

Stability criteria of two-port networks, particularly for thermo-acoustic systems

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Stability criteria of two-port networks, particularly for thermo-acoustic systems

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Abstract

System theory methods are developed and applied within the two-port modeling approach to the problem of thermo-acoustic instability in a combustion appliance. An analogy between thermo-acoustics of combustion and small-signal operation of microwave amplifier is utilized. Notions of unconditional and conditional stabilities of an (active) two-port, representing a burner with flame, are introduced and analyzed. Unconditional stability of two-port means the absence of autonomous oscillation at any embedding of the given two-port by any passive network at the system's upstream (source) and downstream (load) sides. It has been shown that for velocity-sensitive compact burner in the limit of zero Mach number the condition of unconditional stability cannot be fulfilled. The analysis is performed in the spirit of a known criterion in microwave network theory, the so-called Edwards-Sinsky's criterion. Therefore, two methods have been applied to originate the necessary and sufficient conditions of a linear active two-port system to be conditionally stable. The first method is a new algebraic technique to prove and derive the conditional and unconditional stability criteria, and the second method is based on certain properties of Mobius (bilinear) transformations for combinations of reflection coefficients and scattering matrix of (active) two-ports. The developed framework allows formulating design requirements for the stabilization of operation of a combustion appliance via purposeful modifications of either the burner properties or/and of its acoustic embeddings.

Keywords

Edwards-Sinsky's criterion, Rollett factor, Conditional stability, Mobius Transformation, thermoacoustic.

Introduction

Thermo-acoustic combustion instability manifests itself as a high level of tonal noise, vibration, and structural damage. The ability to eliminate and/or control combustion instability at the appliance design phase is one of the main goals of combustion-acoustics research. The low-order (acoustic network) modeling approach is one of the intensively developing tools which has proven its efficiency in performing problem analysis, synthesis, and eventually the appliances design tasks.

Various acoustic network models have been developed (Åbom, 1992; Lavrentjev m.fl., 1995; Munjal, 1987) that are used in analyzing the combustion thermo-acoustic instabilities and the design of combustion equipment. This modeling allows treating combustion appliance components as acoustic two-ports (Keller, 1995; Polifke m.fl., 1997; Stow & Dowling, 2001). Accordingly, the availability of a purely acoustic characterization of the burner with flame is the prerequisite of the model. This is achievable within the concept of the transfer matrix (TM) or scattering matrix (SM) (Paschereit m.fl., 2002). Then, a network model of the combustion system is obtained when all two-ports components are combined.

The methodological similarity of approaches to and the network models equivalence of the electrical circuits and combustion acoustic systems have been shown in various work since 1957 (Merk, 1957). However, the stability

analysis and design methods of the two-port networks have not been developed/applied in the combustion field as much as in microwave theory. The linear two-port network theory has been an intensively developing research subject and the results have been applied ubiquitously in the practice of microwave devices' design. The closest analogy can be established between the combustion thermo-acoustic instability problem and the problem of stability of operation of microwave amplifier. Here, the burner with flame and the amplifier (e.g. transistor) both represent a so-called, "dependent source" or active element. Furthermore, the acoustics of the burner upstream and downstream parts in a combustion appliance are an analogy of the "source" and "load" passive network embeddings of the microwave amplifier. One of the extremely useful and well-developed concepts of the microwave amplifier's design process is the notion of unconditional stability.

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In microwave theory, this means that there is no passive source and passive load combination that can cause the circuit with the given amplifier to oscillate. Correspondingly, the unconditional stability in a thermo-acoustic context means operation stability regardless of the (passive) acoustics of upstream and downstream sides of the burner/flame. The pioneering work on this subject was done by Rollett in 1962 (Rollett, 1962). He introduced a quantity (criterion) to characterize the degree of stability. Later, it was shown that the combination of validity of certain inequality requested from the Rollett factor together with only one other auxiliary condition are necessary and sufficient to provide the unconditional stability (Ha, 1981; Ku, 1966; Meys, 1990; Owens & Woods, 1970).

