{H}_\infty$ controller synthesis

The current limitation in the synthesis of distributed $\mathscr{H}_2$ controllers for linear interconnected systems is scalability due to non-convex or unstructured synthesis conditions. In this chapter we develop convex and structured conditions for the existence of a distributed $\mathscr{H}_2$ controller for discrete-time interconnected systems with an interconnection structure that corresponds to an arbitrary graph. Neutral interconnections and a storage function with a block-diagonal structure are utilized to attain coupling conditions that are of a considerably lower computational complexity compared to the corresponding centralized $\mathscr{H}_2$ controller synthesis problem. A detailed procedure is provided for the construction of the distributed controller, which is applicable to both the distributed $\mathscr{H}_2$ controller, as well as distributed $\mathscr{H}_\infty$ controller existence conditions which are recalled from the literature. The effectiveness and scalability of the developed distributed $\mathscr{H}_2$ controller synthesis method is demonstrated for small- to large-scale oscillator networks on a cycle graph.

4.1 Introduction

Control of interconnected systems is relevant to a wide area of applications in smart grids, communication networks, irrigation networks and chemical plant networks, fueled by the digital industrial revolution, see e.g. (Lunze, 1992) and

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This chapter is based on the publication: T. R. V. Steentjes, M. Lazar, and P. M. J. Van den Hof. Scalable distributed $\mathscr{H}_2$ controller synthesis for interconnected linear discrete-time systems. *IFAC-PapersOnLine*, 54(9):66–71, 2021c. 24^{th} International Symposium on Mathematical Theory of Networks and Systems

(Bullo, 2018). Distributed control is preferred for such systems due to its scalable implementation and it has been a major research topic in recent years for several control objectives, including $\mathscr{H}_2$ and $\mathscr{H}_\infty$ performance criteria.

For continuous-time systems, sufficient conditions for the existence of a controller that admits the same interconnection structure as the plant and that achieves unit $\mathscr{H}_\infty$ performance were developed by Langbort et al. (2004). The basis for these sufficient conditions is laid by dissipativity theory, introduced by Willems (1972), which is also the cornerstone for this work. Van Horssen and Weiland (2016) presented a discrete-time analogue of the work in (Langbort et al., 2004) with additional robust stability and robust $\mathscr{H}_\infty$ performance guarantees. For both the continuous- and discrete-time distributed $\mathscr{H}_\infty$ control problems, the conditions can be stated as linear matrix inequalities (LMIs) (Langbort et al., 2004), (Van Horssen and Weiland, 2016).

The method in (Eilbrecht et al., 2017) provides an approach to solve the discrete-time $\mathscr{H}_2$ output-feedback problem for interconnected systems, by minimizing a linear combination of the closed-loop system's $\mathscr{H}_2$ norm and a cost related to the sparsity of the controller matrices. However, this approach yields a non-convex problem in general. (Vamsi and Elia, 2016) solved the discrete-time $\mathscr{H}_2$ problem for a 'strictly causal' network, via the search for an unstructured controller and a subsequent transformation into a structured one. The structure of systems interconnected over one spatial dimension was exploited by (Rice, 2010) for the efficient design of $\mathscr{H}_2$ controllers interconnected in a string. The distributed $\mathscr{H}_2$ controller synthesis for continuous-time systems with arbitrary interconnection topology was recently considered by (Chen et al., 2019). Unlike the $\mathscr{H}_\infty$ case, however, the feasibility problem for the distributed $\mathscr{H}_2$ controller existence in (Chen et al., 2019) is not convex, but amounts to solving a bilinear optimization problem.

The $\mathscr{H}_2$ norm has a particularly interesting interpretation in the field of data-driven modeling of interconnected systems, where stochastic assumptions on disturbance signals are key (Van den Hof et al., 2013). This is due to the fact that the $\mathscr{H}_2$ norm equals the asymptotic output variance for a white noise excitation (Scherer and Weiland, 2017). The trend for data-driven modeling of interconnected systems asks for accompanying distributed controller design methods that apply to discrete-time systems affected by stochastic disturbance signals. However, the current approaches to distributed $\mathscr{H}_2$ control, reviewed above, do not facilitate the controller synthesis for arbitrarily-structured large-scale systems, due to non-convex or unstructured synthesis conditions, or due to restrictions to systems that are spatially distributed in one dimension. Hence, it is of interest to develop scalable (convex) conditions for the synthesis of distributed $\mathscr{H}_2$ controllers for systems with a general interconnection structure.

In this chapter, we therefore develop sufficient conditions for the existence of a distributed $\mathscr{H}_2$ controller for a discrete-time system with an arbitrary interconnection structure, by adopting the fundamental approach to distributed controller synthesis of (Langbort et al., 2004). Analogous to distributed $\mathscr{H}_2$ controller synthesis for linear continuous-time systems (Chen et al., 2019), the conditions are principally not convex, which is induced by a number of scalar terms that are nonlinear w.r.t. the optimization variables, equal to the number of subsystems. However, we show that the resulting conditions are equivalent to alternative *convex* conditions stated as LMIs, with no loss of generality or scalability.

Basic nomenclature

The basic notation in the Notation section of the front matter of this thesis is considered. We recall the relevant notation for this chapter here for convenience. The integers are denoted by $\mathbb{Z}$. Given $a \in \mathbb{Z}$, $b \in \mathbb{Z}$ such that $a < b$, we denote $\mathbb{Z}_{[a:b]} := \{a, a+1, \dots, b-1, b\}$. Let $I_n \in \mathbb{R}^{n\times n}$, or simply I, denote the identity matrix. The operator $\mathrm{col}(\cdot)$ vertically concatenates its arguments. The block diagonal matrix $\mathrm{diag}(X_1, \dots, X_m)$ has matrices X_i, $i \in \mathbb{N}_{[1:m]}$, in its block diagonal entries. For $S \subseteq \mathbb{Z}$, the block diagonal matrix $\mathrm{diag}_{i\in S} X_i$ has matrices X_i, $i \in S$, in its block diagonal entries. The image and kernel of a matrix $A \in \mathbb{R}^{m\times n}$ are $\operatorname{im} A := \{Ax \mid x \in \mathbb{R}^n\}$ and $\ker A := \{x \in \mathbb{R}^n \mid Ax = 0\}$, with $A_\perp$ a basis matrix of $\ker A$. For a real symmetric matrix X, $X \succ 0$ denotes that X is positive definite.

4.2 Preliminaries

Let the structure of an interconnected system be given by a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is the vertex set of cardinality L and $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ is the edge set. Each vertex $v_i \in \mathcal{V}$, corresponds to a discrete-time system $\mathcal{P}_i$. An edge $(v_i, v_j) \in \mathcal{E}$ exists if subsystems $\mathcal{P}_i$ and $\mathcal{P}_j$ are interconnected. For ease of presentation, self-connections are excluded for all subsystems $\mathcal{P}_i$, $i \in \mathbb{Z}_{[1:L]}$.

Each subsystem $\mathcal{P}_i$ is assumed to admit a state-space representation

$$\begin{bmatrix} x_i(k+1) \\ o_i(k) \\ z_i(k) \end{bmatrix} = \begin{bmatrix} A_i^{\mathrm{TT}} & A_i^{\mathrm{TS}} & B_i^{\mathrm{T}d} \\ A_i^{\mathrm{ST}} & A_i^{\mathrm{SS}} & B_i^{\mathrm{S}d} \\ C_i^{z\mathrm{T}} & C_i^{z\mathrm{S}} & D_i^{zd} \end{bmatrix} \begin{bmatrix} x_i(k) \\ s_i(k) \\ d_i(k) \end{bmatrix}, \tag{4.1}$$

where $x_i : \mathbb{Z} \to \mathbb{R}^{k_i}$ is the subsystem's state, $o_i : \mathbb{Z} \to \mathbb{R}^{n_i}$ and $s_i : \mathbb{Z} \to \mathbb{R}^{n_i}$ are the outgoing and incoming interconnection variables, and $z_i : \mathbb{Z} \to \mathbb{R}^{q_i}$ and $d_i : \mathbb{Z} \to \mathbb{R}^{f_i}$ are the performance output and disturbance input, respectively.

We write the interconnection signals s_i and o_i as $s_i = \mathrm{col}(s_{i1}, s_{i2}, \dots, s_{iL})$ and $o_i = \mathrm{col}(o_{i1}, o_{i2}, \dots, o_{iL})$ so that (s_{ij}, o_{ij}) denotes the interconnection channel between subsystem $\mathcal{P}_i$ and subsystem $\mathcal{P}_j$. For the ease of the interconnection definition, we assume, without loss of generality (Langbort et al., 2004), that o_{ij}, s_{ij}, o_{ji} and s_{ji} are all elements of $\mathbb{R}^{n_{ij}}$, $n_{ij} \geq 0$. The interconnection between system $\mathcal{P}_i$ and $\mathcal{P}_j$ is defined through the interconnection equation

$$\begin{bmatrix} o_{ij}(k) \\ s_{ij}(k) \end{bmatrix} = \begin{bmatrix} s_{ji}(k) \\ o_{ji}(k) \end{bmatrix}, \quad \forall k \in \mathbb{Z}. \tag{4.2}$$

Hence, $\mathcal{P}_i$ and $\mathcal{P}_j$ are interconnected if and only if $n_{ij} > 0$, if and only if $(v_i, v_j) \in \mathcal{E}$.

The interconnected system can be compactly represented by

$$\begin{bmatrix} x(k+1) \\ o(k) \\ z(k) \end{bmatrix} = \begin{bmatrix} A^{\mathrm{TT}} & A^{\mathrm{TS}} & B^{\mathrm{T}} \\ A^{\mathrm{ST}} & A^{\mathrm{SS}} & B^{\mathrm{S}} \\ C^{\mathrm{T}} & C^{\mathrm{S}} & D \end{bmatrix} \begin{bmatrix} x(k) \\ s(k) \\ d(k) \end{bmatrix}$$

and the interconnection equation $o = \Delta s$, with the variables defined as $x := \mathrm{col}(x_1, \ldots, x_L)$, $o := \mathrm{col}(o_1, \ldots, o_L)$, $s := \mathrm{col}(s_1, \ldots, s_L)$, $z := \mathrm{col}(z_1, \ldots, z_L)$ and $d := \mathrm{col}(d_1, \ldots, d_L)$, the block-diagonal system matrices defined as $A^{\mathrm{TT}} := \mathrm{diag}(A_1^{\mathrm{TT}}, \ldots, A_L^{\mathrm{TT}})$, $A^{\mathrm{TS}} := \mathrm{diag}(A_1^{\mathrm{TS}}, \ldots, A_L^{\mathrm{TS}})$, *et cetera*, and the matrix Δ defined by aggregating (4.2) for all corresponding index pairs. Elimination of the interconnection variables s and o yields a state-space representation

$$\mathcal{P}_{\mathcal{I}} : \quad \begin{bmatrix} x(k+1) \\ z(k) \end{bmatrix} = \begin{bmatrix} A_{\mathcal{I}} & B_{\mathcal{I}} \\ C_{\mathcal{I}} & D_{\mathcal{I}} \end{bmatrix} \begin{bmatrix} x(k) \\ d(k) \end{bmatrix} \tag{4.3}$$

where

$$\begin{bmatrix} A_{\mathcal{I}} & B_{\mathcal{I}} \\ C_{\mathcal{I}} & D_{\mathcal{I}} \end{bmatrix} := \begin{bmatrix} A^{\mathrm{TT}} & B^{\mathrm{T}} \\ C^{\mathrm{T}} & D \end{bmatrix} + \begin{bmatrix} A^{\mathrm{TS}} \\ C^{\mathrm{S}} \end{bmatrix} (\Delta - A^{\mathrm{SS}})^{-1} \begin{bmatrix} A^{\mathrm{ST}} & B^{\mathrm{S}} \end{bmatrix}.$$

Consider the interconnection variable subspaces (Langbort et al., 2004)

$$\mathcal{S}_{\mathcal{I}} := \{(o, s) \in \mathbb{R}^{2n} \mid o = \Delta s\} \text{ and}$$
$$\mathcal{S}_{\mathcal{B}} := \{(o, s) \in \mathbb{R}^{2n} \mid \mathrm{col}(o_i, s_i) \in \mathrm{im}\,\mathrm{col}(A_i^{\mathrm{SS}}, I), i \in \mathbb{Z}_{[1:L]}\}.$$

Definition 4.2.1. *An interconnected system described by* (4.1) *and* (4.2) *is said to be well-posed if* $\mathcal{S}_{\mathcal{I}} \cap \mathcal{S}_{\mathcal{B}} = \{0\}$.

Well-posedness of an interconnected system means that the inputs d_i and initial conditions $x_i(0)$ uniquely determine the system variables x_i, o_i, s_i and z_i.

Definition 4.2.2. *A well-posed interconnected system is said to be asymptotically stable (AS) if the roots of* $\det(zI - A_{\mathcal{I}})$ *are in the open unit disk in the complex plane.*

Definition 4.2.3. *The $\mathscr{H}_2$ norm of a well-posed and AS interconnected system with a transfer function $T(z) := C_{\mathcal{I}}(zI - A_{\mathcal{I}})^{-1}B_{\mathcal{I}} + D_{\mathcal{I}}$ is defined by*

$$\|\mathcal{P}_{\mathcal{I}}\|_{\mathscr{H}_2} := \left(\frac{1}{2\pi}\operatorname{trace}\int_{-\pi}^{\pi} T^*(e^{i\omega})T(e^{i\omega})\,\mathrm{d}\omega\right)^{\frac{1}{2}},$$

quantifying the mapping from disturbance inputs to performance outputs.

4.2.1 Dissipative interconnected systems

As a basis for the analysis of the interconnected system and the synthesis of distributed controllers, we employ the theory of dissipative dynamical systems (Willems, 1972).

Definition 4.2.4. *Subsystem $\mathcal{P}_i$ is said to be dissipative with respect to the supply function $\sigma_i : \mathcal{S}_i \times \mathcal{O}_i \times \mathcal{D}_i \times \mathcal{Z}_i \to \mathbb{R}$, if there exists a non-negative storage function $V_i : \mathcal{X}_i \to \mathbb{R}_{\geq 0}$, so that for all $t \in \mathbb{Z}_{\geq 0}$ the inequality*

$$V_i(x_i(t)) - V_i(x_i(0)) \leq \sum_{k=0}^{t-1} \sigma_i(s_i(k), o_i(k), d_i(k), z_i(k))$$

holds for all trajectories $(x_i, s_i, o_i, d_i, z_i)$ of (4.1).

We consider the class of quadratic storage functions:

$$V_i(x_i) := x_i^\top X_i x_i, \quad i \in \mathbb{Z}_{[1:L]},$$

with $X_i \succ 0$. Supply functions are restricted to be quadratic functions of the form

$$\sigma_i(s_i, o_i, d_i, z_i) := \sigma_i^{\mathrm{int}}(s_i, o_i) + \sigma_i^{\mathrm{ext}}(d_i, z_i), \quad i \in \mathbb{Z}_{[1:L]},$$

with 'internal' supply functions

$$\sigma_i^{\mathrm{int}}(s_i, o_i) := \sum_{j=1}^{L} \sigma_{ij}(s_{ij}, o_{ij}), \quad \sigma_{ij}(s_{ij}, o_{ij}) := \begin{bmatrix} o_{ij} \\ s_{ij} \end{bmatrix}^\top X_{ij} \begin{bmatrix} o_{ij} \\ s_{ij} \end{bmatrix},$$

where X_{ij} is a real symmetric matrix, and 'external' supply functions

$$\sigma_i^{\mathrm{ext}}(d_i, z_i) := \rho_i d_i^\top d_i - z_i^\top z_i,$$

where $\rho_i > 0$. For any pair $(i, j) \in \mathbb{Z}_{[1:L]}^2$, $i \neq j$, the interconnection between subsystem $\mathcal{P}_i$ and subsystem $\mathcal{P}_j$ is said to be neutral if the internal supply functions satisfy (Scherer and Weiland, 2017)

$$0 = \sigma_{ij}(s_{ij}, o_{ij}) + \sigma_{ji}(s_{ji}, o_{ji}). \tag{4.4}$$

One can interpret a neutral interconnection as a lossless one; no 'energy' is dissipated or supplied through the interconnection channel (Willems, 1972). The neutrality condition (4.4) is equivalent with

$$0 = X_{ij} + \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} X_{ji} \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}.$$

4.2.2 Interconnected-system analysis

The following result provides sufficient conditions for well-posedness, stability and bounding the $\mathscr{H}_2$ norm of the interconnected system, and provides a discrete-time counterpart of the continuous-time result of (Chen et al., 2019, Theorem 1). Define the matrix

$$T_i := \left[\begin{array}{ccc} I & 0 & 0 \\ A_i^{\mathrm{TT}} & A_i^{\mathrm{TS}} & B_i^{\mathrm{Td}} \\ \hline A_i^{\mathrm{ST}} & A_i^{\mathrm{SS}} & B_i^{\mathrm{Sd}} \\ 0 & I & 0 \\ \hline C_i^{\mathrm{zT}} & C_i^{\mathrm{zS}} & D_i^{zd} \\ 0 & 0 & I \end{array}\right]. \tag{4.5}$$

Proposition 4.2.1. *The interconnected system $\mathcal{P}_{\mathcal{I}}$ is well-posed, AS and $\|\mathcal{P}_{\mathcal{I}}\|_{\mathscr{H}_2} < \gamma$, if $B_i^{\mathrm{S}d} = 0$ for all $i \in \mathbb{Z}_{[1:L]}$ and there exist positive-definite $X_i \in \mathbb{R}^{k_i \times k_i}$, $\rho_i > 0$, symmetric $X_{ij}^{11} \in \mathbb{R}^{n_{ij} \times n_{ij}}$, $(i,j) \in \mathbb{Z}_{[1:L]}^2$, and $X_{ij}^{12} \in \mathbb{R}^{n_{ij} \times n_{ij}}$, $(i,j) \in \mathbb{Z}_{[1:L]}^2$, $i > j$, with*

$$T_i^\top \left[\begin{array}{cc|cc|cc} -X_i & 0 & 0 & 0 & 0 & 0 \\ 0 & X_i & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & Z_i^{11} & Z_i^{12} & 0 & 0 \\ 0 & 0 & (Z_i^{12})^\top & Z_i^{22} & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & -\rho_i I \end{array}\right] T_i \prec 0, \tag{4.6}$$

$$\sum_{i=1}^{L} \operatorname{trace}\left((B_i^{\mathrm{T}d})^\top X_i B_i^{\mathrm{T}d} + (D_i^{zd})^\top D_i^{zd}\right) < \gamma^2, \tag{4.7}$$

where

$$Z_i^{11} := -\operatorname*{diag}_{j \in \mathbb{Z}_{[1:L]}} X_{ij}^{11}, \; Z_i^{22} := \operatorname*{diag}_{j \in \mathbb{Z}_{[1:L]}} X_{ji}^{11},$$

$$Z_i^{12} := \operatorname{diag}\left(-\operatorname*{diag}_{j \in \mathbb{Z}_{[1:i]}} X_{ij}^{12}, \; \operatorname*{diag}_{j \in \mathbb{Z}_{[i+1:L]}} (X_{ji}^{12})^\top \right).$$

Proof. The proof is provided in Appendix 4.B. □

Remark 4.2.1. *Proposition 4.2.1 is a discrete-time version of a continuous-time result in the literature (Chen et al., 2019, Theorem 1)). Both results follow a dissipativity-based approach for the analysis, following the framework by Langbort et al. (2004). One of the differences in the continuous- and discrete-time results for $\mathscr{H}_2$ analysis, is that direct feed-through from disturbance inputs to performance outputs are allowed in the discrete-time setting, which is also revealed by* (4.7). *The main innovation of the results in this chapter is in the convexity of the distributed controller* synthesis *conditions, which will be discussed in Section 4.3.*

Let us illustrate the analysis conditions by a simple example.

Example 4.1. *Consider two identical scalar subsystems described by*

$$x_i(k+1) = \frac{1}{2}x_i(k) + \frac{1}{10}s_i(k) + d_i(k), \quad i = 1, 2, \; k \in \mathbb{Z},$$

and $z_i(k) = o_i(k) = x_i(k)$, *with interconnection constraints* $s_1(k) = o_2(k)$, $s_2(k) = o_1(k)$. *It is easily verified that LMI* (4.6) *holds for* $i = 1, 2$, *with* $X_i = \frac{7}{4}$, $X_{12}^{11} = X_{21}^{11} = -\frac{1}{5}$, $X_{21}^{12} = 0$ *and* $\rho_i = 20$. *By Proposition 4.2.1, the interconnected system is well-posed, asymptotically stable and the expression* $\|\mathcal{P}_\mathcal{I}\|_{\mathscr{H}_2} < \gamma$ *holds for all* $\gamma > \sqrt{X_1 + X_2} = \sqrt{\frac{7}{2}} \approx 1.87$. *The actual* $\mathscr{H}_2$ *norm of the system is* $\|\mathcal{P}_\mathcal{I}\|_{\mathscr{H}_2} = 1.68$. □

4.3 Distributed $\mathscr{H}_2$ and $\mathscr{H}_\infty$ controller synthesis

Let us now address the problem of synthesizing a distributed controller. Consider the case where each subsystem $\mathcal{P}_i$ has a control input u_i and a measured output y_i, such that

$$\begin{bmatrix} x_i(k+1) \\ o_i(k) \\ z_i(k) \\ y_i(k) \end{bmatrix} = \begin{bmatrix} A_i^{\mathrm{TT}} & A_i^{\mathrm{TS}} & B_i^{\mathrm{T}d} & B_i^{\mathrm{T}u} \\ A_i^{\mathrm{ST}} & A_i^{\mathrm{SS}} & B_i^{\mathrm{S}d} & B_i^{\mathrm{S}u} \\ C_i^{z\mathrm{T}} & C_i^{z\mathrm{S}} & D_i^{zd} & D_i^{zu} \\ C_i^{y\mathrm{T}} & C_i^{y\mathrm{S}} & D_i^{yd} & D_i^{yu} \end{bmatrix} \begin{bmatrix} x_i(k) \\ s_i(k) \\ d_i(k) \\ u_i(k) \end{bmatrix}, \tag{4.8}$$

where we assume that $D_i^{yu} = 0$, without loss of generality (Langbort et al., 2004).

The to-be-synthesized distributed controller is also an interconnected system, with subsystems $\mathcal{C}_i$, $i \in \mathbb{Z}_{[1:L]}$, described by

$$\begin{bmatrix} \xi_i(k+1) \\ o_i^{\mathcal{C}}(k) \\ u_i(k) \end{bmatrix} = \begin{bmatrix} (A_i^{\mathrm{TT}})_\mathcal{C} & (A_i^{\mathrm{TS}})_\mathcal{C} & (B_i^{\mathrm{T}})_\mathcal{C} \\ (A_i^{\mathrm{ST}})_\mathcal{C} & (A_i^{\mathrm{SS}})_\mathcal{C} & (B_i^{\mathrm{S}})_\mathcal{C} \\ (C_i^{\mathrm{T}})_\mathcal{C} & (C_i^{\mathrm{S}})_\mathcal{C} & (D_i)_\mathcal{C} \end{bmatrix} \begin{bmatrix} \xi_i(k) \\ s_i^{\mathcal{C}}(k) \\ y_i(k) \end{bmatrix}, \tag{4.9}$$

where $\xi_i : \mathbb{Z} \to \mathbb{R}^{k_i}$ is the controller's state, and $o_i^{\mathcal{C}} : \mathbb{Z} \to \mathbb{R}^{n_i^{\mathcal{C}}}$, $s_i^{\mathcal{C}} : \mathbb{Z} \to \mathbb{R}^{n_i^{\mathcal{C}}}$ are the controller's interconnection (communication) variables. Controller $\mathcal{C}_i$ and $\mathcal{C}_j$ are interconnected only if $\mathcal{P}_i$ and $\mathcal{P}_j$ are interconnected and the interconnection equation is

$$\begin{bmatrix} o_{ij}^{\mathcal{C}}(k) \\ s_{ij}^{\mathcal{C}}(k) \end{bmatrix} = \begin{bmatrix} s_{ji}^{\mathcal{C}}(k) \\ o_{ji}^{\mathcal{C}}(k) \end{bmatrix}, \quad \forall k \in \mathbb{Z}. \tag{4.10}$$

The local closed-loop (controlled) system, $\mathcal{K}_i$ say, can then be represented by

$$\begin{bmatrix} x_i^{\mathcal{K}}(k+1) \\ o_i^{\mathcal{K}}(k) \\ z_i(k) \end{bmatrix} = \underbrace{\begin{bmatrix} (A_i^{\mathrm{TT}})_{\mathcal{K}} & (A_i^{\mathrm{TS}})_{\mathcal{K}} & (B_i^{\mathrm{T}})_{\mathcal{K}} \\ (A_i^{\mathrm{ST}})_{\mathcal{K}} & (A_i^{\mathrm{SS}})_{\mathcal{K}} & (B_i^{\mathrm{S}})_{\mathcal{K}} \\ (C_i^{\mathrm{T}})_{\mathcal{K}} & (C_i^{\mathrm{S}})_{\mathcal{K}} & (D_i)_{\mathcal{K}} \end{bmatrix}}_{=:\Gamma_i} \begin{bmatrix} x_i^{\mathcal{K}}(k) \\ s_i^{\mathcal{K}}(k) \\ d_i(k) \end{bmatrix}, \tag{4.11}$$

where $x_i^{\mathcal{K}} := \mathrm{col}(x_i, \xi_i)$, $o_i^{\mathcal{K}} := \mathrm{col}(o_i, o_i^{\mathcal{C}})$ and $s_i^{\mathcal{K}} := \mathrm{col}(s_i, s_i^{\mathcal{C}})$. Such a representation is obtained through elimination of the control variables y_i, u_i, as depicted in Figure 4.1. The state-space matrices of a closed-loop subsystem are affine with respect to the state-space matrices of the local controller:

$$\Gamma_i = U_i^{\top} \Theta_i V_i + W_i, \tag{4.12}$$

with

$$\Theta_i := \begin{bmatrix} (A_i^{\mathrm{TT}})_{\mathcal{C}} & (A_i^{\mathrm{TS}})_{\mathcal{C}} & (B_i^{\mathrm{T}})_{\mathcal{C}} \\ (A_i^{\mathrm{ST}})_{\mathcal{C}} & (A_i^{\mathrm{SS}})_{\mathcal{C}} & (B_i^{\mathrm{S}})_{\mathcal{C}} \\ (C_i^{\mathrm{T}})_{\mathcal{C}} & (C_i^{\mathrm{S}})_{\mathcal{C}} & (D_i)_{\mathcal{C}} \end{bmatrix}, \quad V_i := \begin{bmatrix} 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ C_i^{y\mathrm{T}} & 0 & C_i^{y\mathrm{S}} & 0 & D_i^{yd} \end{bmatrix},$$

$$U_i^{\top} := \begin{bmatrix} 0 & 0 & B_i^{\mathrm{T}u} \\ I & 0 & 0 \\ 0 & 0 & B_i^{\mathrm{S}u} \\ 0 & I & 0 \\ 0 & 0 & D_i^{zu} \end{bmatrix}, \quad W_i := \begin{bmatrix} A_i^{\mathrm{TT}} & 0 & A_i^{\mathrm{TS}} & 0 & B_i^{\mathrm{T}d} \\ 0 & 0 & 0 & 0 & 0 \\ A_i^{\mathrm{ST}} & 0 & A_i^{\mathrm{SS}} & 0 & B_i^{\mathrm{S}d} \\ 0 & 0 & 0 & 0 & 0 \\ C_i^{z\mathrm{T}} & 0 & C_i^{z\mathrm{S}} & 0 & D_i^{zd} \end{bmatrix}.$$

4.3.1 $\mathscr{H}_2$ conditions

The feasibility test provided by Proposition 4.2.1 directly induces a feasibility test for well-posedness, stability and $\mathscr{H}_2$ performance for the closed-loop system, which consists of subsystems (4.11), as stated in the following corollary. Define

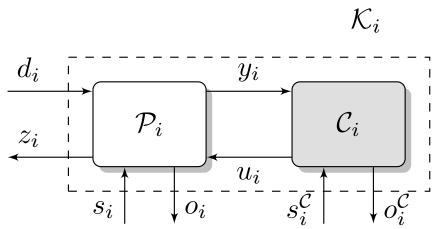


Figure 4.1: Interconnection visualization of a locally controlled system $\mathcal{K}_i$, $i \in \mathbb{Z}_{[1:L]}$.

the matrix

$$T_i^{\mathcal{K}} := \left[\begin{array}{ccc} I & 0 & 0 \\ (A_i^{\mathrm{TT}})_{\mathcal{K}} & (A_i^{\mathrm{TS}})_{\mathcal{K}} & (B_i^{\mathrm{T}})_{\mathcal{K}} \\ \hline (A_i^{\mathrm{ST}})_{\mathcal{K}} & (A_i^{\mathrm{SS}})_{\mathcal{K}} & (B_i^{\mathrm{S}})_{\mathcal{K}} \\ 0 & I & 0 \\ \hline (C_i^{\mathrm{T}})_{\mathcal{K}} & (C_i^{\mathrm{S}})_{\mathcal{K}} & (D_i)_{\mathcal{K}} \\ 0 & 0 & I \end{array}\right].$$

Corollary 4.3.1. *The interconnected system $\mathcal{K}_{\mathcal{I}}$ of* (4.11) *is well-posed, AS and $\|\mathcal{K}_{\mathcal{I}}\|_{\mathscr{H}_2} < \gamma$, if $(B_i^{\mathrm{S}})_{\mathcal{K}} = 0$ for all $i \in \mathbb{Z}_{[1:L]}$ and there exist positive-definite $X_i^{\mathcal{K}} \in \mathbb{R}^{2k_i \times 2k_i}$, $\rho_i > 0$, symmetric $(X_{ij}^{11})_{\mathcal{K}} \in \mathbb{R}^{(n_{ij}+n_{ij}^{\mathcal{C}})\times(n_{ij}+n_{ij}^{\mathcal{C}})}$, $(i,j) \in \mathbb{Z}_{[1:L]}^2$, and $(X_{ij}^{12})_{\mathcal{K}} \in \mathbb{R}^{(n_{ij}+n_{ij}^{\mathcal{C}})\times(n_{ij}+n_{ij}^{\mathcal{C}})}$, $(i,j) \in \mathbb{Z}_{[1:L]}^2$, $i > j$, with*

$$(T_i^{\mathcal{K}})^\top \left[\begin{array}{cc|cc|cc} -X_i^{\mathcal{K}} & 0 & 0 & 0 & 0 & 0 \\ 0 & X_i^{\mathcal{K}} & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & (Z_i^{11})_{\mathcal{K}} & (Z_i^{12})_{\mathcal{K}} & 0 & 0 \\ 0 & 0 & (Z_i^{12})_{\mathcal{K}}^\top & (Z_i^{22})_{\mathcal{K}} & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & -\rho_i I \end{array}\right] T_i^{\mathcal{K}} \prec 0, \tag{4.13}$$

$$\sum_{i=1}^{L} \operatorname{trace}\left((B_i^{\mathrm{T}})_{\mathcal{K}}^\top X_i^{\mathcal{K}} (B_i^{\mathrm{T}})_{\mathcal{K}} + (D_i)_{\mathcal{K}}^\top (D_i)_{\mathcal{K}}\right) < \gamma^2, \tag{4.14}$$

with the closed-loop scales

$$(Z_i^{11})_{\mathcal{K}} := \begin{bmatrix} (Z_i^{11})_{\mathcal{P}} & (Z_i^{11})_{\mathcal{PC}} \\ (Z_i^{11})_{\mathcal{PC}}^\top & (Z_i^{11})_{\mathcal{C}} \end{bmatrix}, \quad (Z_i^{22})_{\mathcal{K}} := \begin{bmatrix} (Z_i^{22})_{\mathcal{P}} & (Z_i^{22})_{\mathcal{PC}} \\ (Z_i^{22})_{\mathcal{PC}}^\top & (Z_i^{22})_{\mathcal{C}} \end{bmatrix},$$
$$(Z_i^{12})_{\mathcal{K}} := \begin{bmatrix} (Z_i^{12})_{\mathcal{P}} & (Z_i^{12})_{\mathcal{PC}} \\ (Z_i^{12})_{\mathcal{CP}} & (Z_i^{12})_{\mathcal{C}} \end{bmatrix},$$

and the submatrices defined in Appendix 4.C. □

Given the affine dependence of the closed-loop state-space matrices Γ_i of subsystems $\mathcal{K}_i$ with respect to the controller parameters Θ_i, it follows that (4.13) is not an LMI with respect to the decision variables Θ_i, $X_i^{\mathcal{K}}$, $(X_{ij}^{11})_{\mathcal{K}}$ and $(X_{ij}^{12})_{\mathcal{K}}$. Through elimination of the controller parameters, the conditions can be transformed into LMIs, which will be discussed in Section 4.3.3.

4.3.2 $\mathscr{H}_\infty$ conditions

The related distributed $\mathscr{H}_\infty$ control problem has been addressed in the literature, where sufficient conditions for robust $\mathscr{H}_\infty$ performance of discrete-time interconnected systems were derived in (Van Horssen and Weiland, 2016). The definition of the $\mathscr{H}_\infty$ norm is given in Appendix 4.D. We recall the robust result from Van Horssen and Weiland (2016) for the nominal case, i.e., for the case that the parametric uncertainty is zero.

Corollary 4.3.2. *The interconnected system $\mathcal{K}_{\mathcal{I}}$ of* (4.11) *is well-posed, AS and $\|\mathcal{K}_{\mathcal{I}}\|_{\mathscr{H}_\infty} < \gamma$, if there exist positive-definite $X_i^{\mathcal{K}} \in \mathbb{R}^{2k_i \times 2k_i}$, symmetric $(X_{ij}^{11})_{\mathcal{K}} \in \mathbb{R}^{(n_{ij}+n_{ij}^{\mathcal{C}})\times(n_{ij}+n_{ij}^{\mathcal{C}})}$, $(i,j) \in \mathbb{Z}^2_{[1:L]}$, and $(X_{ij}^{12})_{\mathcal{K}} \in \mathbb{R}^{(n_{ij}+n_{ij}^{\mathcal{C}})\times(n_{ij}+n_{ij}^{\mathcal{C}})}$, $(i,j) \in \mathbb{Z}^2_{[1:L]}$, $i > j$, with*

$$(T_i^{\mathcal{K}})^\top \left[\begin{array}{cc|cc|cc} -X_i^{\mathcal{K}} & 0 & 0 & 0 & 0 & 0 \\ 0 & X_i^{\mathcal{K}} & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & (Z_i^{11})_{\mathcal{K}} & (Z_i^{12})_{\mathcal{K}} & 0 & 0 \\ 0 & 0 & (Z_i^{12})_{\mathcal{K}}^\top & (Z_i^{22})_{\mathcal{K}} & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \frac{1}{\gamma} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\gamma I \end{array}\right] T_i^{\mathcal{K}} \prec 0, \qquad (4.15)$$

4.3.3 Distributed $\mathscr{H}_2$ controller existence conditions

Recall the definition of T_i in (4.5) and define

$$S_i = \left[\begin{array}{ccc} (A_i^{\mathrm{TT}})^\top & (A_i^{\mathrm{ST}})^\top & (C_i^{z\mathrm{T}})^\top \\ -I & 0 & 0 \\ \hline 0 & -I & 0 \\ (A_i^{\mathrm{TS}})^\top & (A_i^{\mathrm{SS}})^\top & (B_i^{\mathrm{S}d})^\top \\ \hline 0 & 0 & -I \\ (B_i^{\mathrm{T}d})^\top & (B_i^{\mathrm{S}d})^\top & (D_i^{zd})^\top \end{array}\right].$$

We are now ready to state the main result, which provides necessary and sufficient conditions for the existence of a distributed controller that satisfies the conditions in Corollary 4.3.1, in the form of LMIs.

Proposition 4.3.1. *Let $B_i^{\mathrm{S}d} = 0$, $D_i^{yd} = 0$ for all $i \in \mathbb{Z}_{[1:L]}$. The following statements are equivalent:*

- *There exist controllers $\mathcal{C}_i$, with $n_{ij}^{\mathcal{C}} = 3n_{ij}$ for all $(i,j) \in \mathbb{Z}_{[1:L]}^2$ so that the controlled interconnected system described by* (4.2), (4.10) *and* (4.11) *admits $\rho_i > 0$, matrices $X_i^{\mathcal{K}} \succ 0$, $i \in \mathbb{Z}_{[1:L]}$, symmetric $(X_{ij}^{11})_{\mathcal{K}}$, $(i,j) \in \mathbb{Z}_{[1:L]}^2$, and $(X_{ij}^{12})_{\mathcal{K}}$, $(i,j) \in \mathbb{Z}_{[1:L]}^2$, $i > j$, that satisfy inequalities* (4.13) *and* (4.14).

- *There exist X_i, Y_i, symmetric $(X_{ij}^{11})_{\mathcal{P}}$, $(Y_{ij}^{11})_{\mathcal{P}}$, $\alpha_i, \beta_i > 0$ for all $(i,j) \in \mathbb{Z}_{[1:L]}^2$, and $(X_{ij}^{12})_{\mathcal{P}}$, $(Y_{ij}^{12})_{\mathcal{P}}$ for all $(i,j) \in \mathbb{Z}_{[1:L]}^2$, $i > j$, that satisfy*

$$\begin{bmatrix} X_i & I \\ I & Y_i \end{bmatrix} \succ 0, \tag{4.16}$$

$$\sum_{i=1}^{L} \operatorname{trace}\left((B_i^{\mathrm{T}d})^\top X_i B_i^{\mathrm{T}d} + (D_i^{zd})^\top D_i^{zd}\right) < \gamma^2, \tag{4.17}$$

$$\Psi_i^\top T_i^\top \left[\begin{array}{cc|cc|cc} -X_i & 0 & 0 & 0 & 0 & 0 \\ 0 & X_i & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & (Z_i^{11})_{\mathcal{P}} & (Z_i^{12})_{\mathcal{P}} & 0 & 0 \\ 0 & 0 & (Z_i^{12})_{\mathcal{P}}^\top & (Z_i^{22})_{\mathcal{P}} & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & -\alpha_i I \end{array}\right] T_i \Psi_i \prec 0, \tag{4.18}$$

$$\Phi_i^\top S_i^\top \left[\begin{array}{cc|cc|cc} -Y_i & 0 & 0 & 0 & 0 & 0 \\ 0 & Y_i & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & (W_i^{11})_{\mathcal{P}} & (W_i^{12})_{\mathcal{P}} & 0 & 0 \\ 0 & 0 & (W_i^{12})_{\mathcal{P}}^\top & (W_i^{22})_{\mathcal{P}} & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & -\beta_i I \end{array}\right] S_i \Phi_i \succ 0, \tag{4.19}$$

where the columns of Ψ_i and Φ_i form a basis of $\ker(C_i^{yT}\ C_i^{yS}\ D_i^{yd})$ *and* $\ker((B_i^{\mathrm{T}u})^\top\ (B_i^{\mathrm{S}u})^\top\ (D_i^{zu})^\top)$*, respectively, and*

$$(W_i^{11})_{\mathcal{P}} := -\operatorname*{diag}_{j \in \mathbb{Z}_{[1:L]}} (Y_{ij}^{11})_{\mathcal{P}}, \quad (W_i^{22})_{\mathcal{P}} := \operatorname*{diag}_{j \in \mathbb{Z}_{[1:L]}} (Y_{ji}^{11})_{\mathcal{P}},$$

$$(W_i^{12})_{\mathcal{P}} := \operatorname{diag}\left(-\operatorname*{diag}_{j \in \mathbb{Z}_{[1:i]}} (Y_{ij}^{12})_{\mathcal{P}}, \operatorname*{diag}_{j \in \mathbb{Z}_{[i+1,L]}} (Y_{ji}^{12})_{\mathcal{P}}^\top \right).$$

Proof. We first show that the existence of positive scalars α_i and β_i such that (4.18) and (4.19) hold is equivalent with the existence of a positive scalar ρ_i such that

$$\Psi_i^\top T_i^\top \Lambda_i(\rho_i) T_i \Psi_i \prec 0 \text{ and } \Phi_i^\top S_i^\top \Pi_i(\rho_i^{-1}) S_i \Phi_i \succ 0, \tag{4.20}$$

with

$$\begin{aligned}&\Lambda_i : \xi \mapsto \operatorname{diag}(-X_i, X_i, (Z_i)_\mathcal{P}, I, -\xi I) \text{ and}\\ &\Pi_i : \xi \mapsto \operatorname{diag}(-Y_i, Y_i, (W_i)_\mathcal{P}, I, -\xi I).\end{aligned}$$

For sufficiency, let α_i and β_i satisfy (4.18) and (4.19). We distinguish two cases. First, if $\alpha_i\beta_i \geq 1$, then

$$\begin{aligned}&\underbrace{\Phi_i^\top S_i^\top \Pi_i(\beta_i) S_i \Phi_i}_{\succ 0} + \underbrace{\Phi_i^\top S_i^\top \operatorname{diag}(0,0,0,0,(\beta_i - \alpha_i^{-1})I) S_i \Phi_i}_{\succeq 0}\\ &= \Phi_i^\top S_i^\top \Pi_i(\alpha_i^{-1}) S_i \Phi_i \succ 0.\end{aligned}$$

Hence, (4.20) holds for $\rho_i = \alpha_i$. In the other case $\alpha_i\beta_i < 1$, thus it follows that

$$\begin{aligned}&\underbrace{\Psi_i^\top T_i^\top \Lambda_i(\alpha_i) T_i \Psi_i}_{\prec 0} + \underbrace{\Psi_i^\top T_i^\top \operatorname{diag}(0,0,0,0,(\alpha_i - \beta_i^{-1})T_i \Psi_i}_{\preceq 0}\\ &= \Psi_i^\top T_i^\top \Lambda_i(\beta_i^{-1}) T_i \Psi_i \prec 0.\end{aligned}$$

Hence, (4.20) holds for $\rho_i = \beta_i^{-1}$. Necessity follows directly by taking $\alpha_i = \rho_i$ and $\beta_i = \rho_i^{-1}$.

For a proof that the existence of X_i, Y_i, $(Z_i)_\mathcal{P}$, $(W_i)_\mathcal{P}$ and ρ_i that satisfy (4.20) and (4.16) is equivalent with the existence of $X_i^\mathcal{K}$, $(Z_i)_\mathcal{K}$ and ρ_i that satisfy (4.13), we refer the reader to (Langbort et al., 2004) due to space limitations.

Finally, we will show that (4.17) is equivalent with (4.14). We note that for necessity X_i can be taken as the upper-left block of $X_i^\mathcal{K}$, while for sufficiency, $X_i^\mathcal{K}$ can be taken such that its upper-left block equals X_i (Langbort et al., 2004). Thus, by (4.12), we have that

$$\begin{aligned}(B_i^{\mathrm{T}d})^\top X_i B_i^{\mathrm{T}d} + (D_i^{zd})^\top D_i^{zd} = {}&(B_i^\mathrm{T})_\mathcal{K}^\top X_i^\mathcal{K} (B_i^\mathrm{T})_\mathcal{K}\\ &+ (D_i)_\mathcal{K}^\top (D_i)_\mathcal{K}\end{aligned}$$

for all $i \in \mathbb{Z}_{[1:L]}$, since $D_i^{yd} = 0$. It therefore follows that (4.17) $\Leftrightarrow$ (4.14), which concludes the proof. □

Remark 4.3.1. *The equivalence between the convex conditions* (4.18), (4.19) *and non-convex conditions* (4.20) *can be transferred to the continuous-time case (Chen*

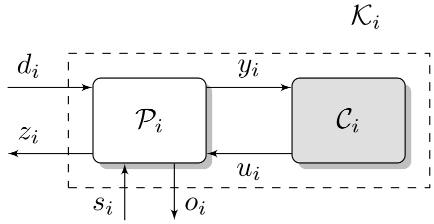


Figure 4.2: Locally controlled system $\mathcal{K}_i$ for a decentralized controller.

et al., 2019, Theorem 2) mutatis mutandis. *The continuous-time distributed $\mathscr{H}_2$ controller existence problem can then be solved via equivalent LMIs, instead of the equivalent bilinear optimization problem with L additional LMIs in (Chen et al., 2019), with L the cardinality of the vertex set $\mathcal{V}$.*

4.3.4 Decentralized $\mathscr{H}_2$ controller existence conditions

A special distributed controller is a decentralized controller, where no controller interconnections are present. This is depicted in Figure 4.2 for a locally controlled system. The synthesis of decentralized controllers is motivated by interconnected systems where no communication between controllers is possible. In this case $n_i^{\mathcal{C}} = 0$, hence Proposition 4.3.1 cannot be applied for the construction of a decentralized controller, since it guarantees the existence of a controller with $n_{ij}^{\mathcal{C}} = 3n_{ij}$ only.

Therefore, we provide conditions for the existence of a controller with $n_{ij}^{\mathcal{C}} = 0$ which achieves global $\mathscr{H}_2$ performance by fixing the supply functions related to the interconnection variables. Given symmetric X_{ij}^{11}, $(i,j) \in \mathbb{Z}_{[1:L]}^2$, and X_{ij}^{12}, $(i,j) \in \mathbb{Z}_{[1:L]}^2$, $i > j$, we have the following result.

Proposition 4.3.2. *Let $B_i^{\mathrm{S}d} = 0$, $D_i^{yd} = 0$ for all $i \in \mathbb{Z}_{[1:L]}$. The following statements are equivalent:*

- *There exist controllers $\mathcal{C}_i$, with $n_i^{\mathcal{C}} = 0$ for all $i \in \mathbb{Z}_{[1:L]}$ so that the controlled interconnected system described by* (4.2), (4.10) *and* (4.11) *admits $\rho_i \in \mathbb{R}_{>0}$, $X_i^{\mathcal{K}} \succ 0$, $i \in \mathbb{Z}_{[1:L]}$ that satisfy* (4.13) *and* (4.14) *for $(X_{ij}^{11})_{\mathcal{K}} = X_{ij}$ and $(X_{ij}^{12})_{\mathcal{K}} = X_{ij}^{12}$.*

- *There exist X_i, Y_i and $\alpha_i, \beta_i > 0$, $i \in \mathbb{Z}_{[1:L]}$, so that* (4.16), (4.17), (4.18) *and* (4.19) *are satisfied for $(Z_i^{11})_{\mathcal{P}} = Z_i^{11}$, $(Z_i^{12})_{\mathcal{P}} = Z_i^{12}$, $(Z_i^{22})_{\mathcal{P}} = Z_i^{22}$*

and

$$\begin{bmatrix}(W_i^{11})_{\mathcal{P}} & (W_i^{12})_{\mathcal{P}} \\ (W_i^{12})_{\mathcal{P}}^\top & (W_i^{22})_{\mathcal{P}}\end{bmatrix} := \begin{bmatrix}Z_i^{11} & Z_i^{12} \\ (Z_i^{12})^\top & Z_i^{22}\end{bmatrix}^{-1}. \tag{4.21}$$

Proof. ($\Leftarrow$) Take an arbitrary $i \in \mathbb{Z}_{[1:L]}$. By (4.16), there exist extended matrices $X_i^{\mathcal{K}}, Y_i^{\mathcal{K}}$, so that $X_i^{\mathcal{K}} = (Y_i^{\mathcal{K}})^{-1}$. Define $\Lambda_i := \operatorname{diag}(-X_i^{\mathcal{K}}, X_i^{\mathcal{K}}, Z_i, I, -\rho_i I)$. Then by (4.18) and (4.19), a permutation of Λ_i gives a matrix P_i which satisfies

$$\begin{gathered}(V_i)_\perp^\top \begin{bmatrix} I \\ W_i \end{bmatrix}^\top P_i \begin{bmatrix} I \\ W_i \end{bmatrix} (V_i)_\perp \prec 0 \text{ and} \\ (U_i)_\perp^\top \begin{bmatrix} -W_i^\top \\ I \end{bmatrix}^\top P_i^{-1} \begin{bmatrix} -W_i^\top \\ I \end{bmatrix} (U_i)_\perp \succ 0.\end{gathered} \tag{4.22}$$

Hence, by the elimination lemma (Scherer, 2001), there exists a Θ_i so that

$$\begin{bmatrix} I \\ U_i^\top \Theta_i V_i + W_i \end{bmatrix}^\top P_i \begin{bmatrix} I \\ U_i^\top \Theta_i V_i + W_i \end{bmatrix} \prec 0, \tag{4.23}$$

which is equivalent with (4.13) for $(X_{ij}^{11})_{\mathcal{K}} = X_{ij}$ and $(X_{ij}^{12})_{\mathcal{K}} = X_{ij}^{12}$.

($\Rightarrow$) To show necessity, observe again that (4.13) is equivalent with (4.23), which is equivalent with (4.22). Then, by taking X_i and Y_i as the upper-left blocks of $X_i^{\mathcal{K}}$ and $Y_i^{\mathcal{K}}$, respectively, we obtain (4.18) and (4.19).

The equivalence of (4.17) and (4.14) was shown in the proof of Proposition 4.3.1, which concludes the proof. □

The main feature of Proposition 4.3.2 is that the existence of a decentralized controller is guaranteed if the conditions hold true, which is crucial if communication between subcontrollers is infeasible. However, this feature comes at the cost of supply functions for the interconnection channels that are assumed to be fixed, which can introduce conservatism regarding the existence of a decentralized controller for the interconnected system under consideration.

Remark 4.3.2. *Fixing the supply functions for the closed-loop subsystems as $\sigma_{ij}(s_{ij}, o_{ij}) = o_{ij}^\top s_{ij}$, corresponding to $X_{ij}^{11} = 0$ and $X_{ij}^{12} = \frac{1}{2}I$, implies that the closed-loop subsystems are required to be passive with respect to the interconnection variables. The design of passive systems holds an important place in control theory (van der Schaft, 2016) and is a classical method for guaranteeing stability of interconnected systems (Arcak et al., 2016); see e.g. (Cucuzzella et al., 2019) for a recent development of passivity-based distributed control for DC microgrids.*

4.3.5 Distributed $\mathscr{H}_\infty$ controller existence conditions

For distributed $\mathscr{H}_\infty$ control, the following convex existence conditions follow from the robust result (Van Horssen and Weiland, 2016, Theorem 2), that we state here for reference:

Proposition 4.3.3. *The following statements are equivalent:*

- *There exist controllers $\mathcal{C}_i$, with $n_{ij}^{\mathcal{C}} = 3n_{ij}$ for all $(i,j) \in \mathbb{Z}^2_{[1:L]}$ so that the controlled interconnected system described by* (4.2), (4.10) *and* (4.11) *admits matrices $X_i^{\mathcal{K}} \succ 0$, $i \in \mathbb{Z}_{[1:L]}$, symmetric $(X_{ij}^{11})_{\mathcal{K}}$, $(i,j) \in \mathbb{Z}^2_{[1:L]}$, and $(X_{ij}^{12})_{\mathcal{K}}$, $(i,j) \in \mathbb{Z}^2_{[1:L]}$, $i > j$, that satisfy inequalities* (4.15).

- *There exist X_i, Y_i, symmetric $(X_{ij}^{11})_{\mathcal{P}}$, $(Y_{ij}^{11})_{\mathcal{P}}$ for all $(i,j) \in \mathbb{Z}^2_{[1:L]}$, and $(X_{ij}^{12})_{\mathcal{P}}$, $(Y_{ij}^{12})_{\mathcal{P}}$ for all $(i,j) \in \mathbb{Z}^2_{[1:L]}$, $i > j$, that satisfy*

$$\begin{bmatrix} X_i & I \\ I & Y_i \end{bmatrix} \succ 0, \tag{4.24}$$

$$\Psi_i^\top T_i^\top \left[\begin{array}{cc|cc|cc} -X_i & 0 & 0 & 0 & 0 & 0 \\ 0 & X_i & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & (Z_i^{11})_{\mathcal{P}} & (Z_i^{12})_{\mathcal{P}} & 0 & 0 \\ 0 & 0 & (Z_i^{12})_{\mathcal{P}}^\top & (Z_i^{22})_{\mathcal{P}} & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \frac{1}{\gamma} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\gamma I \end{array}\right] T_i \Psi_i \prec 0, \tag{4.25}$$

$$\Phi_i^\top S_i^\top \left[\begin{array}{cc|cc|cc} -Y_i & 0 & 0 & 0 & 0 & 0 \\ 0 & Y_i & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & (W_i^{11})_{\mathcal{P}} & (W_i^{12})_{\mathcal{P}} & 0 & 0 \\ 0 & 0 & (W_i^{12})_{\mathcal{P}}^\top & (W_i^{22})_{\mathcal{P}} & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \gamma I & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{\gamma} I \end{array}\right] S_i \Phi_i \succ 0. \tag{4.26}$$

4.3.6 Controller construction

In essence, the controller construction consists of three parts: (i) the extension of the matrices X_i, Y_i, defining the storage functions, to their closed-loop versions $X_i^{\mathcal{K}}$, $Y_i^{\mathcal{K}}$, (ii) the extension of the matrices $(X_{ij}^{11})_{\mathcal{P}}$, $(Y_{ij}^{11})_{\mathcal{P}}$, $(X_{ij}^{12})_{\mathcal{P}}$ and $(Y_{ij}^{12})_{\mathcal{P}}$, defining the 'internal' supply functions, to their closed-loop versions $(Z_i^{11})_{\mathcal{K}}$, $(Z_i^{12})_{\mathcal{K}}$ and $(Z_i^{22})_{\mathcal{K}}$ and (iii) the computation of controller matrices Θ_i

such that the conditions in Corollary 4.3.1 are satisfied. One procedure to construct the distributed controller is provided in this section. The details of this procedure are provided in Appendix 4.E. The main steps for the construction are summarized in the following algorithm.

Algorithm 4.3.1. *For each pair $(i,j) \in \mathbb{Z}^2_{[1:L]}$, let X_i, Y_i, ρ_i, $(X^{11}_{ij})_{\mathcal{P}}$, $(Y^{11}_{ij})_{\mathcal{P}}$, and for each pair $(i,j) \in \mathbb{Z}^2_{[1:L]}$, $i > j$, let $(X^{12}_{ij})_{\mathcal{P}}$, $(Y^{12}_{ij})_{\mathcal{P}}$, be computed to satisfy* (4.16)-(4.17), (4.20).

For each $i \in \mathbb{Z}_{[1:L]}$, the synthesis of controller $\mathcal{C}_i$ proceeds as follows:

1. *Let M_i and N_i be non-singular and such that $M_i N_i^\top = I - X_i Y_i$. Compute $X_i^{\mathcal{K}}$ as the unique solution to the linear equation*

$$X_i^{\mathcal{K}} \begin{bmatrix} Y_i & I \\ N_i^\top & 0 \end{bmatrix} = \begin{bmatrix} I & X_i \\ 0 & M_i^\top \end{bmatrix}.$$

2. *Define*

$$X^{\mathcal{P}}_{ij} := \begin{bmatrix} (X^{11}_{ij})_{\mathcal{P}} & (X^{12}_{ij})_{\mathcal{P}} \\ (X^{12}_{ij})^\top_{\mathcal{P}} & -(X^{11}_{ji})_{\mathcal{P}} \end{bmatrix}, \; Y^{\mathcal{P}}_{ij} := \begin{bmatrix} (Y^{11}_{ij})_{\mathcal{P}} & (Y^{12}_{ij})_{\mathcal{P}} \\ (Y^{12}_{ij})^\top_{\mathcal{P}} & -(Y^{11}_{ji})_{\mathcal{P}} \end{bmatrix}.$$

and compute an eigendecomposition $X^{\mathcal{P}}_{ij} - (Y^{\mathcal{P}}_{ij})^{-1} = V_{ij}\Lambda_{ij}V_{ij}^\top$, with Λ_{ij} a diagonal matrix with the eigenvalues on its diagonal in a descending order. Scale the eigenvectors as $\bar{V}_{ij} = V_{ij}|\Lambda_{ij}|^{\frac{1}{2}}$ such that

$$X^{\mathcal{P}}_{ij} - (Y^{\mathcal{P}}_{ij})^{-1} = (\bar{V}^+_{ij} \; \bar{V}^-_{ij}) \operatorname{diag}(I, -I)(\bar{V}^+_{ij} \; \bar{V}^-_{ij})^\top,$$

with $\bar{V}_{ij} =: (\bar{V}^+_{ij} \; \bar{V}^-_{ij})$. Let $M^{12}_{ij} := \frac{1}{\sqrt{3}}(\bar{V}^+_{ij} \; \bar{V}^+_{ij} \; \bar{V}^+_{ij} \; \bar{V}^-_{ij} \; \bar{V}^-_{ij} \; \bar{V}^-_{ij})$ and $M^{22}_{ij} := \operatorname{diag}(I_{3n_{ij}}, -I_{3n_{ij}})$, and define

$$M^{12}_{ij} =: \begin{bmatrix} (X^{11}_{ij})_{\mathcal{PC}} & (X^{12}_{ij})_{\mathcal{PC}} \\ (X^{12}_{ij})^\top_{\mathcal{CP}} & -(X^{11}_{ji})_{\mathcal{PC}} \end{bmatrix}, \quad M^{22}_{ij} =: \begin{bmatrix} (X^{11}_{ij})_{\mathcal{C}} & (X^{12}_{ij})_{\mathcal{C}} \\ (X^{12}_{ij})^\top_{\mathcal{C}} & -(X^{11}_{ji})_{\mathcal{C}} \end{bmatrix}.$$

3. *Construct the closed-loop scales defined in Appendix 4.C and let*

$$P_i := \left[\begin{array}{ccc|ccc} -X_i^{\mathcal{K}} & 0 & 0 & 0 & 0 & 0 \\ 0 & (Z^{22}_i)_{\mathcal{K}} & 0 & 0 & (Z^{12}_i)^\top_{\mathcal{K}} & 0 \\ 0 & 0 & -\rho_i I & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & X_i^{\mathcal{K}} & 0 & 0 \\ 0 & (Z^{12}_i)_{\mathcal{K}} & 0 & 0 & (Z^{11}_i)_{\mathcal{K}} & 0 \\ 0 & 0 & 0 & 0 & 0 & I \end{array}\right].$$

Solve the following inequality for Θ_i*:*

$$\begin{bmatrix} I \\ U_i^\top \Theta_i V_i + W_i \end{bmatrix}^\top P_i \begin{bmatrix} I \\ U_i^\top \Theta_i V_i + W_i \end{bmatrix} \prec 0. \tag{4.27}$$

The quadratic matrix inequality (4.27) *can be solved by computing an eigendecomposition and solving a linear equation, see Appendix 4.E for details.*

The controller construction is not limited to the discrete-time $\mathscr{H}_2$ distributed control problem; it can also be used for the continuous-time $\mathscr{H}_2$ (Chen et al., 2019), continuous-time $\mathscr{H}_\infty$ (Langbort et al., 2004) and discrete-time $\mathscr{H}_\infty$ (Van Horssen and Weiland, 2016) distributed control problem. We emphasize that the controller construction procedure is performed for each controller $\mathcal{C}_i$ individually, while the LMIs (4.16), (4.17), (4.18) and (4.19) are solved centrally, due to coupling in inequalities (4.17), (4.18) and (4.19).

When the conditions in Proposition 4.3.1 are feasible, Algorithm 4.3.1 will provide a distributed controller such that the closed-loop interconnected system satisfies the conditions in Corollary 4.3.1. The steps in the algorithm are computationally attractive, because these can be performed through the solutions to linear quations, the computation of eigendecompositions and basic matrix operations. The resulting distributed controller is not necessarily well posed, however; a problem that has also been observed in the literature for LPV controller construction (Apkarian and Gahinet, 1995) and distributed $\mathscr{H}_\infty$ controller contruction (Langbort et al., 2004). Langbort et al. (2004) noted that if, for all i, $C_i^{y\mathrm{S}} = 0$ or $B_i^{\mathrm{S}u} = 0$, then well-posedness of the closed-loop interconnected system is equivalent with well-posedness of both the plant and distributed controller. The interpretation of these constraints on the system matrices, is that the control inputs do not directly affect the interconnection signals, or that the interconnection signals do not directly affect the sensors' measurements.

4.4 Numerical examples

To illustrate the distributed $\mathscr{H}_2$ controller synthesis method, we consider a linear coupled-oscillator network consisting of L oscillators. For each node $i \in \mathbb{Z}_{[1:L]}$, the dynamics are described by

$$m_i\ddot{\theta}_i + b_i\dot{\theta}_i = u_i - \sum_{j\in\mathcal{N}_i} k_{ij}(\theta_i - \theta_j) + d_i, \tag{4.28}$$

with inertia m_i, damping b_i and coupling coefficient $k_{ij} = k_{ji}$. The mechanical analogue of a linear coupled-oscillator network is a network of masses that are interconnected through linear springs and have linear damping. A typical system that is modeled as a linear oscillator network is a linearized power network, consisting of generators ($m_i \neq 0$) and loads ($m_i = 0$) (Bergen and Hill, 1981; Dörfler et al., 2013). The local measurement is assumed to be $y_i := \theta_i$ and the performance output is set equal to the state $z_i := x_i := \mathrm{col}(\theta_i, \dot{\theta}_i)$. We use a zero-order hold discretization with sampling time $T = 0.1$ seconds for each subsystem and an approximation $e^M \approx I + M$, so that each subsystem $\mathcal{P}_i$ has an input/state/output representation (4.1) with matrices

$$\begin{aligned}
A_i^{\mathrm{TT}} &= \begin{bmatrix} 1 & T \\ -\sum_{j\in\mathcal{N}_i} \frac{k_{ij}}{m_i}T & 1 - \frac{b_i}{m_i}T \end{bmatrix}, A_i^{\mathrm{TS}} = \underset{j\in\mathcal{N}_i}{\mathrm{row}} \begin{bmatrix} 0 \\ \frac{k_{ij}}{m_i}T \end{bmatrix}, \\
A_i^{\mathrm{ST}} &= C_i^{y\mathrm{T}} = \underset{j\in\mathcal{N}_i}{\mathrm{col}} \begin{bmatrix} 1 & 0 \end{bmatrix},\ A_i^{\mathrm{SS}} = 0_{n_i \times n_i}, \\
B_i^{\mathrm{S}d} &= B_i^{\mathrm{S}u} = 0_{n_i \times 1},\ B_i^{\mathrm{T}d} = B_i^{\mathrm{T}u} = \mathrm{col}(0, \frac{T}{m_i}),\ C_i^{z\mathrm{T}} = I_2, \\
C_i^{z\mathrm{S}} &= 0_{2\times n_i}, D_i^{zd} = D_i^{zu} = 0_{2\times 1}, D_i^{yd} = D_i^{yu} = 0.
\end{aligned}$$

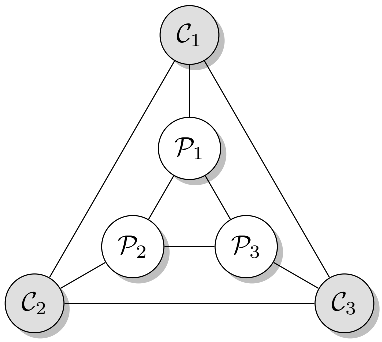


Figure 4.3: Structure of the oscillator network represented by a triangle graph ($L = 3$). The synthesized distributed $\mathscr{H}_2$ controller modules are depicted in gray.

4.4.1 Triangle network ($L = 3$)

Let us consider a network with a triangular structure, as depicted in Figure 4.3. The systems' inertia, damping and coupling coefficients are $m_1 = 3$, $m_2 = 1$, $m_3 = 2$, $b_1 = 2$, $b_2 = 1$, $b_3 = 4$ and $k_{12} = k_{23} = k_{31} = 1$. The open-loop system is not AS. We aim for disturbance attenuation via the synthesis of a distributed controller that achieves unit $\mathscr{H}_2$ performance for the controlled network. We therefore verify the feasibility of the LMIs in Proposition 4.3.1 for $\gamma = 1$. We find that the LMIs are feasible, hence there exists a distributed controller that

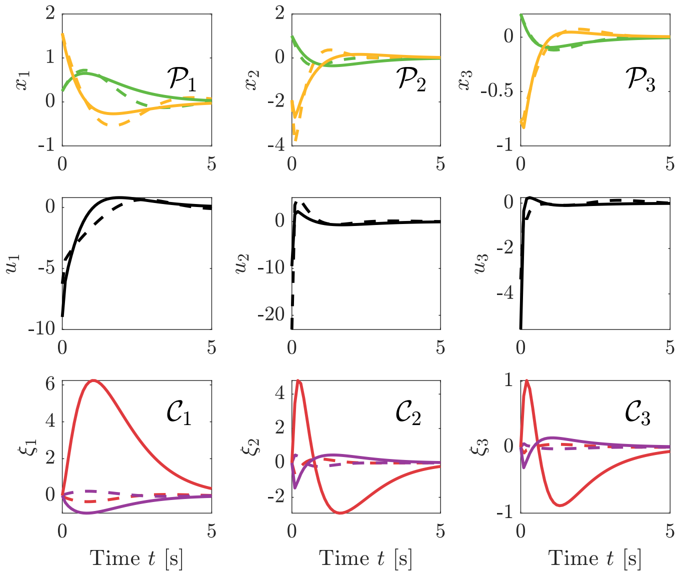
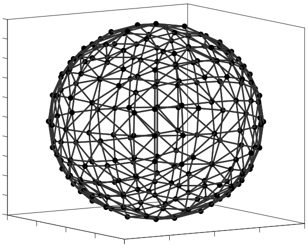


Figure 4.4: Subsystem states $[x_i]_1$ (green) and $[x_i]_2$ (yellow), controller states $[\xi_i]_1$ (red) and $[\xi_i]_2$ (violet) and control inputs u_i (black), $i \in \{1, 2, 3\}$, for the distributed (solid) and central (dashed) controller.

achieves $\|\mathcal{K}_{\mathcal{I}}\|_{\mathscr{H}_2} < 1$. The distributed controller is constructed according to Algorithm 4.3.1 and results in a closed-loop $\mathscr{H}_2$ norm of 0.22. Simulation of the controlled network with zero disturbance, with the subsystems' initial conditions drawn from a normal distribution $\mathcal{N}(0, 1)$ and the controllers' initial conditions set identical to zero, results in the trajectories depicted in Figure 4.4. We observe that the subsystems' and controllers' states asymptotically converge to zero, illustrating asymptotic stability of the closed-loop system. For validation, we also compute a central controller via the feasibility problem in (Scherer and Weiland, 2017) for an $\mathscr{H}_2$ upper-bound equal to 0.22. The resulting controller achieves an $\mathscr{H}_2$ norm of 0.18 and the trajectories are shown in Figure 4.4 (the central controller state $\xi \in \mathbb{R}^6$ is denoted $\xi = \mathrm{col}(\xi_1, \xi_2, \xi_3)$).

