

Decomposition of a Positive Definite Matrix Function that is Continuously Differentiable

Citation for published version (APA):

Verhoek, C., Koelewijn, P. J. W., Tóth, R., & Haesaert, S. (2022). *Decomposition of a Positive Definite Matrix Function that is Continuously Differentiable*. (Technical Report TUE CS). Eindhoven University of Technology.

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Document status and date:

Published: 13/01/2022

Document Version:

Accepted manuscript including changes made at the peer-review stage

Please check the document version of this publication:

- A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
- The final author version and the galley proof are versions of the publication after peer review.
- The final published version features the final layout of the paper including the volume, issue and page numbers.

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Decomposition of a Positive Definite Matrix Function that is Continuously Differentiable

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I. INTRODUCTION

When studying nonlinear problems in a quadratic form, it can be of interest to decompose a continuously differentiable, positive definite matrix function A , which maps from $\mathcal{X} \subseteq \mathbb{R}^n$ to $\mathbb{R}^{n \times n}$ into a constant matrix that is pre- and post multiplied with another matrix function that maps from $\mathcal{X} \subseteq \mathbb{R}^n$ to $\mathbb{R}^{n \times n}$, i.e., for all $x \in \mathcal{X}$ we have $A(x) = B(x)^\top Q B(x)$. In this document, we show that this can be accomplished under certain assumptions.

II. PROBLEM DEFINITION

Consider the matrix function $A : \mathcal{X} \mapsto \mathbb{R}^{n \times n}$, where $\mathcal{X} \subseteq \mathbb{R}^n$, $n \in \mathbb{N}$, and assume that A admits the following properties

- $\forall x \in \mathcal{X}$, we have that $A(x)$ is real, symmetric and bounded.
- $A(x)$ is a \mathcal{C}_1 function, i.e., $A(x)$ and $\frac{\partial A(x)}{\partial x}$ are continuous functions.
- There exist two constants $0 < c_1 \leq c_2$ such that $\forall x \in \mathcal{X}$, $c_1 I \preceq A(x) \preceq c_2 I$.

Note that the last property ensures that $A(x)$ is positive definite for all $x \in \mathcal{X}$. The problem is to show that $A(x)$ can always be decomposed as

$$A(x) = B(x)^\top Q B(x),$$

where the matrix Q is positive definite, i.e., $0 \prec Q \in \mathbb{R}^{n \times n}$, and the matrix function $B : \mathcal{X} \mapsto \mathbb{R}^{n \times n}$ is a \mathcal{C}_1 function which is non-singular for all $x \in \mathcal{X}$, i.e., $\forall x \in \mathcal{X}$, $\det(B(x)) \neq 0$.

III. RESULT

We now show that this decomposition is possible under the assumptions in Section II. First, denote the set of all real, symmetric matrices by \mathcal{S} and the set of all positive definite matrices by \mathcal{P} . Note that \mathcal{P} is open in the $\frac{n(n+1)}{2}$ -dimensional real vector space \mathcal{S} . The positive definite square-root function $g(A) = A^{\frac{1}{2}}$ is well-defined on \mathcal{P} [1]. In fact, given A , its positive definite square-root is uniquely determined by the Lagrange interpolation polynomial that maps each distinct eigenvalue λ of A to $\sqrt{\lambda}$.

Now, let A_0 be any positive definite matrix and let $B_0 = A_0^{\frac{1}{2}}$. Note that B_0 is positive definite. Define $f(B) = B^2$ on \mathcal{S} . Then f is \mathcal{C}_1 and its Fréchet derivative $Df : X \mapsto BX + XB$ is non-singular at B_0 . Note that X is the argument of Df here, and serves as a dummy variable. As the Fréchet derivative of f is non-singular at B_0 , by the inverse function theorem, f has a \mathcal{C}_1 local inverse defined on some neighbourhood \mathcal{W} containing A_0 . Since \mathcal{P} is open, we may assume that $\mathcal{W} \subseteq \mathcal{P}$. As this

local inverse of f gives a positive definite square-root function on \mathcal{W} , it must agree with g . Hence, g is \mathcal{C}_1 on \mathcal{W} . It follows that g is \mathcal{C}_1 on \mathcal{P} because A_0 is arbitrary.

Finally, when A is a \mathcal{C}_1 function of x , its square root $A(x)^{\frac{1}{2}} = g(A(x))$ is also \mathcal{C}_1 , by the chain rule. Therefore, the matrix function $A(x)$ can always be decomposed as $A(x) = B(x)^\top Q B(x)$ by taking $B(x) := A(x)^{\frac{1}{2}}$ and $Q = I$.

ACKNOWLEDGEMENT

We gracefully thank a user, who wishes to remain anonymous, from the `math.SE.com`-forum that really helped us out with the proof of this problem.

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