In 1992, Edwards and Sinsky proposed a single parameter, instead of two of Rollett's conditions, to determine the necessary and sufficient active element unconditional stability requirements. The arguments and analysis were based on a geometrical approach. Various applications and design tools based on the Edwards-Sinsky criteria were developed and discussed (Balsi m.fl., 2006; Marietti m.fl., 2006; Olivieri m.fl., 2005). Particularly, Balsi et al. extended the geometrical approach and derived the necessary and sufficient conditions for a linear active two-port to be conditionally stable. A recent work of Lombardi and Neri (Lombardi & Neri, 2019) presented the existence of a duality mapping between the input and the output of the two-port network; then by using certain properties of Mobius Transformation (MT), they demonstrated all possible cases of mapping between the input and the output of the system. MT is the bilinear rational transformation (Kühnau, 1988) as one of the mathematical concepts named after A.F. Mobius. It is well-known that the MT maps a line or circle into another line or circle (Özgür, 2009). Çakmak et al. derived explicit formulas relating the centres and the radiuses of that mapped circles (Çakmak m.fl., 2018).

On the other hand, the beginning of active development of the acoustic network modeling approach to the problem of combustion instability falls in the period after ~1990. The main focus was on formulating thermo-acoustic network models, predicting the instability, searching for unstable frequencies, calculating/measuring the growth rate, searching methods for stabilizing systems, etc. References to most of the performed research can be found in the review paper (Schuller m.fl., 2020).

The conventional methodology for analyzing the stability of thermo-acoustic systems consists of measuring/modeling a Flame Transfer Function (FTF). Then, a wave-based 1D linear two-port network approach is applied to provide the system matrix. The eigenfrequencies of the system (zeros of the matrix' determinant) determine the (in-)stability and growth/decay rates. Therefore, the common practice is to create a system matrix each time. Then, the system is altered to check the corresponding effect on the eigenfrequencies. This kind of modeling strategy allows resolving the dilemma of the (in-)stability of a particular system and provides information about the frequencies of

oscillation. However, it does not provide a guideline for the premeditated design. The crux is in the absence of specific parameters or criteria to determine the system stability and lack of tools (rules) on how to manipulate the system design as it is done in microwave theory.

A new impetus in the development of system-level analysis of thermo-acoustic network models was given by the discovery of the phenomenon of the burner intrinsic thermo-acoustic mode of instability (Emmert m.fl., 2015; Hoeijmakers m.fl., 2014). This and further research on the subject use system theory. Particularly, the derived system instability conditions are based on the gain and phase of TFF for only ITA mode (Hoeijmakers m.fl., 2016). A review of literature on this subject can be found in the recent publication (Yong m.fl., 2021).

Another research direction was introduced by Kornilov and de Goey who showed the analogy between the thermo-acoustic and linear two-port networks (V. N. Kornilov & de Goey, 2015) and use it to investigate two unconditional stability criteria, 'Rollett' and 'Edwards-Sinsky', for the purpose of evaluation of a burner/flame figure of merit (V. Kornilov & de Goey, 2017). In turn, this work gave the inspiration to develop a prospective method to assess thermo-acoustic instabilities based on reflection coefficients measured only from the upstream side of the burner (cold side) by Kojourimanesh et al. (Kojourimanesh, Kornilov, Arteaga, m.fl., 2021). In this approach, two reflection coefficients, R_{up} and R_{in} , at the cold side of the flame are measured. As displayed in Fig. 1, R_{up} is the reflection coefficient of the upstream side of the burner and R_{in} is the input reflection coefficient of the burner terminated by R_{dn} . In other words, if we disconnect the network in Fig. 1 from the R_{up} , send in wave f_1 and measure reflected wave g_1 , then the corresponding reflection coefficient would be: $R_{in} \triangleq g_1/f_1$.

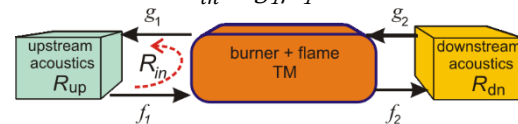


Fig. 1. Thermo-acoustic model of a combustion system.

In this method, the stability of the system can be determined by the Nyquist plot of the measured R_{up} times R_{in} . They showed that the condition applied to the magnitude of $R_{up}R_{in}(i\omega)$ to be less than 1 for all frequencies from 0 to infinity is sufficient (but not necessary) to result in thermo-acoustic stability of the system. Accordingly, in another study, they applied the MT properties to provide the necessary conditions of R_{dn} to ensure that the magnitude of R_{in} becomes less than 1 (Kojourimanesh, Kornilov, Lopez Arteaga, m.fl., 2021a).