4.4.2 Large-scale network ($L = 218$)

Next, we consider a large-scale oscillator network, consisting of $L = 218$ subsystems, with parameters m_i, b_i and $k_{ij} = k_{ji}$ random variables drawn from uniform distributions $\mathcal{U}(2, 3)$, $\mathcal{U}(2, 3)$ and $\mathcal{U}(1, 2)$, respectively. The interconnection structure is described by the graph $\mathcal{G}$, which is visualized in Figure 4.5. This graph has 218 vertices and 648 edges.



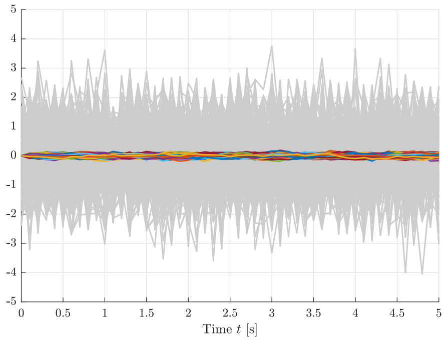


Figure 4.5: Left: Interconnection structure of the considered large-scale oscillator network ($L = 218$). Right: Disturbances d_i (gray) and corresponding performance output components $[z_i]_1$ and $[z_i]_2$ (coloured), $i \in \mathbb{Z}_{[1:218]}$

The goal is to synthesize a distributed controller that achieves $\|\mathcal{K}_{\mathcal{I}}\|_{\mathscr{H}_2} < \gamma$ for $\gamma = 1$. For each $i \in \mathbb{Z}_{[1:L]}$, we select $\alpha_i = \beta_i^{-1} = 10$ and consider the LMIs from Proposition 4.3.1. The corresponding feasibility problem, a semidefinite programming problem consisting of 873 matrix variables, 2593 scalar variables and 5196 constraints, was solved in 0.73 seconds using MOSEK Optimization Suite (MOSEK ApS, 2019) on a PC with a 2.3GHz Intel Core i5 processor and 16GB memory. Interconnection of $\mathcal{P}_{\mathcal{I}}$ with the computed distributed controller $\mathcal{C}_{\mathcal{I}}$ results in the interconnected system $\mathcal{K}_{\mathcal{I}}$, which is asymptotically stable and $\|\mathcal{K}_{\mathcal{I}}\|_{\mathscr{H}_2} = 0.80$.

For illustration of the controlled network's ability to reduce output variance in the case of stochastic disturbance signals, we initialize the system with $x(0) = 0$, $\xi(0) = 0$, and apply signals d_i, that are mutually uncorrelated Gaussian white-noise processes with unit variance. Asymptotically, the obtained $\mathscr{H}_2$ norm for the controlled network is directly related to the output variance through $\lim_{k\to\infty} \mathrm{E} z^\top(k) z(k) = \|\mathcal{K}_{\mathcal{I}}\|^2_{\mathscr{H}_2}$ (Scherer and Weiland, 2017). This stochastic interpretation gives rise to the assessment of the variance of the output on a finite interval. Figure 4.5 shows the two components of all performance outputs z_i, $i \in \mathbb{Z}_{[1:218]}$, which illustrate a significant attenuation of the stochastic disturbances by the distributed controller.

4.4.3 Computation times

To demonstrate the scalability of the developed synthesis method, we consider the controller construction for the oscillator network on cycle graphs with increased values of L. For each graph, the constants m_i, b_i and $k_{ij} = k_{ji}$ are drawn from uniform distributions $\mathcal{U}(1,2)$, $\mathcal{U}(2,3)$ and $\mathcal{U}(1,2)$, respectively. Table 4.1 summarizes the times required to solve the controller existence LMIs in Proposition 4.3.1. The performance bound is chosen as $\gamma = 10$, such that the LMIs are feasible for all values of L in Table 4.1. Computations were performed using MOSEK Optimization Suite (MOSEK ApS, 2019) on a PC with a 2.3GHz Intel Core i5 processor and 16GB memory. We observe that for a cycle graph of moderate size ($L = 50$), the computation time is considerably lower for the distributed controller compared to the central controller. For $L \geq 100$, no solution was obtained for the central controller after 4 hours of computation, while the distributed controller problem was solved for up to $L = 10,000$ in less than 6 seconds.

L	Central controller	Distributed controller
3	0.44s	0.24s
10	0.78s	0.29s
50	831.57s	0.34s
100	†	0.42s
$1,000$	†	1.35s
$10,000$	†	5.77s

Table 4.1: Computation times for solving the LMIs in Proposition 4.3.1 for the distributed $\mathscr{H}_2$ controller and the corresponding LMIs for the central $\mathscr{H}_2$ controller for L interconnected systems on a cycle graph. †: No solution after 4 hours.

4.5 Conclusions

In this chapter, methods have been developed to compute distributed controllers that achieve an $\mathscr{H}_2$ performance bound for interconnected linear discrete-time systems with arbitrary interconnection structure. Convex controller existence conditions have been derived in the form of LMIs, which provide a scalable approach to the construction of distributed $\mathscr{H}_2$ controllers. We have observed a considerable reduction in computation time with respect to centralized $\mathscr{H}_2$ controller synthesis for moderately-sized networks and efficient computation for large-scale networks for which the centralized $\mathscr{H}_2$ synthesis is not tractable.

Appendix

4.A $\mathscr{H}_2$-norm analysis results

Consider a linear discrete-time system Σ described by an input/state/output representation

$$\Sigma : \begin{cases} x(k+1) &= Ax(k) + Bd(k), \\ z(k) &= Cx(k) + Dd(k), \end{cases}$$

with state variable $x : \mathbb{Z} \to \mathbb{R}^n$, (disturbance) input variable $d : \mathbb{Z} \to \mathbb{R}^m$ and output variable $z : \mathbb{Z} \to \mathbb{R}^p$.

Definition 4.A.1. *System Σ is called asymptotically stable (AS) if the roots of* $\det(zI - A)$ *are contained in the open unit disk on the complex plane.*

Definition 4.A.2. *The $\mathscr{H}_2$ norm of an AS system Σ having transfer function $T(z) := C(zI - A)^{-1}B + D$ is defined by*

$$\|\Sigma\|_{\mathscr{H}_2} := \left(\frac{1}{2\pi} \operatorname{trace} \int_{-\pi}^{\pi} T^*(e^{i\omega}) T(e^{i\omega}) \, \mathrm{d}\omega \right)^{\frac{1}{2}}.$$

Lemma 4.A.1. *For an AS system Σ, $\|\Sigma\|_{\mathscr{H}_2}^2 = \operatorname{trace}\left(B^\top M B + D^\top D\right)$ with $M \succeq 0$ satisfying*

$$A^\top M A - M + C^\top C = 0.$$

Proof. By Parseval's theorem we infer that

$$\begin{aligned} \|\Sigma\|_{\mathscr{H}_2}^2 &= \operatorname{trace}\left(\frac{1}{2\pi} \int_{-\pi}^{\pi} T^*(e^{i\omega}) T(e^{i\omega}) \, \mathrm{d}\omega \right) \\ &= \operatorname{trace}\left(D^\top D + \sum_{k=0}^{\infty} B^\top (A^k)^\top C^\top C A^k B \right) \\ &= \operatorname{trace}\left(D^\top D + B^\top M B \right), \end{aligned}$$

with $M := \sum_{k=0}^{\infty}(A^k)^\top C^\top C A^k \succeq 0$ the observability Gramian, that satisfies the matrix equation

$$A^\top M A - M + C^\top C = 0.$$

□

The following result is a discrete-time version of one of the equivalence results in (Scherer and Weiland, 2017, Proposition 3.13), and is instrumental for the proof of Proposition 4.2.1.

Proposition 4.A.1. *Let system Σ be AS and let $\gamma \in \mathbb{R}_{>0}$. The following statements are equivalent:*

(i) $\|\Sigma\|_{\mathscr{H}_2} < \gamma$.

(ii) There exists $X \succ 0$ so that

$$A^\top X A - X + C^\top C \prec 0 \textit{ and } \operatorname{trace}\left(B^\top X B + D^\top D\right) < \gamma^2.$$

Proof. We first show (i) $\Rightarrow$ (ii). Since Σ is AS, there exists $P \succ 0$ so that $A^\top P A - P \prec 0$. Then by Lemma 4.A.1, there exists an $\varepsilon \in \mathbb{R}_{>0}$ so that $X := M + \varepsilon P$ satisfies

$$\operatorname{trace} B^\top X B + D^\top D = \operatorname{trace} B^\top M B + D^\top D + \varepsilon B^\top P B < \gamma^2,$$

with $M \succeq 0$ so that $A^\top M A - M + C^\top C = 0$. Hence, $X \succ 0$ and

$$A^\top X A - X + C^\top C = A^\top M A - M + C^\top C + \varepsilon(A^\top P A - P) \prec 0.$$

Next, we show (ii) $\Rightarrow$ (i). If (ii) is true, then there exists a matrix Γ so that

$$0 = A^\top X A - X + C^\top C + \Gamma^\top \Gamma = A^\top X A - X + \begin{bmatrix} C \\ \Gamma \end{bmatrix}^\top \begin{bmatrix} C \\ \Gamma \end{bmatrix}.$$

Hence, with $T_\Gamma(z) := \Gamma(zI - A)^{-1}B$, we use Lemma 4.A.1 to conclude that

$$\gamma^2 > \|\operatorname{col}(T, T_\Gamma)\|_{\mathscr{H}_2}^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} T^*(e^{i\omega})T(e^{i\omega}) + T_\Gamma^*(e^{i\omega})T_\Gamma(e^{i\omega})\,\mathrm{d}\omega \geq \|T\|_{\mathscr{H}_2}^2,$$

which concludes the proof. □

4.B Proof of Proposition 4.2.1

Proof. Well-posedness is identically defined for continuous-time systems (Langbort et al., 2004) and the proof for well-posedness of $\mathcal{P}_\mathcal{I}$ is identical to the first part of the proof of (Langbort et al., 2004, Theorem 1), since (4.6) implies the condition used therein. Let (4.6) and (4.7) be true. We define the candidate local storage functions $V_i(x_i) := x_i^\top X_i x_i$ and the candidate global storage function $V(x) := \sum_{i=1}^L V_i(x_i)$. Multiplication of inequality (4.6) from the right and from the left with $\operatorname{col}(x_i(k), s_i(k), d_i(k))$ and its transpose yields

$$\begin{aligned}
0 &> x_i^\top(k+1)X_i x_i(k+1) - x_i^\top(k)X_i x_i(k) \\
&\quad + \begin{bmatrix} o_i(k) \\ s_i(k) \end{bmatrix}^\top \begin{bmatrix} Z_i^{11} & Z_i^{12} \\ (Z_i^{12})^\top & Z_i^{22} \end{bmatrix} \begin{bmatrix} o_i(k) \\ s_i(k) \end{bmatrix} \\
&\quad + z_i^\top(k) z_i(k) - \varepsilon_i d_i^\top(k) d_i(k) \\
&= V_i(x(k+1)) - V_i(x(k)) - \sigma_i^{\text{int}}(s_i(k), o_i(k)) - \sigma_i^{\text{ext}}(d_i(k), z_i(k)).
\end{aligned}$$

Thus system $\mathcal{P}_i$ is dissipative with respect to the supply function σ_i. Summing the latter inequality over i yields

$$V(x(k+1)) - V(x(k)) < \sum_{i=1}^L \sigma_i^{\text{int}} + \sigma_i^{\text{ext}}.$$

From the neutrality condition (4.4), we observe that $\sum_{i=1}^L \sigma_i^{\text{int}} = 0$, and thus

$$V(x(k+1)) - V(x(k)) < \sum_{i=1}^L \sigma_i^{\text{ext}}. \tag{4.29}$$

To prove stability, consider the case that $d(k) = 0$. Then

$$V(x(k+1)) - V(x(k)) < -\sum_{i=1}^L z_i^\top(k) z_i(k) \leq 0.$$

Therefore, V is a Lyapunov function for the interconnected system $\mathcal{P}_\mathcal{I}$ with $d(k) = 0$, from which we conclude asymptotic stability of the interconnected system (Kalman and Bertram, 1960, Corollary 1.2).

Next, we prove $\mathscr{H}_2$ performance for $\mathcal{P}_\mathcal{I}$. From (4.3) and inequality (4.29), it follows that for all (x, d)

$$\begin{aligned}
&\begin{bmatrix} x \\ d \end{bmatrix}^\top \begin{bmatrix} I & 0 \\ A_\mathcal{I} & B_\mathcal{I} \end{bmatrix}^\top \begin{bmatrix} -X_\mathcal{I} & 0 \\ 0 & X_\mathcal{I} \end{bmatrix} \begin{bmatrix} I & 0 \\ A_\mathcal{I} & B_\mathcal{I} \end{bmatrix} \begin{bmatrix} x \\ d \end{bmatrix} \\
&\quad < -\begin{bmatrix} x \\ d \end{bmatrix}^\top \begin{bmatrix} C_\mathcal{I} & D_\mathcal{I} \\ 0 & I \end{bmatrix}^\top \begin{bmatrix} I & 0 \\ 0 & -E \end{bmatrix} \begin{bmatrix} C_\mathcal{I} & D_\mathcal{I} \\ 0 & I \end{bmatrix} \begin{bmatrix} x \\ d \end{bmatrix},
\end{aligned}$$

with $X_{\mathcal{I}} := \operatorname{diag}_{i\in\mathbb{Z}_{[1:L]}} X_i$ and $E := \operatorname{diag}_{i\in\mathbb{Z}_{[1:L]}} \varepsilon_i I$. Hence

$$\begin{bmatrix} A_{\mathcal{I}}^\top X_{\mathcal{I}} A_{\mathcal{I}} - X_{\mathcal{I}} + C_{\mathcal{I}}^\top C_{\mathcal{I}} & A_{\mathcal{I}}^\top X_{\mathcal{I}} B_{\mathcal{I}} + C_{\mathcal{I}}^\top D_{\mathcal{I}} \\ B_{\mathcal{I}}^\top X_{\mathcal{I}} A_{\mathcal{I}} + D_{\mathcal{I}}^\top C_{\mathcal{I}} & B_{\mathcal{I}}^\top X_{\mathcal{I}} B_{\mathcal{I}} + D_{\mathcal{I}}^\top D_{\mathcal{I}} - E \end{bmatrix} \prec 0,$$

which implies

$$A_{\mathcal{I}}^\top X_{\mathcal{I}} A_{\mathcal{I}} - X_{\mathcal{I}} + C_{\mathcal{I}}^\top C_{\mathcal{I}} \prec 0. \tag{4.30}$$

Since $B_i^{\mathrm{S}d} = 0$ for all $i \in \mathbb{Z}_{[1:L]}$, we have

$$\begin{aligned} &\operatorname{trace}\left(B_{\mathcal{I}}^\top X_{\mathcal{I}} B_{\mathcal{I}} + D_{\mathcal{I}}^\top D_{\mathcal{I}}\right) && (4.31)\\ &= \operatorname{trace}\left(\sum_{i=1}^{L} (B_i^{\mathrm{T}d})^\top X_i B_i^{\mathrm{T}d} + (D_i^{zd})^\top D_i^{zd}\right) && (4.32)\\ &= \sum_{i=1}^{L} \operatorname{trace}\left((B_i^{\mathrm{T}d})^\top X_i B_i^{\mathrm{T}d} + (D_i^{zd})^\top D_i^{zd}\right) < \gamma^2. && (4.33) \end{aligned}$$

Hence, by Proposition 4.A.1, inequalities (4.30) and (4.33) imply $\|\mathcal{P}_{\mathcal{I}}\|_{\mathscr{H}_2} < \gamma$ and the proof is completed. □

4.C Closed-loop scales

$$\begin{aligned}
(Z_i^{11})_{\mathcal{P}} &:= -\operatorname*{diag}_{j\in\mathbb{Z}_{[1:L]}} (X_{ij}^{11})_{\mathcal{P}}, (Z_i^{22})_{\mathcal{P}} := \operatorname*{diag}_{j\in\mathbb{Z}_{[1:L]}} (X_{ji}^{11})_{\mathcal{P}},\\
(Z_i^{12})_{\mathcal{P}} &:= \operatorname{diag}\left(-\operatorname*{diag}_{j\in\mathbb{Z}_{[1:i]}} (X_{ij}^{12})_{\mathcal{P}}, \operatorname*{diag}_{j\in\mathbb{Z}_{[i+1:L]}} (X_{ji}^{12})_{\mathcal{P}}^\top\right),\\
(Z_i^{11})_{\mathcal{C}} &:= -\operatorname*{diag}_{j\in\mathbb{Z}_{[1:L]}} (X_{ij}^{11})_{\mathcal{C}}, (Z_i^{22})_{\mathcal{C}} := \operatorname*{diag}_{j\in\mathbb{Z}_{[1:L]}} (X_{ji}^{11})_{\mathcal{C}},\\
(Z_i^{12})_{\mathcal{C}} &:= \operatorname{diag}\left(-\operatorname*{diag}_{j\in\mathbb{Z}_{[1:i]}} (X_{ij}^{12})_{\mathcal{C}}, \operatorname*{diag}_{j\in\mathbb{Z}_{[i+1:L]}} (X_{ji}^{12})_{\mathcal{C}}^\top\right),\\
(Z_i^{11})_{\mathcal{PC}} &:= -\operatorname*{diag}_{j\in\mathbb{Z}_{[1:L]}} (X_{ij}^{11})_{\mathcal{PC}}, (Z_i^{22})_{\mathcal{PC}} := \operatorname*{diag}_{j\in\mathbb{Z}_{[1:L]}} (X_{ji}^{11})_{\mathcal{PC}},\\
(Z_i^{12})_{\mathcal{PC}} &:= \operatorname{diag}\left(-\operatorname*{diag}_{j\in\mathbb{Z}_{[1:i]}} (X_{ij}^{12})_{\mathcal{PC}}, \operatorname*{diag}_{j\in\mathbb{Z}_{[i+1:L]}} (X_{ji}^{12})_{\mathcal{CP}}^\top\right),\\
(Z_i^{12})_{\mathcal{CP}} &:= \operatorname{diag}\left(-\operatorname*{diag}_{j\in\mathbb{Z}_{[1:i]}} (X_{ij}^{12})_{\mathcal{CP}}, \operatorname*{diag}_{j\in\mathbb{Z}_{[i+1:L]}} (X_{ji}^{12})_{\mathcal{PC}}^\top\right).
\end{aligned}$$

4.D Definition $\mathscr{H}_\infty$ norm

Let ℓ_2^m be the set of all Lebesgue measurable functions $d : \mathbb{N} \to \mathbb{R}^m$ for which (Stoorvogel, 1992)

$$\|d\|_{\ell_2}^2 := \sum_{k=0}^{\infty} \|d(k)\|^2 < \infty.$$

Definition 4.D.1. *The $\mathscr{H}_\infty$ norm of an AS system Σ is defined by*

$$\|\Sigma\|_{\mathscr{H}_\infty} := \sup_{0 \neq d \in \ell_2^m} \frac{\|Td\|_{\ell_2}}{\|d\|_{\ell_2}}.$$

4.E Controller reconstruction details

Let X_i, Y_i, ρ_i, $(X_{ij}^{11})_{\mathcal{P}}$, $(Y_{ij}^{11})_{\mathcal{P}}$, $(X_{ij}^{12})_{\mathcal{P}}$ and $(Y_{ij}^{12})_{\mathcal{P}}$ satisfy LMIs (4.16), (4.17) and (4.20). Let $i \in \mathbb{Z}_{[1:L]}$. First, we construct the closed-loop matrices

$$X_i^{\mathcal{K}} := \begin{bmatrix} X_i & X_i^{\mathcal{PC}} \\ (X_i^{\mathcal{PC}})^\top & X_i^{\mathcal{C}} \end{bmatrix}, \quad Y_i^{\mathcal{K}} := \begin{bmatrix} Y_i & Y_i^{\mathcal{PC}} \\ (Y_i^{\mathcal{PC}})^\top & Y_i^{\mathcal{C}} \end{bmatrix}, \tag{4.34}$$

so that $X_i^{\mathcal{K}} = (Y_i^{\mathcal{K}})^{-1} \succ 0$. The extension of X_i and Y_i to their closed-loop counterparts $X_i^{\mathcal{K}} \in \mathbb{R}^{2k_i \times 2k_i}$ and $Y_i^{\mathcal{K}} \in \mathbb{R}^{2k_i \times 2k_i}$ is well-known for the centralized quadratic performance problem (including the $\mathscr{H}_\infty$ control problem), see e.g. (Scherer and Weiland, 2017, Theorem 4.2), (Gahinet and Apkarian, 1994), and can be performed as follows. Inequality (4.16) is equivalent to $I - X_iY_i \prec 0$, hence $I - X_iY_i$ is of rank k_i. Take non-singular matrices $M_i, N_i \in \mathbb{R}^{k_i \times k_i}$ so that $M_iN_i^\top = I - X_iY_i$. Now, we find $Y_i^{\mathcal{K}}$ as the unique solution to the linear equation

$$\begin{bmatrix} Y_i & I \\ N_i^\top & 0 \end{bmatrix} = Y_i^{\mathcal{K}} \begin{bmatrix} I & X_i \\ 0 & M_i^\top \end{bmatrix}, \tag{4.35}$$

and set $X_i^{\mathcal{K}} := (Y_i^{\mathcal{K}})^{-1}$. It is clear that $X_i^{\mathcal{K}}$ and $Y_i^{\mathcal{K}}$ are of the form (4.34). Observe that $X_i^{\mathcal{K}} \succ 0$ and $(Y_i^{\mathcal{K}}) \succ 0$ is equivalent to $I - X_iY_i \prec 0$, by application of the Schur complement to the explicit expression of the solution $Y_i^{\mathcal{K}}$ to (4.35).

Let $(i,j) \in \mathbb{Z}_{[1:L]}^2$, $i > j$ and let $X_{ij}^{\mathcal{P}}, Y_{ij}^{\mathcal{P}} \in \mathbb{R}^{2n_{ij} \times 2n_{ij}}$ be defined by

$$X_{ij}^{\mathcal{P}} := \begin{bmatrix} (X_{ij}^{11})_{\mathcal{P}} & (X_{ij}^{12})_{\mathcal{P}} \\ (X_{ij}^{12})_{\mathcal{P}}^\top & -(X_{ji}^{11})_{\mathcal{P}} \end{bmatrix}, \quad Y_{ij}^{\mathcal{P}} := \begin{bmatrix} (Y_{ij}^{11})_{\mathcal{P}} & (Y_{ij}^{12})_{\mathcal{P}} \\ (Y_{ij}^{12})_{\mathcal{P}}^\top & -(Y_{ji}^{11})_{\mathcal{P}} \end{bmatrix}.$$

By (Dullerud and D'Andrea, 2004, Lemma 21), there exist matrices $M_{ij}^{12}, N_{ij}^{12} \in \mathbb{R}^{2n_{ij} \times l_{ij}}$ and $M_{ij}^{22}, N_{ij}^{22} \in \mathbb{R}^{l_{ij} \times l_{ij}}$ so that

$$\begin{bmatrix} X_{ij}^{\mathcal{P}} & M_{ij}^{12} \\ (M_{ij}^{12})^\top & M_{ij}^{22} \end{bmatrix} = \begin{bmatrix} Y_{ij}^{\mathcal{P}} & N_{ij}^{12} \\ (N_{ij}^{12})^\top & N_{ij}^{22} \end{bmatrix}^{-1}$$

with

$$\mathrm{in}\begin{bmatrix} X_{ij}^{\mathcal{P}} & M_{ij}^{12} \\ (M_{ij}^{12})^\top & M_{ij}^{22} \end{bmatrix} = (\iota_{ij}^-, 0, \iota_{ij}^+),$$

if and only if

$$\mathrm{in}^- \begin{bmatrix} X_{ij}^{\mathcal{P}} & I \\ I & M_{ij}^{12} \end{bmatrix} \leq \iota_{ij}^- \text{ and } \mathrm{in}^+ \begin{bmatrix} X_{ij}^{\mathcal{P}} & I \\ I & M_{ij}^{12} \end{bmatrix} \leq \iota_{ij}^+.$$

For $l_{ij} = 6n_{ij}$ and $i_{ij}^- = i_{ij}^+ = 4n_{ij}$, the latter inertia requirements are satisfied Langbort et al. (2004). The construction of such M_{ij}^{12}, N_{ij}^{12} and M_{ij}^{22}, N_{ij}^{22} follows from the constructive proof for (Dullerud and D'Andrea, 2004, Lemma 21). Let $M_{ij}^{22} := \mathrm{diag}(I, -I) \in \mathbb{R}^{6n_{ij} \times 6n_{ij}}$ and $M_{ij}^{12} \in \mathbb{R}^{2n_{ij} \times 6n_{ij}}$ so that $\mathrm{in}\, M_{ij}^{22} = (\iota_{ij}^-, 0, \iota_{ij}^+) - \mathrm{in}\, Y_{ij}^{\mathcal{P}}$ and

$$X_{ij}^{\mathcal{P}} - (Y_{ij}^{\mathcal{P}})^{-1} = M_{ij}^{12} M_{ij}^{22} (M_{ij}^{12})^\top = M_{ij}^{12} \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} (M_{ij}^{12})^\top. \tag{4.36}$$

Since $X_{ij}^{\mathcal{P}} - (Y_{ij}^{\mathcal{P}})^{-1}$ is symmetric, it commutes with itself and hence it admits an eigendecomposition (Bernstein, 2011, Corollary 5.4.4)

$$X_{ij}^{\mathcal{P}} - (Y_{ij}^{\mathcal{P}})^{-1} = V_{ij} \Lambda_{ij} V_{ij}^\top,$$

with $\Lambda_{ij} = \mathrm{diag}_{k \in \mathbb{Z}_{[1:2n_{ij}]}} (\lambda_{ij})_k$, $(\lambda_{ij})_1 \geq (\lambda_{ij})_2 \geq \cdots \geq (\lambda_{ij})_{2n_{ij}}$ and V_{ij} a unitary matrix whose columns are corresponding eigenvectors. Clearly, if we let $\bar{V}_{ij} = V_{ij} |\Lambda_{ij}|^{\frac{1}{2}}$, then $X_{ij}^{\mathcal{P}} - (Y_{ij}^{\mathcal{P}})^{-1} = (\bar{V}_{ij}^+ \ \bar{V}_{ij}^-) \,\mathrm{diag}(I, -I) (\bar{V}_{ij}^+ \ \bar{V}_{ij}^-)^\top$, with $\bar{V}_{ij} =: (\bar{V}_{ij}^+ \ \bar{V}_{ij}^-)$. Thus we take

$$M_{ij}^{12} := \frac{1}{\sqrt{3}} \begin{bmatrix} \bar{V}_{ij}^+ & \bar{V}_{ij}^+ & \bar{V}_{ij}^+ & \bar{V}_{ij}^- & \bar{V}_{ij}^- & \bar{V}_{ij}^- \end{bmatrix}$$

such that (4.36) holds. Hence, by defining

$$M_{ij}^{12} =: \begin{bmatrix} (X_{ij}^{11})_{\mathcal{PC}} & (X_{ij}^{12})_{\mathcal{PC}} \\ (X_{ij}^{12})_{\mathcal{CP}}^\top & -(X_{ji}^{11})_{\mathcal{PC}} \end{bmatrix}, \quad M_{ij}^{22} =: \begin{bmatrix} (X_{ij}^{11})_{\mathcal{C}} & (X_{ij}^{12})_{\mathcal{C}} \\ (X_{ij}^{12})_{\mathcal{C}}^\top & -(X_{ji}^{11})_{\mathcal{C}} \end{bmatrix},$$

we can construct the scales

$$Z_i^{\mathcal{K}} := \begin{bmatrix} (Z_i^{11})_{\mathcal{K}} & (Z_i^{12})_{\mathcal{K}} \\ (Z_i^{12})_{\mathcal{K}}^\top & (Z_i^{22})_{\mathcal{K}} \end{bmatrix}, \quad W_i^{\mathcal{K}} := \begin{bmatrix} (W_i^{11})_{\mathcal{K}} & (W_i^{12})_{\mathcal{K}} \\ (W_i^{12})_{\mathcal{K}}^\top & (W_i^{22})_{\mathcal{K}} \end{bmatrix}, \tag{4.37}$$

such that $Z_i^{\mathcal{K}} = (W_i^{\mathcal{K}})^{-1}$, with $(W_i^{11})_{\mathcal{K}}$, $(W_i^{12})_{\mathcal{K}}$ and $(W_i^{22})_{\mathcal{K}}$ analogously defined as $(Z_i^{11})_{\mathcal{K}}$, $(Z_i^{12})_{\mathcal{K}}$ and $(Z_i^{22})_{\mathcal{K}}$ in Appendix 4.C.

For each $i \in \mathbb{Z}_{[1:L]}$, let $\bar{P}_i = \mathrm{diag}(-X_i^{\mathcal{K}}, X_i^{\mathcal{K}}, Z_i^{\mathcal{K}}, I, -\rho_i I)$. Permute the rows and columns of $\bar{P}_i$ to obtain

$$P_i := \left[\begin{array}{ccc|ccc} -X_i^{\mathcal{K}} & 0 & 0 & 0 & 0 & 0 \\ 0 & (Z_i^{22})_{\mathcal{K}} & 0 & 0 & (Z_i^{12})_{\mathcal{K}}^{\top} & 0 \\ 0 & 0 & -\rho_i I & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & X_i^{\mathcal{K}} & 0 & 0 \\ 0 & (Z_i^{12})_{\mathcal{K}} & 0 & 0 & (Z_i^{11})_{\mathcal{K}} & 0 \\ 0 & 0 & 0 & 0 & 0 & I \end{array}\right], \tag{4.38}$$

such that

$$\begin{gathered} (V_i)_{\perp}^{\top} \begin{bmatrix} I \\ W_i \end{bmatrix}^{\top} P_i \begin{bmatrix} I \\ W_i \end{bmatrix} (V_i)_{\perp} \prec 0 \text{ and} \\ (U_i)_{\perp}^{\top} \begin{bmatrix} -W_i^{\top} \\ I \end{bmatrix}^{\top} P_i^{-1} \begin{bmatrix} -W_i^{\top} \\ I \end{bmatrix} (U_i)_{\perp} \succ 0. \end{gathered} \tag{4.39}$$

By the elimination lemma (Scherer, 2001, Lemma A.2), there exists a controller matrix Θ_i so that (4.13) is satisfied, or, equivalently, so that

$$\begin{bmatrix} I \\ U_i^{\top}\Theta_i V_i + W_i \end{bmatrix}^{\top} P_i \begin{bmatrix} I \\ U_i^{\top}\Theta_i V_i + W_i \end{bmatrix} \prec 0. \tag{4.40}$$

To construct such a Θ_i, let H_i and J_i be non-singular matrices such that

$$V_i H_i =: \begin{bmatrix} \bar{V}_i & 0 \end{bmatrix}, \quad U_i J_i =: \begin{bmatrix} \bar{U}_i & 0 \end{bmatrix},$$

with $\bar{V}_i$ and $\bar{U}_i$ having full column rank. Then with $Q_i := J_i^{\top} W_i H_i$, we can rewrite inequality (4.40) as (Scherer, 2001)

$$(\star)^{\top} \underbrace{\begin{bmatrix} H_i^{\top} & 0 \\ 0 & J_i^{-1} \end{bmatrix} P_i \begin{bmatrix} H_i^{\top} & 0 \\ 0 & J_i^{-1} \end{bmatrix}^{\top}}_{=:\Pi_i} \begin{bmatrix} I & 0 \\ 0 & I \\ \bar{U}_i^{\top}\Theta_i \bar{V}_i + Q_i^{11} & Q_i^{12} \\ Q_i^{21} & Q_{22} \end{bmatrix} \prec 0,$$

and, hence, as

$$\begin{bmatrix} R_i \begin{bmatrix} I \\ E_i \end{bmatrix} & S_i \end{bmatrix}^{\top} \Pi_i \begin{bmatrix} R_i \begin{bmatrix} I \\ E_i \end{bmatrix} & S_i \end{bmatrix} \prec 0, \tag{4.41}$$

with $E_i := \bar{U}_i^{\top}\Theta_i \bar{V}_i + Q_i^{11}$ and

$$R_i := \begin{bmatrix} I & 0 \\ 0 & 0 \\ 0 & I \\ Q_i^{21} & 0 \end{bmatrix}, \quad S_i := \begin{bmatrix} 0 \\ I \\ Q_i^{12} \\ Q_i^{22} \end{bmatrix}.$$

Now, because E_i is an unrestricted unknown in (4.41), a suitable solution is given by $E_i = (E_2)_i(E_1)_i^{-1}$ (Scherer, 2001), with $F_i := \mathrm{col}((E_1)_i, (E_2)_i)$ solving the quadratic inequality

$$F_i^\top \underbrace{\left(R_i^\top \Pi_i R_i - R_i^\top \Pi_i S_i (S_i^\top \Pi_i S_i)^{-1} S_i^\top \Pi_i R_i\right)}_{=:\Omega_i} F_i \prec 0. \tag{4.42}$$

Let the columns of F_i be vectors that span the eigenspaces of Γ_i that are associated with negative eigenvalues, such that (4.42) is satisfied. If the resulting $(E_1)_i$ is singular, one can always choose a $\delta_i > 0$ such that $(E_1)_i + \delta_i I$ is non-singular and

$$\begin{bmatrix} (E_1)_i + \delta_i I \\ (E_2)_i \end{bmatrix}^\top \Omega_i \begin{bmatrix} (E_1)_i + \delta_i I \\ (E_2)_i \end{bmatrix} \prec 0.$$

Finally, a suitable controller matrix Θ_i can then be constructed by solving the linear equation

$$\bar{U}_i^\top \Theta_i \bar{V}_i = (E_2)_i \left((E_1)_i + \delta_i I\right)^{-1} - Q_i^{11}. \tag{4.43}$$

Chapter 5

Distributed control in a behavioral setting

Control in a classical transfer function or state-space setting typically views a controller as a signal processor: sensor outputs are mapped to actuator inputs. In behavioral system theory, control is simply viewed as interconnection; the interconnection of a plant with a controller. In this chapter we consider the problem of control of interconnected systems in a behavioral setting. The behavioral setting is especially fit for modeling interconnected systems, because it allows for the interconnection of subsystems without imposing inputs and outputs. The aim is to find a distributed controller that is explicit in the plant dynamics, such that it can also serve as a basis for distributed model-reference control in Part II. We introduce a so-called *canonical* distributed controller that implements a given interconnected behavior that is desired. The controller is distributed and its design is, given the desired behavior's subsystems, decentralized, in the sense that a local controller only depends on the local system behavior, by definition. Regularity of interconnections is an important property in behavioral control that yields feedback interconnections. We provide conditions under which the interconnection of this distributed controller with the plant is regular. Furthermore, we show that the interconnections of subsystems of the canonical distributed controller are regular if and only if the interconnections of the plant and desired behavior are regular.

5.1 Introduction

When physical systems are interconnected, no distinction between inputs and outputs is made. Think for example of the interconnection of two RLC-circuits through their terminals or the interconnection of two mass-spring-damper systems. Typical transfer function and input-state-output representations inherently impose an input-output partition of system variables. One of the main features of the *behavioral* approach to system theory, is that it does not take an input-output structure as a starting point to describe systems: a mathematical model is simply the relation between system variables. In the case of dynamical systems, the set of all time trajectories that are compatible with the model is called the behavior. The behavioral approach has been advocated as a convenient starting point in several applications, among which in the context of interconnected systems (Willems, 2007) and the context of control (Willems, 1997).

In the context of interconnected systems, modeling can be performed through tearing (viewing the interconnected system as an interconnection of subsystems), zooming (modeling the subsystems), and linking (modeling the interconnections) (Willems, 2007). Interconnection of systems in a behavioral setting means variable sharing. When two masses are physically interconnected, the laws of motion for the first mass involve the position of the second mass and vice versa; the laws of motion of both masses together dictate the behavior of the interconnected system. Thinking of system interconnections makes the modeling of interconnected systems remarkably simple. Partitioning variables into input and output variables is appropriate in signal processing, feedback control based on sensor outputs and other unilateral systems, but often unnecessary for physical system variables (Willems, 2007).

Feedback control based on sensor outputs to generate actuator inputs, where the controller is viewed as a signal processor (Trentelman, 2011), holds an important place in control theory. It has been argued that many practical control devices cannot be interpreted as feedback controllers, such as passive-vibration control systems, passive suspension systems or operational amplifiers (Willems, 1997). Indeed, such control systems do not inherit a signal flow, but can be interpreted as an interconnection in a behavioral setting. More specifically, control in a behavioral setting means restricting the behavior of the system that is to be controlled, by interconnecting it with a controller. By specifying a behavior that is desired for the controlled system, the basic control problem is to determine the existence of a controller such that the controlled system's behavior is equal to the desired behavior. This is called the implementability problem (Trentelman, 2011). The *canonical* controller plays a major role in the implementability problem: the canonical controller implements the desired behavior if and only if the desired behavior is implementable (Julius et al., 2005), (van der Schaft, 2003).

In this chapter, we will consider *distributed* control in a behavioral setting. In particular, we will consider distributed control of interconnected linear time-invariant systems. As a natural consequence of behavioral interconnections, we consider a distributed controller to be an interconnected system itself, i.e., we consider it to consist of subsystems that are interconnected without imposing signal flows between subsystems. Several types of interconnections become of interest in this problem: interconnections between subsystems of the to-be-controlled interconnected system (plant), interconnections between subsystems of the plant and subsystems of the distributed controller, and interconnections between subsystems of the distributed controller. Given a desired behavior for the controlled interconnected system that has the same interconnection structure as the plant, the considered distributed control problem is to determine the existence of a distributed controller that implements the desired behavior. We introduce a distributed canonical controller which implements the desired behavior if it is implementable. The distributed canonical controller has an attractive interconnection structure, in the sense that two of its subsystems are interconnected only if two subsystems of the plant or desired behavior are interconnected.

Distributed control with input-output partitioning and communication between subsystems of the distributed controller (considered in, e.g., Chapter 4 and Chapter 6) follows as an important special case of distributed control in a behavioral setting. An important question is: when can the canonical distributed controller be implemented with feedback interconnections? Following up on this question: When can the interconnections between controller subsystems be implemented as communication channels? The main concept in the solution to these problems is *regularity* of the corresponding interconnections. We will analyze regularity of the canonical distributed controller. In particular, we show that the connections between subsystems of this distributed controller are regular if and only if connections between subsystems of the plant and desired behavior are regular.

5.2 Preliminaries

For the notions of systems in the behavioral setting, we will follow the notation in (Trentelman, 2011). A dynamical system is defined as a triple $\Sigma = (T, W, \mathfrak{B})$, where $T \subseteq \mathbb{R}$ is the time space, W is the signal space and $\mathfrak{B} \subseteq W^T$ is the behavior. Consider two dynamical systems $\Sigma_1 = (T, W_1 \times W_3, \mathfrak{B}_1)$ and $\Sigma_2 = (T, W_2 \times W_3, \mathfrak{B}_2)$ with the same time space, and trajectories $(w_1, w_3) \in \mathfrak{B}_1$ and $(w_2, w_3) \in \mathfrak{B}_2$, respectively. The interconnection of Σ_1 and Σ_2 through w_3 yields the dynamical system

$$\Sigma_1 \wedge_{w_3} \Sigma_2 := (T, W_1 \times W_2 \times W_3, \mathfrak{B}),$$

with

$$\mathfrak{B} := \{(w_1, w_2, w_3) : T \to W_1 \times W_2 \times W_3 \mid (w_1, w_3) \in \mathfrak{B}_1 \text{ and } (w_2, w_3) \in \mathfrak{B}_2\}.$$

The manifest behavior of Σ_1 w.r.t. w_1 is

$$(\mathfrak{B}_1)_{w_1} := \{w_1 : T \to W_1 \mid \exists w_3 \text{ so that } (w_1, w_3) \in \mathfrak{B}\}.$$

The set $\mathfrak{L}^{\mathtt{w}}$ denotes the set of all linear differential systems $\Sigma = (\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathfrak{B})$, with $\mathtt{w} \in \mathbb{N}$ variables, where the behavior is

$$\mathfrak{B} := \{w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid R(\frac{\mathrm{d}}{\mathrm{d}t})w = 0\},$$

with a polynomial matrix $R \in \mathbb{R}^{\mathtt{g} \times \mathtt{w}}[\xi]$, $\mathtt{g} \in \mathbb{N}_{>0}$, and $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ denotes the set of infinitely often differentiable functions from $\mathbb{R}$ to $\mathbb{R}^{\mathtt{w}}$.

Consider a behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$. The components of $w \in \mathfrak{B}$ allow for a component-wise partition[1] such that $w = (u, y)$, with u input and y output. The partition $w = (u, y)$ is called an input-output partition if u is free, i.e., for all u there exists a y so that $(u, y) \in \mathfrak{B}$ (Polderman and Willems, 1998). Such a partition is not unique, but always exists. The *number* of components in the input and output, called the input and output cardinality, is invariant, i.e., independent of the input-output partition. Henceforth, $\mathtt{m}(\mathfrak{B})$ denotes the input cardinality and $\mathtt{p}(\mathfrak{B})$ denotes the output cardinality, which implies that $\mathtt{p}(\mathfrak{B}) + \mathtt{m}(\mathfrak{B}) = \mathtt{w}$. For a kernel representation $R\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) w = 0$ of $\mathfrak{B}$, the output cardinality is $\mathtt{p}(\mathfrak{B}) = \operatorname{rank} R$.

5.2.1 Control in a behavioral setting

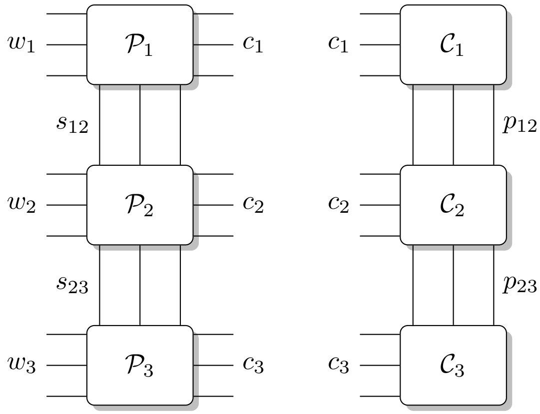
A controlled interconnection is the interconnection of a plant $\Sigma_p = (T, W \times C, \mathcal{P})$ and a controller $\Sigma_c = (T, C, \mathcal{C})$, with the same time space, and trajectories $(w, c) \in \mathcal{P}$ and $c \in \mathcal{C}$, respectively. The plant has two types of variables: w is the to-be-controlled variable and c is the control variable. The controlled interconnection is thus $\mathcal{P} \wedge_c \mathcal{C}$. A general control problem can now be formulated as: Given the plant behavior $\mathcal{P} \subseteq (W \times C)^T$ and a desired behavior $\mathcal{K} \subseteq W^T$, does there exist a controller $\mathcal{C}$ so that $\mathcal{K} = (\mathcal{P} \wedge_c \mathcal{C})_w$, i.e., is $\mathcal{K}$ implementable? The implementability problem has been extensively studied in (Trentelman, 2011).

5.3 Control of interconnected systems

5.3.1 Plant interconnections

For the design of a distributed controller, we consider plants $\Sigma_{p_i} = (T, W_i \times S_i \times C_i, \mathcal{P}_i)$, $i \in \mathbb{Z}_{[1:L]}$, having trajectories $(w_i, s_i, c_i) \in \mathcal{P}_i$, with w_i the to-be-controlled variable, s_i the inter-plant connection variable and c_i the control

[1]Up to re-ordering of the components in w.



(a) System interconnection. (b) Controller interconnection.

Figure 5.1: Behavioral approach to control of interconnected systems.

variable. Partition the inter-plant connection variable s_i into s_{ij}, the variable that behavior $\mathcal{P}_i$ shares with $\mathcal{P}_j$. The interconnection of $\mathcal{P}_i$ and $\mathcal{P}_j$ is given by

$$\mathcal{P}_i \wedge_{s_{ij}} \mathcal{P}_j = \{(w_i, w_j, s_{ij}, c_i, c_j) \,|\, (w_i, s_{ij}, c_i) \in \mathcal{P}_i \text{ and } (w_j, s_{ij}, c_j) \in \mathcal{P}_j\}.$$

We denote the straightforward generalization of the interconnection of $\mathcal{P}_i$, $i \in \mathbb{Z}_{[1:L]}$ as

$$\begin{aligned}\mathcal{P}_{\mathcal{I}} &:= \wedge_{s_i, i\in\mathbb{Z}_{[1:L]}} \mathcal{P}_i \\ &= \{(w_1, \ldots, w_L, s_1, \ldots, s_L, c_1, \ldots, c_L) \,|\, (w_i, s_i, c_i) \in \mathcal{P}_i \text{ for all } i \in \mathbb{Z}_{[1:L]}\}.\end{aligned}$$

Figure 5.1a depicts an interconnection example of three behaviors $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}_3$, through s_{12} and s_{23}, i.e., $\mathcal{P}_1 \wedge_{s_{12}} \mathcal{P}_2 \wedge_{s_{23}} \mathcal{P}_3$. When we eliminate the interconnection variables $(s_i)_{i\in\mathbb{Z}_{[1:L]}}$ from the behavior of the interconnected system, $\mathcal{P}_{\mathcal{I}}$, we obtain the manifest behavior of $\mathcal{P}_{\mathcal{I}}$ with respect to (w, c). This manifest behavior of the plant interconnection w.r.t. (w, c) is

$$\begin{aligned}(\mathcal{P}_{\mathcal{I}})_{(w,c)} &= (\wedge_{s_i, i\in\mathbb{Z}_{[1:L]}} \mathcal{P}_i)_{(w,c)} \\ &= \{(w_1, \ldots, w_L, c_1, \ldots, c_L) \,|\, \exists s_i \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{s}_i}), i \in \mathbb{Z}_{[1:L]}, \\ &\qquad \text{so that } (w_i, s_i, c_i) \in \mathcal{P}_i \text{ for all } i \in \mathbb{Z}_{[1:L]}\}.\end{aligned}$$

5.3.2 Distributed control problem

In the following, we will consider an interconnection of linear systems $\mathcal{P}_i \in \mathfrak{L}^{\mathtt{w}_i+\mathtt{s}_i+\mathtt{c}_i}$, $i \in \mathbb{Z}_{[1:L]}$. Given $\mathcal{K}_i \in \mathfrak{L}^{\mathtt{w}_i+\mathtt{k}_i}$, $i \in \mathbb{Z}_{[1:L]}$, let the desired behavior of the interconnected system be equal to the manifest behavior of the interconnection of $\mathcal{K}_i$ w.r.t. w, i.e.,

$$\begin{aligned}(\mathcal{K}_{\mathcal{I}})_w = (\wedge_{k_i, i\in\mathbb{Z}_{[1:L]}}\mathcal{K}_i)_w = &\{(w_1,\dots,w_L)\,|\,\exists k_i \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{k}_i}), i \in \mathbb{Z}_{[1:L]},\\ &\text{so that } (w_i,k_i)\in\mathcal{K}_i, i\in\mathbb{Z}_{[1:L]}\}.\end{aligned}$$

Complementary to the interconnected plant, we are looking for another interconnected behavior, the controller, such that the interconnection of the plant with the controller yields the desired manifest behavior w.r.t. w, i.e., $(\mathcal{K}_{\mathcal{I}})_w$. The controller behavior is the interconnection of $\mathcal{C}_i \in \mathfrak{L}^{\mathtt{c}_i+\mathtt{p}_i}$, $i \in \mathbb{Z}_{[1:L]}$, through inter-controller connection variable p_i. The controller interconnection is distributed in the following sense: if $\mathcal{P}_i$ and $\mathcal{P}_j$ do not share a variable (they cannot be interconnected), then $\mathcal{C}_i$ and $\mathcal{C}_j$ do not share a variable, i.e., for each pair $(i,j) \in \mathbb{Z}^2_{[1:L]}$, it holds that $\mathtt{s}_{ij} = 0 \Rightarrow \mathtt{p}_{ij} = 0$. In this way, the controller structure will reflect the plant structure and, hence, the structure of the "closed-loop" interconnection. This idea is exemplified in Figure 5.1b for the plant interconnection in Figure 5.1a. The chosen controller structure is a design choice that is natural in the sense that the interconnection structure of the plant is respected. Therefore, this choice is commonly considered in the distributed control literature, cf. (D'Andrea and Dullerud, 2003), (Langbort et al., 2004), (Camponogara et al., 2002), (Cantoni et al., 2007), (Rice and Verhaegen, 2009), (Chen et al., 2019). Alternative distributed controller structures are, for example, hierarchical and multilayer structures, which are designed according to multi-level or multi-resolution models (Christofides et al., 2013) or through optimization (Gusrialdi, 2012).

Considering the control problem described in Section 5.2.1, we can now analogously state the distributed control problem: Given the plant interconnection $\mathcal{P}_{\mathcal{I}} = \wedge_{s_i, i\in\mathbb{Z}_{[1:L]}}\mathcal{P}_i$ and a desired behavior defined by $\mathcal{K}_{\mathcal{I}} = \wedge_{k_i, i\in\mathbb{Z}_{[1:L]}}\mathcal{K}_i$, do there exist controllers $\mathcal{C}_i \in \mathfrak{L}^{\mathtt{c}_i+\mathtt{p}_i}$, $i \in \mathbb{Z}_{[1:L]}$, so that $(\mathcal{K}_{\mathcal{I}})_w = ((\wedge_{s_i, i\in\mathbb{Z}_{[1:L]}}\mathcal{P}_i) \wedge_c (\wedge_{p_i, i\in\mathbb{Z}_{[1:L]}}\mathcal{C}_i))_w$? That is, does there exist a distributed controller such that the desired behavior is equal to the controlled interconnection? Figure 5.2 illustrates this controlled interconnection.

Definition 5.3.1. *Let $\mathcal{K}_i$, $i \in \mathbb{Z}_{[1:L]}$, be given and consider the desired interconnected system behavior $(\mathcal{K}_{\mathcal{I}})_w$. If there exists a distributed controller such that the controlled interconnected behavior equals the desired interconnected behavior,* *i.e., if there exist $\mathcal{C}_i$, $i \in \mathbb{Z}_{[1:L]}$, such that*

$$
\begin{gathered}
(\wedge_{k_i, i\in\mathbb{Z}_{[1:L]}} \mathcal{K}_i)_w = \left([\wedge_{s_i, i\in\mathbb{Z}_{[1:L]}} \mathcal{P}_i] \wedge_c [\wedge_{p_i, i\in\mathbb{Z}_{[1:L]}} \mathcal{C}_i]\right)_w \\
\Updownarrow \\
(\mathcal{K}_{\mathcal{I}})_w = \left(\mathcal{P}_{\mathcal{I}} \wedge_c [\wedge_{p_i, i\in\mathbb{Z}_{[1:L]}} \mathcal{C}_i]\right)_w
\end{gathered}
\tag{5.1}
$$

then $\mathcal{K}_I$ is called implementable via distributed control.

Consequently, a distributed controller with controller behaviors $\mathcal{C}_i$, $i \in \mathbb{Z}_{[1:L]}$, is said to implement $\mathcal{K}_{\mathcal{I}}$ if (5.1) holds.

Remark 5.3.1. *A natural question that comes to mind is: what prevents a desired behavior from being implementable? The necessary conditions of the controller implementability theorem for 'centralized' control by Willems and Trentelman (2002) reveal that there are two restrictions: (i) since control means that the behavior of the plant is restricted, the desired behavior must be a subset of the (manifest) behavior of the plant and (ii) since the hidden behavior of the plant (for $c = 0$) should remain possible, the hidden behavior of the plant must be subset of the desired behavior (Willems and Trentelman, 2002).*

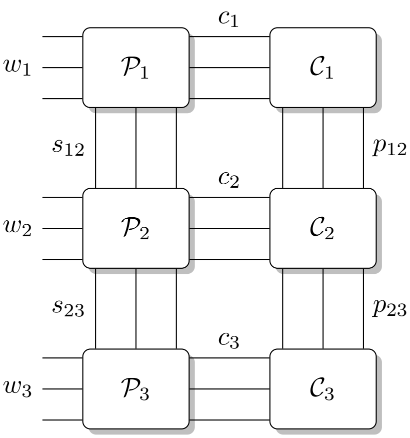


Figure 5.2: Controlled interconnection.

5.4 Canonical distributed controller

Let $\mathcal{K}_i \in \mathfrak{L}^{\mathtt{w}_i+\mathtt{k}_i}$, $i \in \mathbb{Z}_{[1:L]}$, and consider its interconnected manifest behavior $(\mathcal{K}_\mathcal{I})_w \in \mathfrak{L}^{\mathtt{w}+\mathtt{k}}$. We define the controller

$$\begin{aligned}\mathcal{C}_i^{\text{can}} &:= (\mathcal{P}_i \wedge_{w_i} \mathcal{K}_i)_{(c_i,s_i,k_i)} && (5.2)\\ &= \{(c_i,s_i,k_i) \mid \exists w_i \text{ so that } (w_i,s_i) \in \mathcal{P}_i \text{ and } (w_i,k_i) \in \mathcal{K}_i\}, && (5.3)\end{aligned}$$

i.e., the manifest behavior w.r.t. (c_i,s_i,k_i) of the interconnection of the local plant behavior $\mathcal{P}_i$ and desired behavior $\mathcal{K}_i$ through w_i. This interconnection is depicted in Figure 5.3. We call $\mathcal{C}_i^{\text{can}}$ a local canonical controller. By the elimination theorem (Polderman and Willems, 1998, Theorem 6.2.6), we have that $\mathcal{C}_i^{\text{can}} \in \mathfrak{L}^{\mathtt{c}_i+\mathtt{s}_i+\mathtt{k}_i}$.

Notice that by construction of the plant interconnection and interconnection defining the desired behavior, we can interconnect two canonical controllers $\mathcal{C}_i^{\text{can}}$ and $\mathcal{C}_j^{\text{can}}$ through the variables (s_{ij},k_{ij}), i.e., $\mathcal{C}_i^{\text{can}} \wedge_{(s_{ij},k_{ij})} \mathcal{C}_j^{\text{can}}$. In order to construct a distributed controller, we interconnect the local canonical controllers $\mathcal{C}_i^{\text{can}}$, $i \in \mathbb{Z}_{[1:L]}$, through (s_i,k_i). The behavior of the interconnection of the local canonical controllers is

$$\mathcal{C}_\mathcal{I}^{\text{can}} = \wedge_{(s_i,k_i),i\in\mathbb{Z}_{[1:L]}} \mathcal{C}_i^{\text{can}},$$

which is called the canonical distributed controller.

We will now provide conditions on the interconnected system and desired interconnected behavior under which the canonical distributed controller implements $\mathcal{K}_I$. The hidden behavior of $\mathcal{P}_\mathcal{I}$ is defined as

$$\mathcal{N}(\mathcal{P}_I) := \{w \mid (w,0) \in (\mathcal{P}_\mathcal{I})_{(w,c)}\}.$$

Proposition 5.4.1. *The controller $\mathcal{C}_\mathcal{I}^{can}$ implements the desired behavior $\mathcal{K}_\mathcal{I} \in \mathfrak{L}^{\mathtt{w}+\mathtt{k}}$ if*

$$\mathcal{N}(\mathcal{P}_\mathcal{I}) \subseteq (\mathcal{K}_\mathcal{I})_w \subseteq (\mathcal{P}_\mathcal{I})_w.$$

Proof. The proof can be separated in two parts: (i) show that the distributed canonical controller satisfies $(\mathcal{C}_\mathcal{I}^{\text{can}})_c = ((\mathcal{P}_\mathcal{I})_{(w,c)} \wedge_w (\mathcal{K}_\mathcal{I})_w)_c$ and (ii) application of the implementability proof for the centralized canonical controller (Julius et al., 2005). We will prove both parts (i) and (ii) for completeness.

We will first show that $(\wedge_{(s_i,k_i)}\mathcal{C}_i^{\text{can}})_c = \big((\wedge_{s_i}\mathcal{P}_i)_{(w,c)} \wedge_w (\wedge_{k_i}\mathcal{K}_i)_w\big)\, c$, i.e., that $(\mathcal{C}_I^{\text{can}})_c = ((\mathcal{P}_\mathcal{I})_{(w,c)} \wedge_w (\mathcal{K}_\mathcal{I})_w)_c$. The manifest behavior of $\wedge_{k_i}\mathcal{K}_i$ with respect to w_i is

$$(\wedge_{k_i}\mathcal{K}_i)_w = \{(w_1,\ldots,w_L) \mid \exists k_i,\ i \in \mathbb{Z}_{[1:L]},\ \text{so that } (w_i,k_i) \in \mathcal{K}_i,\ i \in \mathbb{Z}_{[1:L]}\}$$

and the manifest behavior of $\wedge_{s_i}\mathcal{P}_i$ with respect to (w,c) is

$$(\wedge_{s_i}\mathcal{P}_i)_{(w,c)} = \{(w_1,\ldots,w_L,c_1,\ldots,c_L)\,|\,\exists s_i,\ i\in\mathbb{Z}_{[1:L]},\ \text{so that}\ (w_i,s_i,c_i)\in\mathcal{P}_i,\ i\in\mathbb{Z}_{[1:L]}\}.$$

Hence, we have

$$\big((\wedge_{s_i}\mathcal{P}_i)_{(w,c)}\wedge_w(\wedge_{k_i}\mathcal{K}_i)_w\big)_c = \{(c_1,\ldots,c_L)\,|\,\exists(w_i,s_i,k_i),\ i\in\mathbb{Z}_{[1:L]},\ \text{so that}\ (w_i,k_i)\in\mathcal{K}_i \text{ and } (w_i,s_i,c_i)\in\mathcal{P}_i\}.$$

Furthermore, the manifest behavior of $\mathcal{C}_{\mathcal{I}}^{\mathrm{can}}$ with respect to c is

$$\begin{aligned}(\mathcal{C}_{\mathcal{I}}^{\mathrm{can}})_c &= (\wedge_{(s_i,k_i),\ i\in\mathbb{Z}_{[1:L]}}\mathcal{C}_i^{\mathrm{can}})_c\\ &= \{(c_1,\ldots,c_L)\,|\,\exists(s_i,k_i),\ i\in\mathbb{Z}_{[1:L]},\ \text{so that}\ (c_i,s_i,k_i)\in\mathcal{C}_i^{\mathrm{can}}\},\\ &= \{(c_1,\ldots,c_L)\,|\,\exists(w_i,s_i,k_i),\ i\in\mathbb{Z}_{[1:L]},\ \text{so that}\\ &\qquad (w_i,k_i)\in\mathcal{K}_i \text{ and } (w_i,s_i,c_i)\in\mathcal{P}_i\}.\end{aligned}$$

Hence, it follows that $(\mathcal{C}_I^{\mathrm{can}})_c = ((\mathcal{P}_{\mathcal{I}})_{(w,c)}\wedge_w(\mathcal{K}_{\mathcal{I}})_w)_c$.

With this expression for the behavior of the canonical distributed controller, we find that the behavior of the interconnection of the manifest behavior of the canonical distributed controller and the manifest behavior of the plant is equal to

$$((\mathcal{P}_{\mathcal{I}})_{(w,c)}\wedge_c((\mathcal{C}_I^{\mathrm{can}})_c)_w = ((\mathcal{P}_{\mathcal{I}})_{(w,c)}\wedge_c((\mathcal{P}_{\mathcal{I}})_{(w,c)}\wedge_w(\mathcal{K}_{\mathcal{I}})_w)_c)_w.$$

We will now show that this behavior is in fact equal to $(\mathcal{K}_{\mathcal{I}})_w$. Consider minimal kernel representations for $(\mathcal{P}_{\mathcal{I}})_{(w,c)}$ and $(\mathcal{K}_{\mathcal{I}})_w$, respectively:

$$R\left(\tfrac{\mathrm{d}}{\mathrm{d}t}\right)w + M\left(\tfrac{\mathrm{d}}{\mathrm{d}t}\right)c = 0,\quad K\left(\tfrac{\mathrm{d}}{\mathrm{d}t}\right)w = 0.$$

We therefore have that

$$\exists w \text{ so that } \begin{bmatrix}R\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) & M\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\\ K\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) & 0\end{bmatrix}\begin{bmatrix}w\\ c\end{bmatrix} = 0$$

is a latent variable representation for $(\mathcal{C}_{\mathcal{I}}^{\mathrm{can}})_c$. Since $\mathcal{N}(\mathcal{P}_{\mathcal{I}})\subseteq(\mathcal{K}_{\mathcal{I}})_w$ and $(\mathcal{K}_{\mathcal{I}})_w\subseteq(\mathcal{P}_{\mathcal{I}})_w$, there exists a polynomial matrix $F(\xi)$ so that $K(\xi)=F(\xi)R(\xi)$. Consider the unimodular matrix $U(\xi) := \begin{bmatrix}F(\xi) & -I\\ I & 0\end{bmatrix}$. Post-multiplication of $U(\xi)$ with $\mathrm{col}(M(\xi),0)$ and $\mathrm{col}(-R(\xi),K(\xi))$ yields $U(\xi)\begin{bmatrix}M(\xi)\\ 0\end{bmatrix} = \begin{bmatrix}F(\xi)M(\xi)\\ M(\xi)\end{bmatrix}$ and

$$U(\xi)\begin{bmatrix}-R(\xi)\\ K(\xi)\end{bmatrix} = \begin{bmatrix}-F(\xi)R(\xi)+K(\xi)\\ -R(\xi)\end{bmatrix} = \begin{bmatrix}0\\ -R(\xi)\end{bmatrix}.$$

We thus have $(\mathcal{C}_{\mathcal{I}})_c = \{c \,|\, F\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) M\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)) \, c = 0\}$ so that

$$\begin{aligned}(\mathcal{P}_{\mathcal{I}})_{(w,c)} \wedge_c ((\mathcal{C}_I^{\text{can}})_c &= \{(w,c) \,|\, \begin{bmatrix} R\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) & M\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) \\ 0 & F\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) M\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) \end{bmatrix} \begin{bmatrix} w \\ c \end{bmatrix} = 0\} \\ &= \{(w,c) \,|\, \begin{bmatrix} R\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) \\ 0 \end{bmatrix} w = \begin{bmatrix} -M\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) \\ -F\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) M\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) \end{bmatrix} c\}.\end{aligned}$$

Now, since

$$U(\xi) \begin{bmatrix} R(\xi) \\ 0 \end{bmatrix} = \begin{bmatrix} F(\xi)R(\xi) \\ R(\xi) \end{bmatrix} \quad \text{and} \quad U(\xi) \begin{bmatrix} -M(\xi) \\ -F(\xi)M(\xi) \end{bmatrix} = \begin{bmatrix} 0 \\ -M(\xi) \end{bmatrix},$$

we have

$$((\mathcal{P}_{\mathcal{I}})_{(w,c)} \wedge_c ((\mathcal{C}_{\mathcal{I}}^{\text{can}})_c)_w = \{w \,|\, FR\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) w = 0\} = \{w \,|\, K\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) w = 0\} = (\mathcal{K}_{\mathcal{I}})_w$$

and the proof is complete. □

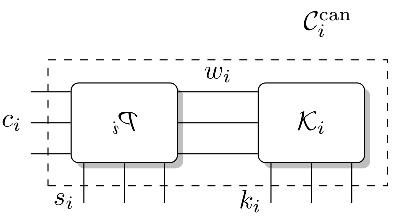


Figure 5.3: Local canonical controller. The mirrored plant notation emphasizes that the control and to-be-controlled variables of $\mathcal{P}_i$ are reversed inside the canonical controller.

Remark 5.4.1. *The manifest behavior of the controller w.r.t. c is equal to the behavior of the "central" canonical controller for the desired interconnected behavior, cf. (Julius et al., 2005). Intuitively, this is sensible, see e.g. the controlled interconnection for the example with 3 subsystems in Figure 5.4. The controllers* $\mathcal{C}_i^{can}$ *are based on "local" behavior* $\mathcal{P}_i$*, while the central canonical controller is based on* $(\mathcal{P}_{\mathcal{I}})_{(w,c)}$*. From a distribution point of view, the control design is decentralized in the sense that only the subsystem* $\mathcal{P}_i$ *of the interconnected system is required to determine* $\mathcal{C}_i^{can}$*, once a desired interconnected behavior has been specified.*

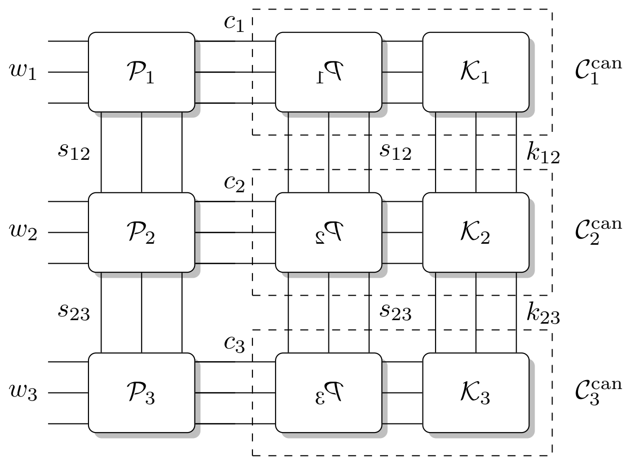


Figure 5.4: Controlled interconnection with local canonical controllers.

5.5 Regularity of the canonical distributed controller

An important type of system interconnections is a *regular interconnection*, introduced by (Willems, 1997). Formally, a regular interconnection of two systems is defined as follows.

Definition 5.5.1. *Consider two behaviors $\mathfrak{B}_1 \in \mathfrak{L}^{\mathtt{w}_1+\mathtt{w}_2}$ and $\mathfrak{B}_2 \in \mathfrak{L}^{\mathtt{w}_2+\mathtt{w}_3}$. The interconnection of $\mathfrak{B}_1$ and $\mathfrak{B}_2$ is said to be regular if*

$$\mathtt{p}(\mathfrak{B}_1 \wedge_{w_2} \mathfrak{B}_2) = \mathtt{p}(\mathfrak{B}_1) + \mathtt{p}(\mathfrak{B}_2),$$

where $\mathfrak{B}_1 \wedge_{w_2} \mathfrak{B}_2 = \{(w_1, w_2, w_3) \,|\, (w_1, w_2) \in \mathfrak{B}_1$ and $(w_2, w_3) \in \mathfrak{B}_2\}$.

