Decomposition of a Positive Definite Matrix Function that is Continuously Differentiable

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I. Introduction

When studying nonlinear problems in a quadratic form, it can be of interest to decompose a continuously differentiable, positive definite matrix function A, which maps from $\mathcal{X} \subseteq \mathbb{R}^n$ to $\mathbb{R}^{n\times n}$ into a constant matrix that is pre- and and post multiplied with another matrix function that maps from $\mathcal{X} \subseteq \mathbb{R}^n$ to $\mathbb{R}^{n\times n}$, i.e., for all $x \in \mathcal{X}$ we have $A(x) = B(x)^\top QB(x)$. In this document, we show that this can be accomplished under certain assumptions.

II. Problem definition

Consider the matrix function $A : \mathcal{X} \mapsto \mathbb{R}^{n\times n}$, where $\mathcal{X} \subseteq \mathbb{R}^n$, $n \in \mathbb{N}$, and assume that A admits the following properties

- $\forall x \in \mathcal{X}$, we have that $A(x)$ is real, symmetric and bounded.
- $A(x)$ is a $\mathcal{C}_1$ function, i.e., $A(x)$ and $\frac{\partial A(x)}{\partial x}$ are continuous functions.
- There exist two constants $0 < c_1 \leq c_2$ such that $\forall x \in \mathcal{X}$, $c_1 I \preccurlyeq A(x) \preccurlyeq c_2 I$.

Note that the last property ensures that $A(x)$ is positive definite for all $x \in \mathcal{X}$. The problem is to show that $A(x)$ can always be decomposed as

$$A(x) = B(x)^\top QB(x),$$

where the matrix Q is positive definite, i.e., $0 \prec Q \in \mathbb{R}^{n\times n}$, and the matrix function $B : \mathcal{X} \mapsto \mathbb{R}^{n\times n}$ is a $\mathcal{C}_1$ function which is non-singular for all $x \in \mathcal{X}$, i.e., $\forall x \in \mathcal{X}$, $\det(B(x)) \neq 0$.

III. Result

We now show that this decomposition is possible under the assumptions in Section II. First, denote the set of all real, symmetric matrices by $\mathcal{S}$ and the set of all positive definite matrices by $\mathcal{P}$. Note that $\mathcal{P}$ is open in the $\frac{n(n+1)}{2}$–dimensional real vector space $\mathcal{S}$. The positive definite square-root function $g(A) = A^{\frac{1}{2}}$ is well-defined on $\mathcal{P}$ [1]. In fact, given A, its positive definite square-root is uniquely determined by the Lagrange interpolation polynomial that maps each distinct eigenvalue λ of A to $\sqrt{\lambda}$.

Now, let A_0 be any positive definite matrix and let $B_0 = A_0^{\frac{1}{2}}$. Note that B_0 is positive definite. Define $f(B) = B^2$ on $\mathcal{S}$. Then f is $\mathcal{C}_1$ and its Fréchet derivative $\mathrm{D}f : X \mapsto BX + XB$ is non-singular at B_0. Note that X is the argument of $\mathrm{D}f$ here, and serves as a dummy variable. As the Fréchet derivative of f is non-singular at B_0, by the inverse function theorem, f has a $\mathcal{C}_1$ local inverse defined on some neighbourhood $\mathcal{W}$ containing A_0. Since $\mathcal{P}$ is open, we may assume that $\mathcal{W} \subseteq \mathcal{P}$. As this local inverse of f gives a positive definite square-root function on $\mathcal{W}$, it must agree with g. Hence, g is $\mathcal{C}_1$ on $\mathcal{W}$. It follows that g is $\mathcal{C}_1$ on $\mathcal{P}$ because A_0 is arbitrary.

Finally, when A is a $\mathcal{C}_1$ function of x, its square root $A(x)^{\frac{1}{2}} = g(A(x))$ is also $\mathcal{C}^1$, by the chain rule. Therefore, the matrix function $A(x)$ can always be decomposed as $A(x) = B(x)^\top QB(x)$ by taking $B(x) := A(x)^{\frac{1}{2}}$ and $Q = I$.

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References

[1] M. Koeber and U. Schäfer, "The unique square root of a positive semidefinite matrix," *International Journal of Mathematical Education in Science and Technology*, vol. 37, no. 8, pp. 990–992, 2006.