The present paper contributes to the further development of the research in the direction of the system-level analysis of network modeling of thermo-acoustic instability of combustion systems and utilizes the close analogy with the theory of microwave networks.

The following procedure is followed. The particular goal of the present contribution is to introduce a new analysis methodology which is based on the stability criteria of

active two-ports. The criteria will be derived using the original approach based on properties of a Mobius transformation in combination with some algebraic transformations. The more general goal is to illustrate the power of the system analysis method in the application to the thermo-acoustic network modeling.

The model of thermo-acoustic system is first written in the form of the network of scattering matrix for power waves to show the analogy with the microwave theory and derive system stability conditions. Then, new algebraic proofs of unconditional stability, namely the Edwards-Sinsky criterion, and conditional stability are proposed. Besides, an alternative approach using MT is introduced to determine the stability condition. Next, by combining the outcome of the aforementioned methods, conditional stability criteria for the thermo-acoustic systems are provided.

The results obtained and presented below can be in principle generalized to the case of an arbitrary burner with flame for which the purely acoustic representation in the form of a two-port is known and given, e.g. by the burner transfer matrix. However, here we limit the consideration to one particular type of burner, namely, an acoustic velocity-sensitive dependent source of acoustic velocity (analogy of current sensitive current source in microwave theory). In this case, we may use some internal symmetries of the transfer and scattering matrices. Physically this type of thermo-acoustic properties is appropriate to a wide class of perfectly premixed gaseous fuel burners operating in the limit of low Mach numbers for mean flow when the heat release zone is compact with respect to the acoustic wavelength under consideration.

Furthermore, we will work in frequency domain, and consider only plane longitudinal waves, 1-D acoustics. The network model variables will be represented by the forward and backward traveling waves f and g and the convention for the time dependence is e^{st} where s is the complex frequency $s = i\omega + \sigma$.

2. Stability criteria of thermo-acoustic sys.

It has been shown that the scattering matrix of the thermo-acoustic two-port can be defined, if one rearranges equations of the transfer matrix (T) such that the ingoing waves appear as inputs to the matrix, and the outgoing waves as outputs. In that case, the scattering matrix (S) of the thermo-acoustic system would be (Hoeijmakers, 2014):

$$S = \frac{1}{2T_{11}} \begin{bmatrix} -2T_{21} & 4 \\ T_{11}^2 - T_{21}^2 & 2T_{21} \end{bmatrix}, \quad (1)$$

where T_{ij} are transfer matrix elements. For the compact velocity-sensitive flame in the limit of zero mean Mach number

$$T = 0.5 \begin{bmatrix} \varepsilon + 1 + \theta FTF_S & \varepsilon - 1 - \theta FTF_S \\ \varepsilon - 1 - \theta FTF_S & \varepsilon + 1 + \theta FTF_S \end{bmatrix}. \quad (2)$$

In this notation, $\theta = \frac{T_2}{T_1} - 1$ is the temperature ratio; $T_{1,2}$ being the temperature at upstream and downstream sides of the flame; $\varepsilon = \frac{\rho_1 c_1}{\rho_2 c_2}$ is the jump in specific acoustic impedance across the flame; and FTF_S is the flame transfer function in the complex domain which relates the oscillation of heat release rate to the oscillation of acoustic velocity

and scaled to mean values of heat release and gas velocity. The scattering matrix expression can be used only for none ITA modes ($T_{11} \neq 0$).

2.1. Unconditional stability in thermo-acoustic systems

Lemma 1. The defined thermo-acoustic system in Eq. (1), cannot be unconditionally stable.

Proof. Rollett stability condition says that the combination of Rollett stability factor $K > 1$, Eq. (3), together with any one of the following auxiliary conditions given in Eq. (4) are necessary and sufficient for unconditional stability (Edwards & Sinsky, 1992) of an (active) two-port described by the scattering matrix S .

$$K = \frac{1 - |S_{11}|^2 - |S_{22}|^2 + |\Delta|^2}{2|S_{12}S_{21}|} > 1, \quad (3)$$

$$\begin{cases} B_1 = 1 + |S_{11}|^2 - |S_{22}|^2 - |\Delta|^2 > 0 \\ B_2 = 1 - |S_{11}|^2 + |S_{22}|^2 - |\Delta|^2 > 0 \\ |\Delta| = |S_{11}S_{22} - S_{12}S_{21}| < 1 \\ 1 - |S_{11}|^2 > |S_{12}S_{21}| \\ 1 - |S_{22}|^2 > |S_{12}S_{21}| \end{cases} \quad (4)$$