Regularity of the interconnection of two systems has multiple interpretations. First, regularity means in a sense that the equations describing the dynamics of $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are independent of each other (Willems et al., 2003). For the second interpretation, consider a plant $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$, a controller $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ and their interconnection $\mathcal{K} := \{(w, c) \in \mathcal{P} \,|\, c \in \mathcal{C}\}$. According to Definition 5.5.1, the plant-controller interconnection is regular if

$$\mathtt{p}(\mathcal{K}) = \mathtt{p}(\mathcal{P}) + \mathtt{p}(\mathcal{C}).$$

This interconnection is regular if and only if the controller $\mathcal{C}$ can be realized as a transfer function from an output variable to an input variable of $\mathcal{P}$ for an input/output partitioning of the control variable c (Willems et al., 2003). From a control-point-of-view, regularity of the plant-controller interconnection therefore means that the controller acts as a feedback controller, i.e., it can process sensor outputs to actuator inputs. Notice that, while this is a typical assumption in control theory that is not in the behavioral framework, it is not a matter of course in control in a behavioral setting (Willems, 1997), (Willems et al., 2003), (Julius et al., 2005).

Let us now consider regularity of the interconnections related to the canonical distributed controller, which was introduced in Section 5.4. There are two types of interconnections that are of interest: (i) the interconnection between the canonical distributed controller $\mathcal{C}_I^{\text{can}}$ and the interconnected system $\mathcal{P}_{\mathcal{I}}$, i.e., the plant-controller interconnection and (ii) the interconnection between $\mathcal{C}_i^{\text{can}}$ and $\mathcal{C}_j^{\text{can}}$, $(i,j) \in \mathbb{Z}^2_{[1:L]}$ and $i \neq j$, i.e., the interconnection of local controllers. The interpretation of regularity of the plant-controller interconnection has been considered in the previous paragraph. Regularity of the interconnection of local canonical controllers can be interpreted as follows. If the interconnection between controllers is regular, then the interconnection variable p_{ij} can always be partitioned to achieve a regular feedback interconnection, i.e., such that the transfers from inputs in the partitioning to outputs are proper. In a sense, regularity of interconnection between local controllers thus means that the controllers can communicate with each other, by processing received communication signals (input) into sent communication signals (output).

5.5.1 Regularity of the plant-controller interconnection

Regularity of the interconnection of the interconnected system behavior $\mathcal{P}_{\mathcal{I}}$ and a distributed controller $\mathcal{C}_{\mathcal{I}}$ follows from the regularity of the behaviors with the interconnection variables $(s_1, \ldots, s_L)$ and $(p_1, \ldots, p_L)$ eliminated, i.e., from $(\mathcal{P}_{\mathcal{I}})_{(w,c)}$ and $(\mathcal{C}_{\mathcal{I}})_c$. By definition, the interconnection of $(\mathcal{P}_{\mathcal{I}})_{(w,c)}$ and $(\mathcal{C}_{\mathcal{I}})_c$ is regular if

$$\mathtt{p}((\mathcal{P}_{\mathcal{I}})_{(w,c)}) + \mathtt{p}((\mathcal{C}_{\mathcal{I}})_c) = \mathtt{p}((\mathcal{P}_{\mathcal{I}})_{(w,c)} \wedge_c (\mathcal{C}_{\mathcal{I}})_c). \tag{5.4}$$

If (5.4) holds, then the distributed controller is called regular with respect to c. A sufficient condition for regularity with respect to c of all distributed controllers that implement $\mathcal{K}_{\mathcal{I}}$ follows from (Julius et al., 2005, Theorem 12).

Proposition 5.5.1. *Let $\mathcal{P}_i \in \mathfrak{L}^{\mathtt{w}_i+\mathtt{s}_i+\mathtt{c}_i}$ and $\mathcal{C}_i \in \mathfrak{L}^{\mathtt{c}_i+\mathtt{p}_i}$, $i \in \mathbb{Z}_{[1:L]}$, and consider the interconnected system $\mathcal{P}_{\mathcal{I}} = \wedge_{s_i, i\in\mathbb{Z}_{[1:L]}} \mathcal{P}_i$ and distributed controller $\mathcal{C}_{\mathcal{I}} = \wedge_{p_i, i\in\mathbb{Z}_{[1:L]}} \mathcal{C}_i$. Let $(\mathcal{K}_{\mathcal{I}})_w$ be the desired behavior, with $\mathcal{K}_{\mathcal{I}} = \wedge_{k_i, i\in\mathbb{Z}_{[1:L]}} \mathcal{K}_i$, where $\mathcal{K}_i \in \mathfrak{L}^{\mathtt{w}_i+\mathtt{k}_i}$, $i \in \mathbb{Z}_{[1:L]}$.*

Every distributed controller $\mathcal{C}_{\mathcal{I}}$ that implements $\mathcal{K}_{\mathcal{I}}$, i.e., (5.1) *holds, is regular with respect to c if $(\mathcal{P}_{\mathcal{I}})_c = \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{c}})$, where $(\mathcal{P}_{\mathcal{I}})_c$ is the manifest behavior of* *the interconnected system with respect to c, i.e.,*

$$(\mathcal{P}_{\mathcal{I}})_c = \{c \,|\, \exists (w,s) \text{ so that } (w,s,c) \in \mathcal{P}_{\mathcal{I}}\}.$$

Proof. First, notice that $\mathcal{P}_{\mathcal{I}} = \wedge_{s_i, i \in \mathbb{Z}_{[1:L]}} \mathcal{P}_i \in \mathfrak{L}^{\mathtt{w}+\mathtt{s}+\mathtt{c}}$ and that $(\mathcal{P}_{\mathcal{I}})_{(w,c)} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$. Hence, there exists a minimal kernel representation for $(\mathcal{P}_{\mathcal{I}})_{(w,c)}$:

$$R\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) w + M\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) c = 0.$$

Assume that $(\mathcal{P}_{\mathcal{I}})_c = \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{c}})$. Then R has full row rank. Now, take any distributed controller $\mathcal{C}_{\mathcal{I}} = \wedge_{p_i, i \in \mathbb{Z}_{[1:L]}} \mathcal{C}_i \in \mathfrak{L}^{\mathtt{c}+\mathtt{p}}$ that implements $\mathcal{K}_{\mathcal{I}}$. The manifest behavior of $\mathcal{C}_{\mathcal{I}}$ with respect to c, i.e., $(\mathcal{C}_{\mathcal{I}})_c$, satisfies $(\mathcal{C}_{\mathcal{I}})_c \in \mathfrak{L}^{\mathtt{c}}$ and therefore has a minimal kernel representation

$$C\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) c = 0.$$

Since R has full row rank, we find that

$$\begin{bmatrix} R\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) & M\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) \\ 0 & C\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) \end{bmatrix} \begin{bmatrix} w \\ c \end{bmatrix} = 0$$

is a minimal kernel representation of $(\mathcal{K}_{\mathcal{I}})_{(w,c)}$. We find that

$$\mathtt{p}((\mathcal{K}_{\mathcal{I}})_{(w,c)}) = \operatorname{rank} R + \operatorname{rank} C = \mathtt{p}((\mathcal{P}_{\mathcal{I}})_{(w,c)}) + \mathtt{p}((\mathcal{C}_{\mathcal{I}})_c),$$

which was to be proven. $\square$

Corollary 5.5.1. *Consider an interconnected system* $\mathcal{P}_{\mathcal{I}} = \wedge_{s_i, i \in \mathbb{Z}_{[1:L]}} \mathcal{P}_i$*,* $\mathcal{P}_i \in \mathfrak{L}^{\mathtt{w}_i+\mathtt{s}_i+\mathtt{c}_i}$*, and the desired behavior* $(\mathcal{K}_{\mathcal{I}})_w$*, with* $\mathcal{K}_{\mathcal{I}} = \wedge_{k_i, i \in \mathbb{Z}_{[1:L]}} \mathcal{K}_i$*,* $\mathcal{K}_i \in \mathfrak{L}^{\mathtt{w}_i+\mathtt{k}_i}$*,* $i \in \mathbb{Z}_{[1:L]}$*. Assume that*

$$\mathcal{N}(\mathcal{P}_{\mathcal{I}}) \subseteq (\mathcal{K}_{\mathcal{I}})_w \subseteq (\mathcal{P}_{\mathcal{I}})_w.$$

If $(\mathcal{P}_{\mathcal{I}})_c = \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{c}})$*, then the canonical distributed controller implements* $\mathcal{K}_{\mathcal{I}}$ *and is regular with respect to c.*

5.5.2 Regularity of the interconnection of local canonical controllers

Let us now consider the regularity of the interconnection of local canonical controllers, i.e., the regularity of $\mathcal{C}_i^{\mathrm{can}} \wedge_{(s_{ij}, k_{ij})} \mathcal{C}_j^{\mathrm{can}}$, $(i,j) \in \mathbb{Z}^2_{[1:L]}$ and $i \neq j$. Without loss of generality, we will consider that $L = 2$ in this subsection. The interconnection of $\mathcal{C}_1^{\mathrm{can}}$ and $\mathcal{C}_2^{\mathrm{can}}$ is regular if

$$\mathtt{p}(\mathcal{C}_1^{\mathrm{can}} \wedge_{(s,k)} \mathcal{C}_2^{\mathrm{can}}) = \mathtt{p}(\mathcal{C}_1^{\mathrm{can}}) + \mathtt{p}(\mathcal{C}_2^{\mathrm{can}}).$$

The behaviors $\mathcal{P}_i \in \mathfrak{L}^{\mathtt{w}_i+\mathtt{s}_i+\mathtt{c}_i}$, $i = 1, 2$, admit kernel representations

$$R_i\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) w_i + S_i\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) s + M_i\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) c_i = 0, \quad i = 1, 2. \tag{5.5}$$

Similarly, the behaviors $\mathcal{K}_i \in \mathfrak{L}^{\mathtt{w}_i+\mathtt{k}_i}$, $i = 1, 2$, admit kernel representations

$$W_i\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) w_i + K_i\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) k = 0, \quad i = 1, 2. \tag{5.6}$$

Define the partitioned matrix

$$\left[\begin{array}{cccc|cc} M_1 & 0 & S_1 & 0 & R_1 & 0 \\ 0 & 0 & 0 & K_1 & W_1 & 0 \\ \hline 0 & M_2 & S_2 & 0 & 0 & R_2 \\ 0 & 0 & 0 & K_2 & 0 & W_2 \end{array}\right] =: \left[\begin{array}{c|c} L_1 & N_1 \\ \hline L_2 & N_2 \end{array}\right]. \tag{5.7}$$

Proposition 5.5.2. *Consider the behaviors $\mathcal{P}_i \in \mathfrak{L}^{\mathtt{w}_i+\mathtt{s}_i+\mathtt{c}_i}$ and $\mathcal{K}_i \in \mathfrak{L}^{\mathtt{w}_i+\mathtt{k}_i}$, $i = 1, 2$, and the kernel representations* (5.5) *and* (5.6)*, respectively. The interconnection of $\mathcal{C}_1^{can}$ and $\mathcal{C}_2^{can}$ is regular if and only if*

$$\operatorname{rank}\begin{bmatrix} L_1 & N_1 \end{bmatrix} + \operatorname{rank}\begin{bmatrix} L_2 & N_2 \end{bmatrix} = \operatorname{rank}\begin{bmatrix} L_1 & N_1 \\ L_2 & N_2 \end{bmatrix}. \tag{5.8}$$

Proof. By (5.5) and (5.6), the local canonical controller behavior is represented by the latent variable representation

$$\mathcal{C}_i^{\mathrm{can}} = \{(c_i, s, k) \mid \exists w_i \text{ so that } \begin{bmatrix} R_i(\frac{\mathrm{d}}{\mathrm{d}t}) & S_i(\frac{\mathrm{d}}{\mathrm{d}t}) & M_i(\frac{\mathrm{d}}{\mathrm{d}t}) & 0 \\ W_i(\frac{\mathrm{d}}{\mathrm{d}t}) & 0 & 0 & K_i(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix} \begin{bmatrix} w_i \\ s \\ c_i \\ k \end{bmatrix} = 0\}.$$

Hence, the interconnection of $\mathcal{C}_1^{\mathrm{can}}$ and $\mathcal{C}_2^{\mathrm{can}}$ is

$$\begin{aligned} \mathcal{C}_1^{\mathrm{can}} \wedge_{(s,k)} \mathcal{C}_2^{\mathrm{can}} &= \{(c_1, c_2, s, k) \mid (c_1, s, k) \in \mathcal{C}_1^{\mathrm{can}} \text{ and } (c_2, s, k) \in \mathcal{C}_2^{\mathrm{can}}\} \\ &= \{(c_1, c_2, s, k) \mid \exists (w_1, w_2) \text{ so that } \begin{bmatrix} L_1\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) & N_1\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) \\ L_2\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) & N_2\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ s \\ k \\ w_1 \\ w_2 \end{bmatrix} = 0\}, \end{aligned}$$

which is a latent variable representation for the canonical distributed controller (with latent variable (w_1, w_2). By Lemma 8 in (Belur and Trentelman, 2002), the output cardinality of $\mathcal{C}_1^{\mathrm{can}} \wedge_{(s,k)} \mathcal{C}_2^{\mathrm{can}}$ can be determined from its latent variable representation as

$$\mathtt{p}(\mathcal{C}_1^{\mathrm{can}} \wedge_{(s,k)} \mathcal{C}_2^{\mathrm{can}}) = \operatorname{rank} \begin{bmatrix} L_1 & N_1 \\ L_2 & N_2 \end{bmatrix} - \operatorname{rank} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix}.$$

Similarly, the output cardinality of $\mathcal{C}_1^{\mathrm{can}}$ and $\mathcal{C}_2^{\mathrm{can}}$ is given by

$$\mathtt{p}(\mathcal{C}_i^{\mathrm{can}}) = \operatorname{rank} \begin{bmatrix} M_i & S_i & 0 & R_i \\ 0 & 0 & K_i & W_i \end{bmatrix} - \operatorname{rank} \begin{bmatrix} R_i \\ W_i \end{bmatrix}, \quad i = 1, 2.$$

It follows by (5.7) that

$$\mathtt{p}(\mathcal{C}_1^{\mathrm{can}} \wedge_{(s,k)} \mathcal{C}_2^{\mathrm{can}}) = \operatorname{rank} \begin{bmatrix} L_1 & N_1 \\ L_2 & N_2 \end{bmatrix} - \operatorname{rank} \begin{bmatrix} R_1 \\ W_1 \end{bmatrix} - \operatorname{rank} \begin{bmatrix} R_2 \\ W_2 \end{bmatrix}. \tag{5.9}$$

Hence, by (5.9) and (5.7), we find that

$$\begin{aligned} &\mathtt{p}(\mathcal{C}_1^{\mathrm{can}}) + \mathtt{p}(\mathcal{C}_2^{\mathrm{can}}) \\ &= \operatorname{rank} \begin{bmatrix} L_1 & N_1 \end{bmatrix} - \operatorname{rank} \begin{bmatrix} R_1 \\ W_1 \end{bmatrix} + \operatorname{rank} \begin{bmatrix} L_2 & N_2 \end{bmatrix} - \operatorname{rank} \begin{bmatrix} R_2 \\ W_2 \end{bmatrix} \\ &= \operatorname{rank} \begin{bmatrix} L_1 & N_1 \end{bmatrix} + \operatorname{rank} \begin{bmatrix} L_2 & N_2 \end{bmatrix} - \operatorname{rank} \begin{bmatrix} L_1 & N_1 \\ L_2 & N_2 \end{bmatrix} + \mathtt{p}(\mathcal{C}_1^{\mathrm{can}} \wedge_{(s,k)} \mathcal{C}_2^{\mathrm{can}}). \end{aligned}$$

Therefore, $\mathtt{p}(\mathcal{C}_1^{\mathrm{can}} \wedge_{(s,k)} \mathcal{C}_2^{\mathrm{can}}) = \mathtt{p}(\mathcal{C}_1^{\mathrm{can}}) + \mathtt{p}(\mathcal{C}_2^{\mathrm{can}})$ if and only if (5.8) holds. This concludes the proof. $\square$

Regularity of the interconnection of $\mathcal{C}_1^{\mathrm{can}}$ and $\mathcal{C}_2^{\mathrm{can}}$ turns out to be easily verifiable through regularity of the interconnections of subsystems $\mathcal{P}_1$ and $\mathcal{P}_2$ of the interconnected system that has to be controlled and of the interconnection of $\mathcal{K}_1$ and $\mathcal{K}_2$. We have the following result.

Theorem 5.5.1. *The interconnection of $\mathcal{C}_1^{can}$ and $\mathcal{C}_2^{can}$ is regular if and only if the interconnection of $\mathcal{P}_1$ and $\mathcal{P}_2$ is regular and the interconnection of $\mathcal{K}_1$ and $\mathcal{K}_2$ is regular. That is, the interconnection of $\mathcal{C}_1^{can}$ and $\mathcal{C}_2^{can}$ is regular if and only if*

$$\mathtt{p}(\mathcal{P}_1 \wedge_s \mathcal{P}_2) = \mathtt{p}(\mathcal{P}_1) + \mathtt{p}(\mathcal{P}_2) \quad \textit{and} \quad \mathtt{p}(\mathcal{K}_1 \wedge_k \mathcal{K}_2) = \mathtt{p}(\mathcal{K}_1) + \mathtt{p}(\mathcal{K}_2).$$

Proof. Let $R_i\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) w_i + S_i\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) s + M_i\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) c_i = 0$ be a minimal kernel representation for $\mathcal{P}_i$ and let $W_i\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) w_i + K_i\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) k = 0$ be a minimal kernel representation for $\mathcal{K}_i$, $i = 1, 2$.

($\Rightarrow$) Assume that $\mathtt{p}(\mathcal{P}_1 \wedge_s \mathcal{P}_2) = \mathtt{p}(\mathcal{P}_1) + \mathtt{p}(\mathcal{P}_2)$ and that $\mathtt{p}(\mathcal{K}_1 \wedge_k \mathcal{K}_2) = \mathtt{p}(\mathcal{K}_1) + \mathtt{p}(\mathcal{K}_2)$. We then have that

$$\begin{aligned}\operatorname{rank}\begin{bmatrix} R_1 & M_1 & 0 & 0 & S_1 \\ 0 & 0 & R_2 & M_2 & S_2 \end{bmatrix} &= \mathtt{p}(\mathcal{P}_1 \wedge_s \mathcal{P}_2) = \mathtt{p}(\mathcal{P}_1) + \mathtt{p}(\mathcal{P}_2) \\ &= \operatorname{rank}\begin{bmatrix} R_1 & M_1 & S_1 \end{bmatrix} + \operatorname{rank}\begin{bmatrix} R_2 & M_2 & S_2 \end{bmatrix},\end{aligned} \tag{5.10}$$

and

$$\begin{aligned}\operatorname{rank}\begin{bmatrix} W_1 & 0 & K_1 \\ 0 & W_2 & K_2 \end{bmatrix} &= \mathtt{p}(\mathcal{K}_1 \wedge_s \mathcal{K}_2) = \mathtt{p}(\mathcal{K}_1) + \mathtt{p}(\mathcal{K}_2) \\ &= \operatorname{rank}\begin{bmatrix} W_1 & K_1 \end{bmatrix} + \operatorname{rank}\begin{bmatrix} W_2 & K_2 \end{bmatrix}.\end{aligned} \tag{5.11}$$

By Proposition 5.5.2, $\mathcal{C}_1^{\text{can}} \wedge_{(s,k)} \mathcal{C}_2^{\text{can}}$ is regular if and only if (5.8) holds, i.e., if and only if

$$\operatorname{rank}\begin{bmatrix} A_1 \\ B_1 \\ A_2 \\ B_2 \end{bmatrix} = \operatorname{rank}\begin{bmatrix} A_1 \\ B_1 \end{bmatrix} + \operatorname{rank}\begin{bmatrix} A_2 \\ B_2 \end{bmatrix}, \tag{5.12}$$

where

$$\begin{aligned} A_1 &:= \begin{bmatrix} M_1 & 0 & S_1 & 0 & R_1 & 0 \end{bmatrix}, \quad A_2 := \begin{bmatrix} 0 & M_2 & S_2 & 0 & 0 & R_2 \end{bmatrix}, \\ B_1 &:= \begin{bmatrix} 0 & 0 & 0 & K_1 & W_1 & 0 \end{bmatrix}, \quad B_2 := \begin{bmatrix} 0 & 0 & 0 & K_2 & 0 & W_2 \end{bmatrix}. \end{aligned}$$

Now, by (5.10), A_1 and A_2 do not have rows that are linearly dependent. Similarly, by (5.11), B_1 and B_2 do not have rows that are linearly dependent. Furthermore, B_1 and A_2 do not have rows that are linearly dependent and A_1 and B_2 do not have rows that are linearly dependent, by construction. Hence, $\begin{bmatrix} A_1 \\ B_1 \end{bmatrix}$ and $\begin{bmatrix} A_2 \\ B_2 \end{bmatrix}$ do not have rows that are linearly dependent. Hence, (5.12) holds true and it follows that $\mathcal{C}_1^{\text{can}} \wedge_{(s,k)} \mathcal{C}_2^{\text{can}}$ is regular.

($\Leftarrow$) Let $\mathcal{C}_1^{\text{can}} \wedge_{(s,k)} \mathcal{C}_2^{\text{can}}$ be regular. Then (5.12) holds true. But then A_1 and A_2 cannot contain dependent rows. Hence, $\mathtt{p}(\mathcal{P}_1 \wedge_s \mathcal{P}_2) = \mathtt{p}(\mathcal{P}_1) + \mathtt{p}(\mathcal{P}_2)$. Moreover, by (5.12), B_1 and B_2 cannot contain dependent rows. Hence, $\mathtt{p}(\mathcal{K}_1 \wedge_k \mathcal{K}_2) = \mathtt{p}(\mathcal{K}_1) + \mathtt{p}(\mathcal{K}_2)$. This completes the proof. $\square$

5.6 Conclusions

In this chapter, we have considered the distributed control problem for linear interconnected systems in a behavioral setting. This setting allows to view distributed control from a more general perspective, where controllers are not intrinsically viewed as signal processors. Given a desired behavior represented by a linear interconnected system, the canonical distributed controller implements it, provided that sufficient conditions on the manifest behavior of the plant and desired behavior are satisfied. We have shown that regularity of the interconnections between subsystems in the plant and desired behavior are necessary and sufficient for regularity of the interconnections between subsystems in the canonical distributed controller.

From a design point-of-view, the desired interconnected system behavior is assumed to be known *a priori*, which may not always be at hand in practice. The theory of behavioral distributed control in this chapter, and in particular the canonical distributed controller, forms the basis for distributed model-reference control in Chapter 6 for transfer function representations. The analysis and synthesis of structured reference models (representing the desired interconnected system behavior) will also be addressed in Chapter 6.

Part II

Distributed data-driven model-reference control

Chapter 6

Distributed model-reference control of interconnected systems

Data-driven control methods typically follow a model-reference controller design problem. The basis in a model-reference control problem is formed by given performance specifications for the closed-loop system, captured by a reference model. This reference model yields an ideal controller that is unknown; it is the target object in direct data-driven control. In this chapter, we set the basis for *distributed* data-driven control for interconnected systems. We show that there exists an ideal distributed controller that implements performance specifications captured by a structured reference model. The properness and stability of the distributed controller and reference model are analyzed in this chapter. Finally, a method for synthesizing a structured reference model based on $\mathscr{H}_2$ or $\mathscr{H}_\infty$ performance specifications is developed, hence this chapter also yields an alternative solution to the distributed $\mathscr{H}_2/\mathscr{H}_\infty$ control problems considered in Chapter 4 through the model-reference paradigm.

This chapter is based on the preliminary work: T. R. V. Steentjes, M. Lazar, and P. M. J. Van den Hof. Data-driven distributed control: Virtual reference feedback tuning in dynamic networks. In *Proc. 59th IEEE Conference on Decision and Control (CDC)*, pages 1804–1809, 2020

6.1 Introduction

Model-reference controller design dates back to at least the 1960's, see e.g. (Tyler, 1964). In the model-reference control paradigm, a reference model is employed to describe the desired characteristics of a closed-loop system (Landau, 1978), (Bazanella et al., 2012). In reference tracking, the output cannot attain a reference trajectory exactly, in general, think e.g. of a step reference signal. Since perfect tracking is not possible, the tracking objective is typically relaxed into specifications of what is satisfactory, such as maximum overshoot, rise time, or settling time. The model-reference paradigm collects such specifications (and other specifications) in a linear dynamical system with the corresponding properties. Given a reference signal, the controller design problem is to minimize the difference between the output of the closed-loop system and the output of the reference model, instead of the difference between the output and the reference itself (Bazanella et al., 2012).

A solution to the linear model-reference control problem is given by an ideal controller; a controller that can be written explicitly in terms of dynamics of the plant and reference model, and that achieves exactly the reference model (Campi et al., 2002). Of course, whether this ideal controller can be actually implemented depends on the considered class of controllers. Typical properties that are (implicitly) captured in a controller class are the dynamical order of the controller, properness, and stability of the controller. Stability and properness of the controller yield design restrictions for the reference model (Bazanella et al., 2012) and, hence, should be taken into account in the design of the reference model.

In this chapter, we introduce a model-reference paradigm for interconnected systems. In this paradigm, the structure of the interconnected system (described by a graph), is taken into account in the reference model. More specifically, we develop a framework where the reference model consists of subsystems that are interconnected only if two subsystems of the network under consideration are interconnected. In this way, the interconnection structure of the plant is respected in the controller design in the sense that the closed-loop system attains the same interconnection structure, with possibly removed couplings. The composition of the reference model's subsystems describes the desired behavior for the interconnected system. A special case of the structured reference model is one where all subsystems are disconnected, describing a decoupling control objective.

The natural question that arises is: does there exist a *distributed* controller, such that the closed-loop network dynamics coincide with the structured reference model dynamics? Via the concept of the local canonical controllers described in Chapter 5, we show that there exists a distributed controller with the same structure as the network under consideration that solves the distributed model-reference control problem; an ideal distributed controller.

Properness and stability of a distributed controller are important, or even indispensable properties of a distributed controller. Properness ensures that subsystems of the distributed controller are causal. Stability ensures that control input signals, but also controller interconnection/communication signals, remain bounded. We consider both the properness and stability analysis of the structured reference model in this chapter. Then, the synthesis of a reference model is considered. An ideal distributed controller is inherently non-unique due to the interconnection dynamics, i.e., one can 'shift' dynamics from one controller subsystem to another through dynamic transformations (filtering) of the interconnection dynamics. This freedom is exploited to determine necessary and sufficient conditions for the properness and stability of an ideal distributed controller, leading to a structured procedure for obtaining a structured reference model.

Finally, when the control objective is to guarantee an $\mathscr{H}_2$ or $\mathscr{H}_\infty$ bound on the closed-loop network, it may not be clear how to choose the structured reference model. We show how a decoupled reference model can be designed that corresponds to a system with a given $\mathscr{H}_2$ or $\mathscr{H}_\infty$ bound. As a consequence, the corresponding ideal distributed controller then solves the $\mathscr{H}_2/\mathscr{H}_\infty$ control problem for the interconnected system.

6.2 Preliminaries

Consider an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with vertex set $\mathcal{V}$ of cardinality L and edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. This graph will define the interconnection structure of an interconnected system with bilateral interconnections[1]. The neighbour set of vertex $i \in \mathcal{V}$ is defined as $\mathcal{N}_i := \{j \in \mathcal{V} \,|\, (i,j) \in \mathcal{E}\}$. To each vertex $i \in \mathcal{V}$, we associate a linear discrete-time system with dynamics

$$\begin{aligned} y_i(t) &= G_i(q)u_i(t) + \sum_{j \in \mathcal{N}_i} W_{ij}(q)s_{ij}(t), \\ o_{ij}(t) &= F_{ij}(q)y_i(t), \qquad j \in \mathcal{N}_i, \end{aligned}$$

with G_i, W_{ij}, F_{ij} rational transfer functions, q the forward shift defined as $qx(t) = x(t+1)$, $u_i : \mathbb{Z} \to \mathbb{R}$ is the control input, $y_i : \mathbb{Z} \to \mathbb{R}$ the output, and $o_{ij}, s_{ij} : \mathbb{Z} \to \mathbb{R}$ are variables through which the systems at vertices $(i,j) \in \mathcal{E}$ are interconnected. The problem that we consider is that of reference tracking, i.e., for each system it is desired that the output y_i tracks a reference signal r_i. The tracking error for system i is defined as $z_i := r_i - y_i$. By stacking all incoming and outgoing interconnection variables of system i in vectors s_i and o_i, that is

[1]Unilateral interconnections can be considered by setting the corresponding transfer functions W_{ij} or F_{ji} equal to zero.

$s_i := \mathrm{col}_{j\in\mathcal{N}_i}\, s_{ij}$ and $o_i := \mathrm{col}_{j\in\mathcal{N}_i}\, o_{ij}$, we arrive at the following description for system i, denoted $\mathcal{P}_i$:

$$\mathcal{P}_i : \begin{cases} y_i &= G_i(q)u_i + W_i(q)s_i, \\ o_i &= F_i(q)y_i, \\ z_i &= r_i - y_i, \end{cases} \tag{6.1}$$

where $W_i := \mathrm{row}_{j\in\mathcal{N}_i}\, W_{ij}$ and $F_i := \mathrm{col}_{j\in\mathcal{N}_i}\, F_{ij}$, and the time t is omitted for brevity. The interconnection of system $\mathcal{P}_i$ and $\mathcal{P}_j$, $(i,j) \in \mathcal{E}$, is defined by

$$s_{ij} = o_{ji} \quad \text{and} \quad s_{ji} = o_{ij}. \tag{6.2}$$

We consider a structured reference model described by

$$\mathcal{K}_i : \begin{cases} y_i^d &= T_i(q)r_i + Q_i(q)k_i, \\ p_i &= P_i(q)y_i^d, \end{cases} \tag{6.3}$$

where $Q_i := \mathrm{row}_{j\in\mathcal{N}_i}\, Q_{ij}$ and $P_i := \mathrm{col}_{j\in\mathcal{N}_i}\, P_{ij}$ and the interconnection variables are similarly partitioned as for $\mathcal{P}_i$, i.e., $k_i := \mathrm{col}_{j\in\mathcal{N}_i}\, k_{ij}$ and $p_i := \mathrm{col}_{j\in\mathcal{N}_i}\, p_{ij}$. For each pair $(i,j) \in \mathcal{E}$ the interconnection of $\mathcal{K}_i$ and $\mathcal{K}_j$ is defined by

$$k_{ij} = p_{ji} \quad \text{and} \quad k_{ji} = p_{ij}. \tag{6.4}$$

Hence, $\mathcal{K}_i$ and $\mathcal{K}_j$ can only be interconnected if $\mathcal{P}_i$ and $\mathcal{P}_j$ are interconnected. A particular case of such a reference model occurs when a decoupled closed-loop system is desired, i.e., $Q_{ij} = 0$ and $P_{ij} = 0$, $i, j = 1, 2, \ldots, L$.

For the control of the interconnected system described by (6.1) and (6.2), we consider that each system $\mathcal{P}_i$ is associated with a (parametrized) controller $\mathcal{C}_i$, which is a linear discrete-time system that has the tracking error z_i as an input, control input u_i as an output and is interconnected with other controllers $\mathcal{C}_j$ through interconnection variables η_{ij}, ζ_{ij}:

$$\mathcal{C}_i(\rho_i) : \begin{cases} u_i = C_{ii}(q,\rho_i)z_i + \displaystyle\sum_{j\in\mathcal{N}_i} C_{ij}(q,\rho_i)\eta_{ij}, \\ \zeta_{ij} = K_{ij}(q,\rho_i)z_i + \displaystyle\sum_{h\in\mathcal{N}_i} K_{ijh}(q,\rho_i)\eta_{ih},\; j \in \mathcal{N}_i. \end{cases}$$

The interconnection of $\mathcal{C}_i$ and $\mathcal{C}_j$, $(i,j) \in \mathcal{E}$ is defined by

$$\eta_{ij} = \zeta_{ji} \quad \text{and} \quad \eta_{ji} = \zeta_{ij}. \tag{6.5}$$

An example of a reference model and controlled interconnected system is provided in Figure 6.1 for illustration purposes. By defining $\eta_i := \mathrm{col}_{j\in\mathcal{N}_i}\, \eta_{ij}$ and $\zeta_i := \mathrm{col}_{j\in\mathcal{N}_i}\, \zeta_{ij}$, we compactly represent controller i by

$$\mathcal{C}_i(\rho_i) : \begin{bmatrix} u_i \\ \zeta_i \end{bmatrix} = C_i(q,\rho_i) \begin{bmatrix} z_i \\ \eta_i \end{bmatrix}. \tag{6.6}$$

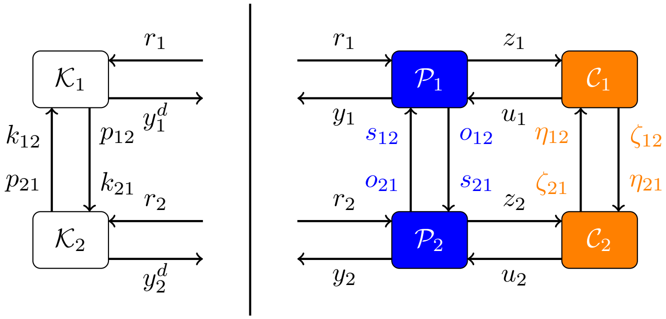


Figure 6.1: Structured reference model (left) and closed-loop network with distributed controller (right) for $L = 2$.

By stacking all the interconnection variables of the interconnected system described by (6.1) and (6.2) as $s := \mathrm{col}(s_1, \dots, s_L)$ and $o := \mathrm{col}(o_1, \dots, o_L)$, we can write

$$y = Gu + Ws, \quad o = Fy, \quad s = \Delta o,$$

with $G = \mathrm{diag}(G_1, \dots, G_L)$, $W = \mathrm{diag}(W_1, \dots, W_L)$, $F = \mathrm{diag}(F_1, \dots, F_L)$ and the matrix Δ defined by aggregating (6.2) for all corresponding index pairs. The input-output behavior of the network is $y = (I - W\Delta F)^{-1}Gu$.

Assumption 6.1. *The interconnected system and reference model satisfy* $\det(I - W\Delta F) \neq 0$ *and* $\det(I - Q\Delta P) \neq 0$.

Assumption 6.2. *The reference model is such that $y^d \neq r$ for all non-zero r, i.e.,* $\det((I - Q\Delta P)^{-1}T - I) \neq 0$.

Assumption 6.1 is required for well-posedness of the interconnected system and structured reference model, in the sense that the transfer matrices from inputs to outputs exist. The rationale behind Assumption 6.2 is that no reference model is chosen that cannot be achieved by a (distributed) controller and ensures that the transfer functions of an ideal distributed controller considered in Section 6.3 are well-defined, cf. (6.9).

Problem 6.2.1. *Given the parametrized controllers $\mathcal{C}_i(\rho_i)$ and the reference models $\mathcal{K}_i$, the considered distributed controller synthesis problem is*

$$\min_{\rho_1, \dots, \rho_L} J_{MR}(\rho_1, \dots, \rho_L), \quad J_{MR}(\rho_1, \dots, \rho_L) := \sum_{i=1}^{L} \bar{E}[y_i^d(t) - y_i(t)]^2, \tag{6.7}$$

where $\bar{E} := \lim_{N\to\infty} \frac{1}{N}\sum_{t=1}^{N} E$ *and* E *is the expectation*[2].

6.3 Ideal distributed controller synthesis

A controller that admits the same structure as the interconnected system and for which the closed-loop network matches the structured reference model exactly, i.e., $y_i = y_i^d$ for all $i = 1, \ldots, L$, is called an *ideal distributed controller.* To derive such an ideal controller, consider the interconnection of a subsystem $\mathcal{K}_i$ of the structured reference model with a subsystem $\mathcal{P}_i$ of the interconnected system, i.e.,

$$\begin{bmatrix} y_i \\ o_i \\ z_i \\ y_i^d \\ p_i \end{bmatrix} = \begin{bmatrix} G_i u_i + W_i s_i \\ F_i y_i \\ r_i - y_i \\ T_i r_i + Q_i k_i \\ P_i y_i^d \end{bmatrix} \quad \text{and} \quad y_i^d = y_i. \tag{6.8}$$

Elimination of the variables y_i^d, y_i and r_i in (6.8) yields a local controller $\mathcal{C}_i^d$, described by (denoting x_i by x_i^c for the interconnection variables to distinguish controller variables from plant variables)

$$\mathcal{C}_i^d : \begin{bmatrix} u_i \\ o_i^c \\ p_i^c \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{T_i}{G_i(1-T_i)} & -\frac{1}{G_i} W_i & \frac{1}{G_i(1-T_i)} Q_i \\ \frac{T_i}{1-T_i} F_i & 0 & \frac{1}{1-T_i} F_i Q_i \\ \frac{T_i}{1-T_i} P_i & 0 & \frac{1}{1-T_i} P_i Q_i \end{bmatrix}}_{=: C_i^d(q)} \begin{bmatrix} z_i \\ s_i^c \\ k_i^c \end{bmatrix}. \tag{6.9}$$

The distributed controller is constructed by interconnecting local controllers $\mathcal{C}_i^d$ and $\mathcal{C}_j^d$, $(i, j) \in \mathcal{E}$, as

$$\begin{bmatrix} s_{ij}^c \\ k_{ij}^c \end{bmatrix} = \begin{bmatrix} o_{ji}^c \\ p_{ji}^c \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} s_{ji}^c \\ k_{ji}^c \end{bmatrix} = \begin{bmatrix} o_{ij}^c \\ p_{ij}^c \end{bmatrix} \tag{6.10}$$

Theorem 6.3.1. *The closed-loop network described by* (6.1) - (6.2) *and the distributed controller* (6.9) - (6.10) *satisfies*

$$y_i = y_i^d, \qquad i = 1, \ldots, L.$$

[2] A deterministic setting is considered in this chapter, but subsequent Chapters 7 and 8 consider the same control problem with a stochastic noise process affecting the output.

Proof. Let the control variables (u_i, z_i) and controller interconnection variables $(s_i^c, o_i^c, k_i^c, p_i^c)$ satisfy (6.9) for all i and (6.10) for all $(i,j) \in \mathcal{E}$, i.e., $s^c = \Delta o^c$ and $k^c = \Delta p^c$. We will first show that there exist latent variables $r_i^c : \mathbb{Z} \to \mathbb{R}$ and $y_i^c : \mathbb{Z} \to \mathbb{R}$ for each i, so that

$$\begin{bmatrix} y_i^c \\ o_i^c \\ z_i \\ y_i^c \\ p_i^c \end{bmatrix} = \begin{bmatrix} G_i u_i + W_i s_i^c \\ F_i y_i^c \\ r_i^c - y_i^c \\ T_i r_i^c + Q_i k_i^c \\ P_i y_i^c \end{bmatrix}. \tag{6.11}$$

Define $y_i^c := G_i u_i + W_i s_i$ and $r_i^c := z_i + y_i^c$. We then have to show that $y_i^c = T_i r_i^c + Q_i k_i^c$, $o_i^c = F_i y_i^c$ and $p_i^c = P_i y_i^c$. By (6.9) we have that

$$\begin{aligned} u_i &= \frac{T_i}{G_i(1-T_i)} z_i + \frac{1}{G_i(1-T_i)} Q_i k_i^c - \frac{1}{G_i} W_i s_i^c \\ &\Leftrightarrow \\ &(1-T_i) G_i u_i = T_i z_i + Q_i k_i^c - (1-T_i) W_i s_i^c, \end{aligned}$$

which, by the definition of y_i^c, is equivalent with

$$(1-T_i) y_i^c = T_i z_i + Q_i k_i^c \tag{6.12}$$

and hence, by the definition of r_i^c, $y_i^c = T_i r_i^c + Q_i k_i^c$. By (6.12) and (6.9), it follows that

$$\begin{aligned} o_i^c &= \frac{T_i}{1-T_i} F_i z_i + \frac{1}{1-T_i} F_i Q_i k_i^c = F_i y_i^c, \\ p_i^c &= \frac{T_i}{1-T_i} P_i z_i + \frac{1}{1-T_i} P_i Q_i k_i^c = P_i y_i^c. \end{aligned}$$

Next, define $y^c := \operatorname{col}(y_1^c, \ldots, y_L^c)$ and $u := \operatorname{col}(u_1, u_2, \ldots, u_L)$. It follows by (6.11) that $y^c = Gu + Ws^c$ and $o^c = Fy$, such that, by $s^c = \Delta o^c$, $y^c = (I - W\Delta F)^{-1} Gu$. Similarly, define $r^c := \operatorname{col}(r_1^c, \ldots, r_L^c)$ to obtain $y^c = (I - Q\Delta P)^{-1} T r^c$ by (6.11), with $Q = \operatorname{diag}(Q_1, \ldots, Q_L)$, $P = \operatorname{diag}(P_1, \ldots, P_L)$ and $T = \operatorname{diag}(T_1, \ldots, T_L)$. Thus, using $z = r^c - y^c$, the controller satisfies

$$\begin{aligned} u = G^{-1}&(I - W\Delta F)(I - Q\Delta P)^{-1} T \\ &\times (I - (I - Q\Delta P)^{-1} T)^{-1} z. \end{aligned} \tag{6.13}$$

Finally, the process $y = (I - W\Delta F)^{-1} Gu$ with $z = r - y$ and the controller (6.13) yield $y = (I - Q\Delta P)^{-1} Tr = y_d$, which concludes the proof. $\square$

If we associate $\rho_1^d, \ldots, \rho_L^d$ with the ideal distributed controller, such that $P_i^\top C_i^d P_i = C_i(\rho_i^d)$ for $i = 1, \ldots, L$, with the permutation matrices $P_i := \mathrm{diag}(1, \overline{P}_i)$, $i = 1, \ldots, L$, such that $\mathrm{col}(s_i^c, k_i^c) = \overline{P}_i \, \mathrm{col}_{j \in \mathcal{N}_i} \, \mathrm{col}(s_{ij}^c, k_{ij}^c)$, then by Theorem 6.3.1, $(\rho_1^d, \ldots, \rho_L^d)$ solves problem (6.7).

The following simple example briefly illustrates the ideal distributed controller constructed in this section.

Example 6.1. *Consider two coupled processes*

$$
\begin{aligned}
y_1(t) &= G_1(q)u_1(t) + G_{12}(q)y_2(t), \\
y_2(t) &= G_2(q)u_2(t) + G_{21}(q)y_1(t),
\end{aligned}
$$

with transfer functions

$$
\begin{aligned}
G_1(q) &= \frac{c_1}{q - a_1}, \quad G_{12}(q) = \frac{d_1}{q - a_1}, \\
G_2(q) &= \frac{c_2}{q - a_2}, \quad G_{21}(q) = \frac{d_2}{q - a_2}.
\end{aligned}
$$

The objective is that the closed-loop interconnected system behaves as two decoupled processes with first-order dynamics, according to

$$
y_i^d(t) = T_i(q)r_i(t), \quad T_i(q) = \frac{1 - \gamma_i}{q - \gamma_i}, \quad i = 1, 2. \tag{6.14}
$$

Now, via (6.9)*, we find that the ideal distributed controller is described by*

$$
\begin{bmatrix} u_1 \\ o_1^c \end{bmatrix} = \begin{bmatrix} C_{11}^d & C_{12}^d \\ K_{12}^d & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ s_1^c \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} u_2 \\ o_2^c \end{bmatrix} = \begin{bmatrix} C_{22}^d & C_{21}^d \\ K_{21}^d & 0 \end{bmatrix} \begin{bmatrix} z_2 \\ s_2^c \end{bmatrix}
$$

with $z_i = r_i - y_i$, the interconnections $s_1^c = o_2^c$, $s_2^c = o_1^c$, and

$$
\begin{aligned}
C_{ii}^d(q) &= \frac{1 - \gamma_i}{c_i} \frac{q - a_i}{q - 1}, \quad C_{ij}^d(q) = -\frac{d_i}{c_i}, \\
K_{ij}^d(q) &= \frac{1 - \gamma_i}{q - 1}, \quad (i, j) \in \mathcal{E}.
\end{aligned}
$$

6.4 Properness and stability: analysis

Theorem 6.3.1 shows that the distributed controller (6.9) yields a closed-loop system with the prescribed structure and dynamics described by the structured reference model. Therefore, given a stable structured reference model, this distributed controller will yield a stable closed-loop network in the sense that the transfer $r \to y_d$ is stable. Internal stability and properness of the distributed controller modules is not naturally guaranteed, however. For the distributed controller to be practically applicable, stability and properness are of paramount importance. In this section we will consider the analysis problem for the existence of *(strictly) proper* and *stable* ideal distributed controllers.

6.4.1 Properness of an ideal distributed controller

In this subsection, we provide conditions on the reference model and plant for which the ideal controller modules $\mathcal{C}_i^d$, $i \in \mathcal{V}$ are causal, i.e., C_i^d is proper. For each node $i \in \mathcal{V}$, let $\mathcal{P}_i$ and $\mathcal{K}_i$ be causal. Consider the following definition of the relative degree of a rational function.

Definition 6.4.1. *The relative degree of a rational function $q \mapsto G(q) = \frac{B(q)}{A(q)}$, with $A(q)$ and $B(q)$ polynomial functions, is defined as $\Delta\deg(G) := \deg(A) - \deg(B)$, with $\deg(A)$ and $\deg(B)$ the degree of, respectively, polynomial A and B.*

We have the following result regarding the causality of the controller $\mathcal{C}_i^d$.

Proposition 6.4.1. *Consider the plant $\mathcal{P}_i$ and reference model $\mathcal{K}_i$. Every non-zero entry of the transfer matrix C_i^d is proper if and only if the following conditions hold:*

- $\Delta\deg(W_{ij}) \geq \Delta\deg(G_i)$ *or* $W_{ij} = 0$ *for all* $j \in \mathcal{N}_i$*,*
- $\Delta\deg(Q_{ij}) \geq \Delta\deg(G_i)$ *or* $Q_{ij} = 0$ *for all* $j \in \mathcal{N}_i$*,*
- $\Delta\deg(T_i) \geq \Delta\deg(G_i)$*.*

Proof. Properness of C_i^d is equivalent with $\Delta\deg[C_i^d]_{jk} \geq 0$ for every entry $(j,k) \in \{1,\ldots,L\}^2$.

It can be shown that

$$\Delta\deg\frac{T_i}{G_i(1-T_i)} = \Delta\deg T_i - \Delta\deg G_i.$$

Hence, $\frac{T_i}{G_i(1-T_i)}$ is proper if and only if $\Delta\deg(T_i) \geq \Delta\deg(G_i)$.

Now, for $j \in \mathcal{N}_i$ we have

$$\Delta\deg\frac{W_{ij}}{G_i} = \Delta\deg W_{ij} - \Delta\deg G_i$$

and

$$\Delta\deg\frac{Q_{ij}}{G_i(1-T_i)} = \Delta\deg Q_{ij} - \Delta\deg G_i.$$

Thus $\frac{W_{ij}}{G_i}$ and $\frac{Q_{ij}}{G_i(1-T_i)}$ are proper if and only if $\Delta\deg(W_{ij}) \geq \Delta\deg(G_i)$ and $\Delta\deg(Q_{ij}) \geq \Delta\deg(G_i)$. Next, for $j \in \mathcal{N}_i$ we have

$$\Delta\deg\frac{T_i}{1-T_i}F_{ij} = \Delta\deg F_{ij} + \Delta\deg T_i \geq 0$$

and

$$\Delta\deg\frac{T_i}{1-T_i}P_{ij} = \Delta\deg P_{ij} + \Delta\deg T_i \geq 0.$$

Hence, $\frac{T_i}{1-T_i}F_{ij}$ and $\frac{T_i}{1-T_i}P_{ij}$ are proper. Finally, for $(j,k) \in \mathcal{N}_i \times \mathcal{N}_i$

$$\Delta\deg\frac{1}{1-T_i}F_{ij}Q_{ik} = \Delta\deg F_{ij} + \Delta\deg Q_{ik} \geq 0$$

and

$$\Delta\deg\frac{1}{1-T_i}P_{ij}Q_{ik} = \Delta\deg P_{ij} + \Delta\deg Q_{ik} \geq 0$$

Hence, $\frac{1}{1-T_i}F_{ij}Q_{ik}$ and $\frac{1}{1-T_i}P_{ij}Q_{ik}$ are proper. This concludes the proof. □

Proposition 6.4.2. *Consider the plant $\mathcal{P}_i$ and reference model $\mathcal{K}_i$. Every non-zero entry of the transfer matrix C_i^d is strictly proper if and only if the following conditions hold:*

- $\Delta\deg(W_{ij}) > \Delta\deg(G_i)$ *or* $W_{ij} = 0$ *for all* $j \in \mathcal{N}_i$,
- $\Delta\deg(Q_{ij}) > \Delta\deg(G_i)$ *or* $Q_{ij} = 0$ *for all* $j \in \mathcal{N}_i$,
- $\Delta\deg(T_i) > \Delta\deg(G_i)$,

Proof. The proof follows directly from the proof for Proposition 6.4.1. □

6.4.2 Stability of an ideal distributed controller

In this subsection, we provide conditions on the reference model and plant for which the ideal controller modules $\mathcal{C}_i^d$, $i \in \mathcal{V}$ are stable, i.e., C_i^d has no poles outside the unit disk. For each node $i \in \mathcal{V}$, let $\mathcal{K}_i$ be stable. We have the following result.

Proposition 6.4.3. *Consider the plant $\mathcal{P}_i$ and reference model $\mathcal{K}_i$ described by* (6.1) *and* (6.3), *respectively. Let the following conditions hold:*

- *for $j \in \mathcal{N}_i$, each pole λ of W_{ij} so that $|\lambda| > 1$ is a pole of G_i,*

- *each zero λ of G_i so that $|\lambda| > 1$ is a zero of W_{ij} for all $j \in \mathcal{N}_i$,*
- *each zero λ of G_i so that $|\lambda| > 1$ is a zero of T_i,*
- *each zero λ of G_i so that $|\lambda| > 1$ is a zero of Q_{ij} for all $j \in \mathcal{N}_i$,*
- *for $j \in \mathcal{N}_i$, each pole λ of F_{ij} so that $|\lambda| > 1$ is a zero of T_i and a zero of Q_{ik} for all $k \in \mathcal{N}_i$,*
- *$T_i(\lambda) - 1 \neq 0$ for all $|\lambda| > 1$.*

Then every entry of the transfer matrix C_i^d is stable.

Proof. Stability of C_i^d is equivalent with: $[C_i^d]_{jk}$ is stable for every entry $(j,k) \in \{1,\ldots,L\}^2$. Let $G_i = \frac{B}{A}$, $W_{ij} = \frac{D}{C}$, with polynomials A, B, C and D. Then, $\frac{W_i}{G_i} = \frac{DA}{CB}$. Hence, $\frac{W_i}{G_i}$ is stable if and only if every zero of B outside the unit disk is a zero of D and every zero of C is a zero of A if and only if every zero of G_i outside the unit disk is a zero of W_{ij} and every pole of W_{ij} is a pole of G_i.

Let $T_i = \frac{Y}{X}$. Then, $\frac{T_i}{G_i(1-T_i)} = \frac{YA}{B(X-Y)}$. Hence, $\frac{W_i}{G_i}$ is stable if every zero of B outside the unit disk is a zero of Y and $X - Y$ has no zero outside the unit disk, which is equivalent with: every zero of G_i outside the unit disk is a zero of T_i and $T_i(\lambda) - 1 \neq 0$ for all $|\lambda| > 1$.

The proof for the other entries follows *mutatis mutandis.* This completes the proof. □

Corollary 6.4.1. *Consider the plant $\mathcal{P}_i$ and reference model $\mathcal{K}_i$. Let the following conditions hold:*

- *G_i and $T_i - 1$ have no zeros outside the unit disk,*
- *F_i and W_i are stable.*

Then every entry of the transfer matrix C_i^d is stable.

Proof. The conditions imply the conditions in Proposition 6.4.3. Hence, the result follows directly. □

6.5 Properness and stability: synthesis

6.5.1 Interconnection variable transformation

Consider any node $i \in \mathcal{V}$. Proposition 6.4.1 provides necessary and sufficient conditions on the transfer functions defining $\mathcal{P}_i$ and $\mathcal{K}_i$ for causality of the ideal distributed controller. The first condition in Proposition 6.4.1 does not depend on the reference model, however: if the condition does not hold, it cannot be satisfied by choosing an appropriate reference model. In this section, we will consider a transformation of the interconnection variables s^c_{ij} and o^c_{ij}, $j \in \mathcal{N}_i$, to transform the entries $C^{Wd}_{ij} := -G_i^{-1} W_{ij}$ of the ideal distributed controller (6.9) in case they are not proper. Properness of C^{Wd}_{ij} is equivalent with $\Delta \deg(W_{ij}) \geq \Delta \deg(G_i)$. Increasing $\Delta \deg(W_{ij})$ may not be feasible, since this requires to change the dynamics of $\mathcal{P}_i$ or its representation. However, if one filters C^{Wd}_{ij} with a filter M_{ij}^{-1}, the resulting relative degree $\Delta \deg(C^{Wd}_{ij} M_{ij}^{-1}) = \Delta \deg(C^{Wd}_{ij}) + \Delta \deg(M_{ij}^{-1})$ can be increased arbitrarily by increasing the relative degree of M_{ij}^{-1}.

Consider a filtered version of s^c_i,

$$\tilde{s}^c_i := M_i s^c_i,$$

together with a filtered version of o^c_i,

$$\tilde{o}^c_i := N_i o^c_i,$$

where M_i and N_i are square transfer matrices. From (6.9), we then have

$$\begin{bmatrix} u_i \\ \tilde{o}^c_i \\ p^c_i \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ 0 & N_i & 0 \\ 0 & 0 & I \end{bmatrix} C^d_i \begin{bmatrix} z_i \\ s^c_i \\ k^c_i \end{bmatrix} = \underbrace{\begin{bmatrix} I & 0 & 0 \\ 0 & N_i & 0 \\ 0 & 0 & I \end{bmatrix} C^d_i \begin{bmatrix} I & 0 & 0 \\ 0 & M_i^{-1} & 0 \\ 0 & 0 & I \end{bmatrix}}_{=:\tilde{C}^d_i} \begin{bmatrix} z_i \\ \tilde{s}^c_i \\ k^c_i \end{bmatrix},$$

which defines an ideal distributed controller

$$\tilde{\mathcal{C}}^d_i : \begin{bmatrix} u_i \\ \tilde{o}^c_i \\ p^c_i \end{bmatrix} = \underbrace{\begin{bmatrix} I & 0 & 0 \\ 0 & N_i & 0 \\ 0 & 0 & I \end{bmatrix} C^d_i \begin{bmatrix} I & 0 & 0 \\ 0 & M_i^{-1} & 0 \\ 0 & 0 & I \end{bmatrix}}_{=\tilde{C}^d_i} \begin{bmatrix} z_i \\ \tilde{s}^c_i \\ k^c_i \end{bmatrix}, \quad i = 1, \ldots, L, \tag{6.15}$$

with the interconnection equations $\tilde{s}^c_{ij} = \tilde{o}^c_{ji}$, $k^c_{ij} = p^c_{ji}$ and $\tilde{s}^c_{ji} = \tilde{o}^c_{ij}$, $k^c_{ji} = p^c_{ij}$

Now, consider M_i and N_i diagonal for every $i \in \mathcal{V}$, so that $M_i := \operatorname{diag}_{j \in \mathcal{N}_i} M_{ij}$ and $N_i := \operatorname{diag}_{j \in \mathcal{N}_i} N_{ij}$. It then follows that the interconnection constraint $\tilde{s}^c_{ij} = \tilde{o}^c_{ji}$ is equivalent with $s^c_{ij} = o^c_{ji}$ if $N_{ji} = M_{ij}$. Indeed, we then have

$$\tilde{s}^c_{ij} = \tilde{o}^c_{ji} \quad \Leftrightarrow \quad M_{ij} s^c_{ij} = N_{ji} o^c_{ji} \quad \Leftrightarrow \quad s^c_{ij} = o^c_{ji}.$$

Hence, it follows that the ideal distributed controller described by (6.9) is equivalent with the one described by (6.15), in the sense that their 'external behavior' is equivalent, i.e., the transfer $z \to u$ is equal.

Proposition 6.5.1. *The closed-loop network described by* (6.1) *and* (6.15) *satisfies*

$$y_i = y^d_i, \quad i = 1, \ldots, L.$$

The interpretation of transforming the interconnection variables between controllers, is that filters are added in 'in between' controller subsystems. In this way, the relative degree of controller transfer functions can be altered through the choice of appropriate filters, while leaving the controller behavior intact.

6.5.2 Synthesis of a structured reference model: existence of a proper ideal distributed controller

From (6.15), we observe that the filter M_{ij} can be chosen such that $\tilde{C}_{ij}^{Wd} := C_{ij}^{Wd} M_{ij}^{-1}$ is proper. Indeed, if we choose the relative degree if M_{ij}^{-1} sufficiently high, i.e.,

$$\Delta \deg M_{ij}^{-1} + \Delta \deg W_{ij} - \Delta \deg G_i \geq 0,$$

then $\tilde{C}_{ij}^{Wd}$ is proper. However, notice that since $N_{ji} = M_{ij}$ for $j \in \mathcal{N}_i$, the relative degree of $\tilde{C}_{ji}^{Fd} := N_{ji} C_{ji}^{Fd}$ and $\tilde{C}_{jik}^{FQd} := N_{ji} C_{jik}^{FQd}$ are equal to

$$\begin{aligned}\Delta \deg \tilde{C}_{ji}^{Fd} &= \Delta \deg F_{ji} + \Delta \deg T_j + \Delta \deg N_{ji} \\ &= \Delta \deg F_{ji} + \Delta \deg T_j - \Delta \deg M_{ij}^{-1}\end{aligned}$$

and

$$\begin{aligned}\Delta \deg \tilde{C}_{jik}^{FQd} &= \Delta \deg F_{ji} + \Delta \deg Q_{jk} + \Delta \deg N_{ji} \\ &= \Delta \deg F_{ji} + \Delta \deg Q_{jk} - \Delta \deg M_{ij}^{-1}.\end{aligned}$$

Therefore, by increasing $\Delta \deg M_{ij}^{-1}$, $\Delta \deg \tilde{C}_{ji}^{Fd}$ and $\Delta \deg \tilde{C}_{jik}^{FQd}$ decrease. Note that $\tilde{C}_{ji}^{Fd}$ and $\Delta \deg \tilde{C}_{jik}^{FQd}$ can always be made proper by choosing $\Delta \deg T_j$ and $\Delta \deg Q_{jk}$ sufficiently high, respectively.

Corollary 6.5.1. *Consider the plant $\mathcal{P}_i$ and reference model $\mathcal{K}_i$. Every entry of the transfer matrix $\tilde{C}_i^d$ is proper if and only if the following conditions hold:*

- $\Delta \deg T_i \geq \Delta \deg M_{ji}^{-1} - \Delta \deg F_{ij}$ *for all* $j \in \mathcal{N}_i$, $\Delta \deg Q_{ik} \geq \Delta \deg M_{ji}^{-1} - \Delta \deg F_{ij}$ *for all* $(j,k) \in \mathcal{N}_i \times \mathcal{N}_i$, *with* M_{ji} *s.t.* $\Delta \deg M_{ji} \leq \Delta \deg W_{ji} - \Delta \deg G_j$,
- $\Delta \deg Q_{ij} \geq \Delta \deg G_i$ *or* $Q_{ij} = 0$ *for all* $j \in \mathcal{N}_i$,
- $\Delta \deg T_i \geq \Delta \deg G_i$.

In order to ensure the existence of an ideal controller (6.15) for which all entries of $\tilde{C}_i^d$, $i \in \mathcal{V}$ are proper, the reference model has to be chosen such that the conditions in Corollary 6.5.1 hold. An algorithm for choosing the structured reference model such that the ideal distributed controller is causal is stated next:

1. For each $i \in \mathcal{V}$, choose T_i so that $\Delta \deg T_i \geq \Delta \deg G_i$ and choose Q_{ij}, $j \in \mathcal{N}_i$, so that $\Delta \deg Q_{ij} \geq \Delta \deg G_i$ or $Q_{ij} = 0$.

2. For each $i \in \mathcal{V}$, for each $j \in \mathcal{N}_i$, if C_{ij}^{Wd} is proper, set $M_{ij} = I$ and $N_{ji} = I$. If C_{ij}^{Wd} is not proper, take an M_{ij} so that

$$\Delta \deg M_{ij} \leq \Delta \deg W_{ij} - \Delta \deg G_i.$$

Choose T_j and Q_{jk} so that

$$\Delta \deg T_j \geq \max\{\Delta \deg M_{ji}^{-1} - \Delta \deg F_{ji},\ \Delta \deg G_j\}$$

and

$$\Delta \deg Q_{jk} \geq \max\{\Delta \deg M_{ji}^{-1} - \Delta \deg F_{ji},\ \Delta \deg G_j\}, \quad k \in \mathcal{N}_j.$$

By following the preceding steps, the structured reference model is constructed systematically in order to ensure the existence of a proper ideal distributed controller.

Example 6.2. *Consider a network with $\mathcal{V} = \{1, 2\}$, $\mathcal{E} = \{(1, 2), (2, 1)\}$. The dynamics of $\mathcal{P}_1$ and $\mathcal{P}_2$ are given by*

$$\begin{aligned} G_1 &= \frac{q - 0.1}{(q - 0.5)(q - 0.6)}, \quad W_{12} = \frac{q - 0.1}{q - 0.5}, \quad F_{12} = 1, \\ G_2 &= \frac{q - 0.2}{(q - 0.4)(q - 0.7)}, \quad W_{21} = \frac{q - 0.2}{q - 0.4}, \quad F_{21} = 1. \end{aligned}$$

We observe that $\Delta \deg W_{12} < \Delta \deg G_1$ and $\Delta \deg W_{21} < \Delta \deg G_2$, thus the conditions in Proposition 6.4.1 cannot be satisfied for both $i = 1$ and $i = 2$. However, by Corollary 6.5.1, we observe that there exists a proper ideal distributed controller, if we choose the reference model such that

$$\Delta \deg T_i \geq \Delta \deg M_{ji}^{-1} = 1, \quad (i, j) \in \mathcal{E}$$

with

$$\deg M_{ji} = -1 \leq \Delta \deg W_{ji} - \Delta \deg G_i = -1,$$

and, for example, $Q_{12} = Q_{21} = 0$. By example, consider the reference model with

$$T_1 = T_2 = \frac{0.5}{q - 0.5}.$$

Then there indeed exists a proper ideal distributed controller.

6.5.3 Synthesis of a structured reference model: existence of a stable ideal distributed controller

The sufficient conditions for stability of the ideal distributed controller in Proposition 6.4.3 are partially independent of the reference model. This implies that there exist plants for which the reference model cannot be chosen in a way that C_i^d is stable for all $i \in \mathcal{V}$. We will now derive conditions for the design of a reference model such that the transformed ideal distributed controller (6.15) is stable.

Consider the entry C_{ij}^{Wd}. This entry is unstable if either there is a zero λ of G_i so that $|\lambda| > 1$ that is not a zero of W_{ij} or there is a pole λ of W_{ij} so that $|\lambda| > 1$ that is not a pole of G_i. The filtered version of C_{ij}^{Wd} in $\tilde{C}_i^d$ is

$$\tilde{C}_{ij}^{Wd} = C_{ij}^{Wd} M_{ij}^{-1} = -\frac{W_{ij}}{G_i M_{ij}}.$$

We observe that the following conditions are sufficient for stability of $\tilde{C}_{ij}^{Wd}$:

- M_{ij}^{-1} is stable,
- each zero λ of G_i so that $|\lambda| > 1$ is a zero of W_{ij} or a pole of M_{ij},
- each pole λ of W_{ij} so that $|\lambda| > 1$ is a pole of G_i or a pole of M_{ij}.

Now, since $N_{ji} = M_{ij}$, we have that $\tilde{C}_{ji}^{Fd} = N_{ji} C_{ji}^{Fd} = M_{ij} C_{ji}^{Fd}$, which is equal to

$$\bar{C}_{ji}^{Fd} = M_{ij} \frac{T_j}{1 - T_j} F_{ji}, \quad j \in \mathcal{N}_i. \tag{6.16}$$

Similarly, we have that

$$\tilde{C}_{jik}^{FQd} = N_{ji} C_{jik}^{FQd} = N_{ji} F_{ji} \frac{1}{1 - T_j} Q_{jk} = M_{ij} F_{ji} \frac{1}{1 - T_j} Q_{jk}, \quad j \in \mathcal{N}_i,\ k \in \mathcal{N}_j. \tag{6.17}$$

Hence, if every pole λ of M_{ij} so that $|\lambda| > 1$ is a zero of T_j, then $\tilde{C}_{ji}^{Fd}$ is stable. Furthermore, if every pole λ of M_{ij} so that $|\lambda| > 1$ is a zero of Q_{jk}, then $\tilde{C}_{jik}^{FQd}$ is stable. In conclusion, we have the following result.

Proposition 6.5.2. *Let the following conditions hold for every $i \in \mathcal{V}$:*

- *each zero λ of G_i so that $|\lambda| > 1$ is a zero of T_i, T_j, $j \in \mathcal{N}_i$ and a zero of Q_{ij}, Q_{jk}, $j \in \mathcal{N}_i$, $k \in \mathcal{N}_j$,*

- *each pole of W_{ij} so that $|\lambda| > 1$ is a zero of T_j, $j \in \mathcal{N}_i$ and a zero of Q_{jk}, $j \in \mathcal{N}_i$, $k \in \mathcal{N}_j$,*
- *for $j \in \mathcal{N}_i$, each pole λ of F_{ij} so that $|\lambda| > 1$ is a zero of T_i and a zero of Q_{ik} for all $k \in \mathcal{N}_i$,*
- *$T_i(\lambda) - 1 \neq 0$ for all $|\lambda| > 1$.*

Then there exist matrices M_i, N_i, $i \in \mathcal{V}$, with $M_{ij} = N_{ji}$ for all $(i,j) \in \mathcal{E}$, so that $\tilde{C}_i^d$ is stable for all $i \in \mathcal{V}$.