Therefore, one can simplify the Rollett stability factor for the thermo-acoustic system as

$$K = \frac{1 - \left| \frac{T_{21}}{T_{11}} \right|^2 - \left| \frac{T_{21}}{T_{11}} \right|^2 + |-1|^2}{2 \left| \frac{T_{11}^2 - T_{21}^2}{T_{11}} \right|} = \frac{1 - \left| \frac{T_{21}}{T_{11}} \right|^2}{\left| 1 - \left(\frac{T_{21}}{T_{11}} \right)^2 \right|}. \quad (5)$$

In Appendix A.1, it is made clear that for any complex number z the function $\frac{1 - |z|^2}{|1 - z^2|} \leq 1$. Accordingly, based on Rollett criteria, the defined thermo-acoustic system cannot be unconditionally stable because one of the necessary conditions of unconditional stability is not satisfied, namely, $K \not> 1$.

Alternately, it is also possible to prove *Lemma 1* via the Edwards and Sinsky parameter, μ . They proved that the necessary and sufficient condition to qualify a two-port as the unconditionally stable element is $\mu > 1$ where

$$\mu = \frac{1 - |S_{11}|^2}{|S_{22} - \bar{S}_{11}\Delta + |S_{12}S_{21}|}. \quad (6)$$

In this notation, the bar symbol is used to denote the conjugate of a complex number.

By substituting parameters, like in Appendix A.2, one can also derive $\mu = \frac{1 - |z|^2}{|2\text{Im}(z)| + |1 - z^2|} \leq 1$ for the thermo-acoustic system which obeys relation (1). Therefore, the thermo-acoustic two-port (a burner with flame) which transfer (scattering) matrix has symmetry as in Eq. (1), cannot be unconditionally stable.

2.2. Conditional stability in thermo-acoustic systems

When considering the notion of conditional stability of a thermo-acoustic system, the upstream and downstream acoustic boundary conditions are playing a role. One needs to search for the range of R_{up} (or R_{dn}) in the complex domain such that the system would be always stable. It is worth mentioning that outside of that range, the system may be stable or unstable. Accordingly, *Lemma 2* with criteria for magnitude R_{up} and R_{dn} is introduced.

Lemma 2: For the defined thermo-acoustic system in Eq. (1), sufficient conditions for the downstream and upstream terminations such that the system would be conditionally stable is

$$|R_{up}| < \frac{1}{|R_{in}|}, \quad (7.a)$$

or

$$|R_{up}| < \frac{-2|S_{11}-|R_{dn}|^2\overline{S_{22}}\Delta|+2|R_{dn}||S_{12}S_{21}|}{2(|R_{dn}|^2|\Delta|^2-|S_{11}|^2)}, \quad (7.b)$$

$$|R_{dn}| < \frac{-2|S_{22}-|R_{up}|^2\overline{S_{11}}\Delta|+2|R_{up}||S_{12}S_{21}|}{2(|R_{up}|^2|\Delta|^2-|S_{22}|^2)}. \quad (7.c)$$

Proof. The main idea of the Edwards-Sinsky criterion is that a system is unconditionally stable if the unit disk in the R_{dn} plane maps inside the unit disk in the R_{in} plane (Edwards & Sinsky, 1992). In other words, the system with passive upstream and downstream terminations is unconditionally stable if and only if $|R_{in}| < 1$. Besides, Kojourimanesh et al. have shown that for none ITA modes, if $|R_{up}R_{in}| < 1$ then the system is conditionally stable (Kojourimanesh, Kornilov, Arteaga, m.fl., 2021; Kojourimanesh, Kornilov, Lopez Arteaga, m.fl., 2021b). So, one can say that for a conditional stable system

$$|R_{up}R_{in}|^2 < 1. \quad (8)$$

Considering $R_{in} = \frac{-\Delta R_{dn} + S_{11}}{-S_{22}R_{dn} + 1}$ and $\Delta = -1$, Appendix B.1 shows how Eq. (8) can be extended in a form of a quadratic inequality as

$$A|R_{dn}|^2 + B|R_{dn}| + C < 0, \quad (9)$$

where $A = |R_{up}|^2|\Delta|^2 - |S_{22}|^2$,

$B = 2(|S_{22} - |R_{up}|^2\overline{S_{11}}\Delta|)$, and $C = |R_{up}|^2|S_{11}|^2 - 1$.

As proven in Appendix B.2, the discriminant, $B^2 - 4AC$, of Eq. (9) would be

$$B^2 - 4AC = 4|R_{up}|^2|S_{12}S_{21}|^2. \quad (10)$$

Because of $B^2 - 4AC \geq 0$, the only way that Eq. (9) becomes always negative is $C < 0$ & $|R_{dn}| < \lambda_1$, where λ_1 is the first root of the quadratic equation, i.e.

$$\lambda_1 = \frac{-B + \sqrt{B^2 - 4AC}}{2A}. \quad (11)$$

Appendix B.3 shows that by substituting A, B, C into λ_1 , the right side of Eq. (7.c) would be the same as λ_1 . Therefore, $|R_{dn}| < \lambda_1$ is the sufficient condition for $|R_{up}R_{in}|^2 < 1$.

Equations (7.b) and (7.c) are almost the same as the conditions suggested by Balsi et al. Appendix C.1 shows how their conditions can be derived from this algebraic method. Moreover, Appendix C.2 provides a new proof of the Edwards-Sinsky criterion from the mentioned algebraic technique.

2.3 Mobius transformation between R_{in} and R_{dn}

Mobius transformation maps a line or a circle into another line or circle. The mapped circle of the unit circle via

$H = \frac{aZ+b}{cZ+d}$, where $ad - bc = 1$, has a specified centre, M , and radius, r (Çakmak m.fl., 2018; Özgür, 2009),

$$M = \frac{b\bar{d}-a\bar{c}}{|d|^2-|c|^2}, \quad r = \frac{1}{||d|^2-|c|^2|}. \quad (12)$$

By looking at the expression for $R_{in} = \frac{-\Delta R_{dn} + S_{11}}{-S_{22}R_{dn} + 1}$, it is clear that it has the form of MT. However, it needs to be normalized first i.e. divided by $\sqrt{ad - bc} = \sqrt{S_{12}S_{21}}$. The downstream side of the burner/flame is an acoustically passive termination, i.e. $|R_{dn}| \leq 1$. Consequently, MT transforms the unit circle of R_{dn} into a circle in the plane of R_{in} with a centre and radius as (Kojourimanesh, Kornilov, Lopez Arteaga, m.fl., 2021a).

$$M = \frac{S_{11} + \overline{S_{22}}}{1 - |S_{22}|^2} = \frac{-2Im(S_{22})}{1 - |S_{22}|^2}i, \quad r = \frac{|S_{12}S_{21}|}{|1 - |S_{22}|^2|}. \quad (13)$$

Furthermore, the centre of R_{dn} unit circle transforms into point $O = \frac{b}{d}$ in R_{in} plane which indicates to where the inside area of the unit circle of R_{dn} is mapped. It can be either the inside or outside area of the circle in R_{in} plane. Figure 2 shows one of the possible cases for this transformation between R_{dn} and R_{in} planes.

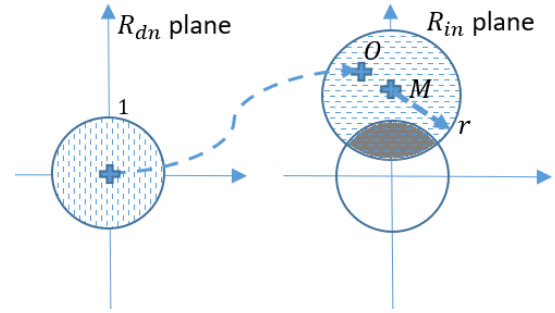


Fig. 2. Mobius transformation of R_{dn} unit circle into R_{in} .