Proof. The third and fourth condition imply that C_{ij}^{Fd} and C_{ij}^{FQd} are stable; this follows from the proof for Proposition 6.4.3. The first and fourth condition imply that $\tilde{C}_{ii}^d = C_{ii}^d$ and $\tilde{C}_{ij}^{Qd} = C_{ij}^{Qd}$ are stable.

Now, for an arbitrary $i \in \mathcal{V}$, consider M_{ij}^{-1}, $j \in \mathcal{N}_i$, with M_{ij}^{-1} stable and so that each zero of G_i outside the unit disk and each pole of W_{ij} outside the unit disk is a pole of M_{ij}. If G_i and and W_{ij} have no zero or pole outside the unit disk, respectively, let $M_{ij} = I$. Then $\tilde{C}_{ij}^{Wd}$ is stable. Furthermore, the first condition then implies that $\tilde{C}_{ji}^{Fd}$ and $\tilde{C}_{jik}^{FQd}$, $k \in \mathcal{N}_j$, are stable. This follows from (6.16) and (6.17). Since, i was chosen arbitrarily the assertion follows. This concludes the proof. □

Proposition 6.5.2 provides not only an existence result. Indeed, by considering the choice for M_i and N_i as in the proof of Proposition 6.5.2, the controllers $\tilde{C}_i^d$ in (6.15) provide a way to choose a controller class that contains an ideal and stable controller. This class can be used subsequently to choose a sensible parametrization in data-driven distributed controller synthesis.

Interconnection dynamics conservation

The conditions in Proposition 6.5.2 specify when there exists an ideal distributed controller (6.15) with stable $\tilde{C}_i^d$. Note, however, that the interconnections in the distributed controller are there to 'control' or 'modify' the interactions dynamics in the plant. In case one decides to maintain or conserve the interactions dynamics W_{ij}, then this can be captured in the reference model interaction dynamics. Consequently, the ideal controller becomes simpler in the sense that it can be realized as a decentralized controller, i.e., it has no interconnections between subcontrollers $\mathcal{C}_i^d$. The choice of reference model interconnection dynamics such that the distributed controller becomes decentralized will be addressed next.

Let us consider the case where $F_{ij} = P_{ij} = 1$ for all $(i,j) \in \mathcal{E}$. Then it follows that $o_{ij} = p_{ij} = y_i$ for all $j \in \mathcal{N}_i$ and thus $s_{ij}^c = k_{ij}^c = y_j$ for all $j \in \mathcal{N}_i$. But from

(6.15) we have, with $M_i = N_i = I$, that

$$u_i = \frac{T_i}{G_i(1-T_i)} z_i + \sum_{j \in \mathcal{N}_i} \underbrace{\left(\frac{Q_{ij}}{G_i(1-T_i)} - \frac{W_{ij}}{G_i} \right)}_{=: C_{ij}} y_j.$$

Now, let $Q_{ij} = \bar{Q}_{ij}(1-T_i)$, with $\bar{Q}_{ij}$ a to be chosen transfer function. Then

$$C_{ij} = \frac{1}{G_i} \left(\bar{Q}_{ij} - W_{ij} \right).$$

Hence, if we set $\bar{Q}_{ij} = W_{ij}$, i.e., if we conserve the interconnection dynamics of the plant, then $C_{ij} = 0$: the ideal distributed controller becomes decentralized. This makes sense intuitively, since this choice of structured reference model implies we do not intend to 'control' or 'modify' the interactions dynamics in the plant. Note that $\bar{Q}_{ij} = W_{ij}$ requires full knowledge of W_{ij}, a transfer function that can be obtained by a network identification procedure, preceding the reference model synthesis and data-driven control procedure.

With $\bar{Q}_{ij} = W_{ij}$, we thus have that $u_i = \frac{T_i}{G_i(1-T_i)} z_i$, leading to an ideal decentralized controller

$$\mathcal{C}_i^d : \quad u_i = \frac{T_i}{G_i(1-T_i)} z_i. \tag{6.18}$$

Corollary 6.5.2. *Let $F_{ij} = P_{ij} = 1$ and $Q_{ij} = W_{ij}(1-T_i)$ for all $(i,j) \in \mathcal{E}$. Then the closed-loop network described by* (6.1) *and the decentralized controller* (6.18) *satisfies*

$$y_i = y_i^d, \quad i = 1, \ldots, L.$$

Sufficient conditions for properness and stability of the decentralized controller modules follow directly from Proposition 6.4.1 and 6.4.3, respectively.

Corollary 6.5.3. *The controller C_{ii}^d is proper if and only if $\Delta \deg(T_i) \geq \Delta \deg(G_i)$.*

Corollary 6.5.4. *Let the following conditions hold:*

- *each zero λ of G_i so that $|\lambda| > 1$ is a zero of T_i,*
- *$T_i(\lambda) - 1 \neq 0$ for all $|\lambda| > 1$.*

Then C_{ii}^d is stable.

We remark that the data-driven procedures DVRFT (Steentjes et al., 2020) and DOCI (Steentjes et al., 2021b) procedures (described in the sequel, in respectively Chapter 7 and 8) are applicable for this choice of reference model and the corresponding decentralized controller class.

6.6 Reference model synthesis from performance specifications

6.6.1 Networks with minimum-phase and stable plants

Consider the case that the network consists of plants $\{\mathcal{P}_i, i \in \mathcal{V}\}$ with minimum-phase and stable dynamics, i.e., G_i, W_{ij} and F_{ij} are stable and minimum-phase for all $j \in \mathcal{N}_i$, $i \in \mathcal{V}$. According to Corollary 6.4.1, any decoupled reference model $\{\mathcal{K}_i, i \in \mathcal{V}\}$ that satisfies "$T_i - 1$ has no zeros outside the unit disk" results in an ideal controller $\{\mathcal{C}_i^d, i \in \mathcal{V}\}$ with C_i^d stable for all $i \in \mathcal{V}$.

Hence, for the stable and minimum-phase case, we consider the synthesis of a decoupled reference model $\{\mathcal{K}_i, i \in \mathcal{V}\}$. Although every reference model with no λ so that $T_i(\lambda) - 1 = 0$ for some $i \in \mathcal{V}$ is sufficient for stability of $\mathcal{C}_i^d$, no performance specifications are guaranteed for the reference model with such a choice for T_i. We will now consider the synthesis of T_i, $i \in \mathcal{V}$, such that performance specifications are satisfied.

$\mathscr{H}_2$ performance compatible reference-model design

The transfer T_i specifies a desired transfer from reference r_i to output y_i. It can therefore be interpreted as a complementary sensitivity function. Given the definition of the tracking error $z_i = r_i - y_i$, the transfer that would ideally be zero is $S_i : r_i \to z_i$, which is equal to $S_i = 1 - T_i$. In the scalar case where T_i is strictly proper, it can be shown that

$$\|S_i\|_{\mathscr{H}_2}^2 = \|T_i\|_{\mathscr{H}_2}^2 + 1.$$

Hence, the minimization of $\|S_i\|_{\mathscr{H}_2}^2$ is equivalent with the minimization of $\|T_i\|_{\mathscr{H}_2}^2$. Let us now consider the following problem: Given $\gamma_i > 0$, $i \in \mathcal{V}$, find $T_i \in \mathcal{T}_i$, with $\mathcal{T}_i$ the set of all feasible reference models (e.g. of fixed order or with minimum relative degree), such that

$$\|T_i\|_{\mathscr{H}_2} < \gamma_i.$$

Let us call this the $\mathscr{H}_2$ compatible problem. For a state-space realization (A_i, B_i, C_i) of T_i, so that $T_i(q) = C_i(qI - A_i)^{-1}B_i$, we recall the following result from Chapter 4.

Proposition 6.6.1. *The following statements are equivalent:*

- $\|T_i\|_{\mathscr{H}_2} < \gamma_i$,
- *there exists* $X_i > 0$ *so that*

$$A_i^\top X_i A_i - X_i + C_i^\top C_i < 0 \quad \textit{and} \quad B_i^\top X_i B_i < \gamma_i^2.$$

Hence, the set

$$\Sigma_{\gamma_i} = \{(A_i, B_i, C_i) \,|\, \exists X_i > 0 : \; A_i^\top X_i A_i - X_i + C_i^\top C_i < 0 \text{ and } B_i^\top X_i B_i < \gamma_i^2\}$$

contains all tuples (A_i, B_i, C_i) so that $\|T_i\|_{\mathscr{H}_2} < \gamma_i$ and therefore, any transfer function T_i that solves the $\mathscr{H}_2$ compatible problem, is in the set

$$\mathcal{T}_{\gamma_i} := \{T_i \in \mathcal{T}_i \,|\, T_i = C_i(qI - A_i)^{-1}B_i, \; (A_i, B_i, C_i) \in \Sigma_{\gamma_i}\}.$$

To find an appropriate reference model, we only have to take one element of the set $\mathcal{T}_{\gamma_i}$. The problem is equivalent with finding $(A_i, B_i, C_i) \in \Sigma_{\gamma_i}$. We will briefly describe one method to find such a tuple.

Let A_i be any stable matrix, i.e., any matrix with all eigenvalues $|\lambda| < 1$. Then there exists a $X_i > 0$ so that

$$A_i^\top X_i A_i - X_i < 0.$$

Now, take any C_i so that $A_i^\top X_i A_i - X_i + C_i^\top C_i < 0$. Finally, $B_i^\top X_i B_i$ is positive definite and quadratic in B_i. Hence, there exists B_i so that $B_i^\top X_i B_i < \gamma_i^2$, which can be found by determining the corresponding sublevel set, $\{B_i \,|\, B_i^\top X_i B_i - \gamma_i^2 < 0\}$.

Mixed sensitivity reference-model design

Consider the reference model for vertex $i \in \mathcal{V}$, T_i. As mentioned, this system defines the desired transfer from $r_i \to y_i^d$, which can be interpreted as a desired complementary sensitivity. Consider again the corresponding sensitivity $S_i = 1 - T_i$ so that $S_i + T_i = 1$.

Tracking problems commonly lead to a mixed sensitivity design, where the shaping of the sensitivity functions is required to meet time-domain constraints and control effort constraints. Such performance requirements are then translated into weighting filters W_i^T and W_i^S, say, so as to achieve

$$\|W_i^T T_i\|_{\mathscr{H}_\infty} < \gamma \tag{6.19}$$

$$\|W_i^S S_i\|_{\mathscr{H}_\infty} < \gamma. \tag{6.20}$$

The problem now becomes to choose T_i so that both (6.19) and (6.20) are satisfied. This problem was considered in (Cerone et al., 2020) for SISO systems. The result for the *decoupled* reference model follows directly from (Cerone et al., 2020, Result 3).

Proposition 6.6.2. *A decoupled reference model that satisfies* (6.19) *and* (6.20), *is given by*

$$T_i = \frac{C_i^F G_i^F}{1 + C_i^F G_i^F}, \quad i \in \mathcal{V},$$

with G_i^F a fictitious plant and C_i^F a controller that solves the $\mathscr{H}_\infty$ control problem:

$$C_i^F \in \{C_i^F \mid \|M_i\|_{\mathscr{H}_\infty} < \gamma\},$$

with $M_i = \mathrm{col}(W_i^T T_i^F, W_i^S S_i^F)$,

$$S_i^F = \frac{1}{1 + C_i^F G_i^F}, \quad T_i^F = 1 - S_i^F.$$

The fictitious plant G_i^F does not have to coincide with the dynamics of the plant $\mathcal{P}_i$ and can be taken as $G_i^F = 1$ for all $i \in \mathcal{V}$ without loss of generality (Cerone et al., 2020). This implies that finding the decoupled reference model boils down to solving L separate $\mathscr{H}_\infty$ control problems.

6.6.2 Networks with non-minimum-phase plants

In the case of non-minimum phase dynamics in the network, by Proposition 6.4.3 and Proposition 6.5.2, we should include the non-minimum phase zeros of G_i in T_i and, possibly T_j, $j \in \mathcal{N}_i$. Consider again the problem of Section 6.6.1, i.e., to synthesize a decoupled reference model with $T_i \in \mathcal{T}_{\gamma_i}$ so that

$$\|T_i\|_{\mathscr{H}_2} < \gamma_i.$$

Proposition 6.6.3. *Let $T_i \in \mathcal{T}_i$ be given by*

$$T_i = \frac{C_i^F G_i^F}{1 + C_i^F G_i^F}, \quad i \in \mathcal{V}, \tag{6.21}$$

with G_i^F a fictitious plant having zeros λ_k^i so that $|\lambda_k^i| > 1$ and C_i^F a fictitious controller that solves the $\mathscr{H}_2$ control problem:

$$C_i^F \in \{C_i^F \, stable \mid \|T_i\|_{\mathscr{H}_2} < \gamma_i\}. \tag{6.22}$$

Then $T_i \in \mathcal{T}_{\gamma_i}$ and every zero λ_k^i is a zero of T_i, $i \in \mathcal{V}$.

Proof. By (6.22), it follows directly that $T_i \in \mathcal{T}_{\gamma_i}$. By the definition of T_i in (6.21), it follows that λ_k^i is a zero of T_i, because $|\lambda_k^i| > 1$ and C_i^F is stable by (6.22). □

Thus, instead of finding a $T_i \in \mathcal{T}_{\gamma_i}$ directly, by Proposition 6.6.3 we can synthesize a fictitious controller C_i^F for a fictitious plant G_i^F so that the closed-loop transfer $T_i = (1 + C_i^F G_i^F)^{-1} C_i^F G_i^F \in \mathcal{T}_{\gamma_i}$. When G_i^F has all zeros outside the unit disk of G_i, respectively of G_i and G_j, $j \in \mathcal{N}_i$, then every zero outside the unit disk of G_i, respectively of G_i and G_j, $j \in \mathcal{N}_i$, is also a zero of $T_i \in \mathcal{T}_{\gamma_i}$.

6.7 Conclusions

In this chapter, we have introduced the distributed model-reference control problem for interconnected systems. We have developed an ideal distributed controller that has the same structure as the network under consideration and solves the structured model-reference control problem. The developed ideal distributed controller depends explicitly on the interconnected system and structured reference model dynamics and forms the basis for the direct data-driven distributed control problem addressed in Chapter 7 and 8. A feature of the ideal distributed controller, is that it can be designed in a decentralized manner, compared with the coupled feasibility problem for the distributed controller design in Chapter 4. How to choose the structured reference model may not be directly clear, however, given rudimentary performance specifications. Properness and stability of the ideal distributed controller have been analyzed and the synthesis of a structured reference model has been considered. In the synthesis, properness and stability of a corresponding ideal distributed controller is taken into account. Quantitative performance specifications through $\mathscr{H}_2$ or $\mathscr{H}_\infty$ norm upper bounds can be taken into account, in the design of a reference model that is decoupled.

Chapter 7

Virtual reference feedback tuning in dynamic networks

In this chapter, we consider the problem of synthesizing a distributed controller directly on the basis of data, with the objective to optimize a model-reference control criterion. In Chapter 6, an explicit ideal distributed controller that solves the model-reference control problem for a structured reference model was introduced. We show how this distributed controller can be obtained through data-driven modeling in a virtual dynamic network, extending virtual reference feedback tuning (VRFT) towards distributed control. When the interconnected system is subject to unmeasured exogenous inputs, we show that the structured model-reference control problem can be approached by solving a dynamic network identification problem with prediction-error filtering and a tailor-made noise model. Sufficient conditions are provided for which the local controller estimates are consistent. Moreover, it is shown how the method can be applied to the single-process case, leading to consistent estimates with standard VRFT as well.

This chapter is based on the publications: T. R. V. Steentjes, M. Lazar, and P. M. J. Van den Hof. Data-driven distributed control: Virtual reference feedback tuning in dynamic networks. In *Proc. 59th IEEE Conference on Decision and Control (CDC)*, pages 1804–1809, 2020 and T. R. V. Steentjes, P. M. J. Van den Hof, and M. Lazar. Handling unmeasured disturbances in data-driven distributed control with virtual reference feedback tuning. *IFAC-PapersOnLine*, 54(7):204–209, 2021d. 19$^{\text{th}}$ IFAC Symposium on System Identification SYSID 2021

7.1 Introduction

Plant models are typically not directly available for controller design. When data from the plant is available, two approaches to controller design can be followed (Hou and Wang, 2013): (i) indirect data-driven control and (ii) direct data-driven control. Indirect data-driven control is model based: first a plant model is estimated on the basis of data and consecutively a controller design is performed on the basis of the plant model. In direct data-driven control, an explicit plant modeling step is omitted; a controller is synthesized directly from data. Typical advantages of direct-data driven controller design are that no loss of data can occur due to undermodeling of the plant and the order of the controller can be fixed. Therefore, direct data-driven control is particularly interesting for the design of distributed controllers for interconnected systems, due to their complex nature and involved data-driven modeling.

State-of-the art methods for direct data-driven controller design are virtual reference feedback tuning (VRFT) (Campi et al., 2002), iterative feedback tuning (IFT) (Hjalmarsson et al., 1998), optimal controller identification (OCI) (Campestrini et al., 2017; Huff et al., 2019), correlation-based tuning (CbT) (van Heusden et al., 2011), asymptotically exact (Formentin et al., 2015) and moment-matching (Breschi et al., 2019) controller tuning. A common feature of these methods is that they are based on the model-reference paradigm, wherein a reference model describes the desired behavior of the closed-loop system (Bazanella et al., 2012). Subsequent estimation of a parametrized controller based on input-output data leads to a closed-loop system that is optimal with respect to the model-reference criterion.

Direct data-driven controller design methods in the literature typically address the design of a centralized controller, where the underlying interconnection structure is not taken into account. The design of a distributed controller directly on the basis of data with a model-reference control objective, is a problem that is unsolved in the literature. This problem yields new aspects that have to be considered in the design, such as the interconnection structure of the system and distributed controller, the distributed model-reference control problem, and the use of measurement data from an interconnected system in the controller design. Given the interconnected nature of the system and controller, network identification methods for data-driven modeling of interconnected systems (Van den Hof et al., 2013), (Gevers et al., 2018), (Bazanella et al., 2019), (Van den Hof and Ramaswamy, 2021), can give relevant insights in the data aspects of the problem.

In this chapter, we first address the problem of designing a distributed controller directly on the basis of data, without exogenous disturbances (noise). We tackle this problem via the structured model-reference paradigm for interconnected systems, presented in Chapter 6. Then, we present an extension of

VRFT to the case of interconnected systems called distributed VRFT (DVRFT) and show that the distributed controller synthesis problem from data can be transformed into a dynamic network identification problem. The contribution to distributed control is the direct data-driven design, as the synthesis of distributed controllers is typically model based. Regarding data-driven control, the contribution is the synthesis of distributed data-driven controllers with a priori defined structure and identification of a distributed controllers via network identification.

When the considered plant is affected by disturbances, VRFT inherently introduces a bias in the controller estimates (Campi et al., 2002), (Bazanella et al., 2012), leading to a degraded closed-loop performance. For DVRFT, also biased estimates are obtained for local controllers in the case that a process noise affects the corresponding subsystem. One approach to solve this problem is the use of an instrumental variable (IV) method, in case the controller model is linear with respect to the parameters. Depending on the choice of IV, however, additional experiments on the system are required (Bazanella et al., 2012) and the parameter variance is increased with a negative effect on the control performance. In the general case, no method for obtaining consistent estimates for VRFT is present in the literature, to the best of the author's knowledge.

Therefore, we develop a method for dealing with noise in both VRFT and DVRFT. The approach that is followed, is to take the modeling of an auxiliary noise filter into account in the estimation of the controller. With the introduction of a tailor-made noise model for VRFT, we provide sufficient conditions under which consistent controller estimates are obtained. The method extends naturally to the DVRFT framework and solves the distributed model reference control problem via DVRFT for a class of interconnected systems with unmeasured exogenous inputs.

7.2 Preliminaries

7.2.1 Dynamical network and distributed controller

Consider a simple and undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with vertex set $\mathcal{V}$ of cardinality L and edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. The neighbour set of vertex $i \in \mathcal{V}$ is defined as $\mathcal{N}_i := \{j \in \mathcal{V} \,|\, (i,j) \in \mathcal{E}\}$. The graph $\mathcal{G}$ describes the structure of a network of linear discrete-time systems, where the dynamics associated with vertex $i \in \mathcal{V}$ are described by

$$y_i(t) = G_i(q)u_i(t) + \sum_{j \in \mathcal{N}_i} G_{ij}(q)y_j(t) + H_i(q)e_i(t), \tag{7.1}$$

with $u_i : \mathbb{Z} \to \mathbb{R}$ the control input, $y_i : \mathbb{Z} \to \mathbb{R}$ the output, e_i an unmeasured zero-mean white-noise proces such that, for all (t,s), $Ee_i(t)e_j(s) = 0$ for $(i,j) \in \mathcal{E}$

and $Ee_i(t)u_j(s) = 0$ for $(i,j) \in \mathcal{V} \times \mathcal{V}$, and q the forward shift defined as $qx(t) = x(t+1)$. The rational transfer functions G_i, G_{ij} and H_i, $(i,j) \in \mathcal{E}$, describe the local dynamics, coupling dynamics and noise dynamics, respectively. The noise filter H_i is assumed to be monic, stable and minimum phase. We omit the time and shift arguments t and q occasionally for brevity, when the context does not yield ambiguity. The network can be compactly written as

$$y = G_I y + Gu + He, \tag{7.2}$$

where $G = \mathrm{diag}(G_1, \ldots, G_L)$, $H = \mathrm{diag}(H_1, \ldots, H_L)$ and

$$G_I = \begin{bmatrix} 0 & G_{12} & \cdots & G_{1L} \\ G_{21} & 0 & \cdots & G_{2L} \\ \vdots & \vdots & \ddots & \vdots \\ G_{L1} & G_{L2} & \cdots & 0 \end{bmatrix}.$$

The transfer between external inputs and outputs is described by

$$y = (I - G_I)^{-1}(Gu + He), \tag{7.3}$$

under the assumption that the network (7.2) is well posed, i.e. $\mathcal{G} := (I - G_I)^{-1}$ exists. The simplest network consists of one node ($L = 1$), so that

$$y = Gu + He, \tag{7.4}$$

which is a standard single-input-single-output process with a disturbance.

Considering a reference tracking problem for the network, each system is equipped with a reference signal r_i and the corresponding tracking error $z_i := r_i - y_i$:

$$\mathcal{P}_i : \begin{cases} y_i &= G_i u_i + \sum_{j \in \mathcal{N}_i} G_{ij} y_j + H_i e_i, \\ z_i &= r_i - y_i. \end{cases} \tag{7.5}$$

The dynamical network is operating in closed-loop with a distributed controller that consists of local controllers

$$\mathcal{C}_i(\rho_i) : \begin{cases} u_i &= C_{ii}(q, \rho_i) z_i + \sum\limits_{j \in \mathcal{N}_i} C_{ij}(q, \rho_i) \eta_{ij}, \\ \zeta_{ij} &= K_{ij}(q, \rho_i) z_i + \sum\limits_{h \in \mathcal{N}_i} K_{ijh}(q, \rho_i) \eta_{ih}, \; j \in \mathcal{N}_i, \end{cases}$$

where each controller is parametrized by a parameter vector ρ_i, and is interconnected with other controllers as:

$$\eta_{ij} = \zeta_{ji} \text{ for } j \in \mathcal{N}_i \quad \text{and} \quad \eta_{ij} = 0 \text{ otherwise.}$$

With the definitions $\eta_i := \mathrm{col}_{j\in\mathcal{N}_i}\,\eta_{ij}$ and $\zeta_i := \mathrm{col}_{j\in\mathcal{N}_i}\,\zeta_{ij}$, we compactly represent $\mathcal{C}_i$ by

$$\mathcal{C}_i(\rho_i) : \begin{bmatrix} u_i \\ \zeta_i \end{bmatrix} = C_i(q,\rho_i) \begin{bmatrix} z_i \\ \eta_i \end{bmatrix}. \tag{7.6}$$

7.2.2 Distributed model-reference control

Distributed model-reference control considers the synthesis of a structured controller such that the closed-loop network dynamics are optimal with respect to a structured reference model. Introduced in Chapter 6, the structured reference model is composed of subsystems $\mathcal{K}_i$, $i \in \mathcal{V}$, that can be interconnected:

$$\mathcal{K}_i : \begin{cases} y_i^d &= T_i(q)r_i + Q_i(q)k_i, \\ p_i &= P_i(q)y_i^d, \end{cases} \tag{7.7}$$

where $Q_i := \mathrm{row}_{j\in\mathcal{N}_i}\,Q_{ij}$ and $P_i := \mathrm{col}_{j\in\mathcal{N}_i}\,P_{ij}$ and the interconnection variables are partitioned as $k_i := \mathrm{col}_{j\in\mathcal{N}_i}\,k_{ij}$ and $p_i := \mathrm{col}_{j\in\mathcal{N}_i}\,p_{ij}$. For each pair $(i,j) \in \mathcal{E}$ the interconnection of $\mathcal{K}_i$ and $\mathcal{K}_j$ is defined by

$$k_{ij} = p_{ji} \quad \text{and} \quad k_{ji} = p_{ij}. \tag{7.8}$$

Hence, $\mathcal{K}_i$ and $\mathcal{K}_j$ can only be interconnected if $\mathcal{P}_i$ and $\mathcal{P}_j$ are interconnected.

Given $e_i = 0$ for all $i \in \mathcal{V}$, the distributed model reference control problem is

$$\min_{\rho_1,\dots,\rho_L} J_{\mathrm{MR}}(\rho_1,\dots,\rho_L) = \min_{\rho_1,\dots,\rho_L} \sum_{i=1}^{L} \bar{E}[y_i^d(t) - y_i(t)]^2, \tag{7.9}$$

where $\bar{E} := \lim_{N\to\infty} \frac{1}{N}\sum_{t=1}^{N} E$ and E is the expectation. A distributed controller that solves (7.9) was developed in Chapter 6. Define the column vector $\mathbf{1} := \mathrm{col}(1,\dots,1)$ and, for $i \in \mathcal{V}$, define $G_{iI} := \mathrm{row}_{j\in\mathcal{N}_i}\,G_{ij}$. We recall Theorem 6.3.1 for the distributed model-reference control problem for the interconnected system with output-interconnected subsystems (7.5).

Proposition 7.2.1. *Consider $e_i = 0$ for all $i \in \mathcal{V}$ and consider a distributed controller described by the subsystems*

$$\mathcal{C}_i^d : \begin{bmatrix} u_i \\ o_i^c \\ p_i^c \end{bmatrix} = \underbrace{\begin{bmatrix} \dfrac{T_i}{G_i(1-T_i)} & -\dfrac{1}{G_i}G_{iI} & \dfrac{1}{G_i(1-T_i)}Q_i \\ \dfrac{T_i}{1-T_i}\mathbf{1} & 0 & \dfrac{1}{1-T_i}\mathbf{1}Q_i \\ \dfrac{T_i}{1-T_i}P_i & 0 & \dfrac{1}{1-T_i}P_iQ_i \end{bmatrix}}_{=:C_i^d(q)} \begin{bmatrix} z_i \\ s_i^c \\ k_i^c \end{bmatrix}, \tag{7.10}$$

for $i \in \mathcal{V}$ and the controller interconnections described by

$$\begin{bmatrix} s^c_{ij} \\ k^c_{ij} \end{bmatrix} = \begin{bmatrix} o^c_{ji} \\ p^c_{ji} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} s^c_{ji} \\ k^c_{ji} \end{bmatrix} = \begin{bmatrix} o^c_{ij} \\ p^c_{ij} \end{bmatrix}, \tag{7.11}$$

for $(i,j) \in \mathcal{E}$. A network with subsystems (7.5) *in closed-loop with the distributed controller* (7.10)-(7.11) *satisfies*

$$y_i = y_i^d, \quad i \in \mathcal{V}.$$

Recall from Chapter 6 (discussion after Theorem 6.3.1), that the parameters $\rho_1^d, \ldots, \rho_L^d$ associated with the distributed controller such that $P_i^\top C_i^d P_i = C_i(\rho_i^d)$, solve (7.9) by Proposition 7.2.1[1], i.e., $C_i(\rho_i^d)$ equals C_i^d up to re-ordering of the interconnection variables. Let $\mathscr{C}_i := \{C_i(q,\rho_i) \,|\, \rho_i \in \mathbb{R}^{l_i}\}$ be the family of parametrized controllers for node $i \in \mathcal{V}$. It is assumed that the controller class $\mathcal{C}_i$ is 'rich' enough in the sense that it contains $C_i(\rho_i^d)$, as formalized in the following assumption.

Assumption 7.1. *$P_i^\top C_i^d P_i \in \mathscr{C}_i$ for each $i = 1, \ldots, L$.*

7.3 Distributed virtual reference feedback tuning

The controller described in Section 6.3 provides a solution to Problem 6.2.1, but requires $\mathcal{P}_i$ to be given. The problem considered in this section, is the direct data-driven synthesis of a distributed controller in the absence of process noise, i.e., given data $\{u_i, y_i\}$, $i = 1, \ldots, L$ for $e_i = 0$, solve problem (6.7). The absence of noise together with Assumption 7.1 corresponds to an ideal situation, in which the main idea is developed for ease of explanation. The case where noise signals are present will be addressed in Section 7.5.

We address the problem in the ideal situation by two steps: virtual reference generation and distributed controller identification.

7.3.1 Virtual reference generation

Consider data $\{u_i, y_i\}$, $i = 1, \ldots, L$ collected from the network described by (7.5). This data can be obtained in closed loop with a stabilizing controller or in open loop if the network is stable, i.e., if $(I - G_I)^{-1}G$ is stable. For the reference model described by (7.7)-(7.8), we recall that $y_d = (I - Q\Delta P)^{-1}Tr$. Now,

[1] The matrices P_i, $i \in \mathcal{V}$, are permutation matrices $P_i := \mathrm{diag}(1, \overline{P}_i)$, with $\overline{P}_i$ such that $\mathrm{col}(s_i^c, k_i^c) = \overline{P}_i \,\mathrm{col}_{j\in\mathcal{N}_i} \mathrm{col}(s_{ij}^c, k_{ij}^c)$.

given $y_1, y_2, \ldots, y_L$, consider the computation of the *virtual reference* signals $\bar{r}_1, \bar{r}_2, \ldots, \bar{r}_L$ according to the structured reference model as

$$y = (I - Q\Delta P)^{-1} T \bar{r}. \tag{7.12}$$

Then $\bar{r}_1, \bar{r}_2, \ldots, \bar{r}_L$ are such that, when the network described by (7.5) is in closed loop with the ideal distributed controller, fictitiously, the measured outputs $y_1, y_2, \ldots, y_L$ are the corresponding outputs. Solving (7.12) requires the data $y_1, y_2, \ldots, y_L$ to be collected by a central governor. Because central data collection is not favourable, we propose to generate the virtual reference signals locally by using the reference model $\mathcal{K}_i$. This can always be done for the considered reference model, by determining the virtual reference signals $\bar{r}_1, \bar{r}_2, \ldots, \bar{r}_L$ and the *virtual interconnection* signals $\bar{p}_1, \bar{p}_2, \ldots, \bar{p}_L$ according to (7.7) and (7.8) so that

$$y_i = T_i \bar{r}_i + \sum_{j \in \mathcal{N}_i} Q_{ij} \bar{p}_{ji} \quad \text{and} \quad \bar{p}_{ij} = P_{ij} y_i, \quad j \in \mathcal{N}_i.$$

Given a virtual reference signal $\bar{r}_i$, the corresponding virtual tracking error and, hence, the input to the ideal controller, is $\bar{z}_i = \bar{r}_i - y_i$. The virtual reference generation can thus be distributed, as summarized in Algorithm 7.3.1.

Algorithm 7.3.1 Distributed virtual reference computation

Input: Reference model transfer functions T_i, Q_i, P_i and output data y_i for $i = 1, \ldots, L$

Output: Virtual signals $\bar{r}_i$, $\bar{z}_i$, $\bar{p}_i$ for $i = 1, \ldots, L$

1: **for** $i = 1$ to L **do**
2: Compute $\bar{p}_i$ such that $\bar{p}_i(t) = P_i(q) y_i(t)$.
3: **end for**
4: **for** $i = 1$ to L **do**
5: Receive $\bar{p}_{ji}$ from nodes $j \in \mathcal{N}_i$. Compute $\bar{r}_i$ such that

$$T_i(q) \bar{r}_i(t) = y_i(t) - \sum_{j \in \mathcal{N}_i} Q_{ij}(q) \bar{p}_{ji}(t).$$

6: $\bar{z}_i \leftarrow \bar{r}_i - y_i$
7: **end for**
8: **return** $\bar{r}_i$, $\bar{z}_i$, $\bar{p}_i$, $i = 1, \ldots, L$

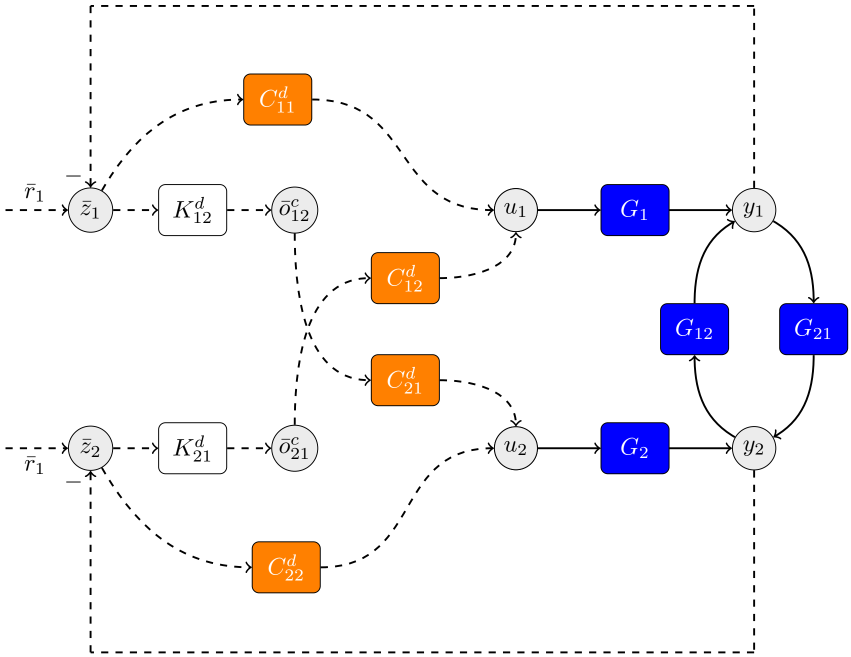


Figure 7.1: Virtual experiment setup for identification of the ideal distributed controller ($L = 2$), with unknown plant dynamics G_i, G_{ij} (blue), unknown ideal distributed controller dynamics C_{ii}^d, C_{ij}^d (orange) and known ideal distributed controller dynamics K_{ij}^d (white). The virtual and real part of the network are visualized respectively by dashed and solid lines.

7.3.2 Identification of the ideal distributed controller in a virtual network

Consider Example 6.1, introduced in Chapter 6. Figure 7.1 shows the constructed virtual network that is obtained by following Algorithm 7.3.1. The task of determining the controllers $\mathcal{C}_1^d$ and $\mathcal{C}_2^d$ now essentially becomes a dynamic network identification problem (Van den Hof et al., 2013), where $\{C_{11}^d, C_{12}^d, K_{12}^d\}$ and $\{C_{22}^d, C_{21}^d, K_{21}^d\}$ are the modules to be identified (strictly speaking $\{C_{11}^d, C_{12}^d\}$ and $\{C_{22}^d, C_{21}^d\}$, since K_{12}^d and K_{21}^d are known). The signals u_1 and u_2 are directly available from the measurements, while $\bar{z}_1$ and $\bar{z}_2$ are virtual and obtained by Algorithm 7.3.1. The virtual controller interconnection signals $\bar{o}_{ij}^c$ are obtained by filtering $\bar{z}_i$ as $\bar{o}_{12}^c = K_{12}^d \bar{z}_1$ and $\bar{o}_{21}^c = K_{21}^d \bar{z}_2$.

To illustrate the identification, consider the sets of parametrized controllers

$\{C_{11}(\rho_1), C_{12}(\rho_1)\}$, $\{C_{22}(\rho_2), C_{21}(\rho_2)\}$, the control-input predictors

$$\hat{u}_1(\rho_1) = C_{12}(\rho_1)\bar{o}^c_{21} + C_{11}(\rho_1)\bar{z}_1, \tag{7.13}$$

$$\hat{u}_2(\rho_2) = C_{21}(\rho_2)\bar{o}^c_{12} + C_{22}(\rho_2)\bar{z}_2 \tag{7.14}$$

and the identification criterion

$$J_{\mathrm{VR}}(\rho_1, \rho_2) = \bar{E}[\varepsilon_1(\rho_1)]^2 + \bar{E}[\varepsilon_2(\rho_2)]^2$$

with $\varepsilon_i := u_i - \hat{u}_i(\rho_i)$. We will now analyze the minima of J_{VR}. Since $\bar{o}^c_{12} = K^d_{12}\bar{z}_1$ and $\bar{o}^c_{21} = K^d_{21}\bar{z}_2$, it follows that

$$\begin{aligned}\varepsilon_1(\rho_1) &= (C^d_1 - C_1(\rho_1))\bar{z}_1 + (C^d_{12} - C_{12}(\rho_1))K^d_{21}\bar{z}_2,\\ \varepsilon_2(\rho_2) &= (C^d_2 - C_2(\rho_1))\bar{z}_2 + (C^d_{21} - C_{21}(\rho_2))K^d_{12}\bar{z}_1.\end{aligned}$$

Then, since $\bar{z} = (T^{-1} - I)(I - G_I)^{-1}Gu$, where $T = \mathrm{diag}(T_1, T_2)$, the prediction errors are

$$\begin{aligned}\begin{bmatrix}\varepsilon_1(\rho_1)\\ \varepsilon_2(\rho_2)\end{bmatrix} &= \begin{bmatrix} C^d_{11} - C_{11}(\rho_1) & (C^d_{12} - C_{12}(\rho_1))K^d_{21}\\ (C^d_{21} - C_{21}(\rho_2))K^d_{12} & C^d_{22} - C_{22}(\rho_2)\end{bmatrix}\\ &\quad \times (T^{-1} - I)(I - G_I)^{-1}Gu.\end{aligned} \tag{7.15}$$

It now appears that a global minimum of J_{VR} is $(\rho_1, \rho_2) = (\rho^d_1, \rho^d_2)$ and that this minimum is unique if the control input signal $u = \mathrm{col}(u_1, u_2)$ from the experiment is persistently exciting of a sufficient order. Hence, the global minimum of $J_{\mathrm{VR}}(\rho_1, \rho_2)$ is then the same as the global minimum of $J_{\mathrm{MR}}(\rho_1, \rho_2)$, where J_{VR} is quadratic in ρ when the sub-controllers are parametrized linearly in ρ. The distributed-controller synthesis problem is therefore reformulated as a network identification problem.

The latter reasoning for Example 6.1 leads us to the following result for a general interconnected system:

Theorem 7.3.1. *Consider the predictor* $\hat{u}_i(\rho_i) := C_{ii}(\rho_i)\bar{z}_i + \sum_{j\in\mathcal{N}_i} C^W_{ij}(\rho_i)\bar{o}^c_{ji} + C^Q_{ij}(\rho_i)\bar{p}_{ji}$ *with* $\bar{o}^c_{ji} = (1-T_j)^{-1}T_j\bar{z}_j + \sum_{h\in\mathcal{N}_j}(1-T_j)^{-1}Q_{jh}\bar{p}_{hj}$. *The identification criterion*

$$J^{VR}_i(\rho_i) = \bar{E}[u_i - \hat{u}_i(\rho_i)]^2$$

has a global minimum point at ρ^d_i *and this minimum is unique if the spectrum of* $w_i = \mathrm{col}(\bar{z}_i, \mathrm{col}_{j\in\mathcal{N}_i}\bar{o}^c_{ji}, \mathrm{col}_{j\in\mathcal{N}_i}\bar{p}_{ji})$, *denoted* $\Phi_{w_i}(\omega)$, *is positive definite for almost all* $\omega \in [-\pi, \pi]$.

Proof. First, we note that $\bar{p}_{ji} = p^c_{ji}$ and $\bar{o}^c_{ji} = o^c_{ji}$, where p^c_{ji} and o^c_{ji} satisfy (6.9) and (6.10) for $z_i = \bar{z}_i$, $i = 1, \ldots, L$. Consequently, by Corollary 1 in (Van den Hof et al., 2013), it follows that ρ^d_i is the unique global minimum point of J_{VR}. $\square$

When the reference model is decoupled, the spectrum condition can be translated directly to the spectrum of the input. Indeed, in this case the predictor inputs are $\bar{z}_i$ and $\bar{o}^c_{ji} = (1 - T_j)^{-1} T_j \bar{z}_j$, where $\bar{z}_i$, $i \in \mathcal{V}$, are related to u_i, $i \in \mathcal{V}$, through $\bar{z} = (T^{-1} - I)(I - G_I)^{-1} G u$ with $T = \mathrm{diag}_{i\in\mathcal{V}} T_i$.

Corollary 7.3.1. *Let $P_i = 0$, $Q_i = 0$ and consider the predictors $\hat{u}^D_i(\rho_i) := C_{ii}(\rho_i)\bar{z}_i + \sum_{j\in\mathcal{N}_i} C_{ij}(\rho_i)\bar{o}^c_{ji}$, $i = 1, 2, \dots, L$. The identification criterion*

$$J_{VR}(\rho_1, \dots, \rho_L) = \sum_{i=1}^{L} \bar{E}[u_i - \hat{u}^D_i(\rho_i)]^2$$

has a global minimum point at $(\rho^d_1, \dots, \rho^d_L)$ and this minimum is unique if $\Phi_u(\omega)$ is positive definite for almost all $\omega \in [-\pi, \pi]$.

The condition on Φ_u in Corollary 7.3.1 can be realized by appropriate experiment design. The condition on Φ_{w_i} in Theorem 7.3.1, however, cannot always be realized by an appropriate design of u. For instance, consider Example 6.1, but now with non-zero Q_i, P_i. Then the number of entries of $w_1 = \mathrm{col}(\bar{z}_1, \bar{o}_{21}, \bar{p}_{21})$ is larger than the number of inputs in $u = \mathrm{col}(u_1, u_2)$, hence $\Phi_{w_1}(\omega)$ cannot be positive definite. We observe that the excitation condition can be relaxed if we do not require ρ^d_i to be the only global minimum[2] of J^{VR}_i.

Corollary 7.3.2. *Each global minimum point ρ^*_i of J^{VR}_i satisfies $C_{ii}(\rho^*_i) = C_{ii}(\rho^d_i)$ and for all $j \in \mathcal{N}_i$:*

$$(C^W_{ij}(\rho^*_i) - C^W_{ij}(\rho^d_i)) + (C^Q_{ij}(\rho^*_i) - C^Q_{ij}(\rho^d_i))P_{ji} = 0 \tag{7.16}$$

if $\Phi_{\xi_i}(\omega)$, $\xi_i = \mathrm{col}(\bar{z}_i, \mathrm{col}_{j\in\mathcal{N}_i} \bar{o}^c_{ji})$, is positive definite for almost all $\omega \in [-\pi, \pi]$.

It can be verified that $(\rho^*_1, \dots, \rho^*_L)$, satisfying $C_{ii}(\rho^*_i) = C_{ii}(\rho^d_i)$ and (7.16) for all $(i, j) \in \mathcal{E}$, is also a global minimum point of J_{MR} and hence solves problem (7.9). Indeed, by the definition of o^c_{ij} and p^c_{ij} in (7.10) and (7.11), the control input $u^*_i := C_{ii}(\rho^*_i)z_i + \sum_{j\in\mathcal{N}_i} C^W_{ij}(\rho^*_i)s^c_{ij} + C^Q_{ij}(\rho^*_i)k^c_{ij} = C_{ii}(\rho^*_i)z_i + \sum_{j\in\mathcal{N}_i}(C^W_{ij}(\rho^*_i) + C^Q_{ij}(\rho^*_i)P_{ji})y_j$. Hence, if $C_{ii}(\rho^*_i) = C_{ii}(\rho^d_i)$ and (7.16) hold true, then u^*_i is equal to the control input of the ideal controller in (7.10), i.e., $u^*_i = C_{ii}(\rho^d_i)z_i + \sum_{j\in\mathcal{N}_i} C^W_{ij}(\rho^d_i)s^c_{ij} + C^Q_{ij}(\rho^d_i)k^c_{ij}$.

Each identification criterion $J^{\mathrm{VR}}_i(\rho_i)$, $i = 1, \dots, L$, can be minimized separately. The required predictor inputs for node i are the virtual signals obtained in Algorithm 7.3.1, which are available locally ($\bar{z}_i$) or communicated by nodes

[2]The situation of having a non-unique global minimum can have consequences for the optimization procedure and requires a careful choice of the optimization algorithm or, alternatively, a re-parametrization. These considerations fall outside the scope of the research presented here.

$j \in \mathcal{N}_i$ ($\bar{o}^c_{ji}$ and $\bar{p}_{ji}$). Of course, J_i^{VR} is not to be considered in practice since it involves expectations; for a finite number (N) of data, a solution ρ_i is obtained by minimizing $\bar{J}_i^{\mathrm{VR}}(\rho_i) = \frac{1}{N}\sum_{t=1}^{N}(u_i(t) - \hat{u}_i(t,\rho_i))^2$. Observe the following difference with respect to multi-variable VRFT (Campestrini et al., 2016). Instead of identifying the transfer from $\bar{z}$ to u, we identify the local controller dynamics C_i^d by exploiting the structure of the interconnected system.

7.4 VRFT: a tailor-made noise model for consistent estimation

In this section we consider the modeling of a noise filter for consistent controller estimation with VRFT for a single process. Consider the single process described by (7.4). The input u and e are assumed to be independent, corresponding to an open-loop experiment in which the data (u, y) are collected. The tracking error for this process is denoted $z := r - y$, with r the reference. A reference model for the process is assumed to be given and described by

$$y^d = T_d r.$$

Given the reference model, it is known that the ideal controller is (Bazanella et al., 2012)

$$C_d = \frac{T_d}{G(1 - T_d)}.$$

We will now discuss the identification of C_d from the data (u, y).

The virtual reference $\bar{r}$ and tracking error $\bar{z}$ are given by

$$\bar{r} := T_d^{-1} y \quad \text{and} \quad \bar{z} := \bar{r} - y. \tag{7.17}$$

We can rewrite (7.4) in terms of C_d, such that

$$u = C_d \bar{z} - G^{-1} H e = C_d \bar{z} + \bar{H}_d e, \tag{7.18}$$

with $\bar{H}_d := -G^{-1}H$. This leads to the virtual control loop shown in Figure 7.2.

It is known that if we follow the standard VRFT procedure (Campi et al., 2002) for the identification of C_d, we will obtain a biased estimate due to the noise e (Campi et al., 2002), (Bazanella et al., 2012). We will now consider the direct identification of C_d together with the identification of $\bar{H}_d$. The question is: by including the estimation of the auxiliary noise filter, can we obtain consistent estimates of C_d?

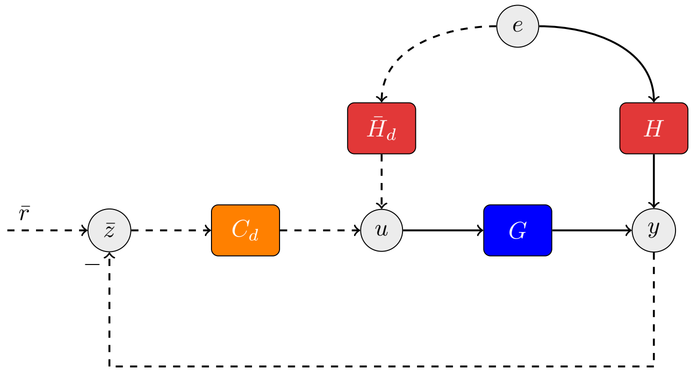


Figure 7.2: Virtual control loop – noisy case.

7.4.1 Modeling $\bar{H}_d$ directly

Consider a parametrized model $C(q,\rho)$ for $C_d(q)$ and a model $\bar{H}(q,\rho)$ for $\bar{H}_d(q)$. Definition of the predictor

$$\hat{u}(t,\rho) := \bar{H}(q,\rho)^{-1}C(q,\rho)\bar{z}(t) + (1 - \bar{H}(q,\rho)^{-1})u(t),$$

leads to the prediction error $\varepsilon(t,\rho) = u(t) - \hat{u}(t,\rho)$:

$$\begin{aligned}\varepsilon(t,\rho) &= \bar{H}(\rho)^{-1}(u - C(\rho)\bar{z}) \\ &= \bar{H}(\rho)^{-1}(C_d\bar{z} + \bar{H}_d e - C(\rho)\bar{z}) \\ &= \bar{H}(\rho)^{-1}\left((C_d - C(\rho))\frac{1-T_d}{T_d}y + \bar{H}_d e\right).\end{aligned} \tag{7.19}$$

After manipulation it can be shown that

$$\varepsilon(t,\rho) = \frac{1}{\bar{H}(\rho)C_d}(C_d - C(\rho))u + \frac{C(\rho)}{C_d}\frac{\bar{H}_d}{\bar{H}(\rho)}e. \tag{7.20}$$

Consider now the asymptotic parameter estimate

$$\rho^* = \arg\min_{\rho} \bar{V}(\rho), \quad \bar{V}(\rho) := \bar{E}\varepsilon^2(t,\rho). \tag{7.21}$$

The cost σ_e^2 is obtained if $C(\rho) = C_d$ and $\bar{H}(\rho) = \bar{H}_d$, but we cannot conclude that this is the minimum, since $\frac{C(\rho)}{C_d}\frac{\bar{H}_d}{\bar{H}(\rho)}$ is not necessarily monic. Hence, we cannot conclude that $C(\rho^*) = C_d$ and $\bar{H}(\rho^*) = H_d$ for the minimizing argument ρ^*.

7.4.2 Tailor-made noise model with prediction-error filtering

Let us return to the (virtual) data-generating system

$$u = C_d \bar{z} + \bar{H}_d e. \tag{7.22}$$

We have seen in the previous subsection that by modeling the auxiliary noise filter directly, consistent estimates cannot be guaranteed. Note that if we filter the prediction error (7.19) with G, then a noise filter $-H$ is obtained in the prediction error $\varepsilon_G = G\varepsilon$, with H monic. The plant G is, however, assumed to be unknown.

A more attractive solution is obtained as follows. By the definition of C_d, it follows that

$$\frac{T_d}{1-T_d} = C_d G.$$

Hence, by filtering the prediction error with $L := C_d G$ instead of G, we have

$$\varepsilon_F(t,\rho) := C_d G \varepsilon(t,\rho) = \frac{T_d}{1-T_d}\varepsilon(t,\rho). \tag{7.23}$$

The filter L depends only on the reference model T_d, which is known. Rewriting ε_F yields

$$\begin{aligned}\varepsilon_F(t,\rho) &= L\varepsilon(t,\rho)\\ &= \frac{T_d}{1-T_d}\bar{H}(\rho)^{-1}\left((C_d - C(\rho))\frac{1-T_d}{T_d}y + \bar{H}_d e\right)\\ &= \bar{H}(\rho)^{-1}\left((C_d - C(\rho))y + \frac{T_d}{1-T_d}\bar{H}_d e\right)\\ &= \bar{H}(\rho)^{-1}\left((C_d - C(\rho))y - C_d H e\right).\end{aligned} \tag{7.24}$$

Substituting the relation $y = Gu + He$ yields

$$\begin{aligned}\varepsilon_F(t,\rho) &= \bar{H}(\rho)^{-1}\left((C_d - C(\rho))(Gu + He) - C_d H e\right)\\ &= \bar{H}(\rho)^{-1}\left((C_d - C(\rho))Gu - C(\rho)He\right).\end{aligned}$$

By selecting a tailor-made parametrization $\bar{H}(\rho) = -C(\rho)\breve{H}(\rho)$ with $\breve{H}(\rho)$ monic, we have

$$\begin{aligned}\varepsilon_F(t,\rho) &= -\bar{H}(\rho)^{-1}\bar{H}(\rho)e + e + \bar{H}(\rho)^{-1}\left((C_d - C(\rho))Gu - C(\rho)He\right)\\ &= \bar{H}(\rho)^{-1}\left(\Delta C(\rho)Gu + C(\rho)\Delta H(\rho)e\right) + e,\end{aligned}$$

with $\Delta C(\rho) := C_d - C(\rho)$ and $\Delta H(\rho) := \breve{H}(\rho) - H$. Now, since u and e are independent, $\Delta C(\rho)Gu$ and e are uncorrelated[3]. Furthermore, since H and $\breve{H}(\rho)$ are both monic, $\Delta H(\rho)$ is strictly proper so that $C(\rho)\Delta H(\rho)e$ and e are uncorrelated. Therefore,

$$\begin{aligned}\bar{V}_F(\rho) &:= \bar{E}\varepsilon_F^2(t,\rho)\\ &= \bar{E}\left[\left(\bar{H}(\rho)^{-1}\left(\Delta C(\rho)Gu + C(\rho)\Delta H(\rho)e\right) + e\right)^2\right]\\ &= \bar{E}\left[\left(\bar{H}(\rho)^{-1}\left(\Delta C(\rho)Gu + C(\rho)\Delta H(\rho)e\right)\right)^2\right] + \sigma_e^2,\end{aligned}$$

which implies $\bar{V}_F(\rho) \geq \sigma_e^2$ for all ρ. The minimum of $\bar{V}_F$ is σ_e^2 and, if u is persistently exciting of sufficient order, then $\bar{V}_F(\rho^*) = \sigma_e^2$ if and only if $\Delta C(\rho^*) = 0$ and $\Delta H(\rho^*) = 0$. We conclude that consistent estimates of C_d are obtained.

Theorem 7.4.1. *Consider the filtered prediction error ε_F and let ρ^* be a minimizing argument of $\bar{V}_F$. Let the following conditions be satisfied:*

- *the spectral density of u, Φ_u, is positive definite for almost all $\omega \in [-\pi, \pi]$,*
- *there exists a ρ^d such that $C(\rho^d) = C^d$ and $\breve{H}(\rho^d) = H$.*

Then $C(\rho^*) = C^d$ and $\breve{H}(\rho^*) = H$.

Remark 7.4.1. *The consequence of employing a tailor-made noise model is that the parametrized controller needs to be incorporated in the noise model. This implies that standard identification software, such as the identification toolbox in* MATLAB*, cannot be readily applied. The generally non-convex identification problem can, however, be solved using standard non-linear least squares solvers, which will be demonstrated in a numerical example in Section 7.6.*

Remark 7.4.2. *In the case that G is non-minimum phase, it is well known that the zeroes outside the closed unit disk should be zeroes of the reference model T_d as well (Bazanella et al., 2012), cf. Proposition 6.4.3, in order to ensure stability of C_d. The consequence for implementation is that $\bar{r}$ is determined by the unstable inverse of T_d. Examples of solutions to this problem, include a mapping of the unstable zeroes of T_d inside the unit circle and filtering ε with an all-pass filter, or simply filtering ε with T_d itself (Bazanella et al., 2012, Section 3.2.1). In addition to the problem of noise, the method presented in this section also provides a solution to this problem. Indeed, the filtering of ε with $\frac{T_d}{1-T_d}$ in* (7.23) *obviates filtering with the unstable inverse of T_d, as shown in the expression of the prediction error in* (7.24)*.*

[3]If u and e are not independent, e.g., if data is collected in a closed-loop configuration, then $\Delta C(\rho)Gu$ and e can still be uncorrelated. In this situation, it has to be ensured that $\Delta C(\rho)G$ contains a delay, which is satisfied if the controller (model) is proper and G contains a delay.

For the single-process case, a tailor-made noise model was considered in (van Heusden et al., 2011) for correlation-based tuning (CbT) of a linearly parametrized controller. To the best of the authors' knowledge, modeling the noise filter to obtain consistent estimates for VRFT is new. This approach also provides a method to deal with noise for distributed VRFT, which will be discussed in the following section.

7.5 Distributed VRFT in dynamic networks: consistent estimation

For simplicity, we will consider the structured model-reference control problem with a decoupled reference model in this section:

$$y_i = T_i \bar{r}_i \quad \text{and } \bar{z}_i = \bar{r}_i - y_i. \tag{7.25}$$

We note however, that the results in this section apply *mutatis mutandis* to a general structured reference model as considered in Section 7.3 for DVRFT without noise modeling. Now, as was done in Section 7.4 for the single process, we can similarly form an inverse model of the network (7.1). From (7.1), we can write

$$\begin{aligned} u_i &= G_i^{-1} y_i - \sum_{j \in \mathcal{N}_i} G_i^{-1} G_{ij} y_j - G_i^{-1} H_i e_i \\ &= C_{ii}^d \bar{z}_i + \sum_{j \in \mathcal{N}_i} C_{ij}^d \bar{o}_{ji}^c + \bar{H}_i^d e_i, \quad i \in \mathcal{V}, \end{aligned} \tag{7.26}$$

where we have used the definition of the ideal controller modules, (7.25), and, in accordance with (6.9), $K_{ji}^d := T_i(1 - T_i)^{-1}$ for $j \in \mathcal{N}_i$ such that

$$\begin{aligned} \bar{o}_{ji}^c &:= K_{ji}^d \bar{z}_j = y_j, \quad j \in \mathcal{N}_i, \\ \bar{H}_i^d &:= -G_i^{-1} H_i. \end{aligned} \tag{7.27}$$

In conjunction with the network dynamics described by (7.1), equation (7.26) describes a virtual network, with the transfer functions describing the ideal controller C_i^d being unknown modules in this network. Figure 7.3 provides an interpretation of this network for $L = 2$, with the real data from the underlying system depicted by the solid lines and the virtual data depicted by dashed lines. The relation between the output data y_i and virtual signals $\bar{z}_i$ (through $\bar{r}_i$) is described by (7.25), but is implicit in Figure 7.3. This relation is made explicit in Figure 7.4, which gives an alternative interpretation[4] of the virtual network and shows how u acts as an external excitation source for the network.

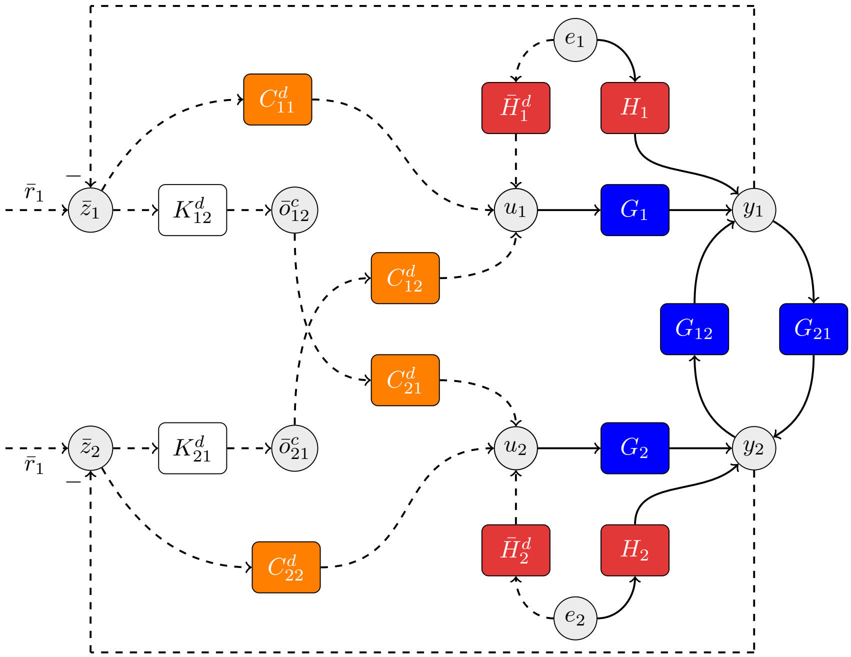


Figure 7.3: Virtual network for $L = 2$ – noisy case.

Consider now the predictor

$$\hat{u}_i(t, \rho_i) = \bar{H}_i(\rho_i)^{-1} \left(C_{ii}(\rho_i)\bar{z}_i + \sum_{j \in \mathcal{N}_i} C_{ij}(\rho_i)\bar{o}_{ji}^c \right) + (1 - \bar{H}_i(\rho_i)^{-1})u_i,$$

which leads to the prediction error

$$\begin{aligned} \varepsilon_i(t, \rho_i) &:= u_i(t) - \hat{u}_i(t, \rho_i) \\ &= \bar{H}_i(\rho_i)^{-1} \big(u_i - C_{ii}(\rho_i)\bar{z}_i - \sum_{j \in \mathcal{N}_i} C_{ij}(\rho_i)\bar{o}_{ji}^c \big). \end{aligned}$$

Then, by filtering the prediction error ε_i with a filter $L_i := C_{ii}^d G_i = T_i(1 - T_i)^{-1}$, the filtered prediction error $\varepsilon_i^F(t, \rho_i) := L_i \varepsilon_i(t, \rho_i)$ is obtained. We can now formulate conditions for the corresponding filtered network identification

[4] While the interpretation is different, the dynamical relations between variables in Figure 7.3 and Figure 7.4 are the same and follow from (7.1), (7.25) and (7.26).

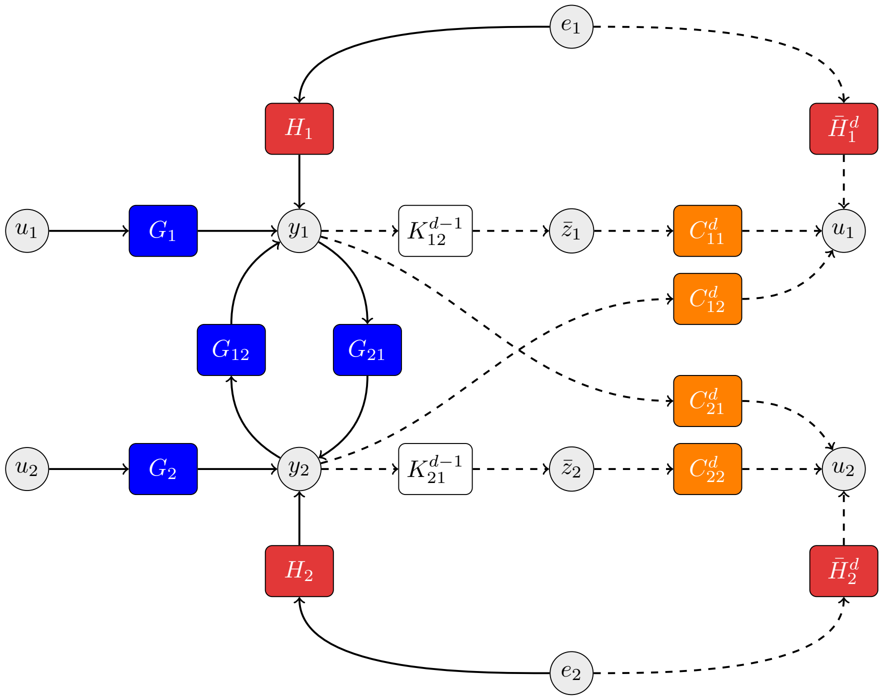


Figure 7.4: Alternative network interpretation for $L = 2$ – noisy case.

problem. The asymptotic parameter estimate for controller $i \in \mathcal{V}$ is given by

$$\rho_i^* = \arg\min_{\rho_i} \underbrace{\bar{E}\varepsilon_i^F(t,\rho_i)^2}_{=:\bar{V}_i^F(\rho_i)}.$$

Theorem 7.5.1. *Consider a tailor-made noise model defined by*

$$\bar{H}_i(\rho_i) := -C_{ii}(\rho_i)\breve{H}_i(\rho_i),$$

with $\breve{H}_i$ a monic transfer function and let the following conditions be satisfied:

- *the spectral density of $\zeta_i := \mathrm{col}(\bar{z}_i, u_i, \mathrm{col}_{j\in\mathcal{N}_i}\, \bar{o}_{ji}^c)$, Φ_{ζ_i}, is positive definite for almost all $\omega \in [-\pi, \pi]$,*
- *there exists a ρ_i^d such that $C_{ii}(\rho_i^d) = C_{ii}^d$, $C_{ij}(\rho_i^d) = C_{ij}^d$ and $\breve{H}_i(\rho_i^d) = H_i$,*
- *G_{ji} contains a delay for every $j \in \mathcal{N}_i$.*

Then it holds that $C_{ii}(\rho_i^*) = C_{ii}^d$, $C_{ij}(\rho_i^*) = C_{ij}^d$, $j \in \mathcal{N}_i$, *and* $\breve{H}_i(\rho_i^*) = H_i$.

Proof. A proof for Theorem 7.5.1 is given in Appendix 7.B, which is preceded by two instrumental lemmas in Appendix 7.A. □

Remark 7.5.1. *The condition that* G_{ji} *contains a delay for every* $j \in \mathcal{N}_i$ *is a sufficient condition for the transfers* $\bar{\mathcal{G}}_{ii}$ *and* $\bar{\mathcal{G}}_{ji}$, $j \in \mathcal{N}_i$, *with* $\bar{\mathcal{G}}_{ji}$ *denoting the* (j,i)*-th element of* $\mathcal{G}G_I$, *to have a delay (by Lemma 7.A.2). The last condition in Theorem 7.5.1 can be replaced with "*$\bar{\mathcal{G}}_{ii}$ *and* $\bar{\mathcal{G}}_{ji}$, $j \in \mathcal{N}_i$, *contain a delay", if the situation that* G_{ji} *contains a delay for every* $j \in \mathcal{N}_i$ *is not at hand.*

The filter L_i is known, since T_i is a known transfer function that describes the reference model $\mathcal{K}_i$. Hence, the distributed model reference control problem (6.7) can be solved using data, via a DVRFT framework by (i) filtering a prediction-error with a known filter and (ii) a tailor-made noise model. Consistent estimates are guaranteed under the conditions in Theorem 7.5.1, but one should note that the identification criteria $\bar{V}_i^F$ are not convex, in general. 'Standard' DVRFT, i.e., without noise modeling, leads to convex identification criteria for linearly-parametrized controllers, but does not provide consistent estimates in the presence of noise.

Remark 7.5.2. *Analogous to the reasoning in Remark 7.4.2, filtering the prediction error with* L_i *additionally provides a solution to the problem of a non-minimum phase reference model* T_i. *The filtering implies that the filtered prediction error can be computed, without deriving* $\bar{r}$ *and* $\bar{z}$ *through an unstable inverse of* T_i *directly from* (7.25).

7.6 Numerical examples

7.6.1 9-systems network (DVRFT)

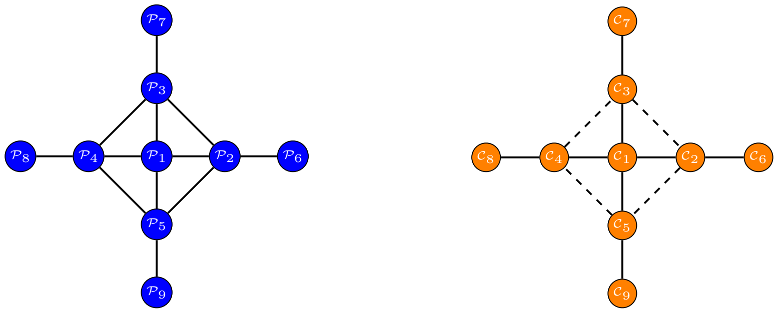
Consider an interconnected system with $L = 9$ subsystems $\mathcal{P}_i$ described by (7.5) and an interconnection structure as depicted in Figure 7.5a. The transfer functions describing the dynamics are of order one and given by

$$G_i = \frac{1}{q - a_i}, \quad G_{ij} = \frac{0.1}{q - a_i}, \quad i = 1, \ldots, 9,$$

with $a_i \in (0, 1)$. It is desired to decouple the interconnected system and to have the same step response for every output channel. Hence the reference model is chosen as $y_i^d = T_i^d(q) r_i$, where

$$T_i(q) = \frac{0.4}{q - 0.6}, \quad i = 1, \ldots, 9.$$

We collect the data $\{u_i(t), y_i(t), t = 1, 2, \ldots, 100\}$ from (7.2) in open-loop, with mutually uncorrelated Gaussian white-noise input signals u_i having a standard deviation of $\sigma_{u_i} = 1$. Hence, we are in the situation of Corollary 7.3.1. Each controller C_i, $i = 1, \ldots, 9$, is parametrized such that Assumption 7.1 holds: $C_{ii}(\rho_i) = \rho_{ia} + \rho_{ib}\frac{1}{q-1}$ and $C_{ij}(\rho_i) = \rho_{ij}$, with parameter vector $\rho_i = \mathrm{col}(\rho_{ia}, \rho_{ib}, \mathrm{col}_{j\in\mathcal{N}_i}\, \rho_{ij})$. Because of the linear parametrization of the sub-controllers, the optimization of $\bar{J}_i^{\mathrm{VR}}$ with predictors (7.13) is a linear least-squares problem. Since there is no noise present in the output, the optimization of $\bar{J}_i^{\mathrm{VR}}$ yields the parameters ρ_i^d and therefore J_{MR} is equal to zero.



(a) Interconnected system (b) Distributed controller

Figure 7.5: Graph for the interconnected system (a) and distributed controller (b) that satisfies Assumption 7.1 (solid and dashed edges), that has reduced number of links (solid edges) and that is decentralized (no edges).

Next, we will analyze a realistic situation where noise affects the system, by considering disturbed outputs $\tilde{y}_i(t) = y_i(t) + v_i(t)$ for the synthesis, with v_i white-noise processes with standard deviations $\sigma_{v_i} = 0.1$ that are mutually uncorrelated and uncorrelated with u_i. The method of generating virtual references and predictors is kept the same, i.e., the noise (filter) is not taken into account in the predictor. The resulting distributed controller is interconnected with the plant and a step reference is applied to each subsystem simultaneously, with an amplitude between zero and one. Figure 7.6 shows the output response of the closed-loop network in red together with the response of the reference model (in black) on the left. We observe only a minor difference between the responses, due to the noise added to the data for identification, as shown in Figure 7.6 on the right.

The distribution of the error between the achieved closed-loop network with the identified distributed controller and the structured reference model resulting from 100 Monte Carlo runs, i.e., from 100 different data sequences, is presented in Figure 7.7. Because $P_i^{\top} C_i^d P_i \in \mathscr{C}_i$ for all i, the error between the achieved

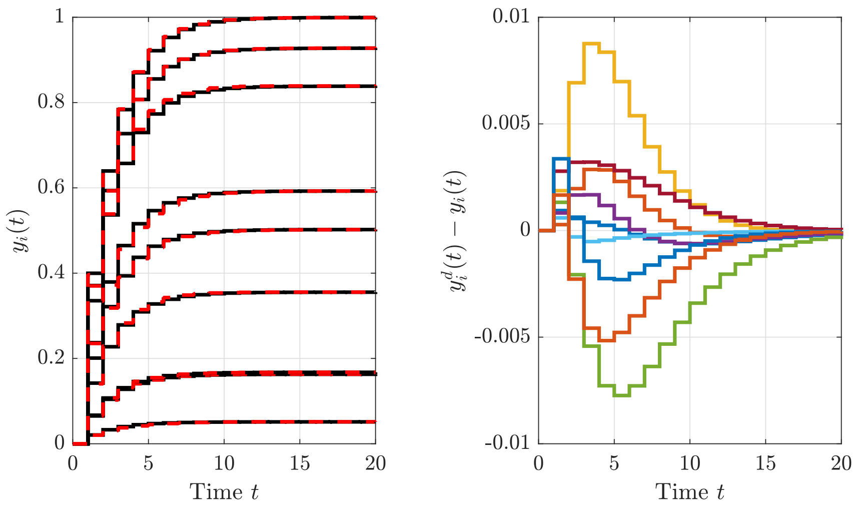


Figure 7.6: Left: Closed-loop response of the network with the data-driven distributed controller (red) and the desired response of the structured reference model (black), for step reference signals applied to each subsystem simultaneously with an amplitude between zero and one. Right: Response errors with respect to the desired outputs.

closed-loop system and the reference model is only due to the noise. The assumption that the controller class is rich enough, i.e., Assumption 7.1, does not always hold in practice. To illustrate such a situation, consider the case where four communication links are not present, represented in Figure 7.5b by the dashed edges. The implication is that in the parametrization $C_{ij}(\rho_i) = 0$ for the corresponding edges $(i,j) \in \mathcal{E}$ and, e.g., $\hat{u}_2(\rho_2) = C_{22}(\rho_2)\bar{z}_2 + C_{21}(\rho_2)\bar{o}_{12}^c + C_{26}(\rho_2)\bar{o}_{62}^c$. Note that links are thus not removed *a posteriori*, but the interconnection structure for the distributed controller is induced by the controller class. As shown by Figure 7.7, there is a significant performance degradation, because the controller class is not 'rich' enough, although the graph for the controller remains connected. We finally consider the data-driven synthesis of a decentralized controller, corresponding to $C_{ij}(\rho_i) = 0$ for all $(i,j) \in \mathcal{E}$. The resulting discrepancy between reference model and closed-loop network is plotted in Figure 7.7 and shows a further decrease in performance.

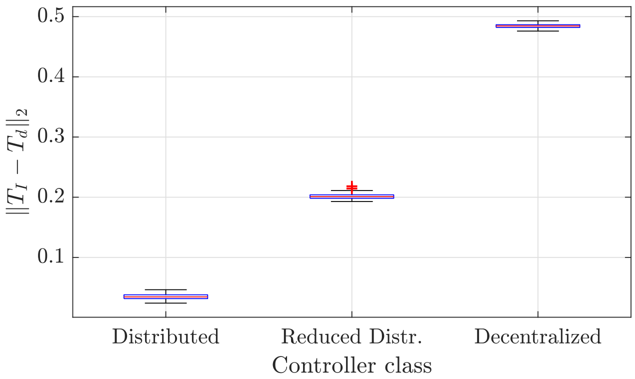


Figure 7.7: Distribution of the achieved performance for various controller classes, where T_I and T_d denote the transfers $r \to y$ and $r \to y_d$, respectively.

7.6.2 2-systems network (DVRFT + noise modeling)

Let us now illustrate the inclusion of noise modeling in DVRFT for obtaining consistent controller estimates, as described in Section 7.5. Consider a two-node network, described by

$$\begin{aligned} y_1 &= G_1 u_1 + G_{12} y_2 + H_1 e_1, \\ y_2 &= G_2 u_2 + G_{21} y_1 + H_2 e_2, \end{aligned}$$

with e_1 and e_2 Gaussian white-noise processes with variance $\sigma_{e_1}^2 = \sigma_{e_2}^2 = \sigma_e^2$, u_1 and u_2 white-noise processes with distribution $\mathcal{U}(0,1)$ and

$$\begin{aligned} G_1 &= \frac{1}{q-0.8}, \quad G_{12} = \frac{0.1}{q-0.8}, \quad H_1 = \frac{q}{q-0.8}, \\ G_2 &= \frac{1}{q-0.6}, \quad G_{21} = \frac{0.1}{q-0.6}, \quad H_2 = \frac{q}{q-0.6}. \end{aligned}$$

In order to synthesize the data-driven controller, we consider that for each $i = 1, 2$, data $(u_i(t), y_i(t))$, $t = 1, \ldots, N$, is collected for $N = 500$ samples. We choose the reference models $T_1 = T_2 = 0.2(q-0.8)^{-1}$ and the parametrization as

$$C_{ii}(\rho_i) = \frac{\rho_{i1} q + \rho_{i2}}{q-1}, \; C_{ij}(\rho_i) = \rho_{i3}, \; \bar{H}_i(\rho_i) = -\frac{C_{ii}(\rho_i) q}{q + \rho_{i4}},$$

such that the second condition in Theorem 7.5.1 is satisfied. Each controller $\mathcal{C}_i(\rho_i)$, $i = 1, 2$, is obtained by minimizing the identification criterion $V_i^F(\rho_i) :=$

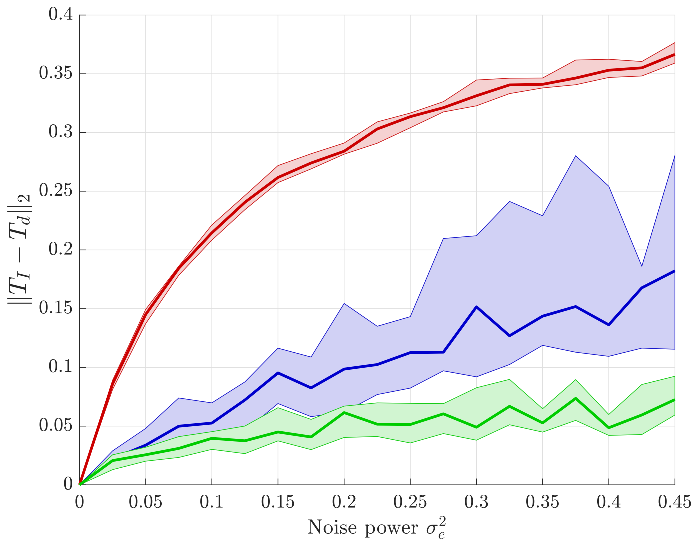


Figure 7.8: Achieved performance versus noise power σ_e^2 for DVRFT (red), DVRFT with IVs (blue) and DVRFT with tailor-made noise modeling and PE filtering (green). The solid lines indicate the median performance and the shaded areas are bounded by the 25$^{\text{th}}$ and 75$^{\text{th}}$ percentiles.

$\sum_{t=1}^{N} \varepsilon_i^F(t, \rho_i)^2$, using the `lsqnonlin.m` function in MATLAB, with initial parameters $\rho_i^{\text{init}} = 0.1\rho_i^d$ relatively 'far' from ρ_i^d.

To illustrate the obtained closed-loop performance with respect to the influence of the disturbances, a Monte-Carlo simulation over 25 experiments is performed for each noise power level $\sigma_e^2 \in \{0, 0.05, \ldots, 0.45\}$. The obtained closed-loop transfer T_I from $r \to y$ results in a performance for DVRFT with the tailor-made noise model and filtered prediction error as depicted in Figure 7.8, in green. For comparison, we compute two distributed controllers from the same experiments via (i) DVRFT without noise modellling and (ii) DVRFT with IVs (obtained by performing an additional experiment for each estimation), as described in Appendix 7.C. As shown in Figure 7.8 in red, the results confirm the expectation that the noise-induced bias for DVRFT degrades the closed-loop performance considerably. The use of IVs in combination with DVRFT provides consistent estimates, but leads to an increased variance for the estimates. This can be observed for the corresponding closed-loop performance for DVRFT with IVs as well, as illustrated in blue in Figure 7.8. By Theorem 7.5.1, DVRFT with a tailor-made noise model and prediction-error filtering also provides consistent estimates. However, due to a decrease in estimator variance, the method described in this paper outperforms DVRFT with IVs considerably for higher noise levels.

7.7 Conclusions

In this chapter we have considered virtual reference feedback tuning in dynamics networks. We have first presented a method for the direct data-driven synthesis of distributed controllers in the model-reference framework of Chapter 6, DVRFT in short. In the case that no noise affects the interconnected system, a distributed controller that solves the model-reference control problem can be identified through network identification in a virtual network with DVRFT. Both the synthesis (identification) and implementation of the constructed data-driven distributed controllers can be distributed, which improves scalability compared to the standard VRFT method.

In the case that noise affects the system, standard VRFT and also DVRFT yield biased controller estimates. We have shown for the single-process case that by including the direct modeling of the auxiliary noise filter, it cannot be concluded that consistent estimates are obtained. However, a filtered prediction-error identification problem can be formulated for which consistent estimates are obtained when a tailor-made noise model is used. For DVRFT applied to dynamic networks, a similar approach is obtained for estimating a local controller that is part of a distributed controller. Sufficient conditions have been given for obtaining consistent estimates in the considered framework, thereby solving the distributed model-reference control problem in the presence of noise. Through an example network consisting of two subsystems, we have shown that the developed method provides a substantial closed-loop performance improvement for increasing levels of noise power.

The main advantage of modeling the noise in (D)VRFT with a tailor-made noise model and prediction-error filtering compared to using an IV method, is that the variance can be reduced significantly, although consistent estimates can be obtained through both approaches. A disadvantage is that the identification problem with a tailor-made noise model becomes more involved, in the sense that standard identification software cannot be used and that the problem is not convex, in general.

Appendix

7.A Instrumental lemmas

In this appendix, we provide two lemmas that will be instrumental in the proof for Theorem 7.5.1.

Lemma 7.A.1. *$\mathcal{G} = I + \bar{\mathcal{G}}$, where $\bar{\mathcal{G}} := \mathcal{G} G_I$.*

Proof. The proof follows directly by the identity

$$I = \mathcal{G}^{-1}\mathcal{G} = (I - G_I)^{-1} - (I - G_I)^{-1} G_I = \mathcal{G} - \mathcal{G} G_I.$$

□

Lemma 7.A.2. *If G_{ji} is strictly proper for each $j \in \mathcal{N}_i$, then*

- *$\bar{\mathcal{G}}_{ii}$ is strictly proper,*
- *$\bar{\mathcal{G}}_{ji}$ is strictly proper for $j \in \mathcal{N}_i$.*

Proof. Assume that G_{ji} is strictly proper for $j \in \mathcal{N}_i$. Then

$$\bar{\mathcal{G}}_{ii} = \sum_{j \in \mathcal{V} \setminus \{i\}} \mathcal{G}_{ij} G_{ji} = \sum_{j \in \mathcal{N}_i} \mathcal{G}_{ij} G_{ji}$$

is strictly proper, because the summands are strictly proper. Pick an arbitrary $j \in \mathcal{N}_i$. Then

$$\bar{\mathcal{G}}_{ji} = \sum_{k \in \mathcal{V} \setminus \{i\}} \mathcal{G}_{jk} G_{ki} = \sum_{k \in \mathcal{N}_i} \mathcal{G}_{jk} G_{ki}$$

is strictly proper, because the summands are strictly proper. This concludes the proof. □

7.B Proof of Theorem 7.5.1

Proof. To prove Theorem 7.5.1, we start by writing the prediction error as

$$\varepsilon_i(t,\rho_i) = \bar{H}_i(\rho_i)^{-1}\big(\Delta C_{ii}(\rho_i)\bar{z}_i + \sum_{j\in\mathcal{N}_i}\Delta C_{ij}(\rho_i)\bar{o}^c_{ji} + \bar{H}^d_i e_i\big),$$

using (7.26), where $\Delta C(\rho_i) := C^d_{ii} - C_{ii}(\rho_i)$, $\Delta C_{ij}(\rho_i) := C^d_{ij} - C_{ij}(\rho_i)$, $j\in\mathcal{N}_i$. The virtual signals are related to y_i and y_j by (7.25) and (7.27), leading to

$$\varepsilon_i(t,\rho_i) = \bar{H}_i(\rho_i)^{-1}\big(\Delta C_{ii}(\rho_i)\frac{1-T_i}{T_i}y_i + \sum_{j\in\mathcal{N}_i}\Delta C_{ij}(\rho_i)y_j + \bar{H}^d_i e_i\big).$$

Now consider the filtered prediction error ε^F_i, with filter $L_i = C^d_{ii}G_i$. By definition, the filter is

$$L_i = C^d_{ii}G_i = \frac{T_i}{1-T_i}.$$

Hence, we have that

$$\begin{aligned}
\varepsilon^F_i(t,\rho_i) &= L_i\varepsilon_i(t,\rho_i)\\
&= \bar{H}_i(\rho_i)^{-1}\big(\Delta C_{ii}(\rho_i)y_i + \frac{T_i}{1-T_i}\sum_{j\in\mathcal{N}_i}\Delta C_{ij}(\rho_i)y_j\\
&\qquad + C^d_{ii}G_i\bar{H}^d_i e_i\big)\\
&= \bar{H}_i(\rho_i)^{-1}\big(\Delta C_{ii}(\rho_i)y_i + \sum_{j\in\mathcal{N}_i}\Delta C_{ij}(\rho_i)K^d_{ij}y_j - C^d_{ii}H_i e_i\big).
\end{aligned}$$

We proceed by writing the 'node' variables y_j in terms of 'external' variables. By (7.3) we have that

$$y = \mathcal{G}(Gu + He) = \mathcal{G}(\bar{u} + v),$$

where $\bar{u} := Gu$ and $v = He$. Hence, by Lemma 7.A.1, it follows that

$$\mathcal{G}H = (I - G_I)^{-1}H = H + \mathcal{G}G_I H.$$

Therefore, we can write the node variables in y as

$$y = \mathcal{G}\bar{u} + \bar{\mathcal{G}}v + He, \tag{7.28}$$

where $\bar{\mathcal{G}} = \mathcal{G}G_I$, or, equivalently,

$$y_i = H_i e_i + \sum_{j\in\mathcal{V}}\mathcal{G}_{ij}\bar{u}_j + \bar{\mathcal{G}}_{ij}v_j, \quad i\in\mathcal{V}.$$

It follows that

$$\varepsilon_i^F(t,\rho_i) = \bar{H}_i(\rho_i)^{-1}\left[x_i(\rho_i) - C_{ii}(\rho_i)H_i e_i\right],$$

where

$$\begin{aligned} x_i(\rho_i) := {} & \Delta C_{ii}(\rho_i)\sum_{j\in\mathcal{V}}\mathcal{G}_{ij}\bar{u}_j + \sum_{j\in\mathcal{N}_i} K_{ij}^d \Delta C_{ij}(\rho_i)\sum_{k\in\mathcal{V}}\mathcal{G}_{jk}\bar{u}_k \\ & + \Delta C_{ii}(\rho_i)\sum_{j\in\mathcal{V}}\bar{\mathcal{G}}_{ij}v_j + \sum_{j\in\mathcal{N}_i} K_{ij}^d \Delta C_{ij}(\rho_i)\left(v_j + \sum_{k\in\mathcal{V}}\bar{\mathcal{G}}_{jk}v_k\right). \end{aligned}$$

Now, considering the tailor-made noise model $\bar{H}_i(\rho_i) = -C_{ii}(\rho_i)\breve{H}_i(\rho_i)$, we obtain the filtered prediction error

$$\begin{aligned} \varepsilon_i^F(t,\rho_i) &= \bar{H}_i(\rho_i)^{-1}\Big[x_i + C_{ii}(\rho_i)\breve{H}_i(\rho_i)e_i - C_{ii}(\rho_i)H_i e_i\Big] \\ &\quad + e_i \\ &= \bar{H}_i^{-1}\left[x_i(\rho_i) + C_{ii}(\rho_i)\Delta H_i(\rho_i)e_i\right] + e_i, \end{aligned}$$

where $\Delta H_i(\rho_i) := \breve{H}_i(\rho_i) - H_i$.

We now show that the noise e_i is uncorrelated with $x_i(\rho_i)$ and $C_{ii}(\rho_i)\Delta H_i(\rho_i)e_i$:

- Since both $\breve{H}_i(\rho_i)$ and H_i are monic, $\Delta H_i(\rho_i)$ is strictly proper. Hence, $C_{ii}(\rho_i)\Delta H_i(\rho_i)$ has a delay, because $C_{ii}(\rho_i)$ is proper, which implies that $C_{ii}(\rho_i)\Delta H_i(\rho_i)e_i$ is uncorrelated with e_i;

- $\Delta C_{ii}(\rho_i)\sum_{j\in\mathcal{V}}\mathcal{G}_{ij}\bar{u}_j$ is uncorrelated with e_i, since it is a filtered linear combination of u_j, $j\in\mathcal{V}$, which are uncorrelated with e_i by assumption;

- $\sum_{j\in\mathcal{N}_i} K_{ij}^d\Delta C_{ij}(\rho_i)\sum_{k\in\mathcal{V}}\mathcal{G}_{jk}\bar{u}_k$ is uncorrelated with e_i, since it is a filtered linear combination of u_j, $j\in\mathcal{V}$, which are uncorrelated with e_i by assumption;

- $\Delta C_{ij}(\rho_i)\sum_{j\in\mathcal{V}}\bar{\mathcal{G}}_{ij}v_j$ is uncorrelated with e_i, because (i) $\bar{\mathcal{G}}_{ii}$ is strictly proper by Lemma 7.A.2 and (ii) e_j, $j\in\mathcal{V}\setminus\{i\}$ is uncorrelated with e_i by assumption;

- $\sum_{j\in\mathcal{N}_i} K_{ij}^d\Delta C_{ij}(\rho_i)\left(v_j + \sum_{k\in\mathcal{V}}\bar{\mathcal{G}}_{jk}v_k\right)$ is uncorrelated with e_i, because (i) $\bar{\mathcal{G}}_{ji}$ is strictly proper for $j\in\mathcal{N}_i$ by Lemma 7.A.2 and (ii) e_j, $j\in\mathcal{V}\setminus\{i\}$ is uncorrelated with e_i by assumption.

Hence,

$$
\begin{aligned}
\bar{V}_i^F(\rho_i) &= \bar{E}\varepsilon_i^F(t,\rho_i)^2 \\
&= \bar{E}\left[\left(\bar{H}_i^{-1}\left[x_i(\rho_i) + C_{ii}(\rho_i)\Delta H_i(\rho_i)e_i\right] + e_i\right)^2\right] \\
&= \bar{E}\left[\left(\bar{H}_i^{-1}\left[x_i(\rho_i) + C_{ii}(\rho_i)\Delta H_i(\rho_i)e_i\right]\right)^2\right] + \sigma_{e_i}^2.
\end{aligned}
\tag{7.29}
$$

But then the minimum of $\bar{V}_i^F(\rho_i)$ must be $\sigma_{e_i}^2$ and a minimizing argument is ρ_i^d by the second condition.

Next, we will show that the minimizing argument is unique. A minimizing argument ρ_i^* must satisfy $V_i^F(\rho_i^*) = \sigma_{e_i}^2$, which, by (7.29), is equivalent with

$$
\begin{aligned}
0 &= \bar{E}\left[\bar{H}_i(\rho_i^*)^{-1}\left[x_i(\rho_i^*) + C_{ii}(\rho_i^*)\Delta H_i(\rho_i^*)e_i\right]\right]^2 \\
&= \bar{E}\left[\Delta x_i(\rho_i^*)\underbrace{\begin{bmatrix} \frac{T_i}{H_i(1-T_i)} & -\frac{G_i}{H_i} & -\frac{1}{H_i}G_{i\mathcal{N}} \\ 0 & G_i & G_{i\mathcal{N}} \\ 0 & 0 & K_{i\mathcal{N}}^d \end{bmatrix}}_{=:\Gamma_i}\begin{bmatrix} \bar{z}_i \\ u_i \\ \bar{o}_{\mathcal{N}i}^c \end{bmatrix}\right]^2,
\end{aligned}
$$

with $K_{i\mathcal{N}}^d := \operatorname{diag}_{j\in\mathcal{N}_i} K_{ij}^d$ and

$$
\Delta x_i(\rho_i^*) = \frac{1}{\bar{H}_i(\rho_i^*)}\begin{bmatrix}\Delta H_i C_{ii} & \Delta C_{ii} & \operatorname{row}_{j\in\mathcal{N}_i}\Delta C_{ij}\end{bmatrix}(\rho_i^*).
$$

Hence, by Parseval's theorem,

$$
\frac{1}{2\pi}\int_{-\pi}^{\pi}\Delta x_i(e^{i\omega},\rho_i^*)^\top\Gamma_i\Phi_{\zeta_i}(\omega)\Gamma_i^*\Delta x_i(e^{-i\omega},\rho_i^*)\,\mathrm{d}\omega = 0.
$$

Now, $\Gamma_i(e^{i\omega})$ has full rank for almost all ω and, by the first condition, $\Phi_{\zeta_i}(\omega)$ is positive definite for all ω. Hence, $[\Delta H_i C_{ii}\ \Delta C_{ii}\ \operatorname{row}_{j\in\mathcal{N}_i}\Delta C_{ij}](\rho_i^*)$ is equal to the zero row for almost all ω, which implies $C_{ii}(\rho_i^*) = C_{ii}^d$, $C_{ij}(\rho_i^*) = C_{ij}^d$, $j\in\mathcal{N}_i$, and $\breve{H}_i(\rho_i^*) = H_i$. This concludes the proof. □

7.C IV approach to DVRFT

Let the parametrization of the controllers in Section 7.6.2 be linear such that

$$
C_{ii}(\rho_i) = \operatorname{col}(\rho_{i1},\rho_{i2})^\top\operatorname{col}(1,\frac{1}{q-1}) \quad \text{and} \quad C_{ij}(\rho_i) = \rho_{i3}.
$$

This leads to a regression vector $\varphi_i(t) = \text{col}(\bar{C}_{ii}\bar{z}_i(t), K^d_{ji}\bar{z}_j(t))$, $j \in \mathcal{N}_i$, with $\bar{C}_{ii} := \text{col}(1, \frac{1}{q-1})$. The estimate of the controller parameter for DVRFT (without the use of an instrumental variable approach) is obtained as

$$\hat{\rho}_i := [\sum_{t=1}^{N} \varphi_i(t)\varphi_i^\top(t)]^{-1} \sum_{t=1}^{N} \varphi_i(t)u_i(t).$$

For DVRFT with an IV approach, an additional experiment is performed, where the same inputs (u_1, u_2) are applied to the network and the corresponding output data (y_1^p, y_2^p) is collected. The virtual references and tracking errors corresponding to the additional experiment are defined as

$$\bar{r}_i^p := T_i^{-1} y_i^p, \quad \bar{z}_i^p := \bar{r}_i^p - y_i^p.$$

The instrumental variable is chosen as $\zeta_i(t) = \text{col}(\bar{C}_{ii}\bar{z}_i^p(t), K^d_{ji}\bar{z}_j^p(t))$, such that it is correlated to $\varphi_i(t)$, but independent of $e_i(t)$ (the noise signal corresponding to the first experiment with the data (u, y)). The controller parameters are then obtained as (Bazanella et al., 2012)

$$\hat{\rho}_i^{\text{IV}} := [\sum_{t=1}^{N} \zeta_i(t)\varphi_i^\top(t)]^{-1} \sum_{t=1}^{N} \zeta_i(t)u_i(t).$$

Chapter 8

Distributed controller identification for data-driven model-reference control

The distributed model-reference control problem introduced in Chapter 6 can be solved directly on the basis of data via DVRFT, as shown in Chapter 7. DVRFT yields consistent estimates of the ideal distributed controller introduced in Chapter 6, provided that no noise signals affect the interconnected system. If process noise signals are present, biased controller estimates are obtained; a problem that can be solved by the use of an instrumental variable method or by using a tailor-made noise model with prediction-error filtering. In this chapter, we also consider the problem of estimating the ideal distributed controller for an interconnected system subject to process noise. We develop a method to solve this problem with network identification methods, by constructing an augmented dynamic network. The resulting method extends a state-of-the-art data-driven control method, called optimal controller identification (OCI), to distributed controller design. The OCI method uses the prediction-error identification method for obtaining controller estimates, which naturally extends to prediction-error identification in dynamic networks for its extension to distributed controller design. Therefore, the method provides a practical alternative to DVRFT, because the distributed model-reference control problem is solved by 'standard' network identification methods. The method has an advantage with respect to DVRFT

This chapter is based on the publication: T. R. V. Steentjes, M. Lazar, and P. M. J. Van den Hof. Controller identification for data-driven model-reference distributed control. In *Proc. 2021 European Control Conference (ECC)*, pages 2358–2363, Rotterdam, The Netherlands, 2021b in the presence of noise, in the sense that no tailor-made noise model has to be included or instrumental variable methods are employed in the identification. We show how the ideal distributed controller can be identified from the augmented network by both direct and indirect methods for network identification.

8.1 Introduction

The distributed model-reference control problem considers the problem of determining a distributed controller for an interconnected system, such that the closed-loop dynamics match a given structured reference model; a reference model with dynamics that are desired for the to-be-controlled interconnected system. An explicit solution to the distributed model-reference control problem was determined and analyzed in Chapter 6. This solution, called an ideal distributed controller, depends explicitly on the subsystem dynamics of the to-be-controlled interconnected system and the subsystem dynamics of the structured reference model. Such an ideal distributed controller is therefore a model-based solution to the distributed model-reference control problem, analogous to the (SISO or MIMO) ideal controller for the 'standard' model-reference control problem, cf. (Campi et al., 2002), (Bazanella et al., 2012), (Huff et al., 2019).

When a model is not known, the distributed model-reference control problem can be solved using data collected from the interconnected system. Distributed virtual reference feedback tuning (DVRFT), introduced in Chapter 7, is a method that solves the distributed model-reference control problem from data, through the identification of an ideal distributed controller in a virtual reference network. Two important cases for DVRFT regarding assumptions on the network are: (i) the network is noiseless, i.e., no unmeasured exogenous signals are present, and (ii) process noise enters the network, i.e., unmeasured exogenous signals are present. In the first case, consistent estimates of the ideal distributed controller can be obtained through DVRFT. In the second case, consistent estimates are only obtained through DVRFT if an instrumental variable (IV) method is employed or a tailor-made noise model is estimated consistently.

When not considered properly, noise can cause a significant performance degradation of the closed-loop network with the resulting controller, due to inconsistent estimates, cf. Chapter 7. Even for scalar systems, VRFT inherently introduces a bias in the controller estimates when disturbances affect the system (Bazanella et al., 2012), leading to a degraded closed-loop performance. For both VRFT and DVRFT, this problem can be solved by using an IV approach in case the controller model is linear with respect to the parameters. Depending on the choice of IV, the introduction of IVs can require additional experiments on the system (Bazanella et al., 2012) and increase the parameter variance with a negative effect on the control performance. For a controller model that is not necessarily linear in the parameters, a method to still obtain consistent estimates is to model the noise in VRFT. This requires to solve a non-standard identification problem, however, where the noise model depends on the controller model (tailor-made noise model), as analyzed in Chapter 7.

An alternative approach to direct data-driven controller design is optimal controller identification (OCI) (Campestrini et al., 2017). This method relies on the algebraic relation between the (unknown) plant, ideal controller and reference model, leading to an identification setup in which the inverse of the ideal controller is to be identified. A feature of OCI is that the estimation of the controller parameters is embedded in a standard prediction-error (PE) identification problem, and consistency therefore follows from a 'standard' PE consistency analysis (Campestrini et al., 2017). Furthermore, in the case of approximate modeling, a flexible parametrization of the controller allows for shaping the bias and variance to improve performance.

In this chapter, we solve the distributed model reference control problem in the case that the interconnected system is subject to unmeasured exogenous inputs, by extending the OCI method for distributed controller synthesis. We show how the interconnected system can be transformed to a network with dynamics of the ideal distributed controller introduced in Chapter 6. By using the direct method for identification in dynamic networks (Van den Hof et al., 2013), we provide sufficient conditions for consistent estimation of the distributed controller, solving the distributed model-reference control problem directly from data. Alternatively, we show how indirect identification can be applied, obviating the necessity of modeling the noise for consistency. The local nature of the identification problems imply that a decentralized computation of the distributed controller is possible.

8.2 Preliminaries

We consider again the network dynamics from Chapter 7:

$$y_i(t) = G_i(q)u_i(t) + \sum_{j\in\mathcal{N}_i} G_{ij}(q)y_j(t) + H_i(q)e_i(t), \quad i \in \mathcal{V}, \tag{8.1}$$

with $\mathcal{V} := \{1, 2, \ldots, L\}$, compactly written as $y = G_I y + Gu + He$, describing an interconnected system with subsystems $\mathcal{P}_i$ described by

$$\mathcal{P}_i : \begin{cases} y_i &= G_i u_i + \sum_{j\in\mathcal{N}_i} G_{ij} y_j + H_i e_i, \\ z_i &= r_i - y_i. \end{cases} \tag{8.2}$$

The subsystems are interconnected over an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, with vertex set $\mathcal{V}$ and edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. Two plants $\mathcal{P}_i$ and $\mathcal{P}_j$ are interconnected if $(i, j) \in \mathcal{E}$.

Recall the structured reference model from Chapter 7, composed of L subsystems, described by

$$\mathcal{K}_i : \begin{cases} y_i^d &= T_i(q)r_i + \sum_{j\in\mathcal{N}_i} Q_{ij}(q)k_{ij}, \\ p_{ij} &= P_{ij}(q)y_i^d, \quad j \in \mathcal{N}_i. \end{cases} \tag{8.3}$$

For $(i,j) \in \mathcal{E}$, subsystems $\mathcal{K}_i$ and $\mathcal{K}_j$ are interconnected by $k_{ij} = p_{ji}$ and $k_{ji} = p_{ij}$. A distributed controller that solves the distributed model-reference control problem (7.9) is

$$\mathcal{C}_i^d : \begin{bmatrix} u_i \\ o_i^c \\ p_i^c \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{T_i}{G_i(1-T_i)} & -\frac{1}{G_i}G_{iI} & \frac{1}{G_i(1-T_i)}Q_i \\ \frac{T_i}{1-T_i}\mathbf{1} & 0 & \frac{1}{1-T_i}\mathbf{1}Q_i \\ \frac{T_i}{1-T_i}P_i & 0 & \frac{1}{1-T_i}P_iQ_i \end{bmatrix}}_{=:C_i^d(q)} \begin{bmatrix} z_i \\ s_i^c \\ k_i^c \end{bmatrix}, \quad i \in \mathcal{V}, \tag{8.4}$$

with the interconnection equations $s_{ij}^c = o_{ji}^c$, $k_{ij}^c = p_{ji}^c$ and $s_{ji}^c = o_{ij}^c$, $k_{ji}^c = p_{ij}^c$. We recall that, by Theorem 6.3.1, the plants $\{P_i,\, i \in \mathcal{V}\}$ in closed loop with the controllers $\{\mathcal{C}_i^d,\, i \in \mathcal{V}\}$ yield a closed-loop network behavior that is equivalent with that of the structured reference model $\{\mathcal{K}_i,\, i \in \mathcal{V}\}$ in the sense that $y_i = y_i^d$ for all $i \in \mathcal{V}$.

8.3 Problem formulation

The problem that we consider in this chapter is to determine the ideal distributed controller described by (8.4) in the case that the network described by (8.1) is unknown, i.e., in the case that the transfer functions G_i, G_{ij} and H_i, $(i,j) \in \mathcal{E}$, are unknown. The local controller modules $\mathcal{C}_i^d$ contain known modules, depending solely on the reference model dynamics $\mathcal{K}_i$ and unknown modules describing the top row in (8.4):

$$u_i = C_{ii}^d(q)z_i + \sum_{j\in\mathcal{N}_i} C_{ij}^d(q)s_{ij}^c + \sum_{j\in\mathcal{N}_i} C_{ij}^{Td}(q)k_{ij}^c, \tag{8.5}$$

where $C_{ii}^d := (G_i(1-T_i))^{-1}T_i$ and $C_{ij}^d := -G_i^{-1}G_{ij}$, $C_{ij}^{Td} := (G_i(1-T_i))^{-1}Q_{ij}$, $(i,j) \in \mathcal{E}$.

Given data collected from the network (8.1), the transfer functions C_{ii}^d, C_{ij}^d and C_{ij}^{Td} can be determined through DVRFT, in the case that no noise is present, i.e., $e_i = 0$ for all $i \in \mathcal{V}$. The application of VRFT for systems with noise leads to biased estimates, both for the single-process case (Bazanella et al., 2012) and the network case, cf. Chapter 7, leading to a degraded closed-loop performance. The bias of distributed controller estimates with VRFT can be characterized, as illustrated in the following example.

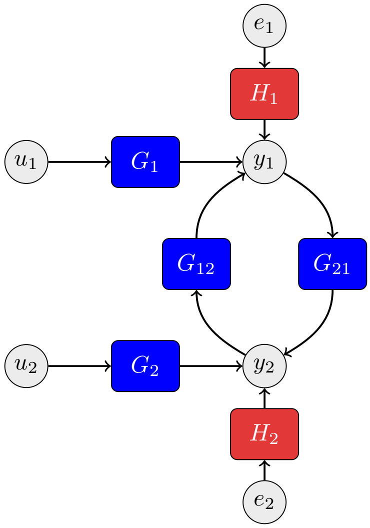


Figure 8.1: Two output coupled process with noise represented by a dynamic network.

Example 8.1. *Consider a network consisting of two subsystems, as depicted in Figure 8.1, described by*

$$y_1 = G_1 u_1 + G_{12} y_2 + H_1 e_1, \tag{8.6}$$
$$y_2 = G_2 u_2 + G_{21} y_1 + H_2 e_2. \tag{8.7}$$

We choose the reference model to be decoupled:

$$y_1^d = T_1 r_1, \quad y_2^d = T_2 r_2. \tag{8.8}$$

The ideal distributed controller is then described by

$$\begin{bmatrix} u_1 \\ o_1^c \end{bmatrix} = \underbrace{\begin{bmatrix} \dfrac{T_1}{G_1(1-T_1)} & -\dfrac{G_{12}}{G_1} \\ \dfrac{T_1}{1-T_1} & 0 \end{bmatrix}}_{=: \begin{bmatrix} C_{11}^d & C_{12}^d \\ K_{12}^d & 0 \end{bmatrix}} \begin{bmatrix} z_1 \\ s_1^c \end{bmatrix}, \tag{8.9}$$

$$\begin{bmatrix} u_2 \\ o_2^c \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{T_2}{G_2(1-T_2)} & -\frac{G_{21}}{G_2} \\ \frac{T_2}{1-T_2} & 0 \end{bmatrix}}_{=: \begin{bmatrix} C_{22}^d & C_{21}^d \\ K_{21}^d & 0 \end{bmatrix}} \begin{bmatrix} z_2 \\ s_2^c \end{bmatrix}, \tag{8.10}$$

with the interconnections $s_1^c = o_2^c$ *and* $s_2^c = o_1^c$. *In the distributed VRFT (DVRFT) approach to distributed controller synthesis described in Chapter 7, virtual reference signals and virtual tracking errors* $\bar{r}_i$ *and* $\bar{z}_i$ *are computed from data, satisfying*

$$y_i = T_i \bar{r}_i \quad \text{and} \quad \bar{z}_i := \bar{r}_i - y_i, \quad i = 1, 2. \tag{8.11}$$

Using the definitions for the virtual signals and ideal controller, the processes (8.6) *and* (8.7) *are rewritten as*

$$u_1 = C_{11}^d \bar{z}_1 + C_{12}^d \bar{o}_2^c - G_1^{-1} H_1 e_1, \tag{8.12}$$

$$u_2 = C_{22}^d \bar{z}_2 + C_{21}^d \bar{o}_1^c - G_2^{-1} H_2 e_2, \tag{8.13}$$

where

$$\bar{o}_i^c := \frac{T_i}{1-T_i} \bar{z}_i = y_i, \quad i = 1, 2.$$

Controller module 1, for example, is then obtained by minimizing the identification criterion

$$J_1^{VR}(\rho_1) = \bar{E}[u_1 - \hat{u}_1(\rho_1)]^2 = \bar{E}[u_1 - \rho_1^\top \varphi_1]^2,$$

where the parametrized controller is chosen to be linear in the parameters:

$$\hat{u}_1(\rho_1) = C_{11}(\rho_1)\bar{z}_1 + C_{12}(\rho_1)\bar{o}_{21}^c = \rho_1^\top \varphi_1,$$

with $\rho_1 := \mathrm{col}(\rho_{11}, \rho_{12})$ *and the regression vector* $\varphi_1 := \mathrm{col}(\bar{C}_{11}\bar{z}_1, \bar{C}_{12}\bar{o}_2^c)$. *Any global minimizer* ρ_1^* *of* J_1^{VR} *satisfies the normal equation*

$$\bar{E}[\varphi_1 \varphi_1^\top]\rho_1^* = \bar{E}[\varphi_1 u_1].$$

If $\bar{E}[\varphi_1 \varphi_1^\top]^{-1}$ *exists, then*

$$\rho_1^* = \bar{E}[\varphi_1 \varphi_1^\top]^{-1} \bar{E}[\varphi_1 u_1]. \tag{8.14}$$

Let ρ_i^d be the parameter that corresponds to C_{ii}^d and C_{ij}^d. The difference between ρ_i^d and its estimate for finite data, $\hat{\rho}_i := [\sum_{t=1}^N \varphi_i(t)\varphi_i^\top(t)]^{-1}\sum_{t=1}^N \varphi_i(t)u_i(t)$, is given by

$$\hat{\rho}_i - \rho_i^d = (\rho_i^* - \rho_i^d) + (\hat{\rho}_i - \rho_i^*),$$

where $\rho_i^* - \rho_i^d$ is the bias error *and $\hat{\rho}_i - \rho_i^*$ the* variance error *(Bazanella et al., 2012).*

Lemma 8.3.1. *The bias error of the estimator $\hat{\rho}_i$, $i = 1, 2$ is*

$$\bar{E}[\varphi_i\varphi_i^\top]^{-1}\bar{E}[\varphi_i G_i^{-1} H_i e_i].$$

Proof. Substitution of (8.12) in (8.14) yields

$$\begin{aligned}\rho_1^* &= \bar{E}[\varphi_1\varphi_1^\top]^{-1}\bar{E}[\varphi_1\left(C_{11}^d\bar{z}_1 + C_{12}^d\bar{o}_{21}^c - G_1^{-1}H_1e_1\right)]\\ &= \rho_1^d - \bar{E}[\varphi_1\varphi_1^\top]^{-1}\bar{E}[\varphi_1 G_1^{-1}H_1e_1],\end{aligned} \tag{8.15}$$

where the linearity of the expectation operator was used to obtain the last equality. The bias expression for $i = 2$ can be obtained analogously. □

Remark 8.3.1. *If there is process noise present at process $i \in \{1, 2\}$, i.e. $e_i \neq 0$, this will influence the quality of the controller parameter estimate $\hat{\rho}_i$ in terms of bias by Lemma 8.3.1. The interpretation is that for DVRFT, e_i acts as a confounding variable affecting both the input and output of the identification problem, while e_j, $j \neq i$, act as excitation sources in the network.*

Due to the bias, controller parameter estimates will in general not minimize the cost function (7.9) and hence lead to a reduced closed-loop performance if the process is subject to noise. Two approaches to tackle this problem have been described in Chapter 7: (i) estimating parameters of a tailor-made noise model simultaneously with the controller parameters or (ii) estimating the controller parameters through an instrumental variable approach (applicable if the controllers are linearly parametrized).

The problem considered in this chapter is to solve the data-driven distributed model-reference control problem in the case that process noises e_i are present in the network (8.1), i.e., the problem that is solved in Section 7.5 with DVRFT. The goal in this chapter is to provide an alternative method to solve this problem, which can be applied using 'standard' prediction error identification methods for dynamic network identification. Recall that OCI provides an alternative to VRFT for solving the standard model-reference control problem in Campestrini et al. (2017). We will consider the development of a method that extends OCI to the distributed model-reference control problem for interconnected systems.

8.4 Distributed optimal controller identification

In order to pose an identification problem, we start by rewriting the network dynamics in terms of the ideal distributed controller dynamics. The approach of rewriting the dynamics of a single-input single-output system in terms of an ideal controller for prediction-error identification was introduced in (Campestrini et al., 2017).

8.4.1 Transformed network dynamics

Let us first consider the case where the reference model is decoupled, i.e., for each $i \in \mathcal{V}$,

$$\mathcal{K}_i : \quad y_i^d = T_i(q) r_i.$$

By (8.4), we observe that the transfer functions in (8.5) are

$$C_{ii}^d = \frac{T_i}{G_i(1 - T_i)}, \quad C_{ij}^d = -\frac{G_{ij}}{G_i}, \quad (i,j) \in \mathcal{E}.$$

Hence, we can write the network dynamics (8.1) in terms of the ideal distributed controller and the reference model as

$$G_i = \frac{1}{C_{ii}^d} \frac{T_i}{1 - T_i}$$

and

$$G_{ij} = -C_{ij}^d G_i = -\frac{C_{ij}^d}{C_{ii}^d} \frac{T_i}{1 - T_i}.$$

Models for the network's transfer functions can then be written in terms of the controller parameters as

$$G_i(\rho_i) := \frac{1}{C_{ii}(\rho_i)} \frac{T_i}{1 - T_i} \text{ and } G_{ij}(\rho_i) := -\frac{C_{ij}(\rho_i)}{C_{ii}(\rho_i)} \frac{T_i}{1 - T_i}.$$

We can thus rewrite the network dynamics (8.1) as

$$y_i = \frac{1}{C_{ii}^d} \frac{T_i}{1 - T_i} u_i - \sum_{j \in \mathcal{N}_i} \frac{C_{ij}^d}{C_{ii}^d} \frac{T_i}{1 - T_i} y_j + H_i e_i, \quad i \in \mathcal{V},$$

or

$$y_i = \bar{C}_{ii}^d \bar{u}_i + \sum_{j \in \mathcal{N}_i} \bar{C}_{ij}^d \bar{y}_{ij} + H_i e_i, \quad i \in \mathcal{V}, \tag{8.16}$$

with

$$\bar{C}_{ii}^d := \frac{1}{C_{ii}^d}, \quad \bar{C}_{ij}^d := \frac{C_{ij}^d}{C_{ii}^d},$$

and the signals

$$\bar{u}_i := \frac{T_i}{1-T_i} u_i = \bar{T}_i u_i, \quad i \in \mathcal{V}, \tag{8.17}$$

$$\bar{y}_{ij} := -\frac{T_i}{1-T_i} y_j = -\bar{T}_i y_j, \quad (i,j) \in \mathcal{E}. \tag{8.18}$$

In the case that T_i is proper, $\bar{T}_i$, $i \in \mathcal{V}$, will be proper and the dynamical relations (8.16)-(8.18) can be interpreted as an (augmented) dynamic network:

$$\begin{bmatrix} y \\ \bar{y} \\ \bar{u} \end{bmatrix} = \begin{bmatrix} 0 & \bar{C}_I^d & \bar{C}^d \\ \bar{T}_{\mathcal{N}} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y \\ \bar{y} \\ \bar{u} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \bar{T} \end{bmatrix} u + \begin{bmatrix} H \\ 0 \\ 0 \end{bmatrix} e, \tag{8.19}$$

with the matrices $\bar{C}_I^d := \operatorname{diag}_{i\in\mathcal{V}} \operatorname{row}_{j\in\mathcal{N}_i} \bar{C}_{ij}^d$, $\bar{C}_d := \operatorname{diag}(C_{11}^d, \ldots, \bar{C}_{LL}^d)$, $\bar{T} := \operatorname{diag}(\bar{T}_1, \ldots, \bar{T}_L)$, $\bar{T}_{\mathcal{N}} := \operatorname{col}_i \operatorname{col}_{j\in\mathcal{N}_i} -\bar{T}_i \mathbf{e}_j^\top$, with $\mathbf{e}_i$ the i-th standard basis vector in $\mathbb{R}^L$, and $\bar{T} := \operatorname{diag}(\bar{T}_1, \ldots, \bar{T}_L)$. This augmented network is visualized in Figure 8.2 for $L = 2$.

To write the model, define $\bar{C}_{ii}(\rho_i) := \frac{1}{C_{ii}(\rho_i)}$ and $\bar{C}_{ij}(\rho_i) := \frac{C_{ij}(\rho_i)}{C_{ii}(\rho_i)}$, $(i,j) \in \mathcal{E}$, such that

$$y_i(\theta_i) = \bar{C}_{ii}(\rho_i)\bar{u}_i + \sum_{j\in\mathcal{N}_i} \bar{C}_{ij}(\rho_i)\bar{y}_{ij} + H_i(\theta_i)e_i, \tag{8.20}$$

where $\theta_i = \operatorname{col}(\rho_i, \eta_i)$, with η_i additional parameters for the noise model $H_i(\theta_i)$[1]. Model (8.20) is visualized in Figure 8.3 for $i = 1$, $L = 2$.

8.4.2 Direct method for controller identification

By using the direct method for dynamic network identification (Van den Hof et al., 2013), the estimates are obtained as

$$\hat{\theta}_i = \arg\min_{\theta_i} V_i(\theta_i), \quad V_i(\theta_i) = \frac{1}{N}\sum_{t=1}^{N} \varepsilon_i^2(t, \theta_i), \tag{8.21}$$

with the prediction error defined by

$$\varepsilon_i(t, \theta_i) := y_i(t) - \hat{y}_i(t, \theta_i)$$

[1]The controller and noise model are not required to be parametrized independently.

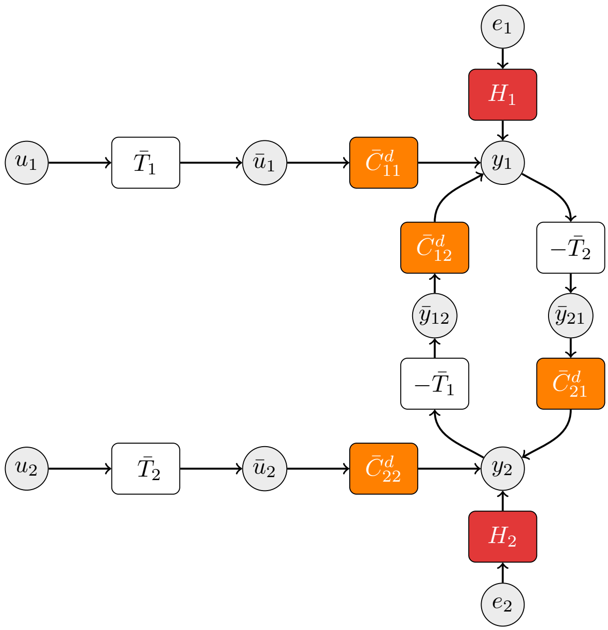


Figure 8.2: The dynamic network represented by modules of the ideal distributed controller and reference model.

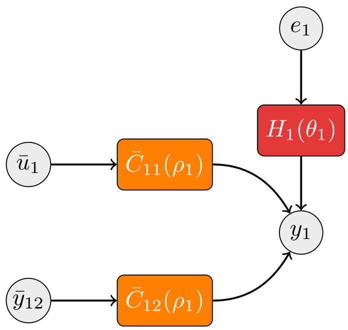


Figure 8.3: Model with the auxiliary controller modules for $i = 1$, $L = 2$.

and the predictor

$$\hat{y}_i(t, \theta_i) := H_i^{-1}(\theta_i) \left(\bar{C}_{ii}(\rho_i)\bar{u}_i + \sum_{j \in \mathcal{N}_i} \bar{C}_{ij}(\rho_i)\bar{y}_{ij} \right) \tag{8.22}$$

$$+ \left(1 - H_i^{-1}(\theta_i)\right) y_i. \tag{8.23}$$

By definition of the auxiliary controller models, the controller estimates are then

$$C_{ii}(\hat{\rho}_i) = \frac{1}{\bar{C}_{ii}(\hat{\rho}_i)}, \quad C_{ij}(\hat{\rho}_i) = \bar{C}_{ij}(\hat{\rho}_i)C_{ii}(\hat{\rho}_i), \quad (i,j) \in \mathcal{E}.$$

Under weak assumptions, cf. Chapter 2, the estimator $\hat{\theta}_i$ converges asymptotically in N (Ljung, 1999):

$$\hat{\theta}_i \to \theta_i^* \quad \text{w.p. 1 as} \quad N \to \infty,$$

where $\theta_i^* = \arg\min_{\theta_i} \bar{V}_i(\theta_i)$ with $\bar{V}_i(\theta_i) := \bar{E}[\varepsilon_i^2(t,\theta_i)]$. The following result provides sufficient conditions for consistent estimation of the controller modules.

Theorem 8.4.1. *Suppose the following conditions hold:*

(1) e_i is uncorrelated to all e_j, $j \in \mathcal{V} \setminus \{i\}$,

(2) e_i is uncorrelated to all u_j, $j \in \mathcal{V}$,

(3) $\mathcal{G}_{ji}$ contains a delay for every $j \in \mathcal{N}_i$, with $\mathcal{G}_{ji} = [(I - G_I)^{-1}]_{ji}$,

(4) the spectral density of $\bar{\xi}_i := \mathrm{col}(y_i, \bar{u}_i, \bar{y}_{ih_1}, \dots, \bar{y}_{ih_L})$, $h_\bullet \in \mathcal{N}_i$, is positive definite for almost all $\omega \in [-\pi, \pi]$,

(5) there exists a $\theta_i^d = (\rho_i^d, \eta_i^d)$ such that $C_i(\rho_i^d) = C_i^d$ and $H_i(\theta_i^d) = H_i$.

Then it holds that $C_{ii}(\theta_i^*) = C_{ii}^d$, $H_i(\theta_i^*) = H_i$ and $C_{ij}(\theta_i^*) = C_{ij}^d$, $j \in \mathcal{N}_i$.

Proof. We will first show that the minimum of the objective function $\bar{V}_i$ is $\sigma_{e_i}^2 = Ee_i^2$. By the definition of the predictor and prediction error, we have

$$\bar{V}_i(\theta_i) = \bar{E}\bigg(H_i(\theta_i)^{-1}\Big(v_i + \sum_{j \in \mathcal{N}_i} \Delta\bar{C}_{ij}(\theta_i)\bar{y}_{ij} + \Delta\bar{C}_{ii}(\theta_i)\bar{u})_i\Big)\bigg)^2.$$

Then, by (8.19) it follows that

$$\bar{V}_i(\theta_i) = \bar{E}\bigg(H_i(\theta_i)^{-1}\Big(v_i - \sum_{j \in \mathcal{N}_i} \Delta\bar{C}_{ij}(\theta_i)\bar{T}_i y_j + \Delta\bar{C}_{ii}(\theta_i)\bar{T}_i u_i\Big)\bigg)^2$$

and by (8.1):

$$\begin{aligned}\bar{V}_i(\theta_i) &= \bar{E}\Bigg(H_i(\theta_i)^{-1}\Bigg(v_i + \Delta\bar{C}_{ii}(\theta_i)\bar{T}_i u_i \\ &\qquad -\sum_{j\in\mathcal{N}_i}\Delta\bar{C}_{ij}(\theta_i)\bar{T}_i\sum_{k\in\mathcal{V}}\mathcal{G}_{jk}(G_k u_k + H_k e_k)\Bigg)\Bigg)^2 \\ &= \bar{E}\Bigg(H_i(\theta_i)^{-1}\Bigg(\Delta H_i(\theta_i)e_i + \Delta\bar{C}_{ii}(\theta_i)\bar{T}_i u_i \\ &\qquad -\sum_{j\in\mathcal{N}_i}\Delta\bar{C}_{ij}(\theta_i)\bar{T}_i\sum_{k\in\mathcal{V}}\mathcal{G}_{jk}(G_k u_k + H_k e_k)\Bigg) + e_i\Bigg)^2.\end{aligned}$$

Since both H_i and $H_i(\theta_i)$ are monic, $\Delta H_i(\theta_i)$ is strictly proper. Hence, $\Delta H_i(\theta_i)e_i$ is uncorrelated with e_i. Also $-\sum_{j\in\mathcal{N}_i}\Delta\bar{C}_{ij}(\theta_i)\bar{T}_i\sum_{k\in\mathcal{V}}\mathcal{G}_{jk}G_k u_k$ is uncorrelated with e_i, since it is a filtered linear combination of u_k, $k\in\mathcal{V}$, which are uncorrelated with e_i by condition (ii). Moreover, $-\sum_{j\in\mathcal{N}_i}\Delta\bar{C}_{ij}(\theta_i)\bar{T}_i\sum_{k\in\mathcal{V}}\mathcal{G}_{jk}H_k e_k$ is uncorrelated with e_i by condition (i) and (v), since $\Delta\bar{C}_{ij}(\theta_i)$ is proper by construction and $\mathcal{G}_{ji}$ is strictly proper for all $j\in\mathcal{N}_i$ by condition (v). Hence,

$$\begin{aligned}\bar{V}_i(\theta_i) &= \bar{E}\Bigg(H_i(\theta_i)^{-1}\Bigg(\Delta H_i(\theta_i)e_i + \Delta\bar{C}_{ii}(\theta_i)\bar{T}_i u_i \\ &\qquad -\sum_{j\in\mathcal{N}_i}\Delta\bar{C}_{ij}(\theta_i)\bar{T}_i\sum_{k\in\mathcal{V}}\mathcal{G}_{jk}(G_k u_k + H_k e_k)\Bigg)\Bigg)^2 + \bar{E}e_i^2 \\ &\geq \sigma_{e_i}^2.\end{aligned}$$

It remains to show that $\bar{V}_i(\theta_i) = \sigma_{e_i}^2 \Rightarrow \theta_i = \theta_i^d$. This follows from conditions (iii) and (iv) and follows *mutatis mutandis* from (Van den Hof et al., 2013, Appendix B). This concludes the proof. $\square$

Remark 8.4.1. *In comparison, both Theorem 7.5.1 and Theorem 8.4.1 provide conditions under which consistent estimates of the ideal controller modules are obtained via, respectively, DVRFT with noise modeling (described in Chapter 7) and the method described in this chapter. The main difference, regarding the identification aspect, is that a tailor-made parametrization for the noise model is utilized in Chapter 7, which is not required in the method described in this chapter, while the parametrization of the controller modules in this chapter can be more involved.*

Remark 8.4.2. *The identification problem for obtaining C_i^d is scalable with respect to the size of the network L. Indeed, each identification criterion $\bar{V}_i$, $i\in\mathcal{V}$,* *can be minimized independently. Furthermore, the number of controller modules that are identified through the minimization of $\bar{V}_i$ is equal to the number of neighbours of node i, i.e., $|\mathcal{N}_i|$, and is therefore independent of the number of nodes L in the network.*

Remark 8.4.3. *Notice that the correlation condition on e_i and u_j, $j \in \mathcal{V}$ (Theorem 8.4.1, condition (ii)) is typically satisfied when data are collected from the network in open loop, i.e., when the interconnected system $(\mathcal{P}_1, \ldots, \mathcal{P}_L)$ is not interconnected with a (preliminary) controller. When the interconnected system is interconnected with a (distributed) controller (such that the input u is a filtered version of z, i.e., $u(t) = C(q)z(t)$), then it is sufficient that the external signals r_j, $j \in \mathcal{V}$, are uncorrelated with e_i and there is a delay in every loop around y_i, cf. Chapter 2. More specifically, the assertion in Theorem 8.4.1 holds true for data collected in closed loop, when conditions (ii) and (iii) are replaced with conditions (ii) and (iii) in Proposition 2.3.1.*

Positive definiteness of the spectrum of the vector of signal $\bar{\xi}_i$, $\Phi_{\bar{\xi}_i}$, is implied by sufficient excitation of the filtered input $\bar{u}_i = \bar{T}_i u_i$ and the signals $\bar{y}_{ij}$, $j \in \mathcal{N}_i$. The condition on $\Phi_{\bar{\xi}_i}$ can be translated to conditions on external signals u_j, e_j, $j \in \mathcal{V}$, and conditions on the augmented network topology, as described in (Van den Hof and Ramaswamy, 2020), cf. Section 2.3.1. The following result is a consequence of Proposition 1 in (Van den Hof and Ramaswamy, 2020).

Lemma 8.4.1. *Consider the vector $\bar{\xi}_i := \mathrm{col}(y_i, \bar{u}_i, \bar{y}_{ih_1}, \ldots, \bar{y}_{ih_L}) \in \mathcal{R}^p$, $h_\bullet \in \mathcal{N}_i$, and let the stacked vector of external signals $\mathrm{col}(u, e)$ have a power spectrum that is positive definite almost everywhere. Then $\Phi_{\bar{\xi}_i}$ is positive definite almost everywhere if there are p vertex-disjoint*[2] *paths from $\mathrm{col}(u, e)$ to $\bar{\xi}_i$.*

Example 8.2. *Consider the augmented network (8.19) for $L = 2$, depicted in Figure 8.2 and consider the power spectrum of $\bar{\xi}_1 = \mathrm{col}(y_1, \bar{u}_1, \bar{y}_{12}) \in \mathbb{R}^3$, $\Phi_{\bar{\xi}_1}$. By Lemma 8.4.1, $\Phi_{\bar{\xi}_1}$ is positive definite almost everywhere if there are three vertex-disjoint paths from $\mathrm{col}(u_1, u_2, e_1, e_2)$ to $\bar{\xi}_1$. Clearly, there are three such vertex disjoint paths: from $e_1 \to y_1$, from $u_1 \to \bar{u}_1$ and from $u_2 \to \bar{y}_{12}$. Hence, we conclude that $\Phi_{\bar{\xi}_1}$ is positive definite almost everywhere. Notice that the same conclusion can be reached by replacing $u_2 \to \bar{y}_{12}$ with $e_2 \to \bar{y}_{12}$ in the reasoning for this example.*

8.4.3 Indirect method for controller identification

In the previous subsection, the estimation of ideal controller modules in the augmented network (8.19) via the direct method for network identification was considered. Alternatively, the controller modules can be identified via an indirect

[2] A set of paths is vertex disjoint if no two of them have one or more vertices in common.

identification method (Van den Hof et al., 2013), (Van den Hof and Ramaswamy, 2021). Compared to the direct method, an indirect method leads to a prediction-error identification problem with different predictor inputs and outputs (Van den Hof and Ramaswamy, 2021) and estimates of C_{ii}^d and C_{ij}^d are obtained through a post-processing step. This different choice of predictor model leads to a different condition for data informativity: the direct method utilizes both input signals in u and noise signals in e for achieving data-informativity (as indicated in Lemma 8.4.1), whereas an indirect method utilizes only signals in u for data-informativity, cf. (Van den Hof and Ramaswamy, 2021). However, an indirect method does not require the inclusion of noise models in the identification, provided that u and e are uncorrelated (Ljung, 1999). Therefore, we will now derive an approach for obtaining consistent controller estimates through the use of an indirect identification method.

The dynamics of the augmented network (8.19) yield a mapping from external signals (u, e) to network signals $(y, \bar{y}, \bar{u})$ as follows:

$$\begin{bmatrix} y \\ \bar{y} \\ \bar{u} \end{bmatrix} = (I - C_T^d)^{-1} \begin{bmatrix} 0 \\ 0 \\ \bar{T} \end{bmatrix} u + (I - C_T^d)^{-1} \begin{bmatrix} H \\ 0 \\ 0 \end{bmatrix} e = \begin{bmatrix} T_y \\ T_{\bar{y}} \\ T_{\bar{u}} \end{bmatrix} u + \bar{H} e$$

with the transfer matrices $(T_y, T_{\bar{y}}, T_{\bar{u}})$ and $\bar{H}$ defined by

$$\begin{bmatrix} T_y \\ T_{\bar{y}} \\ T_{\bar{u}} \end{bmatrix} := (I - C_T^d)^{-1} \begin{bmatrix} 0 \\ 0 \\ \bar{T} \end{bmatrix}, \; \bar{H} := (I - C_T^d)^{-1} \begin{bmatrix} H \\ 0 \\ 0 \end{bmatrix} \text{ and } C_T^d := \begin{bmatrix} 0 & \bar{C}_I^d & \bar{C}^d \\ \bar{T}_{\mathcal{N}} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{8.24}$$

The basic idea of applying the indirect method to the identification of the ideal controller modules, is to first obtain estimates of the transfer matrices $(T_y, T_{\bar{y}}, T_{\bar{u}})$ and subsequently estimate of the controllers from the relation between C_{ii}^d, C_{ij}^d and $(T_y, T_{\bar{y}}, T_{\bar{u}})$. Known modules defined by the structured reference model simplify the procedure, however, as explained next.

From (8.24), we find that

$$\begin{bmatrix} I & -\bar{C}_I^d & -\bar{C}^d \\ -\bar{T}_{\mathcal{N}} & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} T_y \\ T_{\bar{y}} \\ T_{\bar{u}} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \bar{T} \end{bmatrix} \quad \Leftrightarrow \quad \begin{cases} T_y - \bar{C}_I^d T_{\bar{y}} - \bar{C}^d T_{\bar{u}} = 0, \\ T_{\bar{y}} = \bar{T}_{\mathcal{N}} T_y \text{ and } T_{\bar{u}} = \bar{T}. \end{cases} \tag{8.25}$$

Pre-multiplication of the first equation on the right in (8.25) with $C^d = \text{diag}_{i \in \mathcal{V}} C_{ii}^d$, satisfying $C^d \bar{C}^d = I$, yields that (8.25) is equivalent with $T_{\bar{y}} = \bar{T}_{\mathcal{N}} T_y$, $T_{\bar{u}} = \bar{T}$ and

$$C^d T_y = C_I^d T_{\bar{y}} + T_{\bar{u}}, \tag{8.26}$$

where $C_I^d := \mathrm{diag}_{i\in\mathcal{V}}\, \mathrm{row}_{j\in\mathcal{N}_i}\, C_{ij}^d$. We observe the following: given T_y, the controller modules C_{ii}^d, C_{ij}^d, $i \in \mathcal{V}$, $j \in \mathcal{N}_i$, can be obtained by solving the equation (8.26) in C^d and C_I^d, where $T_{\bar{y}} = \bar{T}_{\mathcal{N}} T_y$ and $T_{\bar{u}} = \bar{T}$.