For the case $|R_{up}| \neq 1$, one can apply the same strategy as in Eq. (12) but for MT of

$$R_{up}R_{in} = \frac{(-R_{up}\Delta)R_{dn} + (R_{up}S_{11})}{-S_{22}R_{dn} + 1}. \quad (14)$$

Then, the centre and the radius of the unit circle in R_{dn} plane maps into a circle in $R_{up}R_{in}$ plane. Considering the formula in Eq. (13), one can easily show that the location of the centre, radius and point O in $R_{up}R_{in}$ plane would be the scaled of them in R_{in} plane. Equation (15) and Fig. 3 show the location of the centre, radius and point O of the circle in $R_{up}R_{in}$ plane which is mapped from R_{dn} plane.

$$M_{new} = R_{up}M \quad \& \quad O_{new} = R_{up}\frac{b}{d} \quad \& \quad r_{new} = |R_{up}|r. \quad (15)$$

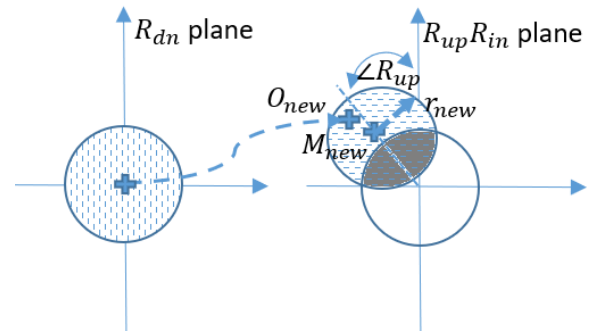


Fig. 3. Mobius transformation of R_{dn} unit circle into $R_{up}R_{in}$

Figure 3 depicts the Mobius transformation of the unit circle in R_{dn} plane into the $R_{up}R_{in}$ plane. Comparing Fig. 2 with

Fig. 3 also Eq. (13) and (15), one can conclude that the properties of the disk in $R_{up}R_{in}$ plane, i.e. M_{new} , O_{new} , and r_{new} , can be created from the properties of the disk in R_{in} plane, i.e. M , O , and r , by scaling them with a factor of $|R_{up}|$ and rotating with the phase of R_{up} .

2.4 Comparing algebraic and MT methods' results

In this section, we are aiming to search for similarities between derived results from the algebraic and MT methods. In the aforementioned thermo-acoustic two-port system (see Eq. (1)), the symmetry implies that $S_{11} = -S_{22}$ and therefore $\Delta = -1$, accordingly $S_{12}S_{21} = 1 - S_{22}^2$. As explained before for any complex z function, $\frac{1-|z|^2}{|1-z^2|} \leq 1$. Then one can readily conclude that the radius of MT in R_{in} plane would be always bigger than 1, i.e.

$$r = \frac{|1-S_{22}^2|}{|1-|S_{22}|^2|} \geq 1. \quad (16)$$

In addition, for the case $|R_{up}| = 1$, the expression $\frac{\sqrt{B^2-4AC}}{2A} = \frac{2|R_{up}||S_{12}S_{21}|}{2(|R_{up}|^2|\Delta|^2-|S_{22}|^2)}$ would be the same as r , i.e.

$$r = \frac{\sqrt{B^2-4AC}}{2A}.$$

The same procedure confirms that $\frac{-B}{2A} = \frac{-|S_{22}+\overline{S_{11}}|}{1-|S_{22}|^2}$.

By comparing the expression of $\frac{-B}{2A}$ with $M = \frac{S_{11}+\overline{S_{22}}}{1-|S_{22}|^2}$ and considering $S_{11} = -S_{22}$ and $C < 0$, it is obvious that

$$\frac{B}{2A} = |M|.$$

Therefore, one can relate the first root of the quadratic equation to the MT parameters as

$$\lambda_1 = \frac{-B}{2A} + \frac{\sqrt{B^2-4AC}}{2A} = -|M| + r. \quad (17)$$

For the case $|R_{up}| = 1$, $|R_{dn}| = 1$, Appendix C.2 proves that the Edwards-Sinsky factor is indeed λ_1 . Hence,

$$\mu = -|M| + r. \quad (18)$$

The other point is that the centre and radius values depend on each other for the system defined in Eq. (1) when $|R_{up}| = 1$. Due to the $A = -C$, one can write the relation between them as

$$r = \sqrt{\frac{B^2-4A(-A)}{4A^2}} = \sqrt{\left(\frac{B}{2A}\right)^2 + 1} = \sqrt{M^2 + 1}. \quad (19)$$

Discussions

Equation (7.a) reveals that for a specific R_{in} , it is possible to design R_{up} such that the system becomes stable, i.e. $|R_{up}| < \frac{1}{|R_{in}|}$. However, for high values of $|R_{in}|$, an upstream damper with a low reflection coefficient, for all frequencies, is needed which may become problematic to design or to produce. This conclusion can be also obtained from the Mobius transformation. Equation (15) confirms that by decreasing the magnitude of R_{up} , the centre of the disk in $R_{up}R_{in}$ plane and the point O_{new} would converge to the origin and the radius of the disk goes to zero as illustrated in Fig. 4. In other words, the whole disk of $R_{up}R_{in}$ will be inside the unit disk, $|R_{up}R_{in}| < 1$. Therefore, decreasing the magnitude of R_{up} causes a high chance of stable system.