Obtaining a consistent estimate of T_y from data measurements of u and y is a standard MIMO identification problem (for data collected in an open-loop setting), (Gevers et al., 2018), cf. (Ljung, 1999). Suppose such an estimate $\hat{T}_y$ of T_y is available. By (8.26), estimates $\hat{C}$ and $\hat{C}_I$ of respectively C^d and C_I^d, can be obtained solving $\hat{C}\hat{T}_y = \hat{C}_I \bar{T}_{\mathcal{N}} \hat{T}_y + \bar{T}$ in $\hat{C}$ and $\hat{C}_I$, where the interconnection structure is taken into account, i.e., for all $(i,j) \in [1,L]^2$, it holds that $[\hat{C}]_{ij} = 0$ if and only if $[C^d]_{ij} = 0$ and for all $(i,j) \in [1,L] \times [1,L_{\mathcal{N}}]$, $[\hat{C}_I]_{ij} = 0$ if and only if $[C_I^d]_{ij} = 0$, where $L_{\mathcal{N}} := \sum_{i\in\mathcal{V}} |\mathcal{N}_i|$ is the number of columns in C_I^d.

Corollary 8.4.1. *Consider that an experiment is performed according to the following conditions:*

(1) all input signals $(u_i)_{i\in\mathcal{V}}$ are chosen such that u is persistently exciting of sufficiently high order and uncorrelated with e,

(2) all output signals $(y_i)_{i\in\mathcal{V}}$ are measured.

Then a consistent estimate $\hat{T}_y$ of T_y can be obtained through standard open-loop identification with a full order model. Consequently, given a consistent estimate $\hat{T}_y$, consistent estimates of the controller matrices C^d and C_I^d are given by $\hat{C}$ and $\hat{C}_I$, obtained from the set of linear equations

$$\hat{C}\hat{T}_y = \hat{C}_I \bar{T}_{\mathcal{N}} \hat{T}_y + \bar{T} \tag{8.27}$$

and the following constraints for incorporation of the interconnection structure of the ideal distributed controller: for all $(i,j) \in [1,L]^2$, it holds that $[\hat{C}]_{ij} = 0$ if and only if $[C^d]_{ij} = 0$ and for all $(i,j) \in [1,L] \times [1,L_{\mathcal{N}}]$, $[\hat{C}_I]_{ij} = 0$ if and only if $[C_I^d]_{ij} = 0$.

Remark 8.4.4. *The method described in this subsection for obtaining consistent controller estimates assumes that experiments are performed in an open-loop setting, i.e., u and e are uncorrelated. This allows us to obtain estimates of T_y with standard MIMO open-loop identification methods, where no noise modeling is required. Practically, it is common that a preliminary (distributed) controller is present in the experiment set-up, e.g. for stabilization of an unstable open-loop system. In such a situation, two approaches can be followed. In the first approach, a consistent estimate $\hat{T}_y$ of the transfer T_y, i.e., from u to y, can be obtained by using the direct method for closed-loop identification (Van den Hof, 1998), provided that the noise filter $\bar{H}$ is estimated consistently, additionally. Consistent estimates of C^d and C_I^d can then be obtained from (8.27). The second approach* *is to use an indirect method, where an estimate of the closed-loop transfer matrix from measured external (dithering/reference) signals r to y is obtained. Consequently, consistent estimates of the transfer functions of interest can be obtained from the consistently estimated closed-loop transfer matrix directly, see e.g. Section 2.3.2.*

8.5 Dealing with a coupled reference model

In the case of a coupled reference model, the ideal distributed controller (8.4) contains additional transfer function modules C_{ij}^{Td}, $(i,j) \in \mathcal{E}$, that are unknown. These modules are not identified in the procedures described in Section 8.4. We will now briefly describe an extension of the procedure in Section 8.4 to obtain consistent estimates of the additional modules C_{ij}^{Td}, $(i,j) \in \mathcal{E}$, via the direct method used in Section 8.4.2.

Consider again the example network (8.6)-(8.7) with a coupled reference model

$$\begin{aligned} y_1^d &= T_1 r_1 + Q_{12} k_{12}, \quad p_{12} = P_{12} y_1^d, \\ y_2^d &= T_2 r_2 + Q_{21} k_{21}, \quad p_{21} = P_{21} y_2^d. \end{aligned}$$

where $k_{12} = p_{21}$ and $k_{21} = p_{12}$. The top row in (8.4) is given by (8.5), with the modules defined in (8.9)-(8.10) and

$$C_{12}^{Td} = \frac{Q_{12}}{G_1(1-T_1)}, \quad C_{21}^{Td} = \frac{Q_{21}}{G_2(1-T_2)}.$$

Now, the network (8.6)-(8.7) can be transformed into the augmented network (8.19) as depicted in Figure 8.2, but by defining $\bar{C}_{12}^{Td} := \frac{1}{C_{12}^{Td}}$, $\bar{C}_{21}^{Td} := \frac{1}{C_{21}^{Td}}$ and the variables

$$\bar{u}_{12} := \frac{Q_{12}}{1-T_1} u_1 = \bar{Q}_{12} u_1, \quad \bar{u}_{21} := \frac{Q_{21}}{1-T_2} u_2 = \bar{Q}_{21} u_2,$$

we can write the network (8.6)-(8.7) also as

$$\begin{aligned} y_1 &= \bar{C}_{12}^{Td} \bar{u}_{12} + \bar{C}_{12}^{d} \bar{y}_{12} + H_1 e_1, \\ y_2 &= \bar{C}_{21}^{Td} \bar{u}_{21} + \bar{C}_{21}^{d} \bar{y}_{21} + H_2 e_2, \end{aligned}$$

as visualized in Figure 8.4.

The unknown modules C_{12}^{Td} and C_{21}^{Td} can now be determined by posing network identification problems for the modules $\bar{C}_{12}^{Td}$ and $\bar{C}_{21}^{Td}$, shown in orange in Figure 8.4. Although the network structure in Figure 8.4 is the same as in Figure 8.2, different signals ($\bar{u}_{12}$ and $\bar{u}_{21}$) and different controller modules ($\bar{C}_{12}^{Td}$ and

$\bar{C}_{21}^{Td}$) are present, where the modules $\bar{C}_{12}^{Td}$ and $\bar{C}_{21}^{Td}$ have not been identified in the methods described in Section 8.4. The modules $\bar{C}_{12}^{d}$ and $\bar{C}_{21}^{d}$ are now depicted in white and assumed to be estimated consistently *a priori* as described in Section 8.4.

For a general network (8.1) we have

$$y_i = \bar{C}_{ij}^{Td}\bar{u}_{ij} + \sum_{k\in\mathcal{N}_i} \bar{C}_{ik}^{d}\bar{y}_{ik} + H_i e_i, \quad (i,j) \in \mathcal{E}. \tag{8.28}$$

Consider now the identification of C_{ij}^{Td} as follows. For a given node $i \in \mathcal{V}$, let C_{ij}^{d}, $j \in \mathcal{N}_i$, and H_i be given. We consider the following model for (8.28):

$$y_{ij}(\rho_{ij}^c) = \bar{C}_{ij}^{c}(\rho_{ij}^c)\bar{u}_{ij} + \sum_{k\in\mathcal{N}_i} \bar{C}_{ik}^{d}\bar{y}_{ik} + H_i e_i, \quad (i,j) \in \mathcal{E},$$

where $\bar{C}_{ij}^{c}(\rho_{ij}^c)$ is a parametrized model for $\bar{C}_{ij}^{Td}$. Define the prediction error

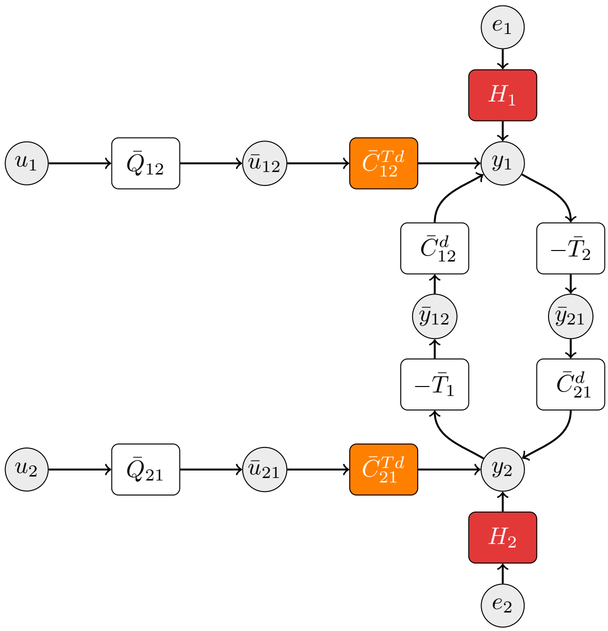


Figure 8.4: The dynamic network represented by modules of the ideal distributed controller and a coupled reference model.

$\varepsilon^c_{ij}(t,\rho^c_{ij}) := y_i(t) - \hat{y}_{ij}(t,\rho^c_{ij})$ with

$$\hat{y}_{ij}(t,\rho^c_{ij}) := H_i^{-1}\left(\bar{C}^c_{ij}(\rho^c_{ij})\bar{u}_{ij} + \sum_{k\in\mathcal{N}_i}\bar{C}^d_{ik}\bar{y}_{ik}\right) + (1-H_i^{-1})y_i.$$

Then by minimizing the identification criterion $V_{ij}(\rho^c_{ij}) := \frac{1}{N}\sum_{t=1}^{N}\varepsilon^c_{ij}(t,\rho^c_{ij})^2$ for each $j\in\mathcal{N}_i$, the parameter estimates $\hat{\rho}^c_{ij}$ are obtained. The asymptotic estimates are $\rho^*_{ij} = \arg\min_{\rho^c_{ij}}\bar{E}[\varepsilon^c_{ij}(t,\rho^c_{ij})^2]$, $j\in\mathcal{N}_i$.

Corollary 8.5.1. *Suppose the following conditions hold:*

- $\Phi_{\bar{u}_{ij}}(\omega)$ *is positive definite for almost all* $\omega\in[-\pi,\pi]$,
- *there exists* ρ^d_{ij} *such that* $C^c_{ij}(\rho^d_{ij}) = C^{Td}_{ij}$.

Then it holds that $C^c_{ij}(\rho^*_{ij}) = C^{Td}_{ij}$.

Notice that one can equivalently minimize $\sum_{j\in\mathcal{N}_i}V_{ij}(\rho^c_{ij})$. Moreover, no additional experiment is required for data acquisition.

Remark 8.5.1. *The controller modules* C^{Td}_{ij} *can also be obtained through an indirect identification method. Indeed, the dynamics of the augmented network described by* (8.28) *yield a mapping from external signals* (u,e) *to network signals (including* $\bar{u}_{ij}$*). Consistent estimates of* C^{Td}_{ij} *can therefore be obtained* mutatis mutandis *via Corollary 8.4.1.*

Remark 8.5.2. *Under the assumption that a consistent estimate* $\hat{C}_{ii}$ *of* C^d_{ii} *has been obtained* a priori, *this estimate can also be used to obtain an estimate* $\hat{C}^{Td}_{ij}$ *of* $C^T_{ij}d$ *without posing an additional identification problem. By the definitions directly after* (8.5), *it follows that* $C^{Td}_{ij} = C^d_{ii}Q_{ij}T_i^{-1}$, *where* Q_{ij} *and* T_i *are transfer functions that describe the reference model and are therefore known. Hence, a consistent estimate of* $C^T_{ij}d$ *can be obtained as* $\hat{C}^{Td}_{ij} = \hat{C}^d_{ii}Q_{ij}T_i^{-1}$.

8.6 Numerical example

Consider the two-node network described by (8.6)-(8.7), with transfer functions

$$G_1(q) = \frac{c_1}{q-a_1},\quad G_{12}(q) = \frac{d_1}{q-a_1},\quad H_1 = 1,$$
$$G_2(q) = \frac{c_2}{q-a_2},\quad G_{21}(q) = \frac{d_2}{q-a_2},\quad H_2 = 1,$$

where $a_1 = 0.5$, $a_2 = 0.2$, $c_1 = c_2 = 1$ and $d_1 = d_2 = 0.1$. The objective is to let the closed-loop interconnected system behave as two decoupled processes with first-order dynamics, according to

$$y_i^d(t) = T_i(q)r_i(t), \quad T_i(q) = \frac{1-\gamma_i}{q-\gamma_i}, \quad i = 1, 2, \tag{8.29}$$

with $\gamma_1 = \gamma_2 = 0.8$.

As described in Section 8.3, the ideal distributed controller is described by (8.9)-(8.10), with the interconnections $s_1^c = o_2^c$, $s_2^c = o_1^c$, and

$$\begin{aligned} C_{11}^d(q) &= \frac{1-\gamma_1}{c_1}\frac{q-a_1}{q-1}, \quad C_{12}^d(q) = -\frac{d_1}{c_1}, \quad K_{12}^d(q) = \frac{1-\gamma_1}{q-1}, \\ C_{22}^d(q) &= \frac{1-\gamma_2}{c_2}\frac{q-a_2}{q-1}, \quad C_{21}^d(q) = -\frac{d_2}{c_2}, \quad K_{21}^d(q) = \frac{1-\gamma_2}{q-1}. \end{aligned}$$

For the experiment, consider that u_1 and u_2 are Gaussian white-noise signals with unit variance and e_1 and e_2 are (unmeasured) Guassian white-noise signals with variance $\sigma_e^2 = 0.25$. As analyzed in Section 8.3, the noise will cause a bias in the controller parameter estimates when the distributed virtual reference feedback tuning (DVRFT) method is applied directly. For the distributed optimal controller identification (DOCI) method, described in Section 8.4, we expect consistent estimates and hence an improved closed-loop performance.

We first represent the network as shown in Figure 8.2, where

$$\bar{C}_{11}^d = \frac{c_1}{1-\gamma_1}\frac{q-1}{q-a_1}, \quad \bar{C}_{12}^d = -\frac{d_1}{c_1}\frac{q-1}{q-a_1}.$$

The modules are therefore parametrized as

$$\begin{aligned} &\bar{C}_{11}(\theta_1) = \theta_{1a}\frac{1-q^{-1}}{1-\theta_{1b}q^{-1}}, \quad \bar{C}_{12}(\theta_1) = \theta_{1c}\frac{1-q^{-1}}{1-\theta_{1b}q^{-1}}, \\ &H_1(\theta_1) = 1, \end{aligned}$$

so that there exists θ_1^d such that $\bar{C}_{11}^d = \bar{C}_{11}(\theta_1^d)$, $\bar{C}_{12}^d = \bar{C}_{12}(\theta_1^d)$ and $H_1 = H_1(\theta_1)$. By forming the predictor

$$\hat{y}_1(t|t-1;\theta_1) := \bar{C}_{11}(\rho_1)\bar{u}_1 + \bar{C}_{12}(\rho_1)\bar{y}_2$$

and minimizing $V_1(\theta_1)$ in (8.21) for $N = 100$ samples, we find the estimate $\hat{\theta}_1$. The estimate $\hat{\theta}_2$ for controller 2 is obtained by following an analogous procedure. Note that V_1 and V_2 are not quadratic functions in the parameters. The corresponding DVRFT cost functions are quadratic and the optimization problems therefore have analytic solutions.

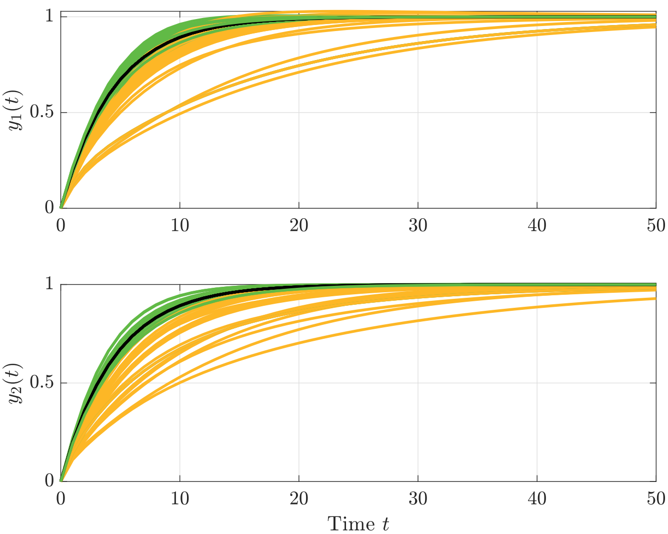


Figure 8.5: Step response of the closed-loop network from (r_1, r_2) to (y_1, y_2) for 20 experiments with DOCI (green), DVRFT (yellow) and the desired closed-loop network (black).

The distributed controller resulting from DOCI leads to a closed-loop network with a step response shown in Figure 8.5 and a frequency response shown in Figure 8.6 in green of the transfer $r \to y$, for 20 experiments. For comparison, we synthesize a DVRFT controller using the same data via the method described in Section 7.3. The corresponding responses are shown in Figure 8.5 and 8.6 in yellow. Note that the controller classes are chosen such that the ideal controller belongs to the controller class for each method, which leads to a convex and non-convex cost function for DVRFT and DOCI, respectively. We observe that the distributed controller synthesized via DOCI leads to a closed-loop network with a response that is closer to the reference model compared to the controller synthesized via VRFT. As discussed in Section 8.3, DVRFT leads to biased controller estimates when the noise terms e_1 and e_2 are non-zero. This bias is illustrated in Figure 8.7b, where the parameter estimates for controller 1 are plotted for 100 experiments. The parameter estimates for controller 1 with DOCI are plotted in Figure 8.7a. Finally, Figure 8.8 shows the distribution of the achieved performance for DOCI and DVRFT in the presence of noise. As described in the introduction, VRFT can yield consistent estimates when instrumental variables (IVs) are used. The construction of IVs for the example network was performed

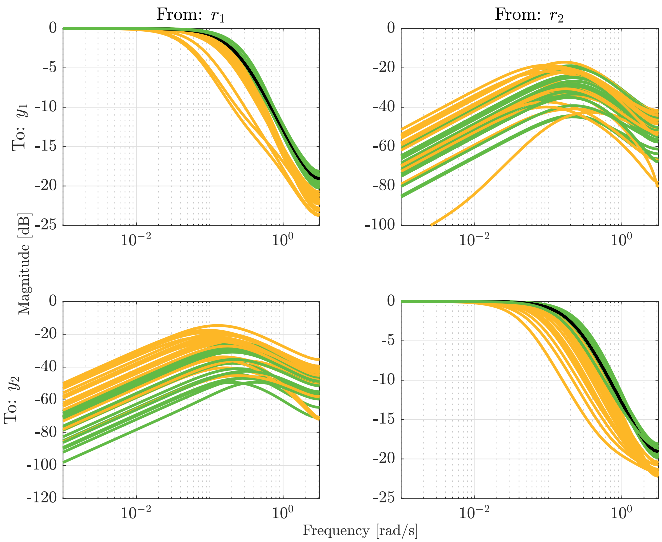
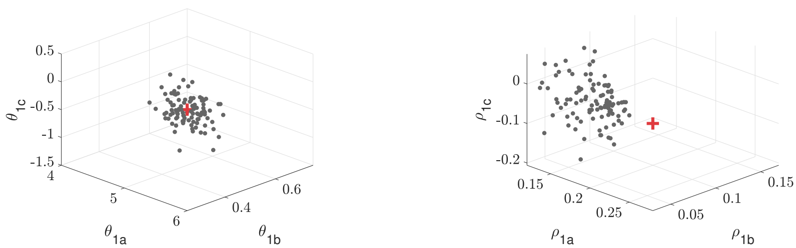


Figure 8.6: Frequency response of the closed-loop network from (r_1, r_2) to (y_1, y_2) for 20 experiments with DOCI (green), DVRFT (yellow) and the desired closed-loop network (black). Notice that the desired network is decoupled, thus the corresponding transfers $r_i \to y_j^d$, $i \neq j$, are identical to zero.



(a) DOCI (b) DVRFT

Figure 8.7: Parameter estimates for 100 experiments (gray) and the true parameter (red) for controller 1.

using an additional experiment (Bazanella et al., 2012), *mutatis mutandis*. The mean value of the performance of DVRFT with IVs is significantly lower compared to DVRFT, while the variance is significantly higher. We observe that

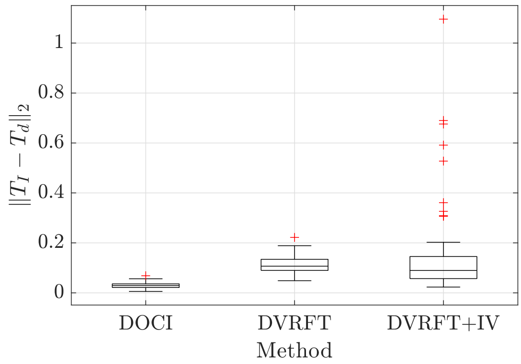


Figure 8.8: Distribution of the achieved performance for DOCI, DVRFT and DVRFT with IVs, where T_I and T_d denote the transfers $r \to y$ and $r \to y_d$, respectively.

the mean value as well as the variance of the performance are considerably lower for DOCI. Hence, although both DOCI and DVRFT with IVs yield consistent estimates, the increased variance due to IVs in DVRFT yields an overall worse performance compared to DOCI.

8.7 Conclusions

We have developed a data-driven method for the construction of a distributed controller for an interconnected system subject to disturbances. This method is enabled by the construction of an augmented network. We have shown how the identification of the ideal distributed controller modules in this network can be solved by direct and indirect identification methods, resulting in consistent controller estimates. The estimated distributed controller therefore solves the model-reference control problem asymptotically in the number of data. Comparatively, the methods can be utilized as an alternative to the DVRFT method described in Chapter 7 and do not require a tailor-made parametrization of the noise model through the direct identification method. In the case of the indirect method, a noise model can be fully omitted if measurement data are collected in an open-loop configuration. By a simple network consisting of two interconnected systems, we have shown the effectiveness and the improvement over biased or high-variance alternative methods on the closed-loop performance.

Part III

Distributed data-driven control with guarantees

Chapter 9

Guaranteed $\mathscr{H}_\infty$ performance analysis and distributed control from noisy input-state data

In this chapter, we extend a recent data-based approach for guaranteed performance analysis to distributed analysis of interconnected linear systems. We present a new set of sufficient LMI conditions based on noisy input-state data that guarantees $\mathscr{H}_\infty$ performance and has a structure that is applicable to distributed controller synthesis from data. Sufficient LMI conditions based on noisy data are provided for the existence of a dynamic distributed controller that achieves $\mathscr{H}_\infty$ performance. The presented approach enables scalable analysis and control of large-scale interconnected systems from noisy input-state data.

9.1 Introduction

Several methods have been developed for data-based system analysis and controller synthesis, we refer to (Hou and Wang, 2013) for a survey on data-based control. Some methods rely on the reference model paradigm, such as virtual

This chapter is based on the publication: T. R. V. Steentjes, M. Lazar, and P. M. J. Van den Hof. $\mathcal{H}_\infty$ performance analysis and distributed controller synthesis for interconnected linear systems from noisy input-state data. In *Proc. 60th IEEE Conference on Decision and Control (CDC)*, pages 3717–3722, Austin, Texas, USA, 2021a reference feedback tuning (Bazanella et al., 2012) and optimal controller identification (Campestrini et al., 2017). Extensions for interconnected systems to data-based distributed controller synthesis include distributed virtual reference feedback tuning in the noiseless (Steentjes et al., 2020), and noisy (Steentjes et al., 2021d) case, described in Chapter 7.

A recent trend in data-based system analysis and control originates from Willems' fundamental lemma (Willems et al., 2005). Applications include data-based predictive control (Coulson et al., 2019), (Allibhoy and Cortés, 2021), the data-based parametrization of stabilizing controllers (De Persis and Tesi, 2020) and robust data-based state-feedback design with noisy data (Berberich et al., 2020). The data-based verification of dissipativity properties was considered in (Koch et al., 2020a), (Koch et al., 2020b), which allows to determine system measures such as the $\mathscr{H}_\infty$ norm or passivity properties from data corrupted by a noise signal satisfying quadratic bounds. A similar noise description was considered in (van Waarde et al., 2022), which extends the data-based controller design results in (van Waarde et al., 2020) to the noisy case. The data-based conditions in (van Waarde et al., 2022) are necessary and sufficient for stabilizing state feedback synthesis, including $\mathscr{H}_2$ or $\mathscr{H}_\infty$ performance specifications.

In this chapter, the data-based $\mathscr{H}_\infty$ performance analysis and *distributed* controller synthesis problem for *interconnected* systems is considered. We extend the data-based framework for parameterizing an unknown system, considered in the two distinct papers(Koch et al., 2020b) and (van Waarde et al., 2022), to the situation of interconnected systems. The analysis in this chapter is enabled by considering a dual parametrization of the set $\Sigma_\mathcal{D}$: the set of systems that are compatible with input-state data $\mathcal{D}$ for unmeasured noise trajectories in a set $\mathcal{W}$ that captures quadratic bounds on the noise sequence. A feature of the dual parametrization is the applicability of standard (primal) conditions for unstructured (Scherer, 2001) and structured (Van Horssen and Weiland, 2016) robust performance analysis. For an interconnected system, we consider sets $\Sigma_\mathcal{D}^i$ of subsystems that are compatible with the local input-state and neighbors' state data, given prior knowledge on the noise signals confined to a set $\mathcal{W}_i$. We develop sufficient data-based conditions for $\mathscr{H}_\infty$ performance analysis and for the existence of a dynamic distributed controller that achieves a given $\mathscr{H}_\infty$ performance level.

A feature of our results is that no model of the interconnected system is identified from the data. The identification of interconnected systems is considered in the field of network identification, which provides structured and consistent methods for identification (Van den Hof et al., 2013). If an identified model is only used for controller synthesis, however, it is arguably more efficient to consider data-based synthesis conditions directly. Additionally, with our data-based method, stability and performance guarantees for the closed-loop interconnected system come with a finite number of data points. This is a consequence of the non-probabilistic noise paradigm that is considered for the data-based analysis and synthesis. Comparatively, system identification based on prediction-error methods comes with consistency results asymptotic in the number of data, but does not provide guarantees for finite data.

Basic nomenclature

The integers are denoted by $\mathbb{Z}$. Given $a \in \mathbb{Z}$, $b \in \mathbb{Z}$ such that $a < b$, we denote $\mathbb{Z}_{[a:b]} := \{a, a+1, \ldots, b-1, b\}$. Let $I_n \in \mathbb{R}^{n\times n}$, or simply I, denote the identity matrix and $\mathbf{1}_n \in \mathbb{R}^n$, or simply $\mathbf{1}$, denote the column vector of all ones. For a subset $A \subset \mathbb{Z}$, the vertical, respectively horizontal, stacking of matrices X_a, $a \in \mathcal{A}$ is denoted $\operatorname{col}_{a\in\mathcal{A}} X_a$, respectively $\operatorname{row}_{a\in\mathcal{A}} X_a$. The kernel of a matrix A is denoted $\ker A$ and a matrix $A_\perp$ denotes a basis matrix of $\ker A$. For a real symmetric matrix X, $X \succ 0$ ($X \succeq 0$) denotes that X is positive (semi-) definite. Matrices that can be inferred from symmetry are denoted by $(\star)$.

9.2 Preliminaries

In this chapter, we consider interconnected systems composed of L linear time-invariant systems of the form

$$\begin{aligned} x_i(k+1) &= A_i x_i(k) + \sum_{j\in\mathcal{N}_i} A_{ij} x_j(k) + B_i u_i(k) + w_i(k), \\ y_i(k) &= C_i x_i(k) + D_i u_i(k) \quad \text{for} \quad i = 1, \ldots, L, \end{aligned} \tag{9.1}$$

where $x_i \in \mathbb{R}^{n_i}$ denotes the state, $u_i \in \mathbb{R}^{m_i}$ the input and $w_i \in \mathbb{R}^{n_i}$ is a noise signal. The set $\mathcal{N}_i := \{j \in \mathcal{V} \,|\, (i,j) \in \mathcal{E}\}$ denotes the neighbours of system i, where $\mathcal{V}$ and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ denote the set of vertices and the set of non-oriented edges defining the connected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$.

Let there exist a true interconnected system defined by the matrices A_i^0, A_{ij}^0 and B_i^0, $(i,j) \in \mathcal{E}$, generating the input-state data $\{(u_i(t),\, x_i(t)),\, t = 0, \ldots, N\}$ for $i \in \mathcal{V}$. This data is collected in the matrices

$$X_i := [x_i(0) \ \cdots \ x_i(N)], \ U_i^- := [u_i(0) \ \cdots \ u_i(N-1)].$$

By defining the matrices

$$\begin{aligned} &X_i^+ := [x_i(1) \ \cdots \ x_i(N)], \ X_i^- := [x_i(0) \ \cdots \ x_i(N-1)], \\ &W_i^- := [w_i(0) \ \cdots \ w_i(N-1)], \end{aligned}$$

we obtain the following data equation for each $i \in \mathcal{V}$:

$$X_i^+ = A_i^0 X_i^- + \sum_{j\in\mathcal{N}_i} A_{ij}^0 X_j^- + B_i^0 U_i^- + W_i^-. \tag{9.2}$$

Consider the stacked input, state and noise variables $u := \mathrm{col}(u_1, \dots, u_L)$, $x := \mathrm{col}(x_1, \dots, x_L)$ and $w := \mathrm{col}(w_1, \dots, w_L)$. Then the interconnected system (9.1) is compactly described by

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) + w(k), \\ y(k) &= Cx(k) + Du(k), \end{aligned} \tag{9.3}$$

with straightforward definitions for A, B, C and D. The corresponding data equation is

$$X_+ = A_0 X_- + B_0 U_- + W_-,$$

with the data matrices defined for system (9.3) as was done for each subsystem. The transfer matrix from u to y of (9.3) is $G(q) := C(qI - A)^{-1}B + D$ and the $\mathscr{H}_\infty$ norm of G is denoted $\|G\|_{\mathscr{H}_\infty}$. For $\gamma > 0$, we say that the interconnected system achieves $\mathscr{H}_\infty$ performance γ if $\|G\|_{\mathscr{H}_\infty} < \gamma$.

9.3 Inferring system performance from noisy data

In this section, we consider the data-based dissipativity analysis for an unstructured system. We recall a parametrization from (Koch et al., 2020b) and introduce a dual parametrization of systems that are compatible with input-state data. The dual parametrization allow us to (i) derive a dual result with respect to (Koch et al., 2020b) for concluding dissipativity properties from data, and (ii) extend the data-based results to structured results for interconnected systems.

Consider the system

$$x(k+1) = A_0 x(k) + B_0 u(k) + w(k), \tag{9.4}$$
$$y(k) = Cx(k) + Du(k) \tag{9.5}$$

with collected data

$$\begin{aligned} &X_+ := [x(1) \ \cdots \ x(N)], \ X_- := [x(0) \ \cdots \ x(N-1)], \\ &U_- := [u(0) \ \cdots \ u(N-1)], \end{aligned}$$

and noise sequence $W_- := [w(0) \ \cdots w(N-1)]$. We assume that the data (U_-, X) are known, while W_- is unknown, but

$$W_- \in \mathcal{W} := \left\{ W \mid \begin{bmatrix} W^\top \\ I \end{bmatrix}^\top \begin{bmatrix} Q_w & S_w \\ S_w^\top & R_w \end{bmatrix} \begin{bmatrix} W^\top \\ I \end{bmatrix} \succeq 0 \right\},$$

with $Q_w \prec 0$ so that $\mathcal{W}$ is bounded. No assumptions on the statistics of w are made. This noise model can represent, e.g., an energy bound $W_- W_-^\top \preceq R_w$ for $Q_w = -I$ and $S_w = 0$ or bounds on individual components $w(k)$ (van Waarde et al., 2022). The square of sample cross-covariance bounds, as considered in (Hakvoort and Van den Hof, 1995) for parameter-bounding identification, can be captured by $\mathcal{W}$ with Q_w generally not strictly negative definite; this is a topic of current research. We assume that the data are informative in the sense that the matrix $\operatorname{col}(X_-, U_-)$ has full row rank.

Because the noise term is unknown, there exist multiple pairs of system matrices that are consistent with the data. The set of all pairs (A, B) that are consistent with the data is defined as

$$\Sigma_{\mathcal{D}} = \{(A, B) \mid X_+ = AX_- + BU_- + W \text{ for some } W \in \mathcal{W}\}.$$

We note that the true system $(A_0, B_0) \in \Sigma_{\mathcal{D}}$ by construction. Furthermore, in the *noiseless* case ($W_- = 0$), $\Sigma_{\mathcal{D}}$ reduces to the singleton $\{(A_0, B_0)\}$ if $\operatorname{col}(X_-, U_-)$ has full rank (van Waarde et al., 2020).

The following result from (van Waarde et al., 2022), cf. (Koch et al., 2020b), provides a parametrization of the set $\Sigma_{\mathcal{D}}$.

Lemma 9.3.1 (Parametrization $\Sigma_{\mathcal{D}}$). *It holds that*

$$\Sigma_{\mathcal{D}} = \{(A, B) \mid \begin{bmatrix} -A^\top \\ -B^\top \\ I \end{bmatrix}^\top \begin{bmatrix} \bar{Q}_{\mathcal{D}} & \bar{S}_{\mathcal{D}} \\ \bar{S}_{\mathcal{D}}^\top & \bar{R}_{\mathcal{D}} \end{bmatrix} \begin{bmatrix} -A^\top \\ -B^\top \\ I \end{bmatrix} \succeq 0\},$$

$$\textit{with} \quad \begin{bmatrix} \bar{Q}_{\mathcal{D}} & \bar{S}_{\mathcal{D}} \\ \bar{S}_{\mathcal{D}}^\top & \bar{R}_{\mathcal{D}} \end{bmatrix} = \begin{bmatrix} X_- & 0 \\ U_- & 0 \\ X_+ & I \end{bmatrix} \begin{bmatrix} Q_w & S_w \\ S_w^\top & R_w \end{bmatrix} \begin{bmatrix} X_- & 0 \\ U_- & 0 \\ X_+ & I \end{bmatrix}^\top.$$

We now present a dual parametrization of $\Sigma_{\mathcal{D}}$.

Lemma 9.3.2 (Dual parametrization $\Sigma_{\mathcal{D}}$). *Let the matrix*

$$\begin{bmatrix} Q_w & S_w \\ S_w^\top & R_w \end{bmatrix}$$

be invertible. Then it holds that

$$\Sigma_{\mathcal{D}} = \{(A, B) \mid \begin{bmatrix} I & 0 \\ 0 & I \\ A & B \end{bmatrix}^\top \begin{bmatrix} Q_{\mathcal{D}} & S_{\mathcal{D}} \\ S_{\mathcal{D}}^\top & R_{\mathcal{D}} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & I \\ A & B \end{bmatrix} \preceq 0\},$$

where $R_{\mathcal{D}} \succ 0$ *with* $\begin{bmatrix} Q_{\mathcal{D}} & S_{\mathcal{D}} \\ S_{\mathcal{D}}^\top & R_{\mathcal{D}} \end{bmatrix} := \begin{bmatrix} \bar{Q}_{\mathcal{D}} & \bar{S}_{\mathcal{D}} \\ \bar{S}_{\mathcal{D}}^\top & \bar{R}_{\mathcal{D}} \end{bmatrix}^{-1}$.

Proof. The proof is provided in Appendix 9.A. □

Since any system that is consistent with the data is an element of $\Sigma_\mathcal{D}$, every such system admits a representation

$$x(k+1) = \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix}, \qquad \text{with } (A,B) \in \Sigma_\mathcal{D}.$$

As it was shown in (Koch et al., 2020b), this uncertain system admits the following linear fractional transformation (LFT) representation

$$\begin{bmatrix} x(k+1) \\ y(k) \\ p(k) \end{bmatrix} = \begin{bmatrix} 0 & 0 & I \\ C & D & 0 \\ I & 0 & 0 \\ 0 & I & 0 \end{bmatrix} \begin{bmatrix} x(k) \\ u(k) \\ l(k) \end{bmatrix}, \ l(k) = \begin{bmatrix} A & B \end{bmatrix} p(k),$$

with $(A,B) \in \Sigma_\mathcal{D}$.

Proposition 9.3.1 (Dissipativity from data)**.** *If there exist a P and α such that $P \succ 0$, $\alpha > 0$ and (9.6) hold (see bottom of this page), then*

$$\left[\begin{array}{cc} I & 0 \\ A & B \\ \hline 0 & I \\ C & D \end{array}\right]^\top \left[\begin{array}{cc|cc} -P & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ \hline 0 & 0 & -Q & -S \\ 0 & 0 & -S^\top & -R \end{array}\right] \left[\begin{array}{cc} I & 0 \\ A & B \\ \hline 0 & I \\ C & D \end{array}\right] \prec 0 \tag{9.7}$$

holds for all $(A,B) \in \Sigma_\mathcal{D}$.

Proof. Let (9.6) hold and let $M = \begin{bmatrix} A & B \end{bmatrix}$. By Lemma 9.3.2, it holds that for $\alpha > 0$,

$$\begin{bmatrix} I \\ M \end{bmatrix}^\top \begin{bmatrix} -\alpha Q_\mathcal{D} & -\alpha S_\mathcal{D} \\ -\alpha S_\mathcal{D}^\top & -\alpha R_\mathcal{D} \end{bmatrix} \begin{bmatrix} I \\ M \end{bmatrix} \succeq 0$$

$$\left[\begin{array}{ccc} I & 0 & 0 \\ 0 & I & 0 \\ \hline 0 & I & 0 \\ I & 0 & 0 \\ 0 & 0 & I \\ \hline 0 & 0 & I \\ C & 0 & D \end{array}\right]^\top \left[\begin{array}{cc|cc|cc} -P & 0 & 0 & 0 & 0 & 0 \\ 0 & P & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & -\alpha R_\mathcal{D} & -\alpha S_\mathcal{D}^\top & 0 & 0 \\ 0 & 0 & -\alpha S_\mathcal{D} & -\alpha Q_\mathcal{D} & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & -Q & -S \\ 0 & 0 & 0 & 0 & -S^\top & -R \end{array}\right] \left[\begin{array}{ccc} I & 0 & 0 \\ 0 & I & 0 \\ \hline 0 & I & 0 \\ I & 0 & 0 \\ 0 & 0 & I \\ \hline 0 & 0 & I \\ C & 0 & D \end{array}\right] \prec 0 \tag{9.6}$$

for all $(A, B) \in \Sigma_{\mathcal{D}}$. Therefore, by the full block S-procedure (Scherer, 2001), it follows that (9.7) holds. $\square$

Inequality (9.7) is the well known condition for dissipativity for a quadratic supply rate matrix $\Pi = -\begin{bmatrix} Q & S \\ S^\top & R \end{bmatrix}$. A special case of the supply rate matrix is $Q = \gamma^2 I$, $S = 0$ and $R = -I$ for $\gamma > 0$. For this specific case there exists a $P \succ 0$ so that (9.7) holds if and only if the channel $u \to y$ achieves $\mathscr{H}_\infty$ performance γ.

We have derived a dual parametrization of $\Sigma_{\mathcal{D}}$, which allows the application of standard robust control tools to the LFT representation. The parametrization from Lemma 9.3.1, see also (Koch et al., 2020b, Lemma 2), requires the application of the dualization lemma on the data-based LMI. A feature of the dual parametrization of $\Sigma_{\mathcal{D}}$ in Lemma 9.3.2, is that robust analysis tools for interconnected systems can be applied *mutatis mutandis*, as we will show in the next section.

9.4 Interconnected system analysis

Let us return to the interconnected system (9.1). We consider the data U_i^-, X_i and X_j, $j \in \mathcal{N}_i$, is available for each system i, while W_i^- is unknown. For each $i \in \mathcal{V}$, we assume

$$W_i^- \in \mathcal{W}_i = \left\{ W_i \mid \begin{bmatrix} W_i^\top \\ I \end{bmatrix}^\top \underbrace{\begin{bmatrix} Q_w^i & S_w^i \\ (S_w^i)^\top & R_w^i \end{bmatrix}}_{=:\Pi_w^i} \begin{bmatrix} W_i^\top \\ I \end{bmatrix} \succeq 0 \right\},$$

with $Q_w^i \prec 0$. We assume that the data are informative enough in the sense that the matrix $\mathrm{col}(X_i^-, X_{\mathcal{N}_i}^-, U_i^-)$ has full row rank for each $i \in \mathcal{V}$.

For each subsystem, there exist multiple tuples $(A_i, A_{\mathcal{N}_i}, B_i)$ that are consistent with the data, i.e., that satisfy

$$X_i^+ = A_i X_i^- + \sum_{j \in \mathcal{N}_i} A_{ij} X_j^- + B_i U_i^- + W_i \tag{9.8}$$

for some $W_i \in \mathcal{W}_i$. Here, we define $A_{\mathcal{N}_i} := \mathrm{row}_{j \in \mathcal{N}_i} A_{ij}$. Hence, for each $i \in \mathcal{V}$, the set of subsystems that are consistent with the data is

$$\Sigma_{\mathcal{D}}^i := \{(A_i, A_{\mathcal{N}_i}, B_i) \mid (9.8) \text{ holds for some } W_i \in \mathcal{W}_i\}$$

We note that under the assumption that $W_i^- \in \mathcal{W}_i$, the true system matrices are in the set $\Sigma_{\mathcal{D}}^i$ by construction.

Lemma 9.4.1 (Parametrization $\Sigma^i_{\mathcal{D}}$). *It holds that*

$$\Sigma^i_{\mathcal{D}} = \{(A_i, A_{\mathcal{N}_i}, B_i) \mid (\star)^\top \begin{bmatrix} \bar{Q}^i_{\mathcal{D}} & \bar{S}^i_{\mathcal{D}} \\ (\bar{S}^i_{\mathcal{D}})^\top & \bar{R}^i_{\mathcal{D}} \end{bmatrix} \begin{bmatrix} -A_i^\top \\ -A_{\mathcal{N}_i}^\top \\ -B_i^\top \\ I \end{bmatrix} \succeq 0\},$$

$$\textit{with} \quad \begin{bmatrix} \bar{Q}^i_{\mathcal{D}} & \bar{S}^i_{\mathcal{D}} \\ (\bar{S}^i_{\mathcal{D}})^\top & \bar{R}^i_{\mathcal{D}} \end{bmatrix} := \begin{bmatrix} X_i^- & 0 \\ X_{\mathcal{N}_i}^- & 0 \\ U_i^- & 0 \\ X_i^+ & I \end{bmatrix} \Pi_i^w \begin{bmatrix} X_i^- & 0 \\ X_{\mathcal{N}_i}^- & 0 \\ U_i^- & 0 \\ X_i^+ & I \end{bmatrix}^\top .$$

Lemma 9.4.2 (Dual parametrization $\Sigma^i_{\mathcal{D}}$). *Let Π_i^w be invertible. It holds that $\Sigma^i_{\mathcal{D}}$ is equal to*

$$\{(A_i, A_{\mathcal{N}_i}, B_i) \mid (\star)^\top \begin{bmatrix} Q^i_{\mathcal{D}} & S^i_{\mathcal{D}} \\ (S^i_{\mathcal{D}})^\top & R^i_{\mathcal{D}} \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \\ A_i & A_{\mathcal{N}_i} & B_i \end{bmatrix} \preceq 0\},$$

where $R^i_{\mathcal{D}} \succ 0$ with $\left[\begin{smallmatrix} Q^i_{\mathcal{D}} & S^i_{\mathcal{D}} \\ (S^i_{\mathcal{D}})^\top & R^i_{\mathcal{D}} \end{smallmatrix}\right] := \left[\begin{smallmatrix} \bar{Q}^i_{\mathcal{D}} & \bar{S}^i_{\mathcal{D}} \\ (\bar{S}^i_{\mathcal{D}})^\top & \bar{R}^i_{\mathcal{D}} \end{smallmatrix}\right]^{-1}$.

The proofs for Lemma 9.4.1 and 9.4.2 follow an analogue reasoning as the proofs for Lemma 9.3.1 and 9.3.2, respectively, and are omitted for brevity.

We note that if any interconnected system with subsystems in $\Sigma^i_{\mathcal{D}}$, i.e., any interconnected system that is consistent with the data, has a certain property, then also the true interconnected system has this property. To show a property for all interconnected systems that are consistent with the data, we use the following LFT representation.

Every interconnected system that is consistent with the data can be described by subsystems $\Sigma^i_{\mathcal{D}}$, $i \in \mathcal{V}$,

$$\begin{bmatrix} x_i(k+1) \\ y_i(k) \\ p_i(k) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & I \\ C_i & 0 & D_i & 0 \\ \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ I \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \end{bmatrix} \begin{bmatrix} x_i(k) \\ \mathrm{col}_{j\in\mathcal{N}_i}\, x_j(k) \\ u_i(k) \\ l_i(k) \end{bmatrix}$$

and $l_i(k) = \begin{bmatrix} A_i & A_{\mathcal{N}_i} & B_i \end{bmatrix} p_i(k)$, with $(A_i, A_{\mathcal{N}_i}, B_i) \in \Sigma^i_{\mathcal{D}}$.

This LFT representation for each subsystem allows us to apply robust analysis results for interconnected systems, to conclude $\mathscr{H}_\infty$ performance for all interconnected systems that are compatible with the data. Consider the matrices Z_i

defined in Appendix 9.B and define the matrix

$$
J_i := \left[\begin{array}{cccc}
I & 0 & 0 & 0 \\
0 & 0 & I & 0 \\
\hline
\mathbf{1} \otimes I & 0 & 0 & 0 \\
0 & I & 0 & 0 \\
\hline
0 & 0 & I & 0 \\
\begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ I \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix} \\
\hline
0 & 0 & 0 & I \\
C_i & 0 & 0 & D_i
\end{array}\right]. \tag{9.9}
$$

Proposition 9.4.1 (Performance from structured data). *Let $Q^i_{\mathcal{D}} \prec 0$ and $\gamma > 0$. If there exist P_i, Z_i and α_i so that $P_i \succ 0$, $\alpha_i > 0$ and*

$$
J_i^\top \left[\begin{array}{cc|cc|cc|cc}
-P_i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & P_i & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & Z_i^{11} & Z_i^{12} & 0 & 0 & 0 & 0 \\
0 & 0 & (Z_i^{12})^\top & Z_i^{22} & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & -\alpha_i R^i_{\mathcal{D}} & -\alpha_i (S^i_{\mathcal{D}})^\top & 0 & 0 \\
0 & 0 & 0 & 0 & -\alpha_i S^i_{\mathcal{D}} & -\alpha_i Q^i_{\mathcal{D}} & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & -\gamma^2 I & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & I
\end{array}\right] J_i \prec 0 \tag{9.10}
$$

holds for all $i \in \mathcal{V}$, then all interconnected systems with subsystems $(A_i, A_{\mathcal{N}_i}, B_i) \in \Sigma^i_{\mathcal{D}}$, $i \in \mathcal{V}$, achieve $\mathscr{H}_\infty$ performance γ.

The proof follows by a similar argument as in Proposition 9.3.1 and the application of (Van Horssen and Weiland, 2016, Theorem 1) to the LFT representation.

9.5 Distributed controller synthesis from data

So far we have considered the performance analysis of (interconnected) systems from data for the channel $u \to y$. We will now consider a distributed control problem for the interconnected system (9.1), where we take u_i and y_i as the control input and measured output respectively. Recall that we assume that input-state data is collected to determine $\Sigma^i_{\mathcal{D}}$ for each i. With the system matrices defined as $C_i = I$, $D_i = 0$, this implies only state-measurements are available for control. We note, however, that C_i is allowed to be chosen arbitrarily in this section and that $D_i = 0$; this implies that output measurements can be utilized for control. Future research will focus on extending the framework to the case when only input-output data is available for synthesis. The problem under consideration is to guarantee that the channel $w \to z$ achieves $\mathscr{H}_\infty$ performance $\gamma > 0$, with performance output

$$z_i = C_i^z x_i + \sum_{j \in \mathcal{N}_i} C_{ij}^z x_j + D_i^z u_i. \tag{9.11}$$

We consider a distributed controller that is an interconnected system with dynamic subsystems

$$\begin{bmatrix} \xi_i(k+1) \\ o_i(k) \\ u_i(k) \end{bmatrix} = \Theta_i \begin{bmatrix} \xi_i(k) \\ s_i(k) \\ y_i(k) \end{bmatrix}, \quad i = 1, \ldots, L, \tag{9.12}$$

where $\xi_i \in \mathbb{R}^{n_i}$ is the state of controller i and $o_i = \operatorname{col}_{j \in \mathcal{N}_i} o_{ij}$, $s_i = \operatorname{col}_{j \in \mathcal{N}_i} s_{ij}$ are interconnection variables satisfying $s_{ij} = o_{ji} \in \mathbb{R}^{n_{ij}}$ for $(i,j) \in \mathcal{E}$. By representing every interconnected system with performance output (9.11) and subsystems $(A_i, A_{\mathcal{N}_i}, B_i) \in \Sigma_{\mathcal{D}}^i$ in LFT form, we obtain conditions on the data for the existence of a distributed controller by (Van Horssen and Weiland, 2016, Theorem 2).

Define the matrices

$$T_i := \left[\begin{array}{cccc} I & 0 & 0 & 0 \\ 0 & 0 & I & I \\ \hline \mathbf{1} \otimes I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ \hline 0 & 0 & I & 0 \\ \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ I \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ \hline 0 & 0 & 0 & I \\ C_i^z & C_{\mathcal{N}_i}^z & 0 & 0 \end{array}\right], \quad i \in \mathcal{V},$$

and let $S_i := (T_i)_\perp$, $i \in \mathcal{V}$.

Theorem 9.5.1. *Let Ψ_i and Φ_i be matrices that are a basis of* $\ker \begin{bmatrix} C_i & 0 \end{bmatrix}$ *and* $\ker \begin{bmatrix} 0 & I & (D_i^z)^\top \end{bmatrix}$*, respectively, and let $n_{ij} = 3n_i$. If there exist P_i, $\bar{P}_i$, Z_i, $\bar{Z}_i$, α_i such that $P_i \succ 0$, $\bar{P}_i \succ 0$, $\alpha_i > 0$,* (9.14)-(9.15) *hold (see next page) with $\beta_i = \alpha_i^{-1}$ and*

$$\begin{bmatrix} P_i & I \\ I & \bar{P}_i \end{bmatrix} \succeq 0, \tag{9.13}$$

then there exist Θ_i, $i \in \mathcal{V}$, so that all closed-loop interconnected systems described by (9.1), (9.11) *and* (9.12) *with subsystems $(A_i, A_{\mathcal{N}_i}, B_i) \in \Sigma_{\mathcal{D}}^i$ achieve $\mathscr{H}_\infty$ performance γ.*

$$
\Psi_i^\top T_i^\top \left[\begin{array}{cc|cc|cc|cc}
-P_i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & P_i & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & Z_i^{11} & Z_i^{12} & 0 & 0 & 0 & 0 \\
0 & 0 & (Z_i^{12})^\top & Z_i^{22} & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & -\alpha_i R_\mathcal{D}^i & -\alpha_i (S_\mathcal{D}^i)^\top & 0 & 0 \\
0 & 0 & 0 & 0 & -\alpha_i S_\mathcal{D}^i & -\alpha_i Q_\mathcal{D}^i & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & -\gamma^2 I & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & I
\end{array}\right] T_i \Psi_i \prec 0 \tag{9.14}
$$

$$
\Phi_i^\top S_i^\top \left[\begin{array}{cc|cc|cc|cc}
-\bar{P}_i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \bar{P}_i & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & \bar{Z}_i^{11} & \bar{Z}_i^{12} & 0 & 0 & 0 & 0 \\
0 & 0 & (\bar{Z}_i^{12})^\top & \bar{Z}_i^{22} & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & -\beta_i \bar{R}_\mathcal{D}^i & -\beta_i (\bar{S}_\mathcal{D}^i)^\top & 0 & 0 \\
0 & 0 & 0 & 0 & -\beta_i \bar{S}_\mathcal{D}^i & -\beta_i \bar{Q}_\mathcal{D}^i & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & -\gamma^{-2} I & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & I
\end{array}\right] S_i \Phi_i \succ 0 \tag{9.15}
$$

Remark 9.5.1. *The conditions in Proposition 9.5.1 are sufficient for any $\alpha_i > 0$ are LMIs for fixed α_i. Conservatism can be reduced by, e.g., verifying feasibility of the LMIs on a discrete interval for α_i, $i \in \mathcal{V}$.*

In particular, Theorem 9.5.1 implies that the existence of a distributed controller for which the 'true' interconnected system achieves $\mathscr{H}_\infty$ performance, can be verified by checking a set of LMIs based on noisy input-state data. Suitable matrices P_i, $\bar{P}_i$, Z_i, $\bar{Z}_i$ are thus indirectly based on the data; these matrices can be used for the subsequent construction of the controller matrices Θ_i as described in Chapter 4, cf. (Van Horssen and Weiland, 2016). We note that neither our existence conditions, nor the construction of Θ_i is based on the unknown matrices $(A_i, A_{\mathcal{N}_i}, B_i)$.

9.6 Numerical examples

9.6.1 Example 1: $\mathscr{H}_\infty$-norm analysis

Consider a system of the form (9.4) with $L = 3$,

$$A_0 = \begin{bmatrix} 0.5 & 0.1 & 0 \\ 0.1 & 0.4 & 0.1 \\ 0 & 0.1 & 0.6 \end{bmatrix} \quad \text{and} \quad B_0 = I.$$

We choose $y = x$ so that $C = I$ and $D = 0$. The input entries are drawn from a normal distribution with zero mean and unit variance. The noise $w(k)$ is drawn uniformly from the set $\{w \mid \|w\|_2 \leq \sigma\}$, where $\sigma > 0$ determines the noise level. Hence, considering the set $\mathcal{W}$ with $Q_w = -I$, $S_w = 0$ and $R_w = N\sigma^2 I$, we have that the noise satisfies $W_- \in \mathcal{W}$.

The aim is to find an upperbound on the $\mathscr{H}_\infty$ norm of the channel $u \to y$ using the noisy data (U_-, X) with $N = 50$ samples. The true $\mathscr{H}_\infty$ norm is $\gamma_0 = 2.8836$. We choose eleven noise levels σ in the interval $[0.04, 0.25]$ and generate one data set for each noise level. For each data set, we minimize γ subject to (9.6) with $Q = \gamma^2 I$, $S = 0$ and $R = -I$. The results are displayed in Figure 9.1 in blue. By Proposition 9.3.1, the corresponding solutions satisfy (9.7), hence γ is an upperbound on the $\mathscr{H}_\infty$ norm for all systems in $\Sigma_{\mathcal{D}}$ and, therefore, for (A_0, B_0).

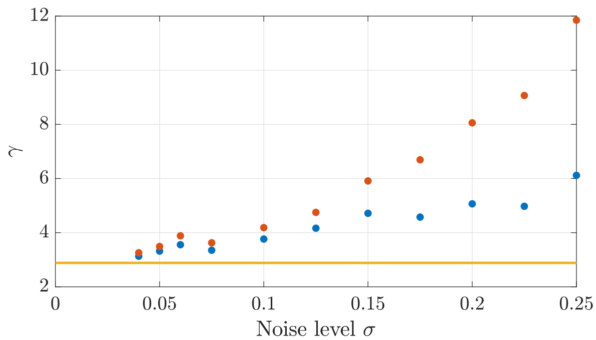


Figure 9.1: Example 1: Upper bound on the $\mathscr{H}_\infty$ norm determined from noisy data with increasing noise levels σ via lumped (blue) and structured (red) data analysis. The $\mathscr{H}_\infty$ norm of the true system is shown in orange.

Next, we perform the analysis through Proposition 9.4.1 using the same data sets. It is clear that $W_i^- \in \mathcal{W}_i$ for each i with $Q_w^i = -I$, $S_w^i = 0$ and $R_w^i = N\sigma^2 I$. For each data set, we minimize γ subject to the LMIs (9.10) for $i = 1, 2, 3$. The resulting γ values provide a guaranteed upper bound on the $\mathscr{H}_\infty$ norm of $u \to y$ and are shown in Figure 9.1 in red.

The computed value of γ using either Proposition 9.3.1 or Proposition 9.4.1 is a guaranteed upper bound for the $\mathscr{H}_\infty$ norm of the true system for all noise levels. The bound provides a good approximation of γ_0 for low noise levels. For increasing noise levels, the bound γ becomes more conservative for both methods. Comparing the results from Proposition 9.3.1 (unstructured data) with Proposition 9.4.1 (structured data), the bounds obtained from (9.10) are conservative with respect to those from (9.6) for higher noise levels, while the difference is small for low noise levels. By solving the unstructured data-based conditions in (Koch et al., 2020b, Theorem 4), we find the same bounds as obtained per Proposition 9.3.1, as expected from the duality of the results.

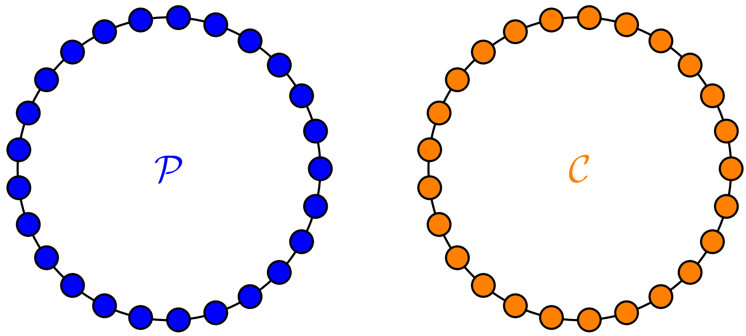


Figure 9.2: The problem in Example 2 is to synthesize a distributed controller ($\mathcal{C}$) from noisy input-state data collected from an interconnected system ($\mathcal{P}$) with $L = 25$ subsystems on a cycle graph, such that $w \to z$ achieves a given $\mathscr{H}_\infty$ norm.

9.6.2 Example 2: Distributed $\mathscr{H}_\infty$ controller synthesis

Consider an interconnected system with $L = 25$ subsystems, each having one state ($n_i = 1$). The subsystems are interconnected according to a *cycle* graph $\mathcal{G}$ (as depicted in Figure 9.2) and the matrices A_i and A_{ij} are drawn uniformly on the interval $[0, 1]$ and $[0, 0.1]$, respectively, and $B_i = 1$. We consider $y_i = x_i$ for all subsystems and consider the performance output $z_i = x_i$, so that $C_i^z = I$ and $C_{ij}^z = D_i^z = 0$. For the data acquisition, the input entries are drawn from a normal distribution with zero mean and unit variance. The noise signals $w_i(k)$ are drawn uniformly from the set $\{w \,|\, |w| \leq \sigma\}$, where $\sigma = 0.05$ is the noise level.

Hence, considering the sets $\mathcal{W}_i$ with $Q_w^i = -I$, $S_w^i = 0$ and $R_w^i = N\sigma^2 I$, we have that the noise sequences satisfy $W_-^i \in \mathcal{W}_i$, $i = 1, \dots, L$.

The goal is to synthesize a distributed controller that yields an upperbound γ on the $\mathscr{H}_\infty$ norm of the channel $w \to z$, without using knowledge of A_i, A_{ij} and B_i. First, we verify what the smallest upperbound γ is, for which there exists a *model-based* distributed controller by the *nominal* LMIs in (Van Horssen and Weiland, 2016, Theorem 2). This smallest upperbound of γ is 1.00 and serves as a benchmark: our data-based method for distributed control cannot perform better than the model-based distributed controller. We generate the data matrices (U_i^-, X_i) for $N = 50$ samples. For $\alpha_i := \alpha = 1$, we observe that the LMIs (9.14) and (9.15) are feasible for $\gamma = 1.10$. Hence, by Theorem 9.5.1, there exists a distributed controller that achieves an $\mathscr{H}_\infty$ norm less than 1.10 in closed-loop with the true interconnected system.

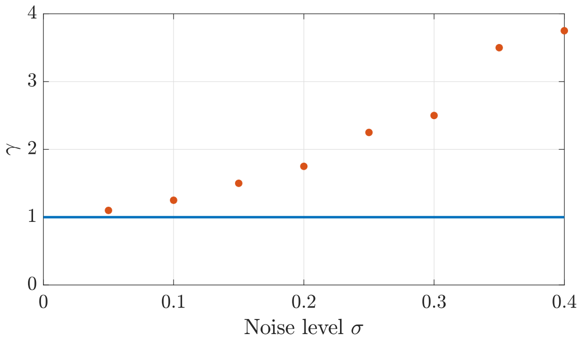


Figure 9.3: Example 2: Achievable $\mathscr{H}_\infty$ norm with distributed control from noisy data with increasing noise levels σ (red) and achievable $\mathscr{H}_\infty$ norm with distributed control computed using the true system (blue).

Next, we increase the noise level, up to $\sigma = 0.4$. The resulting values of γ are shown in Figure 9.3 and obtained from the conditions in Theorem 9.5.1 by varying α in a discrete interval. We observe that the conservatism increases for increasing noise levels. This can be explained by the increasing size of $\Sigma_{\mathcal{D}}^i$, leading to the existence of a more conservative distributed controller that achieves $\mathscr{H}_\infty$ performance for all interconnected systems consistent with the data.

9.7 Conclusions

We have considered the problem of analyzing the $\mathscr{H}_\infty$ norm of an interconnected system and finding a distributed controller that achieves $\mathscr{H}_\infty$ performance based on noisy data. First, we considered a dual parametrization of the set of systems consistent with the data and we presented a dual result for data-based dissipativity analysis, with respect to the results in (Koch et al., 2020b). A dual parametrization of data-compatible subsystems allowed us to introduce an interconnected system with LFT representations of the subsystems. We have presented sufficient LMI conditions based on data that guarantee $\mathscr{H}_\infty$ performance or the existence of a distributed controller that achieves $\mathscr{H}_\infty$ performance.

The noise that affects the system has been characterized by quadratic bounds in this chapter, which can represent, for example, magnitude or energy bounds. This characterization can yield guarantees on the achieved performance for finite data, compared to the methods described in Chapter 7 and Chapter 8 in Part II, as well as Chapter 2 in conjunction with Chapter 4 in Part I, that yield asymptotic properties. Alternative noise characterizations with practical relevance will be utilized in the sequel of Part III for performance analysis and controller synthesis.

Appendix

9.A Proof of Lemma 9.3.2

Proof. Let $M := \begin{bmatrix} A & B \end{bmatrix}$ and

$$\bar{P} := \begin{bmatrix} \bar{Q}_{\mathcal{D}} & \bar{S}_{\mathcal{D}} \\ \bar{S}_{\mathcal{D}}^\top & \bar{R}_{\mathcal{D}} \end{bmatrix} \in \mathbb{R}^{n \times n}.$$

Since $\operatorname{col}(U_-, X_-)$ has full row rank, $(A, B) \in \Sigma_{\mathcal{D}}$ if and only if

$$\begin{bmatrix} -M^\top \\ I \end{bmatrix}^\top \begin{bmatrix} \bar{Q}_{\mathcal{D}} & \bar{S}_{\mathcal{D}} \\ \bar{S}_{\mathcal{D}}^\top & \bar{R}_{\mathcal{D}} \end{bmatrix} \begin{bmatrix} -M^\top \\ I \end{bmatrix} \succeq 0 \quad \text{and} \quad \bar{Q}_{\mathcal{D}} \prec 0,$$

by Lemma 9.3.1. This is equivalent to

$$\bar{P} \succeq 0 \text{ on } \operatorname{im} \begin{bmatrix} -M^\top \\ I \end{bmatrix} \quad \text{and} \quad \bar{P} \prec 0 \text{ on } \operatorname{im} \begin{bmatrix} I \\ 0 \end{bmatrix}. \tag{9.16}$$

Since the direct sum of $\operatorname{im} \begin{bmatrix} -M^\top \\ I \end{bmatrix}$ and $\operatorname{im} \begin{bmatrix} I \\ 0 \end{bmatrix}$ is equal to $\mathbb{R}^n$, it follows by the dualization lemma (Scherer and Weiland, 2017, Lemma 4.9) that (9.16) holds if an only if

$$\bar{P}^{-1} \preceq 0 \text{ on } \operatorname{im} \begin{bmatrix} -M^\top \\ I \end{bmatrix}^\perp \quad \text{and} \quad \bar{P}^{-1} \succ 0 \text{ on } \operatorname{im} \begin{bmatrix} I \\ 0 \end{bmatrix}^\perp,$$

which is equivalent to

$$\bar{P}^{-1} \preceq 0 \text{ on } \operatorname{im} \begin{bmatrix} I \\ M \end{bmatrix} \quad \text{and} \quad \bar{P}^{-1} \succ 0 \text{ on } \operatorname{im} \begin{bmatrix} 0 \\ I \end{bmatrix}.$$

Thus, (9.16) holds if and only if

$$\begin{bmatrix} I \\ M \end{bmatrix}^\top \begin{bmatrix} Q_{\mathcal{D}} & S_{\mathcal{D}} \\ S_{\mathcal{D}}^\top & R_{\mathcal{D}} \end{bmatrix} \begin{bmatrix} I \\ M \end{bmatrix} \preceq 0 \quad \text{and} \quad R \succ 0,$$

which proves the assertion. $\square$

9.B Definition scales

Let X_{ij}^{11} and $\bar{X}_{ij}^{11}$ be symmetric matrices. We define

$$\begin{aligned}
Z_i^{11} &:= -\operatorname*{diag}_{j\in\mathbb{Z}_{[1:L]}} X_{ij}^{11}, Z_i^{22} := \operatorname*{diag}_{j\in\mathbb{Z}_{[1:L]}} X_{ji}^{11},\\
\bar{Z}_i^{11} &:= -\operatorname*{diag}_{j\in\mathbb{Z}_{[1:L]}} \bar{X}_{ij}^{11}, \bar{Z}_i^{22} := \operatorname*{diag}_{j\in\mathbb{Z}_{[1:L]}} \bar{X}_{ji}^{11},\\
Z_i^{12} &:= \operatorname{diag}\left(-\operatorname*{diag}_{j\in\mathbb{Z}_{[1:i]}} X_{ij}^{12}, \operatorname*{diag}_{j\in\mathbb{Z}_{[i+1:L]}} (X_{ji}^{12})^\top\right),\\
\bar{Z}_i^{12} &:= \operatorname{diag}\left(-\operatorname*{diag}_{j\in\mathbb{Z}_{[1:i]}} \bar{X}_{ij}^{12}, \operatorname*{diag}_{j\in\mathbb{Z}_{[i+1:L]}} (\bar{X}_{ji}^{12})^\top\right).
\end{aligned}$$

Chapter 10

Data-informativity for control: ellipsoidal cross-covariance noise bounds

In this chapter, we address the design of controllers based on noisy data that are not necessarily informative for identification, under the assumption that the noise satisfies sample cross-covariance bounds with respect to an instrumental variable. New controller synthesis methods are developed that extend existing frameworks in two relevant directions: a more general noise characterization in terms of cross-covariance bounds and informativity conditions for control based on input-output data. Previous works have derived necessary and sufficient informativity conditions for noisy input-state data with quadratic noise bounds via an S-procedure. Although these bounds do not capture cross-covariance bounds in general, we show that the S-procedure is still applicable for obtaining non-conservative conditions on the data. Informativity-conditions for stability, $\mathscr{H}_\infty$ and $\mathscr{H}_2$ control are developed, which are sufficient for input-output data and also necessary for input-state data. Simulation experiments illustrate that cross-covariance bounds can be less conservative for informativity, compared to norm bounds typically employed in the literature.

This chapter is based on the publication: T. R. V. Steentjes, M. Lazar, and P. M. J. Van den Hof. On data-driven control: Informativity of noisy input-output data with cross-covariance bounds. *IEEE Control Systems Letters*, 6:2192–2197, 2022b

10.1 Introduction

When mathematical models of dynamical systems are not available, data plays an essential role in the process of learning system characteristics. Indeed, data can contain information about the system from which a model of the system can be derived or a controller can be learned, either from a data-based model or directly from the data. A key problem for data-driven control is to determine whether a set of data collected from a system contains enough information to design a controller, independent of the methodology.

An indirect approach for controller design from data consists of two steps: obtaining a model from data through system identification (Ljung, 1999) and subsequently designing a controller via a model-based method. In the field of identification for control, the problem of determining a suitable model for controller design is considered (Van den Hof and Schrama, 1995), (Gevers, 2005), aiming at minimizing performance degradation due to model mismatching. Whether the data used for obtaining a model are sufficiently rich for identification, is determined by a property called informativity.

Even if data are not informative for identification, data can still be informative for controller design. Necessary and sufficient conditions for informativity of data for control were developed in (van Waarde et al., 2020) for noiseless input-state data. These results were extended in (van Waarde et al., 2022) for noisy input-state data with prior knowledge on the noise in the form of quadratic bounds, via a matrix variant of the S-procedure. Quadratic noise bounds play a key role in data-driven controller design (van Waarde et al., 2022), (De Persis and Tesi, 2020), (Berberich et al., 2021), distributed controller design (Chapter 9) and dissipativity analysis (Koch et al., 2020a), (van Waarde et al., 2021) from data, and represent, for example, magnitude, energy and variance bounds on the noise.

In this chapter, we consider the problem of determining informativity of *input-output* and input-state data for control with prior knowledge of the noise in the form of a sample *cross-covariance* type bound with respect to a user-chosen instrumental signal. Bounds on the sample cross-covariance were introduced in (Hakvoort and Van den Hof, 1995) as an alternative to magnitude bounds in parameter bounding identification, given its overly conservative noise characterization, cf. (Bisoffi et al., 2021a) for a comparison of instantaneous and (quadratic) energy type bounds for data-driven control. Our approach to data-driven control extends existing frameworks in two relevant directions: a more general noise characterization in terms of cross-covariance bounds with practical relevance and informativity conditions for control based on input-output data. We provide sufficient conditions for informativity for stabilization, $\mathscr{H}_\infty$ and $\mathscr{H}_2$ control, which are also necessary for input-state data.

10.2 Input-output data: cross-covariance bounds

Consider a class of linear systems described by

$$A(q^{-1})y(t) = B(q^{-1})u(t) + e(t), \tag{10.1}$$

with $A(\xi) \in \mathbb{R}^{p\times p}[\xi]$ and $B(\xi) \in \mathbb{R}^{p\times m}[\xi]$ polynomial matrices, given by $A(\xi) = I + A_1\xi + A_2\xi^2 + \cdots + A_n\xi^l$ and $B(\xi) = B_0 + B_1\xi + B_2\xi^2 + \cdots + B_l\xi^l$ and q^{-1} is the delay operator, i.e., $q^{-1}x(t) = x(t-1)$. By defining $\zeta(t) := \mathrm{col}(y(t-1), \ldots, y(t-l), u(t-1), \ldots, u(t-l))$, $\bar{A} := \mathrm{row}(-A_1, \ldots, -A_l)$ and $\bar{B} := \mathrm{row}(B_1, \ldots, B_l)$, we can write (10.1) equivalently as

$$y(t) = B_0 u(t) + e(t) + \begin{bmatrix}\bar{A} & \bar{B}\end{bmatrix}\zeta(t), \tag{10.2}$$

Hence, with $\zeta \in \mathbb{R}^n$ as a state, a state-space representation is

$$\begin{aligned}
\zeta(t+1) &= \underbrace{\left[\begin{array}{cc|cc} \multicolumn{2}{c|}{\bar{A}} & \multicolumn{2}{c}{\bar{B}} \\ \hline I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 \end{array}\right]}_{=:A_z}\zeta(t) + \underbrace{\begin{bmatrix} B_0 \\ \hline 0 \\ I \\ 0 \end{bmatrix}}_{=:B_z}u(t) + \underbrace{\begin{bmatrix} I \\ \hline 0 \\ 0 \\ 0 \end{bmatrix}}_{=:H_z}e(t), \\
y(t) &= \begin{bmatrix}\bar{A} & \bar{B}\end{bmatrix}\zeta(t) + B_0 u(t) + e(t),
\end{aligned} \tag{10.3}$$

Notice that (10.3) is a non-minimal representation of order $n = (p+m)l$. Defining the data matrices $Z_- := \begin{bmatrix}\zeta(0) & \cdots & \zeta(N-1)\end{bmatrix}$, $Y_- := \begin{bmatrix}y(0) & \cdots & y(N-1)\end{bmatrix}$ and U_-, E_- accordingly, we obtain the data equation

$$Y_- = \begin{bmatrix}\bar{A} & \bar{B}\end{bmatrix} Z_- + B_0 U_- + E_-, \tag{10.4}$$

where $\bar{A}$, $\bar{B}$, B_0 are unknown system matrices. We consider the noise not to be measured, i.e., E_- is unknown, while prior knowledge on the cross-covariance of the noise with respect to an instrumental variable is available.

10.2.1 Cross-covariance noise bounds

Consider the sample cross-covariance with respect to the noise $e \in \mathbb{R}^p$ and a variable $r \in \mathbb{R}^M$, given by $\frac{1}{\sqrt{N}}\sum_{t=0}^{N-1} e(t)r(t)^\top = \frac{1}{\sqrt{N}}E_- R_-^\top$. The variable r is instrumental and can be specified by the user (as discussed at the end of this subsection), i.e., it is a given variable in the upcoming analysis. We assume prior knowledge on the noise of the form

$$\frac{1}{N}E_- R_-^\top R_- E_-^\top \preceq H_u, \tag{10.5}$$

where H_u is an upper-bound on the squared sample cross-covariance matrix $\frac{1}{\sqrt{N}} E_- R_-^\top$. In a generalized form, we write

$$\begin{bmatrix} I \\ R_- E_-^\top \end{bmatrix}^\top \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^\top & Q_{22} \end{bmatrix} \begin{bmatrix} I \\ R_- E_-^\top \end{bmatrix} \succeq 0, \tag{10.6}$$

with $Q_{22} \prec 0$. For $Q_{11} = NH_u$, $Q_{12} = 0$ and $Q_{22} = -I$, the bound (10.5) is recovered. Note that (10.6) can be rewritten as the bound in (van Waarde et al., 2022, Assumption 1) with $\Phi_{22} = R_-^\top Q_{22} R_-$, but in general only $\Phi_{22} \preceq 0$ holds, while $\Phi_{22} \prec 0$ is assumed in (van Waarde et al., 2022). This means that the data informativity results of (van Waarde et al., 2022) cannot be used to establish data informativity for general cross-covariance noise bounds. However, the matrix S-lemma in (van Waarde et al., 2022, Theorem 13) can still be exploited to obtain necessary conditions for informativity of input-state data with cross-covariance bounds, as shown in Proposition 10.3.1 of this chapter.

In the state-space representation (10.3), the state-space matrices A_z and B_z contain unknown parameters $\bar{A}$, $\bar{B}$ and B_0. Write

$$A_z = \underbrace{\left[\begin{array}{c|c} \bar{A} & \bar{B} \\ \hline 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{array}\right]}_{=:\Lambda_e} + \underbrace{\left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ \hline I_{p(l-1)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I_{m(l-1)} & 0 \end{array}\right]}_{=:J_1} \tag{10.7}$$

$$\text{and } B_z = \underbrace{\left[\begin{array}{c} B_0 \\ \hline 0 \\ 0 \\ 0 \end{array}\right]}_{=:B_e} + \underbrace{\left[\begin{array}{c} 0 \\ \hline 0 \\ I_m \\ 0 \end{array}\right]}_{=:J_2}, \tag{10.8}$$

so that Λ_e and B_e are unknown parameter matrices, concatenated with zero rows, and J_1 and J_2 are binary matrices. The set of all pairs (Λ_e, B_e) that are compatible with the data is

$$\Sigma_{(U,Y)}^{RQ} := \left\{ \left[\begin{smallmatrix} \bar{A} & \bar{B} \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} B_0 \\ 0 \end{smallmatrix}\right] \right) \mid \exists E_- \text{ such that (10.6) and (10.4) hold} \}.$$

Remark 10.2.1. *A remark about the notation is in order. To keep the notation compact, the state-space matrices A_z and B_z were defined in* (10.3)*. The matrices Λ_e and B_e relate to A_z and B_z as defined in* (10.7)-(10.8) *and both depend affinely on the unknown matrices $\bar{A}$, $\bar{B}$ and B_0. Finally, the coefficients in $\bar{A}$, $\bar{B}$ and B_0 define the polynomial matrices A and B, describing the linear system* (10.1)*.*

Lemma 10.2.1. *Let* $Q_e := \begin{bmatrix} H_z Q_{11} H_z^\top & H_z Q_{12} \\ Q_{12}^\top H_z^\top & Q_{22} \end{bmatrix}$ *and consider*

$$\begin{bmatrix} I \\ \Lambda_e^\top \\ B_e^\top \end{bmatrix}^\top \begin{bmatrix} I & H_z Y_- R_-^\top \\ 0 & -Z_- R_-^\top \\ 0 & -U_- R_-^\top \end{bmatrix} Q_e \begin{bmatrix} I & H_z Y_- R_-^\top \\ 0 & -Z_- R_-^\top \\ 0 & -U_- R_-^\top \end{bmatrix}^\top \begin{bmatrix} I \\ \Lambda_e^\top \\ B_e^\top \end{bmatrix} \succeq 0. \tag{10.9}$$

It holds that $\Sigma_{(U,Y)}^{RQ} \subseteq \{(\Lambda_e, B_e) \,|\, (10.9) \text{ holds}\}$. *Moreover, if* $R_-[Z_-^\top \; U_-^\top]$ *has full column rank, then* $\Sigma_{(U,Y)}^{RQ} = \{(\Lambda_e, B_e) \,|\, (10.9) \text{ holds}\}$.

Proof. The first statement follows from (10.4), (10.6), and the definition of $\Sigma_{(U,Y)}^{RQ}$. Full-column rank of $R_-[Z_-^\top \; U_-^\top]$ and (10.9) imply that the last $ml + p(l-1)$ rows of $[\Lambda_e \; B_e] = \text{col}(M_1, 0)$ are zero and $E_- := Y_- - M_1[Z_-^\top \; U_-^\top]$ satisfies (10.6). Hence, $(\Lambda_e, B_e) \in \Sigma_{(U,Y)}^{RQ}$ so that $\Sigma_{(U,Y)}^{RQ} = \{(\Lambda_e, B_e) \,|\, (10.9) \text{ holds}\}$. $\square$

We denote all (A_z, B_z) that are compatible with the data by

$$\bar{\Sigma}_{(U,Y)}^{RQ} := \{(\Lambda_e + J_1, B_e + J_2) \,|\, (\Lambda_e, B_e) \in \Sigma_{(U,Y)}^{RQ}\}.$$

We have provided a parametrization of (a superset of) $\Sigma_{(U,Y)}^{RQ}$ based on the data equation (10.4). One can equivalently parametrize $\bar{\Sigma}_{(U,Y)}^{RQ}$ on the basis of the state data equation $Z_+ = A_z Z_- + B_z U_- + H_z E_-$. This leads to an equal set $\bar{\Sigma}_{(U,Y)}^{RQ}$, but the 'repeated' data in the parametrization contained in $Z_+ := \begin{bmatrix} \zeta(1) & \cdots & \zeta(N) \end{bmatrix}$, would render the evaluation numerically sensitive. Design methods in (van Waarde et al., 2022), (Berberich et al., 2021) can yield a parametrization $\bar{\Sigma}_{(U,Y)}^{RQ}$ based on this state data equation, but with limited applicability to cross-covariance bounds (10.6), i.e., only if the dimension of r satisfies $M \geq N$.

Existing guidelines (Hakvoort and Van den Hof, 1995) recommend choosing an instrumental variable r that is correlated with the input u, but uncorrelated with the noise e. Hence, this suggests the choice of (filtered/delayed versions) of the input for r in an open-loop case for data collection, and an external reference/dithering signal for r in a closed-loop case. Moreover, Lemma 10.2.1 provides an additional guideline for the choice of r to reduce conservatism in the case of input-output data, i.e., $R_-[Z_-^\top \; U_-^\top]$ has full column rank only if the number of instrumental variables M satisfies $M \geq pl + m(l+1)$.

10.2.2 Output-feedback control

Consider a (dynamic) output feedback controller described by the difference equation of the form (De Persis and Tesi, 2020)

$$C(q^{-1})u(t) = D(q^{-1})y(t), \tag{10.10}$$

with $C(\xi) \in \mathbb{R}^{m\times m}[\xi]$ and $D(\xi) \in \mathbb{R}^{m\times p}[\xi]$ polynomial matrices given by $C(\xi) = I + C_1\xi + C_2\xi^2 + \cdots + C_n\xi^l$ and $D(\xi) = D_1\xi + D_2\xi^2 + \cdots + D_n\xi^l$. We define a state ζ_c for (10.10) as $\zeta_c := \mathrm{col}(u(t-1), \ldots, u(t-l), y(t-1), \ldots, y(t-l))$, yielding a state-space representation for the controller:

$$\begin{aligned} \zeta_c(t+1) &= \left[\begin{array}{cc|cc} \multicolumn{2}{c|}{\bar{C}} & \multicolumn{2}{c}{\bar{D}} \\ \hline I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 \end{array}\right] \zeta_c(t) + \left[\begin{array}{c} 0 \\ \hline 0 \\ I \\ 0 \end{array}\right] y(t), \\ u(t) &= \begin{bmatrix} \bar{C} & \bar{D} \end{bmatrix} \zeta_c(t), \end{aligned} \tag{10.11}$$

with $\bar{C} := \mathrm{row}(-C_1, \ldots, -C_l)$ and $\bar{D} := \mathrm{row}(D_1, \ldots, D_l)$. It follows that $\zeta_c = \left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right] \zeta$, which implies that $u(t) = \begin{bmatrix} \bar{C} & \bar{D} \end{bmatrix} \zeta_c(t) = \begin{bmatrix} \bar{D} & \bar{C} \end{bmatrix} \zeta(t)$. Hence, the closed-loop system described by (10.3) and (10.11) has a representation

$$\zeta(t+1) = \underbrace{\left[\begin{array}{cc|cc} \multicolumn{2}{c|}{\bar{A} + B_0\bar{D}} & \multicolumn{2}{c}{\bar{B} + B_0\bar{C}} \\ \hline I & 0 & 0 & 0 \\ \hline \multicolumn{2}{c|}{\bar{D}} & \multicolumn{2}{c}{\bar{C}} \\ \hline 0 & 0 & I & 0 \end{array}\right]}_{=:A_{\mathrm{cl}}} \zeta(t) + \left[\begin{array}{c} I \\ \hline 0 \\ 0 \\ 0 \end{array}\right] e(t). \tag{10.12}$$

With $K := \begin{bmatrix} \bar{D} & \bar{C} \end{bmatrix}$, the closed-loop system matrix A_{cl} satisfies $A_{\mathrm{cl}} = A_z + B_z K$. For some $(A_z, B_z) \in \bar{\Sigma}^{RQ}_{(U,Y)}$, we say that the controller (10.10) stabilizes (10.1) if the closed-loop system (10.12) is stable, i.e., if all eigenvalues of $A_z + B_z K$ are in the open unit disk, since this implies stability of the closed-loop system (10.1) and (10.10). The notion of stabilization with respect to the state-space representation (10.3) was introduced in (De Persis and Tesi, 2020) for data-driven stabilization. We note that in the single-input-single-output case, (A_z, B_z) is controllable if and only if $A(\xi)$ and $B(\xi)$ are coprime (De Persis and Tesi, 2020).

10.3 Informativity for stabilization

10.3.1 Informativity of input-output data

Definition 10.3.1. *The data* (U, Y) *are said to be informative for quadratic stabilization by output-feedback controller* (10.10) *if there exist a* K *and* $P \succ 0$ *so that*

$$\bar{\Sigma}^{RQ}_{(U,Y)} \subseteq \{(A, B) \mid (A + BK)P(A + BK)^\top - P \prec 0\}.$$

By (10.7)-(10.8), we find that the existence of K and $P \succ 0$ so that $(A_z + B_z K)P(A_z + B_z K)^\top - P \prec 0$, is equivalent to the existence of K and $P \succ 0$ such that (10.13) holds true. Now, for the data (U, Y) to be informative for quadratic stabilization, we require the existence of K and $P \succ 0$ so that (10.13) holds for all $(\Lambda_e, B_e) \in \Sigma^{RQ}_{(U,Y)}$. This is precisely a problem that can be solved by the S-procedure; more specifically, by the matrix-valued S-lemma (van Waarde et al., 2022).

Theorem 10.3.1. *The data (U, Y) are informative for quadratic stabilization by output feedback controller* (10.10) *if there exist $L \in \mathbb{R}^{m\times n}$, $P \succ 0$, $\alpha \geq 0$ and $\beta > 0$ so that* (10.14) *holds true. Moreover, for L and P such that* (10.14) *is satisfied, $A_{cl} = A_z + B_z K$ is stable for all $(A_z, B_z) \in \bar{\Sigma}^{RQ}_{(U,Y)}$ with $K := LP^{-1}$.*

Proof. Let L, $P \succ 0$, $\alpha \geq 0$ and $\beta > 0$ exist so that (10.14) holds true and consider the matrix Π defined in (10.13). By the Schur complement, (10.14) is equivalent to

$$\Pi - \alpha\Lambda \succeq \begin{bmatrix} \beta I & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{where } K := LP^{-1} \quad \text{and}$$

$$\Lambda := \begin{bmatrix} I & H_z Y_- R_-^\top \\ 0 & -Z_- R_-^\top \\ 0 & -U_- R_-^\top \end{bmatrix} Q_e \begin{bmatrix} I & H_z Y_- R_-^\top \\ 0 & -Z_- R_-^\top \\ 0 & -U_- R_-^\top \end{bmatrix}^\top .$$

Hence, (10.13) holds true for all $(\Lambda_e, B_e) \in \Sigma^{RQ}_{(U,Y)}$, cf. (van Waarde et al., 2022, Theorem 13). This concludes the proof. $\square$

We remark that if there is a Z so that $\bar{Z}^\top \Lambda \bar{Z} \succ 0$ with $\bar{Z} := \operatorname{col}(I, Z)$, called the generalized Slater condition (van Waarde et al., 2022), then (10.14) is also a necessary condition for informativity of input-output data for quadratic stabilization, if $R_-[Z_-^\top \; U_-^\top]$ has full column rank (Lemma 10.2.1). Unlike in the case of input-state data, which will be discussed next, we note that the generalized Slater condition can in general not hold true in the input-output case if $l \geq 1$, since the noise affects a subspace of the extended state space, yielding a degenerate matrix Λ. The combination of noisy and noiseless states in ζ suggests that necessity could potentially be proven in general by a 'fusion' of the matrix S-lemma and matrix Finsler's lemma (van Waarde and Camlibel, 2021).

$$\begin{bmatrix} I \\ \Lambda_e^\top \\ B_e^\top \end{bmatrix}^\top \underbrace{\begin{bmatrix} P - (J_1 + J_2 K) P (\star)^\top & -(J_1 + J_2 K) P & -(J_1 + J_2 K) P K^\top \\ -P (J_1 + J_2 K)^\top & -P & -P K^\top \\ -K P (J_1 + J_2 K)^\top & -K P & -K P K^\top \end{bmatrix}}_{=:\Pi} \begin{bmatrix} I \\ \Lambda_e^\top \\ B_e^\top \end{bmatrix} \succ 0 \tag{10.13}$$

10.3.2 Informativity of input-state data

We will now consider a special case, where input-state data is available instead of input output data. That is, we measure a state $y(t) = x(t)$ and the class of systems considered is

$$x(t+1) = Ax(t) + Bu(t) + e(t), \tag{10.15}$$

with the corresponding data equation

$$X_+ = AX_- + BU_- + E_-. \tag{10.16}$$

All systems that explain the data (U_-, X) for some E_- satisfying the cross-covariance bound (10.6) are in the set

$$\Sigma^{RQ}_{(U_-,X)} := \{(A,B) \mid \exists E_- \text{ such that (10.6) and (10.16) hold}\}.$$

By (10.16), the set of feasible systems is $\Sigma^{RQ}_{(U_-,X)} = \{(A,B) \mid (A,B) \text{ satisfies (10.17)}\}$, where

$$\begin{bmatrix} I \\ A^\top \\ B^\top \end{bmatrix}^\top \underbrace{\begin{bmatrix} I & X_+R_-^\top \\ 0 & -X_-R_-^\top \\ 0 & -U_-R_-^\top \end{bmatrix} Q \begin{bmatrix} I & X_+R_-^\top \\ 0 & -X_-R_-^\top \\ 0 & -U_-R_-^\top \end{bmatrix}^\top}_{=:\Lambda_X} \begin{bmatrix} I \\ A^\top \\ B^\top \end{bmatrix} \succeq 0. \tag{10.17}$$

Remark 10.3.1. *Consider a specific selection of $M = N$ instrumental variables defined by $r_i(t) := \delta(t-i+1)$, $i = 1, \ldots, N$, and $\delta : \mathbb{Z} \to \{0,1\}$ is the unit impulse defined as $\delta(0) = 1$ and $\delta(x) = 0$ for $x \in \mathbb{Z} \setminus \{0\}$. It follows that $R_- = I$ for this choice of instrumental signals. Then, with the generalized quadratic cross-covariance bound* (10.6)*, we observe that for this special choice $R_- = I$, we recover the set of feasible systems in (van Waarde et al., 2022), and, hence, the informativity conditions in (van Waarde et al., 2022).*

Definition 10.3.2. *The data (U_-, X) are said to be informative for quadratic stabilization by state feedback if there exist a feedback gain K and $P \succ 0$ so that*

$$\Sigma^{RQ}_{(U_-,X)} \subseteq \{(A,B) \mid (A+BK)P(A+BK)^\top - P \prec 0\}.$$

$$\begin{bmatrix} P-\beta I & -J_1P - J_2L & 0 & J_1P + J_2L \\ \star & -P & -L^\top & 0 \\ 0 & -L & 0 & L \\ \star & 0 & L^\top & P \end{bmatrix} - \alpha \begin{bmatrix} I & H_zY_-R_-^\top \\ 0 & -Z_-R_-^\top \\ 0 & -U_-R_-^\top \\ 0 & 0 \end{bmatrix} Q_e\ (\star)^\top \succeq 0 \tag{10.14}$$

We will now provide a necessary and sufficient condition for informativity of input-state data for quadratic stabilization, given prior knowledge on the cross-covariance (10.6). Consider the generalized Slater condition

$$\begin{bmatrix} I \\ Z \end{bmatrix}^\top \Lambda_X \begin{bmatrix} I \\ Z \end{bmatrix} \succ 0. \tag{10.18}$$

Proposition 10.3.1. *Suppose that there exists a Z so that* (10.18) *holds true. Then the data (U_-, X) are informative for quadratic stabilization if and only if there exist $L \in \mathbb{R}^{m\times n}$, $P \succ 0$, $\alpha \geq 0$ and $\beta > 0$ so that*

$$\begin{bmatrix} P-\beta I & 0 & 0 & 0 \\ 0 & -P & -L^\top & 0 \\ 0 & -L & 0 & L \\ 0 & 0 & L^\top & P \end{bmatrix} - \alpha \begin{bmatrix} \Lambda_X & 0 \\ 0 & 0 \end{bmatrix} \succeq 0. \tag{10.19}$$

Moreover, K is such that $A+BK$ is stable for all $(A,B) \in \Sigma^{RQ}_{(U_-,X)}$ if $K := LP^{-1}$ with L and $P \succ 0$ satisfying (10.19)*.*

Proof. ($\Leftarrow$) This is proven by the same argument as in the proof of Theorem 10.3.1. ($\Rightarrow$) Let the data be informative for quadratic stabilization, i.e., there exist K and $P \succ 0$ so that, with Π defined in (10.13) with $J_1 = 0$, $J_2 = 0$:

$$\begin{bmatrix} I \\ A^\top \\ B^\top \end{bmatrix}^\top \Pi\,(\star) \succ 0 \text{ for all } (A,B) \text{ with } \begin{bmatrix} I \\ A^\top \\ B^\top \end{bmatrix}^\top \Lambda_X\,(\star) \succeq 0,$$

$$\text{where } \Lambda_X = \left[\begin{array}{c|c} \Lambda^X_{11} & \Lambda^X_{12} \\ \hline \Lambda^X_{21} & \Lambda^X_{22} \end{array}\right] := \left[\begin{array}{cc} I & X_+R_-^\top \\ \hline 0 & -X_-R_-^\top \\ 0 & -U_-R_-^\top \end{array}\right] Q\,(\star)^\top.$$

We will now show that $\ker \Lambda^X_{22} \subseteq \Lambda^X_{12}$, such that necessity follows by the matrix S-lemma (van Waarde et al., 2022). First, notice that $\ker \Lambda^X_{22} = \ker R_- \begin{bmatrix} X_-^\top & U_-^\top \end{bmatrix}$. Now, take any $x \in \ker \Lambda^X_{22}$. Then $R_- \begin{bmatrix} X_-^\top & U_-^\top \end{bmatrix} x = 0$. Clearly, we have that $(X_+R_-^\top Q_{22} + Q_{12})R_- \begin{bmatrix} X_-^\top & U_-^\top \end{bmatrix} x = 0$, which implies that $x \in \ker \Lambda^X_{12}$. Since $x \in \ker \Lambda^X_{22}$ was chosen arbitrary, this shows that $\ker \Lambda^X_{22} \subseteq \Lambda^X_{12}$. By $\ker \Lambda^X_{22} \subseteq \Lambda^X_{12}$ and (10.18), there exist $\alpha \geq 0$ and $\beta > 0$ so that, by (van Waarde et al., 2022, Theorem 13):

$$\Pi - \alpha\Lambda_X \succeq \begin{bmatrix} \beta I & 0 \\ 0 & 0 \end{bmatrix},$$

which is equivalent to (10.19) for $L := KP$ by the Schur complement. This completes the proof. $\square$

10.4 Including performance specifications

We will now consider the problem of finding a controller (10.10) for which the closed-loop system achieves an $\mathscr{H}_\infty$ or $\mathscr{H}_2$ performance bound from the input-output data (U, Y). Consider the performance output z, given by $z(t) = C_z\zeta(t) + D_z u(t)$. For any pair (A_z, B_z), the controller (10.10) yields the closed loop system

$$\begin{aligned}\zeta(t+1) &= (A_z + B_z K)\zeta(t) + H_z e(t),\\ z(t) &= (C_z + D_z K)\zeta(t).\end{aligned}$$

Hence, the transfer matrix from e to z is given by

$$T(q) := (C_z + D_z K)(qI - A_z - B_z K)^{-1} H_z,$$

for which the $\mathscr{H}_\infty$ and $\mathscr{H}_2$ norm are denoted $\|T\|_{\mathscr{H}_\infty}$ and $\|T\|_{\mathscr{H}_2}$, respectively.

For given K, the $\mathscr{H}_\infty$ norm of T is less than γ, $\|T\|_{\mathscr{H}_\infty} < \gamma$, if and only if there exists $X \succ 0$ such that (Scherer and Weiland, 2017, p. 125)

$$\begin{bmatrix} X & 0 & A_K^\top X & C_K^\top \\ 0 & \gamma I & H_z^\top X & 0 \\ X A_K & X H_z & X & 0 \\ C_K & 0 & 0 & \gamma I \end{bmatrix} \succ 0, \tag{10.20}$$

where $A_K := A_z + B_z K$ and $C_K := C_z + D_z K$.

Definition 10.4.1. *The data (U, Y) are said to be informative for common $\mathscr{H}_\infty$ control by output-feedback controller* (10.10) *with performance γ if there exist a K and $X \succ 0$ so that*

$$\bar{\Sigma}^{RQ}_{(U,Y)} \subseteq \{(A_z, B_z) \mid (10.20) \text{ holds true}\}.$$

Theorem 10.4.1. *The data (U, Y) are informative for common $\mathscr{H}_\infty$ control with performance γ if there exist $L \in \mathbb{R}^{m\times n}$, $P \succ 0$, $\alpha \geq 0$ and $\beta > 0$ so that* (10.21)-(10.22) *holds true.*

Proof. By a congruence transformation of (10.20) with $\operatorname{diag}(P, I, P, I)$ with $P := X^{-1}$ and the application of the Schur complement (twice), the existence of K and $X \succ 0$ so that (10.20) holds, is equivalent to the existence of P and L so that $P \succ 0$ and

$$P - V_z \underbrace{(P - \gamma^{-1} F^\top F)^{-1}}_{=:S} V_z^\top - \gamma^{-1} H_z H_z^\top \succ 0 \tag{10.23}$$

and $P - \gamma^{-1} F^\top F \succ 0$, where $V_z := A_z P + B_z L$ and $F := C_z P + D_z L$. We can now rewrite (10.23) as

$$\begin{bmatrix} I \\ A_z^\top \\ B_z^\top \end{bmatrix}^\top \begin{bmatrix} P - \gamma^{-1} H_z H_z^\top & 0 \\ 0 & -\begin{bmatrix} P \\ L \end{bmatrix} S \begin{bmatrix} P \\ L \end{bmatrix}^\top \end{bmatrix} \begin{bmatrix} I \\ A_z^\top \\ B_z^\top \end{bmatrix} \succ 0,$$

which is equivalent to

$$\begin{aligned} \begin{bmatrix} I \\ \Lambda_e^\top \\ B_e^\top \end{bmatrix}^\top \Pi_{\mathscr{H}_\infty} \begin{bmatrix} I \\ \Lambda_e^\top \\ B_e^\top \end{bmatrix} &:= \begin{bmatrix} I \\ \Lambda_e^\top \\ B_e^\top \end{bmatrix}^\top \begin{bmatrix} P - \gamma^{-1} H_z H_z^\top & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I \\ \Lambda_e^\top \\ B_e^\top \end{bmatrix} \\ - \begin{bmatrix} I \\ \Lambda_e^\top \\ B_e^\top \end{bmatrix}^\top \begin{bmatrix} J_1 P + J_2 L \\ P \\ L \end{bmatrix} S \, (\star)^\top \begin{bmatrix} I \\ \Lambda_e^\top \\ B_e^\top \end{bmatrix} &\succ 0. \end{aligned} \tag{10.24}$$

Hence, the data (U, Y) are informative for common $\mathscr{H}_\infty$ control with performance γ if and only if there exist $P \succ 0$ and L such that $P - \gamma^{-1} F^\top F \succ 0$ and (10.24) holds for all $(\Lambda_e, B_e) \in \Sigma_{(U,Y)}^{RQ}$. By assumption, there exist $P \succ 0$, L, $\alpha \geq 0$ and $\beta > 0$ such that (10.21) holds true. By the Schur complement, (10.21) is equivalent to $\Pi_{\mathscr{H}_\infty} - \alpha \Lambda \succeq \left[\begin{smallmatrix} \beta I & 0 \\ 0 & 0 \end{smallmatrix}\right]$, which implies that (10.24) holds for all $(\Lambda_e, B_e) \in \Sigma_{(U,Y)}^{RQ}$. □

The conditions (10.21)-(10.22) are linear with respect to P, L, α and β. By a straightforward additional application of the Schur complement, (10.21) can also be made linear with respect to γ.

By Proposition 4.A.1, we have that, for a given controller parameter matrix K, the $\mathscr{H}_2$ norm of T is less than γ, $\|T\|_{\mathscr{H}_2} < \gamma$, if and only if there exists $X \succ 0$

$$\begin{bmatrix} P - \gamma^{-1} H_z H_z^\top - \beta I & 0 & 0 & J_1 P + J_2 L & 0 \\ 0 & 0 & 0 & P & 0 \\ 0 & 0 & 0 & L & 0 \\ \star & \star & \star & P & F^\top \\ 0 & 0 & 0 & F & \gamma I \end{bmatrix} - \alpha \begin{bmatrix} I & H_z Y_- R_-^\top \\ 0 & -Z_- R_-^\top \\ 0 & -U_- R_-^\top \\ 0 & 0 \\ 0 & 0 \end{bmatrix} Q_e \, (\star)^\top \succeq 0, \tag{10.21}$$

$$\begin{bmatrix} P & F^\top \\ F & \gamma I \end{bmatrix} \succ 0 \tag{10.22}$$

$$\begin{bmatrix} P-\beta I & 0 & 0 & J_1P+J_2L & 0 \\ 0 & 0 & 0 & P & 0 \\ 0 & 0 & 0 & L & 0 \\ \star & \star & \star & P & F^\top \\ 0 & 0 & 0 & \star & I \end{bmatrix} - \alpha \begin{bmatrix} I & H_zY_-R_-^\top \\ 0 & -Z_-R_-^\top \\ 0 & -U_-R_-^\top \\ 0 & 0 \\ 0 & 0 \end{bmatrix} Q_e \begin{bmatrix} I & H_zY_-R_-^\top \\ 0 & -Z_-R_-^\top \\ 0 & -U_-R_-^\top \\ 0 & 0 \\ 0 & 0 \end{bmatrix}^\top \succeq 0 \tag{10.26}$$

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such that

$$\operatorname{trace} H_z^\top X H_z < \gamma^2 \text{ and } \quad X \succ A_K^\top X A_K + C_K^\top C_K. \tag{10.25}$$

Definition 10.4.2. *The data* (U, Y) *are said to be informative for common* $\mathscr{H}_2$ *control by output-feedback controller* (10.10) *with performance* γ *if there exist a* K *and* $X \succ 0$ *so that* $\bar{\Sigma}^{RQ}_{(U,Y)} \subseteq \{(A_z, B_z) \mid (10.25)$ *holds true*$\}$.

Theorem 10.4.2. *The data* (U, Y) *are informative for common* $\mathscr{H}_2$ *control with performance* γ *if there exist* $L \in \mathbb{R}^{m\times n}$*, symmetric* Z*,* $P \succ 0$*,* $\alpha \geq 0$ *and* $\beta > 0$ *so that* $\operatorname{trace} Z < \gamma^2$*,* (10.26) *holds true,*

$$\begin{bmatrix} P & F^\top \\ F & I \end{bmatrix} \succ 0 \quad \textit{and} \quad \begin{bmatrix} Z & H_z^\top \\ H_z & P \end{bmatrix} \succeq 0. \tag{10.27}$$

Proof. By a congruence transformation of (10.25) with $P := X^{-1}$ and the Schur complement (in both directions), it follows that (10.25) is equivalent to $P - F^\top F \succ 0$,

$$P - V_z(P - F^\top F)^{-1}V_z^\top \succ 0 \tag{10.28}$$

and $\operatorname{trace} H_z^\top P^{-1} H_z < \gamma^2$. Now, we can rewrite (10.28) as

$$\begin{bmatrix} I \\ A_z^\top \\ B_z^\top \end{bmatrix}^\top \begin{bmatrix} P & 0 \\ 0 & -\begin{bmatrix} P \\ L \end{bmatrix}(P - F^\top F)^{-1}\begin{bmatrix} P \\ L \end{bmatrix}^\top \end{bmatrix} \begin{bmatrix} I \\ A_z^\top \\ B_z^\top \end{bmatrix} \succ 0,$$

which, by (10.7)-(10.8), holds if and and only if

$$\begin{bmatrix} I \\ \Lambda_e^\top \\ B_e^\top \end{bmatrix}^\top \begin{bmatrix} J_1P+J_2L \\ P \\ L \end{bmatrix} S \begin{bmatrix} J_1P+J_2L \\ P \\ L \end{bmatrix}^\top \begin{bmatrix} I \\ \Lambda_e^\top \\ B_e^\top \end{bmatrix} \prec \begin{bmatrix} I \\ \Lambda_e^\top \\ B_e^\top \end{bmatrix}^\top \begin{bmatrix} P & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I \\ \Lambda_e^\top \\ B_e^\top \end{bmatrix}. \tag{10.29}$$

There exist $P \succ 0$, L so that (10.29), $P - F^\top F \succ 0$ and $\operatorname{trace} H_z^\top P^{-1} H_z < \gamma^2$ if and only if there exist $P \succ 0$, L, Z so that (10.29), $P - F^\top F \succ 0$, $Z - H_z^\top P^{-1} H_z \succeq 0$ and $\operatorname{trace} Z < \gamma^2$. Indeed, for $Z := H_z^\top P^{-1} H_z$ we infer $\operatorname{trace} Z < \gamma^2$. Sufficiency follows from $H_z^\top P^{-1} H_z \preceq Z \;\Rightarrow\; \operatorname{trace} H_z^\top P^{-1} H_z \leq \operatorname{trace} Z$. Hence, the data (U, Y) are informative for common $\mathscr{H}_2$ control with performance γ if and only if there exist $P \succ 0$, L, Z so that $Z - H_z^\top P^{-1} H_z \succeq 0$, $\operatorname{trace} Z < \gamma^2$, $P - F^\top F \succ 0$ and (10.29) hold for all $(\Lambda_e, B_e) \in \Sigma^{RQ}_{(U,Y)}$. By assumption, $\operatorname{trace} Z < \gamma^2$ is satisfied, $P - F^\top F \succ 0$, $Z - H_z^\top P^{-1} H_z \succeq 0$ follow by (10.27) and via an analogue argument as in the proof of Theorem 10.4.1, (10.29) holds for all $(\Lambda_e, B_e) \in \Sigma^{RQ}_{(U,Y)}$ by (10.26). $\square$

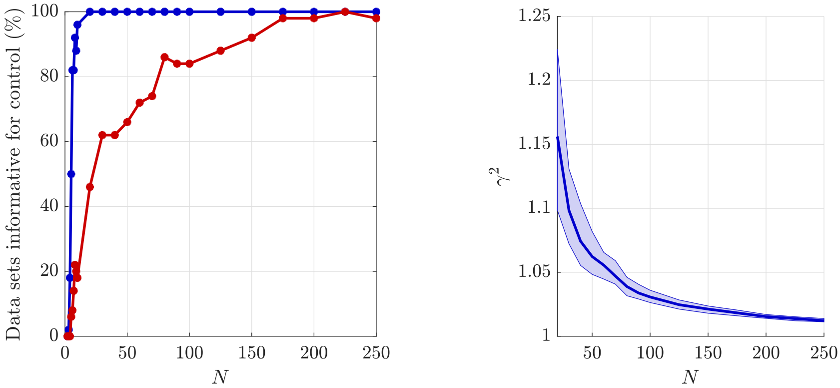
Remark 10.4.1. *The conditions in Theorem 10.4.1/10.4.2 are also necessary for informativity of input-state data for $\mathscr{H}_\infty/\mathscr{H}_2$ control, where $H_z = I$, $J_1 = 0$, $J_2 = 0$ and Y_- and Z_- are replaced by X_+ and X_-, if* (10.18) *holds for some Z.*

10.5 Numerical example

Consider the system (10.15) with true system matrices

$$A_0 = \begin{bmatrix} -0.2414 & -0.8649 & 0.6277 \\ 0.3192 & -0.0301 & 1.0933 \\ 0.3129 & -0.1649 & 1.1093 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}.$$

and consider a performance output $z(t) = [0\ 0\ 1]x(t)$. The objective is to determine if input state data collected from the system are informative for common $\mathscr{H}_2$ control. We consider a noise signal $e(t)$ with a uniform distribution, taking values from the closed ball $\{e \in \mathbb{R}^3 \mid \|e\|_2^2 \leq 0.35\}$. First, we consider this noise bound to be known, represented by the noise model $E_- \in \{E_- \mid E_- E_-^\top \preceq 0.35NI\}$ as described in (van Waarde et al., 2022, Section VI.A). This can be represented by the noise model (10.5) with $R_- = I$, cf. (van Waarde et al., 2022, Equation (5)). We consider the informativity analysis for various data lengths N ranging from $N = 2$ to $N = 250$. For each data length N, we generate 50 data sets. Given the bound on E_-, we can verify informativity for common $\mathscr{H}_2$ control via Theorem 17 in (van Waarde et al., 2022). We find that the generalized Slater condition (van Waarde et al., 2022, Equation (16)), holds true for all data sets, thus the data are informative for common $\mathscr{H}_2$ control with performance γ if and only if the condition (van Waarde et al., 2022, Equation ($\mathscr{H}_2$)) is feasible. The relative number of data sets for which this necessary and sufficient condition is feasible for some $\gamma > 0$ is visualized in Figure 10.1a for each data length N, in red. Naturally, if the condition is not feasible for any $\gamma > 0$, the data are actually not informative for feedback stabilization, although the true system *is* stable.



(a) Informativity for $\mathscr{H}_2$ control

(b) $\mathscr{H}_2$ performance from data

Figure 10.1: (a) Number of input-state data sets that are informative for $\mathscr{H}_2$ control versus data length N for noise-norm bounds (red) and quadratic cross-covariance bounds (blue) and (b) feasible γ^2 obtained versus data length N with quadratic cross-covariance bounds.

Now, we consider the quadratic cross-covariance bound (10.5) for the noise. We choose an instrumental variable that contains lagged versions of the input:

$$r(t) := \mathrm{col}(u(t), u(t-1), u(t-2), \dots, u(t-8), u(t-9)).$$

We assume prior knowledge in the sense that $E_- \in \mathcal{E}_{RQ} = \{E_- \mid E_- R_-^\top R_- E_-^\top \preceq NH_u\}$, where H_u is taken as $H_u = I$, independent of N. The cross-covariance bounds hold true for all generated data sets. We verify that there exists some Z so that (10.18) holds true for all data sets. Hence, by Remark 2, the data are informative for common $\mathscr{H}_2$ control with performance γ if and only if the conditions in Theorem 10.4.2 are feasible. The relative number of data sets for which this necessary and sufficient condition is feasible for some $\gamma > 0$ is visualized in Figure 10.1a for each data length N, in blue. For $N \geq 20$, all data sets are informative for common $\mathscr{H}_2$ control. For these data sets, the smallest $\mathscr{H}_2$ norm upper bounds γ^2 are visualized in Figure 10.1b, where the median performance is indicated by a solid line and the shaded area is bounded by the 25$^{\text{th}}$ and 75$^{\text{th}}$ percentiles. In comparison, the $\mathscr{H}_2$ norm that can be achieved by a state feedback controller with knowledge of (A_0, B_0) is equal to 1.000, which therefore is a benchmark that cannot be outperformed by any data-based controller.

Now, consider that noisy output measurements are available instead of state measurements. Consider the system (10.1) with $A(q^{-1})$ and $B(q^{-1})$ such that $T_0(q^{-1}) = A^{-1}(q^{-1})B(q^{-1})$ with $T_0 := C_0(qI - A_0)^{-1}B_0$, where C_0 is the output matrix. We consider three cases: $C_0 = [1\ 0\ 1]$, $C_0 = [0\ 1\ 0]$, and $C_0 = [1\ 0\ 0]$. The noise is uniformly drawn from $[-0.35, 0.35]$. For each choice of output, we generate 50 data sets for data lengths ranging from $N = 2$ to $N = 250$. We choose an instrumental signal containing lagged input signals as before, which is therefore independent on the choice of output. The upper-bound is chosen $H_u = 0.3$, which holds for all data sets. By Theorem 10.4.2, feasibility of the conditions for informativity for $\mathscr{H}_2$ control for some $\gamma > 0$ is verified for each data set. The results are depicted in Figure 10.2. We observe that the data sets are not informative for low data lengths, which can be expected. For increasing data length, informativity becomes dependent on the choice of output. For $N = 30$, for example, 90% of the data sets yielded feasible informativity conditions for the choice of $C_0 = [1\ 0\ 0]$, compared to less than 50% of the data sets for the other two choices for C_0.

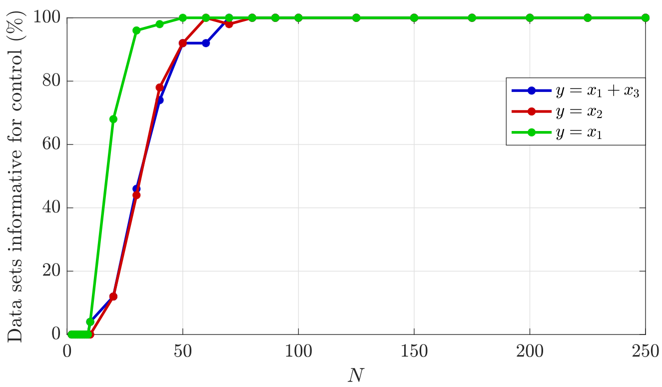


Figure 10.2: Effect of the choice of output on informativity of input-output data for $\mathscr{H}_2$ control, illustrated by the number data sets that satisfy the conditions in Theorem 10.4.2 versus the data length N.

10.6 Conclusions

We have considered the problem of informativity of input-output data for control, with prior knowledge of the noise in the form of quadratic sample cross-covariance bounds. Sufficient informativity conditions for stabilization, $\mathscr{H}_\infty$ and $\mathscr{H}_2$ control via dynamic output feedback were derived, which are also necessary if the state is measured. We have provided a numerical case study where data-informativity can be concluded with cross-covariance bounds, while the data are concluded to be non-informative with magnitude bounds. Finally, we have illustrated how the choice of output affects the informativity of input-output data via a numerical example.

The quadratic sample cross-covariance bounds on the noise have given new insights in the informativity analysis for controller design and can yield less conservative results. The bounds are specified with respect to an instrumental variable, however, that has to be chosen *a priori* and is therefore a crucial variable in the controller design. The specification of cross-covariance bounds is further addressed in Chapter 11, where bounds are specified with respect to individual noise components and the corresponding bounds are linked with the performance analysis of interconnected systems in Chapter 9.

Chapter 11

Data-informativity for control: polyhedral cross-covariance noise bounds

In this chapter, we address the informativity of input-state data for control where noise bounds are defined through the cross-covariance of the noise with respect to an instrumental variable; bounds that were introduced originally as a noise characterization in parameter bounding identification and were considered in a squared form in Chapter 10 in terms of the partial order on positive semi-definite matrices. The cross-covariance bounds considered in this chapter are defined by a finite number of hyperplanes, which induce a (possibly unbounded) polyhedral set of unfalsified systems. An advantage of this noise characterization is that the specification may be done with respect to each noise component separately. We provide informativity conditions for input-state data with polyhedral cross-covariance bounds for stabilization and $\mathscr{H}_2/\mathscr{H}_\infty$ control through vertex/half-space representations of the polyhedral set of unfalsified systems.

11.1 Introduction

In Chapter 10, we have considered the problem of determining informativity of data for controller design, with prior knowledge on the noise in the form of quadratic cross-covariance bounds. The quadratic cross-covariance bounds lead to a data-based parametrization of the set of feasible system matrices, which is an ellipsoid. Informativity conditions for control therefore follow by the application of the matrix S-lemma (van Waarde et al., 2022). However, because the bounds in Chapter 10 are specified in terms of the partial order on positive semi-definite matrices, prior knowledge on the cross covariance of individual noise components cannot be considered directly.

The problem that is considered in this chapter, is to determine if noisy data are informative for controller design, while taking into account bounds on the individual entries of the sample cross-covariance between noise components and instrumental variables. More specifically, we consider informativity of input-state data for controller design in the presence of noise satisfying *polyhedral* cross-covariance bounds. This prior knowledge combined with measurement data leads to sets of feasible system matrices that are intersections of halfspaces and therefore (possibly unbounded) polyhedra. We show how convexity of the sets of feasible system matrices and stability/performance criteria lead to data-based linear matrix inequalities (LMIs) that are necessary and sufficient for quadratic stabilization, $\mathscr{H}_\infty$ and $\mathscr{H}_2$ control in the case the polyhedron is bounded. The technique of using the convexity of polytopes for obtaining a finite set of controller synthesis LMIs is well known in robust control, e.g. for stabilization of systems with polytopic uncertainties, cf. (Kothare et al., 1996), (Scherer and Weiland, 2017, Chapter 5).

When the set of feasible system matrices is unbounded, there is no correspondent from robust control for systems with polytopic uncertainty. An unbounded set of feasible systems implies that data are not informative for system identification in the case of noise-free data, cf. (van Waarde et al., 2020, Example 19), and is therefore particularly interesting for informativity analysis. We provide preliminary results for data informativity for stabilization, in the case of noisy data that lead to a unbounded set of feasible systems.

Finally, utilizing the recently developed matrix S-procedure (van Waarde et al., 2022), we show how conservative approximations obtained through ellipsoidal supersets lead to sufficient but tractable conditions for data-driven control design with polyhedral cross-covariance bounds. For interconnected systems, we show how sets of feasible system matrices are determined per subsystem. The resulting polytopes can be approximated by ellipsoids for each subsystem separately. Application of the method described in Chapter 9 yields conditions for informativity of input-state data for distributed $\mathscr{H}_\infty$ control of interconnected systems.

11.2 Polyhedral cross-covariance bounds

In this chapter, we consider the data-informativity for a class of linear systems that is affected by a noise signal $e(t)$:

$$x(t+1) = Ax(t) + Bu(t) + e(t), \tag{11.1}$$

with state dimension n and input dimension m.

The true system is represented by the pair (A_0, B_0). State and input data generated by the true system are collected in the matrices

$$X := [x(0) \ \cdots \ x(N)], \quad U_- := [u(0) \ \cdots \ u(N-1)].$$

By defining

$$\begin{aligned} X_+ &:= [x(1) \ \cdots \ x(N)], \quad X_- := [x(0) \ \cdots \ x(N-1)], \\ E_- &:= [e(0) \ \cdots \ e(N-1)], \end{aligned}$$

we clearly have

$$X_+ = A_0 X_- + B_0 U_- + E_-. \tag{11.2}$$

In case the noise is *measured*, the set of systems that is consistent with the data (U_-, X) is

$$\Sigma_{(U_-, X, E_-)} = \{(A, B) \mid X_+ = AX_- + BU_- + E_-\}.$$

When the data are informative for system identification, as defined in (van Waarde et al., 2020), the set of feasible system is a singleton $\Sigma_{(U_-,X,E_-)} = \{(A_0, B_0)\}$. This is equivalent with $\operatorname{col}(X_-, U_-)$ having full rank. In the case the data are not informative, the set $\Sigma_{(U_-,X,E_-)}$ is not a singleton, but becomes a line or hyperplane. Even if the data are not informative for system identification, the data can still be informative for other properties, such as stability or feedback stabilization, cf. (van Waarde et al., 2020).

Let $e =: \operatorname{col}(e_1, \ldots, e_n)$ and consider that each noise channel e_j, $j = 1, \ldots, n$, is not measured, i.e., E_j^- is unknown, but that $e_j(t)$ satisfies the bounds

$$c_{ij}^l \leq \frac{1}{\sqrt{N}} \sum_{t=0}^{N-1} r_i(t) e_j(t) \leq c_{ij}^u, \quad i = 1, \ldots, M, \tag{11.3}$$

where r_i are signals that are chosen, typically as a (delayed version of) state or input signal, and c_{ij}^l, c_{ij}^u are specified bounds. Notice that we specify M upper and lower bounds for each noise channel $j \in \{1, \ldots, n\}$, and that the instrumental

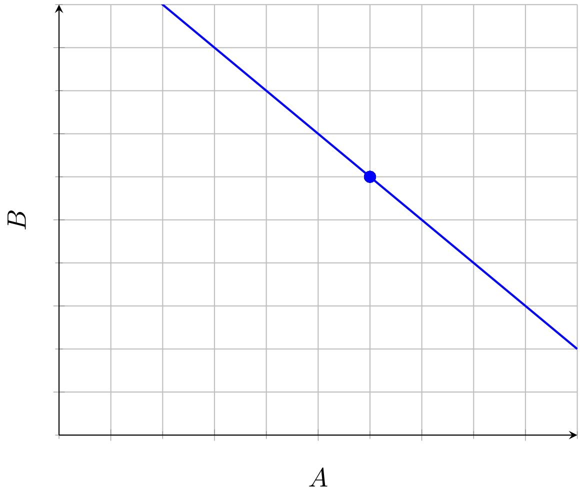


Figure 11.1: Informativity (blue dot) and non-informativity (blue line) of noiseless data (U_-, X) for system identification.

variables r_i, $i \in \{1, \dots, M\}$, are common for all noise channels $j \in \{1, \dots, n\}$. The bounds in (11.3) are satisfied for all j if and only if

$$E_- = \operatorname{col}(E_1^-, \dots, E_n^-) \in \mathcal{E}_R,$$

where

$$\begin{aligned}\mathcal{E}_R &:= \{E \mid C_l \leq \frac{1}{\sqrt{N}} E_- R_-^\top \leq C_u\} \\ &= \{E \mid C_l \leq \frac{1}{\sqrt{N}} \sum_{t=0}^{N-1} e(t) r(t)^\top \leq C_u\}.\end{aligned}$$

with $R_- := \operatorname{col}(R_1^-, \dots R_M^-)$ and with c_{ij}^l and c_{ij}^u the (i,j)-th entry of C_l and C_u, respectively. The inequalities defining $\mathcal{E}_R$ are thus *entry-wise* inequalities.

Remark 11.2.1. *Noise bounds of the type* (11.3) *define upper and lower bounds on the sample cross-covariance of the noise e and an instrumental variable r. These bounds were introduced in (Hakvoort and Van den Hof, 1995) for parameter bounding identification. An 'ellipsoidal' version of these bounds, i.e., a bound on $E_- R_-^\top R_- E_-^\top$ in the terms of the partial order on positive semi-definite matrices, has been considered in Chapter 10 for analyzing informativity for control. The difference in prior knowledge on the noise has two implications: (i)* *the bounds* (11.3) *allow a component-wise specification of bounds on the cross-covariance compared to ellipsoidal bounds, and (ii) incorporating this "polyhedral" (possibly unbounded) prior knowledge on the noise in the informativity analysis requires a fundamentally different approach compared with the application of the matrix S-lemma (van Waarde et al., 2022) used in Chapter 10, as will be discussed in Section 11.3.*

Remark 11.2.2. *Guidelines in the literature recommend choosing an instrumental variable r that is correlated with the input u, but uncorrelated with the noise e (Hakvoort and Van den Hof, 1995). We refer to (Hakvoort and Van den Hof, 1995) for more information on choosing r and estimating the bounds* (11.3) *from data.*

The bounds on the cross-covariance between the noise channels and the instrumental signals induce a restriction on the pairs (A, B) that satisfy the data equation

$$X_+ = AX_- + BU_- + E_-. \tag{11.4}$$

All systems that explain the data (U_-, X) for some $E_- \in \mathcal{E}_R$ are collected in the set $\Sigma^R_{(U_-,X)}$:

$$\Sigma^R_{(U_-,X)} := \{(A, B) \mid \exists E_- \in \mathcal{E}_R \text{ such that (11.4) holds}\}.$$

The following proposition provides a parametrization for the set of feasible systems with cross-covariance bounds.

Proposition 11.2.1. $\Sigma^R_{(U_-,X)} = \{(A, B) \mid (11.5)$ *holds*$\}$, *where*

$$\sqrt{N} C_l \leq X_+ R_-^\top - \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} X_- R_-^\top \\ U_- R_-^\top \end{bmatrix} \leq \sqrt{N} C_u \tag{11.5}$$

Proof. The set of feasible system matrices is

$$\Sigma^R_{(U_-,X)} = \{(A, B) \mid C_l \leq \frac{1}{\sqrt{N}} \sum_{t=0}^{N-1} e(t) r(t)^\top \leq C_u\} \tag{11.6}$$

$$= \{(A, B) \mid C^l \leq R^{N-}_{er} \leq C_u\}, \tag{11.7}$$

where

$$
\begin{aligned}
R_{er}^{N-} &:= \frac{1}{\sqrt{N}} \sum_{t=0}^{N-1} e(t) r(t)^\top \\
&= \frac{1}{\sqrt{N}} \sum_{t=0}^{N-1} \left(x(t+1) - Ax(t) - Bu(t)\right) r(t)^\top \\
&= \frac{1}{\sqrt{N}} X_+ R_-^\top - A \frac{1}{\sqrt{N}} X_- R_-^\top - B \frac{1}{\sqrt{N}} U_- R_-^\top.
\end{aligned}
$$

Hence, the feasible set of systems is

$$
\Sigma_{(U_-,X)}^R = \{(A,B) \,|\, (11.5) \text{ holds}\},
$$

which completes the proof. □

It can be shown that $\Sigma_{(U_-,X)}^R$ is an intersection of half spaces, by observing that

$$
\Sigma_{(U_-,X)}^R = \Sigma_{(U_-,X)}^{R_1} \cap \cdots \cap \Sigma_{(U_-,X)}^{R_M} = \bigcap_{i=1}^{M} \Sigma_{(U_-,X)}^{R_i},
$$

where, for $i = 1, \ldots, M$,

$$
\Sigma_{(U_-,X)}^{R_i} = \{(A,B) \,|\, c_i^l \le R_{xr_i}^{N+} - \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} R_{xr_i}^{N-} \\ R_{ur_i}^{N-} \end{bmatrix} \le c_i^u\},
$$

with $R_{xr_i}^{N+} = \frac{1}{\sqrt{N}} X_+ (R_i^-)^\top$, $R_{xr_i}^{N-} = \frac{1}{\sqrt{N}} X_- (R_i^-)^\top$ and $R_{ur_i}^{N+} = \frac{1}{\sqrt{N}} U_- (R_i^-)^\top$. Hence, the set of feasible subsystems is either an intersection of halfspaces and unbounded (called an $\mathcal{H}$-polyhedron) or it is a bounded polyhedron (called $\mathcal{V}$-polytope). Another way to see that $\Sigma_{(U_-,X)}^R$ is an intersection of halfspaces, is to vectorize the inequalities:

$$
\begin{aligned}
\Sigma_{(U_-,X)}^R = \{(A,B) \,|\, \operatorname{vec}(C_l) \le \operatorname{vec}(R_{xr}^{N+}) - \left(\begin{bmatrix} R_{xr}^{N-} \\ R_{ur}^{N-} \end{bmatrix}^\top \otimes I_n \right) \\
\times \operatorname{vec}\left(\begin{bmatrix} A & B \end{bmatrix} \right) \le \operatorname{vec}(C_u)\}.
\end{aligned}
$$

Lemma 11.2.1. *The set of feasible systems $\Sigma_{(U_-,X)}^R$ is bounded if and only if*

$$
\ker \begin{bmatrix} X_- R_-^\top \\ U_- R_-^\top \end{bmatrix}^\top = \{0\}. \tag{11.8}
$$

Proof. First, we note that $\Sigma^R_{(U_-,X)}$ is not empty. A non-empty polyhedron

$$\Sigma^R_{(U_-,X)} = \{(A,B) \mid M \operatorname{vec}(\begin{bmatrix} A & B \end{bmatrix}) \leq c\}$$

is unbounded if and only if there exists $v \neq 0$ so that $Mv \leq 0$. With

$$M := \begin{bmatrix} -\left(\begin{bmatrix} R^{N-}_{xr} \\ R^{N-}_{ur} \end{bmatrix}^\top \otimes I_n\right) \\ \left(\begin{bmatrix} R^{N-}_{xr} \\ R^{N-}_{ur} \end{bmatrix}^\top \otimes I_n\right) \end{bmatrix},$$

we observe that $Mv \leq 0$ if and only if $Mv = 0$. Hence, $\Sigma^R_{(U_-,X)}$ is unbounded if and only if

$$\ker\left(\begin{bmatrix} R^{N-}_{xr} \\ R^{N-}_{ur} \end{bmatrix}^\top \otimes I_n\right) \neq \{0\} \quad \Leftrightarrow \quad \ker \begin{bmatrix} R^{N-}_{xr} \\ R^{N-}_{ur} \end{bmatrix}^\top \neq \{0\}.$$

We conclude that $\Sigma^R_{(U_-,X)}$ is bounded if and only if $\ker \begin{bmatrix} R^{N-}_{xr} \\ R^{N-}_{ur} \end{bmatrix}^\top = \{0\}$, which concludes the proof. $\square$

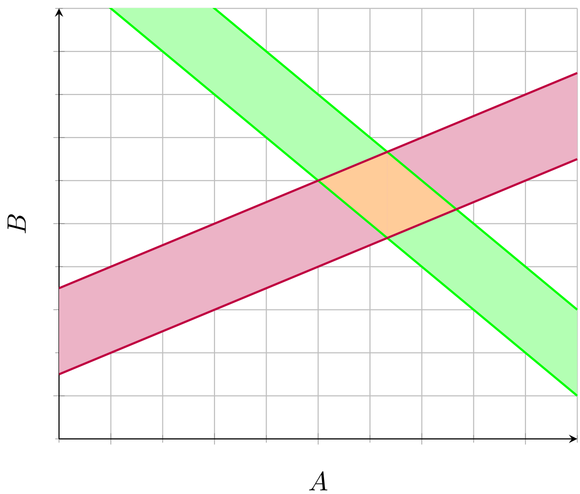


Figure 11.2: Illustration of the set $\Sigma^R_{(U_-,X)}$ for $M = 1$ (green) and for $M > 1$ (orange).

Remark 11.2.3. *The condition for boundedness of $\Sigma^R_{(U_-,X)}$ is equivalent with the matrix* $\operatorname{row}(R_- X_-^\top, R_- U_-^\top)$ *having full column rank. A necessary condition for the rank of this matrix being full, is to have enough instrumental signals. More precisely, a necessary condition for boundedness is that $M \geq n + m$, where we recall that n and m are the state and input dimension, respectively, and M is the dimension of the instrumental signal r. For the scalar case $n = m = 1$, an unbounded set $\Sigma^R_{(U_-,X)}$ is obtained for $M = 1$, as illustrated in Figure 11.2 in green. With $M > 1$ the rank condition can be satisfied (no redundant inequalities) and a polytope is obtained, as illustrated in Figure 11.2 in orange.*

11.3 Informativity for feedback stabilization

Consider the problem of stabilizing the 'true' system (A_0, B_0) using the data (U_-, X). We define the set of systems that are stabilized[1] by K as

$$\Sigma_K := \{(A, B) \mid A + BK \text{ is stable}\}.$$

In line with (van Waarde et al., 2020, Definition 12), we consider the following definition for informativity for stabilization by state feedback.

Definition 11.3.1. *The data (U_-, X) are said to be informative for stabilization by state feedback if there exists a feedback gain K so that*

$$\Sigma^R_{(U_-,X)} \subseteq \Sigma_K.$$

In other words, if there exists a K so that for every system (A, B) in $\Sigma^R_{(U_-,X)}$, $A+BK$ is stable, then the data are informative for stabilization by state feedback.

Definition 11.3.2. *The data (U_-, X) are said to be informative for quadratic stabilization by state feedback if there exist a K and $P \succ 0$ so that*

$$\Sigma^R_{(U_-,X)} \subseteq \{(A, B) \mid (A + BK)P(A + BK)^\top - P \prec 0\}. \tag{11.9}$$

Notice the difference: the data are informative for quadratic stabilization if there exists a common pair (K, P), with $P \succ 0$, such that the inclusion in Definition 11.3.2 holds, while the data are informative for stabilization if there is a common K so that $\Sigma^R_{(U_-,X)} \subseteq \Sigma_K$. Hence, the data (U_-, X) are informative for stabilization by state feedback if the data (U_-, X) are informative for *quadratic* stabilization by state feedback, but the reverse implication is not true, in general.

[1] A matrix is called stable if all its eigenvalues are in the open unit disk.

11.3.1 $\Sigma^R_{(U_-,X)}$ is an unbounded polyhedron

We consider here the scalar case, i.e., $m = n = 1$. In the case that there is one instrumental signal $r = r_1$, the set $\Sigma^R_{(U_-,X)}$ is described by two linear inequalities

$$\begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} R^{N-}_{xr} \\ R^{N-}_{ur} \end{bmatrix} \leq R^{N+}_{xr} - c^l, \ \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} R^{N-}_{xr} \\ R^{N-}_{ur} \end{bmatrix} \geq R^{N+}_{xr} - c^u.$$

We observe that $\Sigma^R_{(U_-,X)}$ is the intersection of two closed half-spaces. The following result states that a sufficient condition for data informativity for stabilization, is the existence of a K that stabilizes all systems on the "boundaries", i.e., the defining hyperplanes of $\Sigma^R_{(U_-,X)}$.

Proposition 11.3.1. *Let R^{N-}_{xr} be non-zero and let there exist $(R^{N-}_{xr})^\dagger$ such that*[2] *$R^{N-}_{xr}(R^{N-}_{xr})^\dagger = 1$ and*

$$(R^{N+}_{xr} - c^l)(R^{N-}_{xr})^\dagger \quad \text{and} \quad (R^{N+}_{xr} - c^u)(R^{N-}_{xr})^\dagger$$

are stable. Then the data (U_-, X) are informative for stabilization by state feedback. Moreover, K is such that $\Sigma^R_{(U_-,X)} \subseteq \Sigma_K$ if $K = R^{N-}_{ur}(R^{N-}_{xr})^\dagger$, with $(R^{N-}_{xr})^\dagger$ as described above.

Proof. Let $(R^{N-}_{xr})^\dagger$ be non-zero and such that

$$(R^{N+}_{xr} - c^l)(R^{N-}_{xr})^\dagger \quad \text{and} \quad (R^{N+}_{xr} - c^u)(R^{N-}_{xr})^\dagger$$

are stable. We will first show that

$$-1 < \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} R^{N-}_{xr} \\ R^{N-}_{ur} \end{bmatrix} (R^{N-}_{xr})^\dagger < 1.$$

Consider the case that $(R^{N-}_{xr})^\dagger$ is positive. Then

$$\begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} R^{N-}_{xr} \\ R^{N-}_{ur} \end{bmatrix} (R^{N-}_{xr})^\dagger \leq (R^{N+}_{xr} - c^l)(R^{N-}_{xr})^\dagger,$$
$$\begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} R^{N-}_{xr} \\ R^{N-}_{ur} \end{bmatrix} (R^{N-}_{xr})^\dagger \geq (R^{N+}_{xr} - c^u)(R^{N-}_{xr})^\dagger.$$

Furthermore, $R^{N+}_{xr} - c^u \leq R^{N+}_{xr} - c^l$ implies that

$$-1 < (R^{N+}_{xr} - c^u)(R^{N-}_{xr})^\dagger \leq (R^{N+}_{xr} - c^l)(R^{N-}_{xr})^\dagger < 1.$$

[2] Note that in this case ($n = 1$), $(R^{N-}_{xr})^\dagger$ is a scalar and is unique.

Hence, any $(A, B) \in \Sigma_{(U_-,X)}$ satisfies

$$-1 < \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} R_{xr}^{N-} \\ R_{ur}^{N-} \end{bmatrix} (R_{xr}^{N-})^{\dagger} < 1. \tag{11.10}$$

Similarly, if $(R_{xr}^{N-})^{\dagger}$ is negative, then $R_{xr}^{N+} - c^u \leq R_{xr}^{N+} - c^l$ implies that

$$-1 < (R_{xr}^{N+} - c^l)(R_{xr}^{N-})^{\dagger} \leq (R_{xr}^{N+} - c^u)(R_{xr}^{N-})^{\dagger} < 1,$$

and we again find that (11.10) for any $(A, B) \in \Sigma_{(U_-,X)}^R$.

Now, define $K := R_{ur}^{N-}(R_{xr}^{N-})^{\dagger}$ to observe that $-1 < A + BK < 1$ for any $(A, B) \in \Sigma_{(U_-,X)}^R$. Hence, there exists a K so that $\Sigma_{(U_-,X)}^R \subseteq \Sigma_K$, which completes the proof. □

Example 11.1. *Consider that data $X = [0\ 1.2\ 3\ 4.1\ 4.25]$, $U_- = [1\ 1\ -0.5\ -2]$ have been collected from a system with system matrices $A_0 = 1.5$ and $B_0 = 1$. The corresponding noise $E_- = [0.2\ 0.2\ 0.1\ 0.1]$ is unknown, but satisfies $E_- \in \mathcal{E}_R$ for $R_- = U_-$ with $C_u = -C_l = 0.25$. For this example, $(R_{xr}^{N+} - c^l)(R_{xr}^{N-})^{\dagger} = 0.6882$ and $(R_{xr}^{N+} - c^u)(R_{xr}^{N-})^{\dagger} = 0.8059$, hence the data are informative for stabilization by state feedback by Proposition 11.3.1 and $K = R_{ur}^{N-}(R_{xr}^{N-})^{\dagger} = -0.7353$ is indeed such that $A_0 + B_0 K$ is stable.*

Alternatively, the sufficient conditions for the data (U_-, X) to be informative for feedback stabilization can be stated in terms of linear matrix inequalities.

Proposition 11.3.2. *Let there exist a Θ satisfying $R_{xr}^{N-}\Theta = (R_{xr}^{N-}\Theta)^{\top}$ so that*

$$\begin{bmatrix} R_{xr}^{N-}\Theta & (R_{xr}^{N+} - c^l)\Theta \\ \Theta^{\top}(R_{xr}^{N+} - c^l)^{\top} & R_{xr}^{N-}\Theta \end{bmatrix} \succ 0 \quad \text{and} \tag{11.11}$$

$$\begin{bmatrix} R_{xr}^{N-}\Theta & (R_{xr}^{N+} - c^u)\Theta \\ \Theta^{\top}(R_{xr}^{N+} - c^u)^{\top} & R_{xr}^{N-}\Theta \end{bmatrix} \succ 0. \tag{11.12}$$

Then the data (U_-, X) are informative for stabilization by state feedback. Moreover, K is such that $\Sigma_{(U_-,X)}^R \subseteq \Sigma_K$ if $K = R_{ur}^{N-}\Theta((R_{xr}^{N-})^{\dagger}\Theta)^{-1}$.

Proof. The inequalities in (11.11)-(11.12) imply that $R_{xr}^{N-}\Theta$ is positive definite and that

$$[(R_{xr}^{N+} - c^l)\Theta(R_{xr}^{N-}\Theta)^{-1}](R_{xr}^{N-}\Theta)[\star]^{\top} - R_{xr}^{N-}\Theta \prec 0 \quad \text{and}$$
$$[(R_{xr}^{N+} - c^u)\Theta(R_{xr}^{N-}\Theta)^{-1}](R_{xr}^{N-}\Theta)[\star]^{\top} - R_{xr}^{N-}\Theta \prec 0.$$

Hence, $(R_{xr}^{N+} - c^l)\Theta(R_{xr}^{N-}\Theta)^{-1}$ and $(R_{xr}^{N+} - c^u)\Theta(R_{xr}^{N-}\Theta)^{-1}$ are stable. That is, there exists a right inverse $(R_{xr}^{N-})^{\dagger} := \Theta(R_{xr}^{N-}\Theta)^{-1}$ so that$(R_{xr}^{N+} - c^l)\Theta(R_{xr}^{N-})^{\dagger}$ and $(R_{xr}^{N+} - c^u)\Theta(R_{xr}^{N-})^{\dagger}$ are stable. Therefore, it follows by Proposition 11.3.1 that the data (U_-, X) are informative for stabilization by state feedback. □

11.3.2 $\Sigma^R_{(U_-,X)}$ is a bounded polyhedron

By Lemma 11.2.1, we observe that $\Sigma^R_{(U_-,X)}$ is a convex polytope with a finite number of vertices $\sigma^i_{(U_-,X)}$, $i = 1, \ldots, L$, if the data (U_-, X) and instrumental signals R_- satisfy (11.8). In the scalar case, for example, the set $\Sigma^R_{(U_-,X)}$ is then described by $L = 4$ vertices with $M = 2$ instrumental variables, as depicted in Figure 11.2.

By Definition 11.3.1, the data (U_-, X) are informative for stabilization by state feedback if there exists a K so that $A+BK$ is stable for all $(A, B) \in \Sigma^R_{(U_-,X)}$. If (11.8) holds true, then $\Sigma^R_{(U_-,X)} = \operatorname{conv}\{\sigma^1_{(U_-,X)}, \ldots, \sigma^L_{(U_-,X)}\}$. The following lemma allows us to verify stability conditions for all matrices (A, B) that are compatible with the data, by verifying the conditions at the extreme points of $\Sigma^R_{(U_-,X)}$.

Lemma 11.3.1. *Let $\Gamma \in \mathbb{S}^{n\times n}$,[3] let $\mathcal{S}_0$ be a set and let $F : \mathcal{S} \to \mathbb{S}^{n\times n}$ be a function with domain $\mathcal{S} = \operatorname{conv} \mathcal{S}_0$. Then $F(x) \prec \Gamma$ for all $x \in \mathcal{S}$ if and only if $F(x) \prec \Gamma$ for all $x \in \mathcal{S}_0$.*

Proof. The assertion is a strict version of the assertion in (Scherer and Weiland, 2017, Proposition 1.14). The proof follows *mutatis mutandis* by the proof of (Scherer and Weiland, 2017, Proposition 1.14). □

Now, given the (known) vertices $\sigma^i_{(U_-,X)}$, $i = 1, \ldots, L$, the problem of verifying informativity for stabilization can be reduced to verifying the stability condition at the extreme points of $\Sigma^R_{(U_-,X)}$, as shown by the following result:

Proposition 11.3.3. *Let (11.8) hold. The data (U_-, X) are informative for quadratic stabilization by state feedback if and only if there exist K and P so that $P \succ 0$ and*

$$\begin{bmatrix} I \\ K \end{bmatrix}^\top (\sigma^i_{(U_-,X)})^\top P \sigma^i_{(U_-,X)} \begin{bmatrix} I \\ K \end{bmatrix} - P \prec 0, \quad i = 1, \ldots, L. \tag{11.13}$$

Proof. Consider the matrix function $F : \Sigma^R_{(U_-,X)} \to \mathbb{S}^{n\times n}$, defined by $F(\sigma) := \operatorname{col}(I, K)^\top \sigma^\top P \sigma \operatorname{col}(I, K)$. Since $\Sigma^R_{(U_-,X)}$ is convex and $P \succ 0$, we infer that F is a convex function. Hence, by Lemma 11.3.1, $F(\sigma) \prec P$ for all $\sigma \in \Sigma^R_{(U_-,X)}$ if and only if $F(\sigma) \prec P$ for all $\sigma \in \{\sigma^1_{(U_-,X)}, \ldots, \sigma^L_{(U_-,X)}\}$. This proves the assertion. □

We note that the conditions in Proposition 11.3.3 are not linear with respect to K and P. The application of the Schur complement yields conditions equivalent to (11.13) that are LMIs:

[3] $\mathbb{S}^{n\times n}$ denotes the set of $n \times n$ symmetric matrices with real entries.

Corollary 11.3.1. *Let* (11.8) *hold. The data* (U_-, X) *are informative for quadratic stabilization by state feedback if and only if there exist* Y *and* M *so that*

$$\begin{bmatrix} Y & Z^\top (\sigma^i_{(U_-,X)})^\top \\ \sigma^i_{(U_-,X)} Z & Y \end{bmatrix} \succ 0, \quad i = 1, \dots, L, \tag{11.14}$$

with $Z := \operatorname{col}(Y, M)$. *Moreover,* K *is such that* $\Sigma^R_{(U_-,X)} \subseteq \Sigma_K$ *if* $K = MY^{-1}$.

Proof. By the Schur complement, the existence of K and $P \succ 0$ such that (11.9) is equivalent with

$$\exists K, P \quad \text{such that} \begin{bmatrix} P & (A+BK)^\top \\ A+BK & P^{-1} \end{bmatrix} \succ 0$$

for all $(A, B) \in \Sigma^R_{(U_-,X)}$. Define $Y := P^{-1}$ and $M := KP^{-1}$ and perform a congruence transformation with $\operatorname{diag}(Y, I)$ to obtain

$$\exists Y, M \quad \text{such that} \begin{bmatrix} Y & (AY+BM)^\top \\ AY+BM & Y \end{bmatrix} \succ 0$$

for all$(A, B) \in \Sigma^R_{(U_-,X)}$. By Lemma 11.3.1, we find that this is equivalent with (11.14), which proves the assertion. $\square$

Corollary 11.3.2. *Let* (11.8) *hold. The data* (U_-, X) *are informative for stabilization by state feedback if one (and therefore all) of the following equivalent statements holds:*

- *the data* (U_-, X) *are informative for quadratic stabilization by state feedback,*
- *there exist* K *and* P *so that* $P \succ 0$ *and* (11.13) *are satisfied,*
- *there exist* Y *and* M *so that* (11.14) *is satisfied.*

Example 11.2. *Consider again the system from Example 11.1 with* $A_0 = 1.5$ *and* $B_0 = 1$. *Consider that the noise* $e(t)$ *is drawn uniformly from the set* $\{e \mid e^2 \leq 0.2\}$ *and data* (U_-, X) *is collected for* $N = 10$. *We select four different instrumental variables* r *based on lagged versions of the input* u *with* $M \in \{2, 3, 4, 5\}$. *These are defined as* $r_M(t) := \operatorname{col}(u(t), u(t-1), \dots, u(t-M+1))$, *i.e.,* $r_2(t) = \operatorname{col}(u(t), u(t-1))$, $r_3(t) = \operatorname{col}(u(t), u(t-1), u(t-2))$, et cetera. *We assume prior knowledge on the cross-covariance through the bounds* (11.3) *with* $c^u_i = -c^l_i = 0.1$, $i = 1, \dots, M$; *these bounds are verified to hold for each of the four choices for* M. *Figure 11.3 shows the set of feasible systems* $\Sigma^R_{(U_-,X)}$ *for each choice of* r_M, *denoted* Σ^R_M, *illustrating a significant reduction in the size of* Σ^R_M *for increasing*

M. We verify that the data (U_-, X) are informative for quadratic stabilization by Corollary 11.3.1, since the LMIs (11.14) *are feasible for $M = 2, \ldots, 5$, yielding $K = -1.4842$ for $M = 5$ such that $\Sigma_K \subseteq \Sigma_5^R$.*

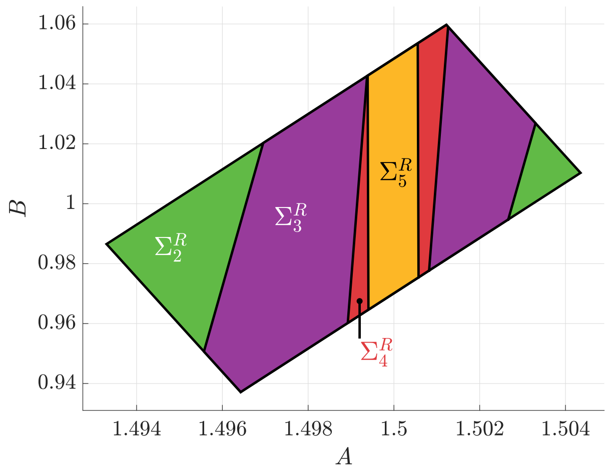


Figure 11.3: Feasible sets of systems $\Sigma^R_{(U_-,X)}$ obtained in Example 11.2 with different choices of R_- for $M \in \{2, 3, 4, 5\}$.

11.4 Including performance specifications

In this section, we will consider the problem of finding a feedback gain from the data (U_-, X), such that the closed-loop system with (A_0, B_0) satisfies a given $\mathscr{H}_\infty$ or $\mathscr{H}_2$ performance bound. Consider the performance output z, given by

$$z(t) = Cx(t) + De(t),$$

where C and D are user-specified matrices. Recall the set Σ_K; the set of systems that are stabilized by K. The set of systems that achieve $\mathscr{H}_\infty$ performance γ with feedback K is defined as

$$\Sigma_K^{\mathscr{H}_\infty}(\gamma) := \Sigma_K \cap \{(A, B) \mid \|T\|_{\mathscr{H}_\infty} < \gamma\},$$

with $T(q) := C(qI - A - BK)^{-1} + D$.

Definition 11.4.1. *The data (U_-, X) are said to be informative for $\mathscr{H}_\infty$ control with performance γ if there exists a feedback gain K so that $\Sigma^R_{(U_-,X)} \subseteq \Sigma_K^{\mathscr{H}_\infty}(\gamma)$.*

Proposition 11.4.1. *Consider a pair (A,B) and $\gamma > 0$. The following statements are equivalent:*

- *there exists K so that $(A,B) \in \Sigma_K^{\mathscr{H}_\infty}(\gamma)$,*
- *there exist K and P so that $P \succ 0$ and*

$$\left[\begin{array}{cc} I & 0 \\ A+BK & I \\ \hline 0 & I \\ C & D \end{array}\right]^\top \left[\begin{array}{cc|cc} -P & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ \hline 0 & 0 & -\gamma^2 I & 0 \\ 0 & 0 & 0 & I \end{array}\right] \left[\begin{array}{cc} I & 0 \\ A+BK & I \\ \hline 0 & I \\ C & D \end{array}\right] \prec 0. \tag{11.15}$$

Definition 11.4.2. *The data (U_-, X) are said to be informative for common $\mathscr{H}_\infty$ control with performance γ if there exist K and P so that $P \succ 0$ and (11.15) holds for all $(A,B) \in \Sigma^R_{(U_-,X)}$.*

Following a similar reasoning as for Proposition 11.3.3, necessary and sufficient conditions on the data for informativity for common $\mathscr{H}_\infty$ control are obtained, by Proposition 11.4.1 in conjunction with Lemma 11.3.1.

Proposition 11.4.2. *The data (U_-, X) are informative for common $\mathscr{H}_\infty$ control with performance γ if and only if there exist K and P so that $P \succ 0$ and for all $i \in \{1, \ldots, L\}$:*

$$\left[\begin{array}{cc} I & 0 \\ \sigma^i_{(U_-,X)} \begin{bmatrix} I \\ K \end{bmatrix} & I \\ \hline 0 & I \\ C & D \end{array}\right]^\top \left[\begin{array}{cc|cc} -P & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ \hline 0 & 0 & -\gamma^2 I & 0 \\ 0 & 0 & 0 & I \end{array}\right] \left[\begin{array}{cc} I & 0 \\ \sigma^i_{(U_-,X)} \begin{bmatrix} I \\ K \end{bmatrix} & I \\ \hline 0 & I \\ C & D \end{array}\right] \prec 0.$$

Application of the Schur complement to the conditions in Proposition 11.4.2 yields necessary and sufficient conditions in the form of LMIs.

Corollary 11.4.1. *The data (U_-, X) are informative for common $\mathscr{H}_\infty$ control with performance γ if and only if there exist Y and M so that for all $i \in \{1, \ldots, L\}$:*

$$\begin{bmatrix} Y & 0 & Z^\top (\sigma^i_{(U_-,X)})^\top & YC^\top \\ 0 & \gamma I & I & D^\top \\ \sigma^i_{(U_-,X)} Z & I & Y & 0 \\ CY & D & 0 & \gamma I \end{bmatrix} \succ 0,$$

with $Z := \operatorname{col}(Y, M)$.

The set of systems that achieve $\mathscr{H}_2$ performance γ with feedback K is defined as

$$\Sigma_K^{\mathscr{H}_2}(\gamma) := \Sigma_K \cap \{(A,B) \mid \|T\|_{\mathscr{H}_2} < \gamma\}.$$

Definition 11.4.3. *The data (U_-, X) are said to be informative for $\mathscr{H}_2$ control with performance γ if there exists a feedback gain K so that $\Sigma^R_{(U_-,X)} \subseteq \Sigma_K^{\mathscr{H}_2}(\gamma)$.*

Proposition 11.4.3. *Consider a pair (A,B) and $\gamma > 0$. The following statements are equivalent:*

- *there exists K so that $(A,B) \in \Sigma_K^{\mathscr{H}_2}(\gamma)$,*
- *there exist K, P and Z so that* trace $Z < \gamma$ *and*

$$\begin{bmatrix} P & P(A+BK) & P \\ \star & P & 0 \\ \star & \star & \gamma I \end{bmatrix} \succ 0, \quad \begin{bmatrix} P & 0 & C^\top \\ 0 & I & D^\top \\ C & D & Z \end{bmatrix} \succ 0. \tag{11.16}$$

Definition 11.4.4. *The data (U_-, X) are said to be informative for common $\mathscr{H}_2$ control with performance γ if there exists K, P and Z so that* trace $Z < \gamma$ *and* (11.16) *holds for all $(A,B) \in \Sigma^R_{(U_-,X)}$.*

The application of Lemma 11.3.1 to the conditions in Proposition 11.4.3 leads to necessary and sufficient conditions for informativity for common $\mathscr{H}_2$ control, as stated in Proposition 11.4.4. These conditions can be stated equivalently as LMIs through a variable transformation, leading to Corollary 11.4.2.

Proposition 11.4.4. *The data (U_-, X) are informative for common $\mathscr{H}_2$ control with performance γ if and only if there exist K, P and Z so that* trace $P < \gamma$ *and for all $i \in \{1, \dots, L\}$:*

$$\begin{bmatrix} P & 0 & C^\top \\ 0 & I & D^\top \\ C & D & Z \end{bmatrix} \succ 0 \quad \text{and} \quad \begin{bmatrix} P & P\sigma^i_{(U_-,X)} \begin{bmatrix} I \\ K \end{bmatrix} & P \\ \star & P & 0 \\ \star & \star & \gamma I \end{bmatrix} \succ 0.$$

Corollary 11.4.2. *The data (U_-, X) are informative for common $\mathscr{H}_2$ control with performance γ if and only if there exist Y, M and P so that* trace $P < \gamma$ *and*

$$\begin{bmatrix} Y & 0 & YC^\top \\ 0 & I & D^\top \\ CY & D & P \end{bmatrix} \succ 0 \quad \text{and} \quad \begin{bmatrix} Y & \sigma^i_{(U_-,X)} Z & I \\ \star & Y & 0 \\ \star & \star & \gamma I \end{bmatrix} \succ 0,$$

holds for all $i \in \{1, \dots, L\}$ with $Z :=$ col(Y, M).

By Proposition 11.4.2 and Proposition 11.4.4, we observe that the extension of the data-informativity conditions for stabilization by state feedback extends directly to conditions for $\mathscr{H}_\infty$ and $\mathscr{H}_2$ control, due to Lemma 11.3.1. Notice that the transfer matrix T under consideration, is the transfer from the disturbances e to the performance output z, i.e., the influence of the disturbance on the performance output z is penalized. If desired, the input u can be included in the performance output *mutatis mutandis.* Furthermore, the results in this section can be modified for a reference tracking problem, by considering the transfer T from a reference r to the corresponding performance output z (tracking error).

11.5 Approximating $\Sigma^R_{(U_-,X)}$ by an ellipsoidal superset

Consider the case where we have multiple instrumental signals such that $M \gg 1$ and that (11.8) holds. From Corollary 11.3.1, 11.4.1 and 11.4.2, we observe that the number of LMIs to be solved for concluding informativity for stabilization, $\mathscr{H}_\infty$ and $\mathscr{H}_2$ control, scales affinely with respect to the number of vertices of $\Sigma^R_{(U_-,X)}$, i.e., with respect to L. The number of vertices grows with the number of instrumental signals. For example, in the scalar case ($n = m = 1$), then for $M = 2$ we have $L = 4$. Similarly, for $M = 3$ we have $L = 6$, as depicted in Figure 11.4.

In the case that L is large, we can, alternatively, approximate $\Sigma^R_{(U_-,X)}$ by a superset, say $\bar{\Sigma}^R_{(U_-,X)}$, which is chosen such that

$$\Sigma^R_{(U_-,X)} \subseteq \bar{\Sigma}^R_{(U_-,X)}. \tag{11.17}$$

We note that this is equivalent with $\sigma^i_{(U_-,X)} \in \bar{\Sigma}^R_{(U_-,X)}$ for all $i = 1, \ldots, L$. Such a superset $\bar{\Sigma}^R_{(U_-,X)}$ can, for example, be an ellipse, as depicted in Figure 11.4 in blue. The closure of an ellipse can be described by a quadratic inequality, as utilized for the parametrization of the feasible set of systems in (van Waarde et al., 2022), (Koch et al., 2020a) and Chapter 9. If $\bar{\Sigma}^R_{(U_-,X)}$ is an ellipse in the scalar case, there exist Q, R and S so that for all $x \in \bar{\Sigma}^R_{(U_-,X)}$:

$$x^\top Q x - 2S^\top x + R \geq 0, \tag{11.18}$$

which is equivalent to

$$\begin{bmatrix} -x \\ I \end{bmatrix}^\top \begin{bmatrix} Q & S \\ S^\top & R \end{bmatrix} \begin{bmatrix} -x \\ I \end{bmatrix} \geq 0.$$

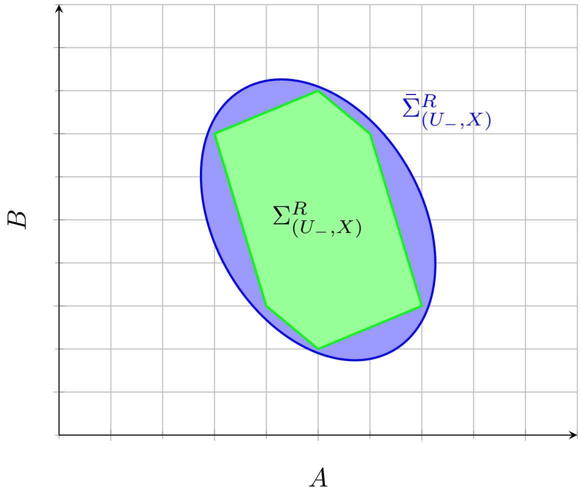


Figure 11.4: Feasible set of systems $\Sigma^R_{(U_-,X)}$ (green) with 6 vertices and the ellipse $\bar{\Sigma}^R_{(U_-,X)}$ (blue) so that $\Sigma^R_{(U_-,X)} \subseteq \bar{\Sigma}^R_{(U_-,X)}$.

In the general case ($n > 1$), we describe $\bar{\Sigma}^R_{(U_-,X)}$ by

$$(\star)^\top \begin{bmatrix} Q & S \\ S^\top & R \end{bmatrix} \begin{bmatrix} -A^\top \\ -B^\top \\ I \end{bmatrix} \succeq 0 \Leftrightarrow (\star)^\top \begin{bmatrix} R & -S^\top \\ -S & Q \end{bmatrix} \begin{bmatrix} I \\ A^\top \\ B^\top \end{bmatrix} \succeq 0, \tag{11.19}$$

i.e., $\bar{\Sigma}^R_{(U_-,X)} := \{(A, B) \,|\, (11.19) \text{ holds}\}$.

Under the condition (11.17), all systems that are compatible with the data are contained in $\bar{\Sigma}^R_{(U_-,X)}$. That is, the superset of systems explaining the data is parametrized by a quadratic matrix inequality. Hence, we can apply the matrix S-lemma (van Waarde et al., 2022) to conclude informativity for stabilization or performance for all systems in $\bar{\Sigma}^R_{(U_-,X)}$, and thus for all systems in $\Sigma^R_{(U_-,X)}$, by using the parametrization in (11.19).

Remark 11.5.1. *In this chapter we consider informativity for stabilization and $\mathscr{H}_2/\mathscr{H}_\infty$ control. The parametrization in* (11.19) *can also be used to determine if the data are informative for other properties, such as dissipativity, as considered in (Koch et al., 2020b), (van Waarde et al., 2021) and Chapter 9.*

Remark 11.5.2. *The frameworks in (van Waarde et al., 2022), (Koch et al., 2020a) and Chapter 9 are based on the assumption that the noise satisfies a quadratic matrix inequality. In this chapter, we assume that the noise satisfies* *cross-covariance bounds with respect to instrumental variables. So, although the assumption on the noise is different, we can still apply the above mentioned results, e.g. the matrix S-lemma, to conclude informativity due to the inclusion* (11.17)*. Some conservatism is introduced, because* $\Sigma^R_{(U_-,X)} \neq \bar{\Sigma}^R_{(U_-,X)}$ *in general, but the computational advantage increases with the number of vertices of* $\Sigma^R_{(U_-,X)}$ *and hence with the number of instrumental variables that is considered.*

11.5.1 Finding the smallest superset $\bar{\Sigma}^R_{(U_-,X)} \supseteq \Sigma^R_{(U_-,X)}$

Finding a minimum volume ellipsoid that covers $\Sigma^R_{(U_-,X)}$ is a convex optimization problem. Consider the vertex description of $\Sigma^R_{(U_-,X)}$:

$$\Sigma^R_{(U_-,X)} = \operatorname{conv}\{\sigma^1_{(U_-,X)}, \dots, \sigma^L_{(U_-,X)}\}$$

and consider the parameterization of an ellipsoid as in (Boyd and Vandenberghe, 2004)

$$\bar{\Sigma}^R_{(U_-,X)} = \{(A,B) \mid \|Hx+g\|_2 \leq 1, x = \operatorname{vec}[A\ B]^\top)\},$$

where the volume of $\bar{\Sigma}^R_{(U_-,X)}$ is proportional to $\det H^{-1}$. The problem of computing the minimum volume ellipsoid that contains $\Sigma^R_{(U_-,X)}$ can be written as (Boyd and Vandenberghe, 2004):

$$\begin{array}{ll} \text{minimize} & \log\det H^{-1} \\ \text{subject to} & \sup\limits_{x \in \bar{\Sigma}^R_{(U_-,X)}} \|Hx+g\|_2 \leq 1, \end{array}$$

with variables H and g. From the vertex representation of $\Sigma^R_{(U_-,X)}$, i.e., $\Sigma^R_{(U_-,X)} = \operatorname{conv}\{\sigma^1_{(U_-,X)}, \dots, \sigma^L_{(U_-,X)}\}$, it follows that this optimization problem is equivalent to

$$\begin{array}{ll} \text{minimize} & \log\det H^{-1} \\ \text{subject to} & \|H\sigma^i_{(U_-,X)} + g\|_2 \leq 1, \quad i = 1, \dots, L. \end{array}$$

The constraints and objective function of this problem are both convex (Boyd and Vandenberghe, 2004).

11.5.2 A practical approach to determine $\bar{\Sigma}^R_{(U_-,X)} \supseteq \Sigma^R_{(U_-,X)}$

Finding the smallest ellipse containing $\Sigma^R_{(U_-,X)}$ is a convex problem. However, the problem assumes that a vertex description of $\Sigma^R_{(U_-,X)}$ is available. For a high number of vertices, this can be tedious. While this is required to determine the smallest ellipsoidal superset, to determine the largest ellipsoidal subset of $\Sigma^R_{(U_-,X)}$, the half-space description (11.6) suffices. Consider a scaled version of $\Sigma^R_{(U_-,X)}$ with $\rho > 0$:

$$\rho\Sigma^R_{(U_-,X)} = \{(A, B) \,|\, a_i^\top \operatorname{vec}[A\ B]^\top \leq b_i, i = 1, \ldots, 2M\}.$$

Now, with a parametrization for the ellipsoid as

$$\bar{\Sigma}^R_{(U_-,X)} = \{Fx + d \,|\, \|x\|_2 \leq 1, x = \operatorname{vec}[A\ B]^\top\},$$

the maximum volume ellipsoid inside $\rho\Sigma^R_{(U_-,X)}$ is obtained by solving the following convex optimization problem for F and d:

$$\begin{aligned} &\text{minimize} && \log\det F^{-1} \\ &\text{subject to} && \|Fa_i\|_2 + a_i^\top d \leq b_i, \quad i = 1, \ldots, 2M. \end{aligned}$$

By taking $\rho > 0$ sufficiently large, the resulting ellipsoid $\bar{\Sigma}^R_{(U_-,X)} \subseteq \rho\Sigma^R_{(U_-,X)}$ will contain $\Sigma^R_{(U_-,X)}$. By reducing ρ, the volume can be decreased while satisfying $\Sigma^R_{(U_-,X)} \subseteq \bar{\Sigma}^R_{(U_-,X)}$, as shown in Figure 11.5.

11.5.3 Variable superset $\bar{\Sigma}^R_{(U_-,X)} \supseteq \Sigma^R_{(U_-,X)}$

Recall that $\Sigma^R_{(U_-,X)} = \operatorname{conv}\{\sigma^1_{(U_-,X)}, \ldots, \sigma^L_{(U_-,X)}\}$. Define

$$\mathcal{W} := \{W = \begin{bmatrix} R & -S^\top \\ -S & Q \end{bmatrix} \,|\, Q \prec 0,\ \begin{bmatrix} I \\ A^\top \\ B^\top \end{bmatrix}^\top W \begin{bmatrix} I \\ A^\top \\ B^\top \end{bmatrix} \succeq 0 \\ \text{for all } \ (A, B) \in \{\sigma^1_{(U_-,X)}, \ldots, \sigma^L_{(U_-,X)}\}.$$

Because any $W \in \mathcal{W}$ has a south-east block that is negative definite, we infer that the map

$$\begin{bmatrix} A^\top \\ B^\top \end{bmatrix} \mapsto \begin{bmatrix} I \\ A^\top \\ B^\top \end{bmatrix}^\top W \begin{bmatrix} I \\ A^\top \\ B^\top \end{bmatrix} =: \Gamma(W, A, B)$$

is concave (Scherer and Weiland, 2017), and, therefore, $\Gamma(W, A, B) \succeq 0$ for all $(A, B) \in \{\sigma^1_{(U_-,X)}, \ldots, \sigma^L_{(U_-,X)}\}$ implies $\Gamma(W, A, B) \succeq 0$ for all $(A, B) \in \Sigma^R_{(U_-,X)}$.

Define the superset with the parameterization in (11.19)

$$\bar{\Sigma}^R_{(U_-,X)} := \{(A, B) \,|\, \Gamma(W, A, B) \succeq 0,\ W \in \mathcal{W}\}.$$

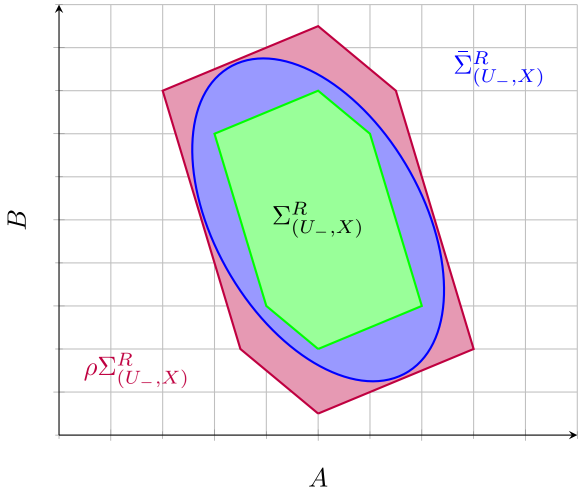


Figure 11.5: Feasible set of systems $\Sigma^R_{(U_-,X)}$ (green) with 6 vertices, scaled version $\rho\Sigma^R_{(U_-,X)}$ (purple) and the ellipse $\bar{\Sigma}^R_{(U_-,X)}$ (blue) so that $\Sigma^R_{(U_-,X)} \subseteq \bar{\Sigma}^R_{(U_-,X)} \subseteq \rho\Sigma^R_{(U_-,X)}$.

We observe now, that we can find a suitable superset by solving a finite set of LMIs

$$\begin{bmatrix} I \\ A^\top \\ B^\top \end{bmatrix}^\top W \begin{bmatrix} I \\ A^\top \\ B^\top \end{bmatrix} \succeq 0, \quad (A, B) \in \{\sigma^1_{(U_-,X)}, \dots, \sigma^L_{(U_-,X)}\}. \tag{11.20}$$

Let us recall the definition for informativity for quadratic feedback stabilization (Definition 11.3.2). It was noted in (van Waarde et al., 2022), that the inequality

$$(A + BK)^\top P(A + BK) - P \prec 0$$

is equivalent to

$$\begin{bmatrix} I \\ A^\top \\ B^\top \end{bmatrix}^\top \begin{bmatrix} P & 0 & 0 \\ 0 & -P & -PK^\top \\ 0 & -KP & -KPK^\top \end{bmatrix} \begin{bmatrix} I \\ A^\top \\ B^\top \end{bmatrix} \succ 0. \tag{11.21}$$

Now, for the data (U_-, X) to be informative for quadratic feedback stabilization, it is sufficient that there exist K and $P \succ 0$ so that (11.21) holds for all pairs $(A,B) \in \bar{\Sigma}^R_{(U_-,X)}$. This is precisely a problem that can be solved by the S-procedure; more specifically, by the matrix-valued S-procedure (van Waarde et al., 2022), which is applicable since $Q \prec 0$.

Application of the matrix-valued S-procedure (van Waarde et al., 2022, Theorem 13) and the Schur complement leads to the following informativity conditions for stabilization by state feedback, $\mathscr{H}_\infty$ control and $\mathscr{H}_2$ control.

Proposition 11.5.1. *The data (U_-,X) are informative for (quadratic) stabilization by state feedback if there exist W, $L \in \mathbb{R}^{m\times n}$, $P \succ 0$ and $\beta > 0$ so that* (11.20) *holds and*

$$\begin{bmatrix} P-\beta I & 0 & 0 & 0 \\ 0 & -P & -L^\top & 0 \\ 0 & -L & 0 & L \\ 0 & 0 & L^\top & P \end{bmatrix} - W \succeq 0. \tag{11.22}$$

Moreover, K is such that $A+BK$ is stable for all $(A,B) \in \Sigma^R_{(U_-,X)}$ if $K = LP^{-1}$ with L and $P \succ 0$ satisfying (11.22)*.*

Proposition 11.5.2. *The data (U_-,X) are informative for (common) $\mathscr{H}_\infty$ control with performance γ if there exist W, $L \in \mathbb{R}^{m\times n}$, $Y \succ 0$ and $\beta > 0$ so that* (11.20) *holds and*

$$\begin{bmatrix} Y-\beta I & 0 & 0 & 0 & U^\top \\ 0 & 0 & 0 & Y & 0 \\ 0 & 0 & 0 & L & 0 \\ 0 & Y & L^\top & Y-\gamma^{-2}I & 0 \\ U & 0 & 0 & U & I \end{bmatrix} - W \succeq 0 \tag{11.23}$$
$$\text{and} \quad Y-\gamma^{-2}I \succ 0$$

where $U := CY+DL$. Moreover, K is such that $\Sigma^R_{(U_-,X)} \subseteq \Sigma^{\mathscr{H}_\infty}_K(\gamma)$ if $K = LY^{-1}$ with L and $Y \succ 0$ satisfying (11.23)*.*

Proposition 11.5.3. *The data (U_-,X) are informative for (common) $\mathscr{H}_2$ control with performance γ if there exist W, $L \in \mathbb{R}^{m\times n}$, $Y \succ 0$, symmetric Z and $\beta > 0$ so that* (11.20) *holds true,* $\operatorname{trace} Z < \gamma^2$,

$$\begin{bmatrix} Y-\beta I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & Y & 0 \\ 0 & 0 & 0 & L & 0 \\ 0 & Y & L^\top & Y & U^\top \\ 0 & 0 & 0 & U & I \end{bmatrix} - W \succeq 0, \tag{11.24}$$

$$\begin{bmatrix} Y & U^\top \\ U & I \end{bmatrix} \succ 0 \quad \text{and} \quad \begin{bmatrix} Z & I \\ I & Y \end{bmatrix} \succeq 0, \tag{11.25}$$

where $U := CY + DL$. *Moreover,* K *is such that* $\Sigma^R_{(U_-,X)} \subseteq \Sigma_K^{\mathscr{H}_2}(\gamma)$ *if* $K = LY^{-1}$ *with* L *and* $Y \succ 0$ *satisfying* (11.24).

Due to the approximation of $\Sigma^R_{(U_-,X)}$ by a superset $\bar{\Sigma}^R_{(U_-,X)}$, the conditions in Proposition 11.5.1, 11.5.2, and 11.5.3 are only sufficient and therefore, in general, more conservative compared to the conditions in Corollary 11.3.1, 11.4.1, and 11.4.2, respectively. From a computational point-of-view, however, the number of LMIs scales affinely with respect to the number of vertices for both approaches, while the size of the LMIs in (11.20) is $n \times n$, which is strictly smaller than the size of the LMIs in Corollary 11.3.1, 11.4.1, and 11.4.2.

11.6 Data-based analysis of interconnected systems with cross-covariance bounds

When the system under consideration is an interconnected system, then the interconnection structure can be taken into account in the data-based analysis with cross-covariance bounds. In this section, we will address the data-based analysis of interconnected systems, where the noise signal corresponding to each subsystem is subject to sample cross-covariance bounds. Consider interconnected systems composed of L linear time-invariant systems of the form

$$x_i(k+1) = A_i x_i(k) + \sum_{j \in \mathcal{N}_i} A_{ij} x_j(k) + B_i u_i(k) + e_i(k), \quad i = 1, \ldots, L, \tag{11.26}$$

where $x_i \in \mathbb{R}^{n_i}$ denotes the state, $u_i \in \mathbb{R}^{m_i}$ the input and $e_i \in \mathbb{R}^{n_i}$ is a noise signal. The set $\mathcal{N}_i := \{j \in \mathcal{V} \,|\, (i,j) \in \mathcal{E}\}$ denotes the neighbours of system i, where $\mathcal{V}$ and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ denote the set of vertices and the set of non-oriented edges defining the connected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$.

We consider the following problem set up. Let there exist a true interconnected system defined by the matrices A_i^0, A_{ij}^0 and B_i^0, $(i,j) \in \mathcal{E}$, generating the input-state data $\{(u_i(t), x_i(t)),\, t = 0, \ldots, N\}$ for $i \in \mathcal{V}$. This data is collected in the matrices

$$X_i^- := [x_i(0) \ \cdots \ x_i(N)], \ U_i^- := [u_i(0) \ \cdots \ u_i(N-1)].$$

By defining the matrices

$$\begin{aligned} &X_i^+ := [x_i(1) \ \cdots \ x_i(N)], \ X_i^- := [x_i(0) \ \cdots \ x_i(N-1)], \\ &E_i^- := [e_i(0) \ \cdots \ e_i(N-1)], \end{aligned}$$

we obtain the data equation

$$X_i^+ = A_i^0 X_i^- + \sum_{j \in \mathcal{N}_i} A_{ij}^0 X_j^- + B_i^0 U_i^- + E_i^-, \tag{11.27}$$

for each $i \in \mathcal{V}$.

11.6.1 Network cross-covariance bounds

We assume here that $e_j(t)$, $j \in \mathcal{V}$, are not measured, i.e., E_j^- is unknown for all $j \in \mathcal{V}$, but that each noise signal $e_j(t)$ satisfies the (element-wise) bounds

$$c_j^l \leq \frac{1}{\sqrt{N}} \sum_{t=0}^{N-1} e_j(t) r_j(t)^\top \leq c_j^u, \quad j = 1, \ldots, L, \tag{11.28}$$

where r_j is a vector that collects measured signals r_{ij}, $i = 1, \ldots, M$, that are chosen, so that $r_j := \operatorname{col}(r_{1j}, \ldots, r_{Mj})$, and c_j^l, c_j^u are specified bounds.. We call r_{ij} instrumental network signals. These instrumental network signals can be any measured signals in the network, e.g., (components of) states x_i, $i \in \mathcal{V}$, or (components of) inputs u_i, $i \in \mathcal{V}$.

For each $j \in \mathcal{V}$, the set of feasible subsystems, i.e., the set of subsystems that are consistent with the data under the assumption on the noise (11.28), is

$$\begin{aligned}
&\Sigma_{R\mathcal{D}}^j := \{(A_j, A_{\mathcal{N}_j}, B_j) \mid \\
&\quad c_j^l \leq \frac{1}{\sqrt{N}} \sum_{t=0}^{N-1} \left(x_j(t+1) - A_j x(t) - \sum_{k \in \mathcal{N}_j} A_{jk} x_k(t) - B_j u_j(t) \right) r_j(t)^\top \leq c_j^u \}.
\end{aligned}$$

We find that the sample cross-covariance between e_j and r_j is

$$\begin{aligned}
R_{e_j r_j}^{N-} &:= \frac{1}{\sqrt{N}} \sum_{t=0}^{N-1} e_j(t) r_j(t)^\top \\
&= \frac{1}{\sqrt{N}} \sum_{t=0}^{N-1} x_j(t+1) r_j(t)^\top - \frac{1}{\sqrt{N}} \sum_{t=0}^{N-1} A_j x_j(t) r_j(t)^\top \\
&\quad - \frac{1}{\sqrt{N}} \sum_{t=0}^{N-1} \sum_{k \in \mathcal{N}_j} A_{jk} x_k(t) r_j(t)^\top - \frac{1}{\sqrt{N}} \sum_{t=0}^{N-1} B_j u_j(t) r_j(t)^\top \\
&= \frac{1}{\sqrt{N}} X_j^+ (R_j^-)^\top - A_j \frac{1}{\sqrt{N}} X_j^- (R_j^-)^\top - \sum_{k \in \mathcal{N}_j} A_{jk} \frac{1}{\sqrt{N}} X_k^- (R_j^-)^\top \\
&\quad - B_j \frac{1}{\sqrt{N}} U_j^- (R_j^-)^\top \\
&= R_{x_j r_j}^{N+} - A_j R_{x_j r_j}^{N-} - \sum_{k \in \mathcal{N}_j} A_{jk} R_{x_k r_j}^{N-} - B_j R_{u_j r_j}^{N-}.
\end{aligned}$$

Hence, the feasible set of systems is

$$\Sigma^j_{R\mathcal{D}} = \{(A_j, A_{\mathcal{N}_j}, B_j)\,|\, c^l_j \leq R^{N+}_{x_jr_j} - \begin{bmatrix} A_j & A_{\mathcal{N}_j} & B_j \end{bmatrix} \begin{bmatrix} R^{N-}_{x_jr_j} \\ R^{N-}_{x_{\mathcal{N}_j}r_j} \\ R^{N-}_{u_jr_j} \end{bmatrix} \leq c^u_j\}.$$

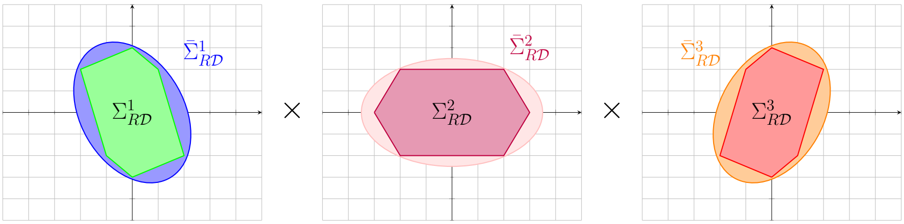


Figure 11.6: Each subsystem has its own feasible set of tuples $(A_i, A_{\mathcal{N}_i}, B_i)$, illustrated by the polytopes. A polytope can be approximated by an ellipsoid for each subsystem separately such that $\Sigma^i_{R\mathcal{D}} \subseteq \bar{\Sigma}^i_{R\mathcal{D}}$ for each $i \in \mathcal{V}$. The set of all feasible tuples $((A_1, A_{\mathcal{N}_1}, B_1), \ldots, (A_L, A_{\mathcal{N}_L}, B_L))$ is the Cartesian product $\Sigma^1_{R\mathcal{D}} \times \cdots \times \Sigma^L_{R\mathcal{D}}$.

We consider the case where the inequalities defining $\Sigma^j_{R\mathcal{D}}$ are not redundant for each $j \in \mathcal{V}$ so that $\Sigma^j_{R\mathcal{D}}$ is a convex polytope with a finite number of vertices $\sigma^{j\nu}_{\mathcal{D}}$, $\nu \in \{1, \ldots, N_j\} =: \mathcal{S}_j$. That is, for each subsystem $j \in \mathcal{V}$, we have a separate set of feasible tuples $(A_i, A_{\mathcal{N}_j}, B_j)$ that is a polytope. This is illustrated in Figure 11.6 for a network with $\mathcal{V} = \{1, 2, 3\}$.

11.6.2 Data-based network analysis with cross-covariance bounds

Since the sets $\Sigma^j_{R\mathcal{D}}$, $j \in \mathcal{V}$, are convex and have a finite number of vertices, Lemma 11.3.1 can again be utilized to analyze the interconnected system performance through verification of performance conditions at the vertices only. Consider an output y_j for each subsystem, given by

$$y_j = C_j x_j + D_j u_j, \quad j \in \mathcal{V}. \tag{11.29}$$

The transfer matrix from $\mathrm{col}(u_1, \ldots, u_L)$ to $\mathrm{col}(y_1, \ldots, y_L)$ is denoted T. We say that the interconnected system (11.26) with output (11.29) achieves $\mathscr{H}_\infty$ performance $\gamma > 0$ if $\|T\|_{\mathscr{H}_\infty} < \gamma$.

We recall the nominal distributed controller existence conditions from Chapter 4, Corollary 4.3.1, which provides sufficient conditions for the $\mathscr{H}_\infty$ performance of the interconnected system.

Theorem 11.6.1. *If, for every $i \in \mathcal{V}$, there exist P_i and Z_i so that $P_i \succ 0$ and*

$$(\star)^\top \left[\begin{array}{cc|cc|cc} -P_i & 0 & 0 & 0 & 0 & 0 \\ 0 & P_i & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & Z_i^{11} & Z_i^{12} & 0 & 0 \\ 0 & 0 & (Z_i^{12})^\top & Z_i^{22} & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & -\gamma^2 I & 0 \\ 0 & 0 & 0 & 0 & 0 & I \end{array}\right] \left[\begin{array}{ccc} I & 0 & 0 \\ A_i & A_{\mathcal{N}_i} & B_i \\ \hline \mathbf{1} \otimes I & 0 & 0 \\ 0 & I & 0 \\ \hline 0 & 0 & I \\ C_i & 0 & D_i \end{array}\right] \prec 0 \quad (11.30)$$

holds, then the interconnected system (11.26) *with output* (11.29) *achieves $\mathscr{H}_\infty$ performance γ.*

Now, by the convexity of the sets of feasible subsystems $\Sigma^j_{R\mathcal{D}}$, $j \in \mathcal{V}$, we can verify the feasibility of the analysis LMIs at each vertex $\sigma^{j\nu}_{\mathcal{D}}$, $\nu = 1, \ldots, N_j$ for each $j \in \mathcal{V}$, to conclude feasibility of the analysis LMIs for *all* data-compatible subsystems. Conversely, if the LMIs are feasible for data-compatible subsystems, they are obviously feasible for subsystems at the vertices of $\Sigma^j_{R\mathcal{D}}$, $j \in \mathcal{V}$.

Proposition 11.6.1 (Performance from structured data)**.** *There exist P_i and Z_i so that $P_i \succ 0$ and*

$$(\star)^\top \left[\begin{array}{cc|cc|cc} -P_i & 0 & 0 & 0 & 0 & 0 \\ 0 & P_i & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & Z_i^{11} & Z_i^{12} & 0 & 0 \\ 0 & 0 & (Z_i^{12})^\top & Z_i^{22} & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & -\gamma^2 I & 0 \\ 0 & 0 & 0 & 0 & 0 & I \end{array}\right] \left[\begin{array}{ccc} I & 0 & 0 \\ \sigma_A^{i\nu} & \sigma_{A_\mathcal{N}}^{i\nu} & \sigma_B^{i\nu} \\ \hline \mathbf{1} \otimes I & 0 & 0 \\ 0 & I & 0 \\ \hline 0 & 0 & I \\ C_i & 0 & D_i \end{array}\right] \prec 0 \quad (11.31)$$

holds for all $(i, \nu) \in \mathcal{V} \times \mathcal{S}_i$, if and only if there exist P_i and Z_i so that $P_i \succ 0$ and

$$(\star)^\top \left[\begin{array}{cc|cc|cc} -P_i & 0 & 0 & 0 & 0 & 0 \\ 0 & P_i & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & Z_i^{11} & Z_i^{12} & 0 & 0 \\ 0 & 0 & (Z_i^{12})^\top & Z_i^{22} & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & -\gamma^2 I & 0 \\ 0 & 0 & 0 & 0 & 0 & I \end{array}\right] \left[\begin{array}{ccc} I & 0 & 0 \\ A_i & A_{\mathcal{N}_i} & B_i \\ \hline \mathbf{1} \otimes I & 0 & 0 \\ 0 & I & 0 \\ \hline 0 & 0 & I \\ C_i & 0 & D_i \end{array}\right] \prec 0 \quad (11.32)$$

holds for all $(A_i, A_{\mathcal{N}_i}, B_i) \in \Sigma^i_{R\mathcal{D}}$, $i \in \mathcal{V}$.

Consequently, if there exist P_i and Z_i so that $P_i \succ 0$ and (11.31) *holds holds for all $(i, \nu) \in \mathcal{V} \times \mathcal{S}_i$, then all interconnected systems with subsystems $(A_i, A_{\mathcal{N}_i}, B_i) \in \Sigma^i_{R\mathcal{D}}$, $i \in \mathcal{V}$, achieve $\mathscr{H}_\infty$ performance γ.*

Proof. Consider $i \in \mathcal{V}$ and the matrix functions $F_i : \Sigma^i_{R\mathcal{D}} \to \mathbb{S}^{n_i \times n_i}$, defined by $F_i(\sigma) := \sigma^\top P_i \sigma_i$. Since $\Sigma^i_{R\mathcal{D}}$ is convex and $P_i \succ 0$, we infer that F_i is a convex function. Hence, by Lemma 11.3.1, (11.32) holds for all $(A_i, A_{\mathcal{N}_i}, B_i) \in \Sigma^i_{R\mathcal{D}}$ if and only if (11.31) holds for all $\nu \in \mathcal{S}_i$. This proves the assertion. □

Now, let us approximate $\Sigma^i_{R\mathcal{D}}$ by a superset, say $\bar{\Sigma}^i_{R\mathcal{D}}$, for each $i \in \mathcal{V}$, which are chosen such that

$$\Sigma^i_{R\mathcal{D}} \subseteq \bar{\Sigma}^i_{R\mathcal{D}}, \quad i \in \mathcal{V}. \tag{11.33}$$

If we choose the supersets as ellipsoids, then there exist triples $(Q^i_{\mathcal{D}}, S^i_{\mathcal{D}}, R^i_{\mathcal{D}})$ for each i, such that

$$\bar{\Sigma}^i_{R\mathcal{D}} = \{(A_i, A_{\mathcal{N}_i}, B_i) \,|\, (\star)^\top \begin{bmatrix} Q^i_{\mathcal{D}} & S^i_{\mathcal{D}} \\ (S^i_{\mathcal{D}})^\top & R^i_{\mathcal{D}} \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \\ A_i & A_{\mathcal{N}_i} & B_i \end{bmatrix} \preceq 0\}.$$

Although, we have used a different assumption on the noise, we have arrived at the same parametrization used in Lemma 9.4.2, but now for a superset of each feasible set of subsystems. Consequently, we have the following result.

Proposition 11.6.2 (Performance from structured data (superset))**.** *Let $Q^i_{\mathcal{D}} \prec 0$ and $\gamma > 0$. Let there exist P_i, Z_i and α_i so that $P_i \succ 0$, $\alpha_i > 0$ and*

$$J_i^\top \left[\begin{array}{cc|cc|cc|cc} -P_i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & P_i & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & Z_i^{11} & Z_i^{12} & 0 & 0 & 0 & 0 \\ 0 & 0 & (Z_i^{12})^\top & Z_i^{22} & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & -\alpha_i R^i_{\mathcal{D}} & -\alpha_i (S^i_{\mathcal{D}})^\top & 0 & 0 \\ 0 & 0 & 0 & 0 & -\alpha_i S^i_{\mathcal{D}} & -\alpha_i Q^i_{\mathcal{D}} & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & -\gamma^2 I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I \end{array}\right] J_i \prec 0 \tag{11.34}$$

holds for all $i \in \mathcal{V}$, with J_i defined in (9.9)*. Then all interconnected systems with subsystems $(A_i, A_{\mathcal{N}_i}, B_i) \in \Sigma^i_{R\mathcal{D}}$, $i \in \mathcal{V}$, achieve $\mathscr{H}_\infty$ performance γ.*

11.6.3 Data-based distributed controller design with cross-covariance bounds

Consider now the step from structured performance analysis from data to distributed controller existence analysis from data. Once the ellipsoidal supersets

$\bar{\Sigma}^i_{R\mathcal{D}}$, $i \in \mathcal{V}$, have been determined, this step can be performed as described in Chapter 9. More specifically, the assertion regarding the existence of a distributed controller in Theorem 9.5.1 holds true with $\Sigma^i_{\mathcal{D}}$ replaced with $\Sigma^i_{R\mathcal{D}}$ and the triples $(Q^i_{\mathcal{D}}, S^i_{\mathcal{D}}, R^i_{\mathcal{D}})$, $i \in \mathcal{V}$, such that the inclusions in (11.33) hold true. If the conditions in Theorem 9.5.1 hold true, then there exists a distributed controller such that the closed-loop network with $(A_i, A_{\mathcal{N}_i}, B_i) \in \Sigma^i_{R\mathcal{D}}$, $i \in \mathcal{V}$, achieves $\mathscr{H}_\infty$ performance γ. A corresponding distributed controller can be constructed as described by the procedure in Section 4.3.6.

11.7 Conclusions

We have addressed the problem of analyzing informativity of data for controller design with prior knowledge on process noise in the form of linear sample cross-covariance bounds. We have established a parametrization of the set of systems that are compatible with data. This set is a (possibly unbounded) polyhedron and the analysis therefore requires a different approach than the application of the S-procedure in Chapter 10, where the noise has been characterized by quadratic sample cross-covariance bounds. First, we have developed sufficient conditions under which the data are informative for feedback stabilization, in the case that the set of feasible systems is unbounded. For the case that this set is bounded, we have used its convexity and the convexity of stability/performance conditions with respect to the system matrices, leading to necessary and sufficient conditions for informativity for stabilization and $\mathscr{H}_2/\mathscr{H}_\infty$ control. To reduce the computational complexity, we have shown how the matrix S-lemma can be applied in the informativity analysis by approximating polytopic sets of feasible systems by ellipsoidal supersets. Finally, the analysis and control of interconnected systems has been addressed, by determining the set of all feasible tuples of subsystem matrices as a Cartesian product of polytopic sets or ellipsoidal supersets. The parametrization of the ellipsoidal supersets is equal to the parametrization in Chapter 9, which allows the application of the approach in Chapter 9 to the analysis of interconnected systems with sample cross-covariance noise characterizations.

Chapter 12

Conclusions and future perspectives

12.1 Conclusions

In this thesis, the use of data from interconnected systems for the design of distributed controllers has been addressed. The problem of using data for distributed controller design has been approached in two philosophically different manners in this thesis: indirect data-driven distributed control and direct data-driven distributed control. The first approach has been considered in Part I, through data-driven modeling of dynamic networks and distributed controller design. In Part II, the direct approach to data-driven distributed control has been addressed. Moreover, fundamental limitations of distributed controller design from data have been addressed, in Part III. The conclusions that follow from the results in this thesis are summarized in this section.

Indirect data-driven distributed control

Different from centralized controller design from data, the design of distributed controllers from data involves an interconnection structure of the controller. Furthermore, the interconnected system that has to be controlled also has an interconnection structure. As reasoned in the introduction, indirect data-driven distributed controller design is concerned with the derivation of a structured model of the interconnected system from data and the model-based synthesis of a structured (distributed or decentralized) controller.

The data-driven modeling of interconnected linear systems has been addressed in Chapter 2. Interconnected linear systems have a state-space representation and a module dynamic network representation that are observationally equivalent. For a dynamic network that is interconnected with a distributed controller, with a clear distinction between interconnected system and distributed controller, it is shown that identification methods developed for dynamic networks can be specified and generalized for obtaining consistent subsystem models. Consistent estimates can be obtained through indirect identification methods, using knowledge of the distributed controller dynamics in the post-processing step. Alternatively, consistent estimates, of one subsystem or the complete interconnected system dynamics, can be obtained through direct methods for identification. The experiment design for indirect methods relies fully on (measured) external excitations, which can be provided through reference signals in the considered experiment setup. This requirement can be relaxed for the direct method, where excitation may be provided through unmeasured exogenous disturbances that affect the underlying plant. In the case that a tailor-made parametrization is used in an indirect method, the network identification criterion can be interpreted as a (closed-loop) $\mathscr{H}_2$ performance degradation that can be used in an iterative scheme of network identification and distributed control design, directly extending the situation of a classical control loop through identification for control.

Measurement data and computational resources may be spatially distributed for network identification. In Chapter 3, a distributed identification framework has been developed for obtaining linearly parametrized MISO models with multiple identification modules that are interconnected through a mutual fusion center. It has been shown that the true parameters can be obtained asymptotically by iteratively updating estimates via a distributed version of recursive least squares, which requires the communication of regressors between identification modules. In the case that process noise affects the output, a noise model can be identified through a separate identification module, leading to asymptotically unbiased estimates.

In Chapter 4, the design of a distributed controller based on a state-space model of an interconnected system, that may be obtained through one of the procedures in Chapter 2, has been considered, with an $\mathscr{H}_2$ or $\mathscr{H}_\infty$ performance measure. Structured $\mathscr{H}_2$ analysis conditions have been presented for interconnected linear systems in a discrete-time setting, through a dissipativity-based approach. It has been shown that these analysis conditions transform to convex synthesis conditions, that are sufficient for the existence of a distributed controller with dynamical sub-controllers. Moreover, when supply functions associated with interconnection variables are fixed *a priori*, it has been shown that the synthesis conditions can be used for the design of a decentralized controller. A detailed controller construction procedure has been provided, which completes the procedure of indirect data-driven distributed control.

Direct data-driven distributed control

Alternative to indirect data-driven distributed control, direct data-driven distributed control omits the modeling of the underling system that is to be controlled. A complete framework for direct data-driven distributed control has been introduced in this thesis, following a structured model-reference approach to control.

In Chapter 6, we have introduced a distributed model-reference control problem, which is to determine a distributed controller that yields a closed-loop interconnected system with a behavior that is prescribed by a structured reference model. It has been shown that this problem is solved by a distributed controller (an ideal distributed controller) that explicitly depends on the plant and reference model, extending the ideal controller in the situation of a classical control loop. The philosophy behind this ideal distributed controller stems from the canonical distributed controller introduced in Chapter 5, where each sub-controller is defined by the interconnection of a reference model and plant subsystem through to-be-controlled variables in a behavioral setting. Analysis and synthesis conditions have been derived for verifying and guaranteeing properness and stability of the distributed controller in Chapter 6.

The distributed model-reference control problem can be solved directly from data, which has been shown in Chapter 7 and Chapter 8. In Chapter 7, a virtual reference framework for distributed control has been developed. The distributed model-reference control problem can be solved by identification of the ideal distributed controller modules in a virtual network, that can be generated with measurement data in combination with the structured reference model. With insights from the direct method in the identification of dynamic networks, it has been found that the identification problem can be performed locally in order to obtain consist estimates of the ideal distributed controller. In the presence of process noise, consistent estimates can be obtained by modeling the noise filter with a tailor-made noise model and by filtering the corresponding prediction error with a module of the reference model. This method is also applicable to classical VRFT, providing an alternative to the instrumental-variable method for obtaining consistent estimates in the presence of process noise.

In Chapter 8 it has been shown that the distributed model-reference control problem can be solved from data by extending the underlying dynamic network, such that transformed versions of ideal distributed controller dynamics appear as modules. It has been shown that the transformed ideal controller modules can be consistently identified in the extended network through indirect and direct identification methods. The network transformation in combination with the direct method for dynamics networks naturally extends the one-shot OCI method for standalone control loops to distributed control. Compared with the method in Chapter 7, no tailor-made noise model is required. The identification criterion is typically non-convex due to the transformed controller modules, however, even if the controller class is linearly parametrized.

Informativity and performance with guarantees

The conclusions that are drawn for the performance in Part I and Part II are based on consistent estimation, i.e., asymptotically in the number of data. In Chapter 9, the design of a dynamical distributed controller based on a finite number of noisy measurement data has been addressed. It has been found that the existence of a distributed controller, that achieves a prescribed $\mathscr{H}_\infty$ performance, can be guaranteed by the feasibility of linear matrix inequalities that are defined by the data. Once the controller existence has been established, a corresponding distributed controller can be constructed via the procedure established in Chapter 4 from the solution to the data-based LMIs.

In Chapter 10, a framework for data informativity for control has been considered, with a characterization of unmeasured exogenous disturbances by their squared sample cross-covariance with respect to instrumental variables. Sufficient informativity conditions for stabilization, $\mathscr{H}_2$ and $\mathscr{H}_\infty$ control via dynamic output feedback were derived, which are also necessary if the state is measured. A numerical case study has been shown where data-informativity can be concluded with cross-covariance bounds, while the data are concluded to be non-informative with magnitude bounds. The choice of output measurements can be of paramount importance for data informativity for control, which has been illustrated via a numerical example. Linear versions of the bound in Chapter 10 have been considered in Chapter 11, leading to a polyhedral set of systems that are compatible with the data. It has been shown that the convexity of the set of feasible systems and analysis conditions, leads to a finite number of LMIs that are necessary and sufficient for stabilization, $\mathscr{H}_2$ and $\mathscr{H}_\infty$ control in the case that the data lead to a bounded polyhedron. Furthermore, it has been shown that informativity for stabilization can even be concluded if the polyhedron is unbounded, but the derived conditions are only sufficient and have been derived for scalar input and state data only. Through ellipsoidal approximations, the informativity conditions have been shown to extend to the distributed control situation via the conditions developed in Chapter 9.

12.2 Future perspectives and extensions

Selective identification for distributed control

In indirect data-driven distributed control, a model of the full underlying plant is identified, to perform a model-based distributed controller synthesis. However, in large-scale interconnected systems, some subsystems may only have a limited contribution to the synthesized distributed controller dynamics, and hence the resulting closed-loop performance. In the scope of identification for control, it is attractive to be able to characterize subsystems that are most 'important' for distributed controller design. In a sense, this would correspond to minimization of the performance degradation (Van den Hof and Schrama, 1995), (Gevers, 2005), by selecting the most important systems for identification.

Local synthesis of sub-controllers from data

Similarly, in direct distributed data-driven control the aim is to find a full distributed controller. It would be attractive to allow the synthesis of sub-controllers that have a relatively large impact on the closed-loop performance. Such a controller could, for example, correspond to a sub-controller attached to subsystem that has altered dynamics. A first approach could be as follows. First, estimate the local 'sensitivity' transfer function from the local reference to local performance output to determine the current performance and construct a reference model with a higher performance (e.g., lower $\mathscr{H}_2$ norm). Consequently, a new local controller could be estimated from data by performing the local identification from Chapter 7 or Chapter 8 such that the closed-loop network achieves the new reference model dynamics.

Synthesis of structured reference models

The structured reference model, considered in direct data-driven distributed control in Chapter 7 and Chapter 8, has been analyzed in Chapter 6. Regarding the synthesis of structured reference models, preliminary steps toward the synthesis of decoupled reference models have been made in Chapter 6. The problem of synthesizing structured reference models in general, however, remains largely open. A systematic procedure for determining structured reference models would improve the applicability of the direct data-driven methods to distributed control described in this thesis. A first step toward this direction could be to investigate the possibility of extending the method in (Gonçalves da Silva et al., 2019) to structured reference models for distributed control.

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