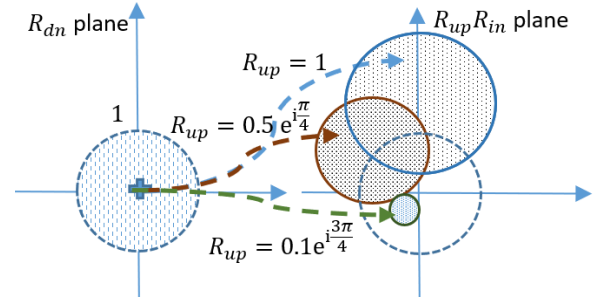


Fig. 4. MT of R_{dn} unit circle into $R_{up}R_{in}$ for 3 different R_{up} .

On the other hand, the system with a high value of the upstream reflection coefficient, $|R_{up}| \sim 1$, cannot be fully stabilized. The reason as expressed in Eq. (16) is the radius of the mapped disk in R_{in} plane is more than one. Therefore, some particular values of R_{dn} can cause $|R_{in}| > 1$. In total, it can be concluded that providing a low reflecting upstream damper will increase the probability of stability for any R_{in} and/or R_{dn} .

Some combustion companies prefer to sell their product (like industrial boilers) without the upstream side (for instance the blower/fan). In this situation, designers would like to optimize their product in terms of thermo-acoustic stability for any upstream appliance. Accordingly, Eq. (7.c) could help them in designing the downstream part such that the system would be thermo-acoustically stable. Besides, the inverse of the transformation mentioned in Eq. (13) could also provide the region of R_{dn} such that $|R_{in}| < 1$ because it maps R_{in} unit disk to the R_{dn} disk i.e. $R_{dn} = \frac{-R_{in}+S_{11}}{-S_{22}R_{in}+\Delta}$.

Equation (17) reveals the relation between Edwards-Sinsky factors, μ & μ^* , and MT parameters as shown in Fig. 5. As can be seen, the Edwards-Sinsky factor, μ , is the closest point of the R_{in} disk from the origin which is $-|M| + r$. Moreover, the second root of the quadratic Eq. (9) which is the same as the second Edwards-Sinsky factor, μ^* , is the farthest point of the R_{in} disk from the origin. Thus, both approaches provide the same results regarding the (in-)stability of the system.

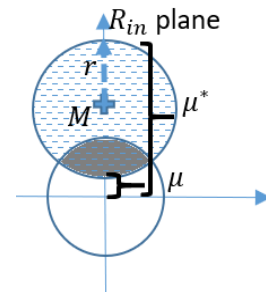


Fig. 5. Relation of Mobius transformation and μ & μ^* .

Conclusions

It is demonstrated that the system-level analysis of a network of two-ports is a very fruitful tool to perform studying on combustion acoustic instability phenomena. Furthermore, it provides promising approaches to the task of system design aiming stability of operation. Original algebraic proofs of conditional and unconditional stability

criteria of linear two-port network systems are proposed. It has been shown that thermo-acoustic systems cannot be unconditionally stable. Hence, the conditional stability criteria have been investigated based on the algebraic technique. Also, a complimentary framework of analysis is proposed which is based on the application of known properties of bilinear Mobius transformation. The comparison of different approaches reveals the relations between the results of the algebraic derivations, the geometrical approach applied in microwave theory, and the Mobius transformation technique. Any of these three approaches can be applied to analyze the stability of the thermo-acoustic system. However, the technique based on the Mobius transformation would provide more insightful information and better visual and intuitive interpretation of results than two other techniques. The elaborated criteria of system stability can be applied for purposeful design of the upstream and downstream sides of the given burner with flame to provide the thermo-acoustic system stability.

Appendix

A.1) Rollett factor for Unconditional stability of TA

Suppose $z = x + iy$ then $|z|^2 = z\bar{z} = x^2 + y^2$,

$|z| = \sqrt{x^2 + y^2}$, and $z^2 = (x^2 - y^2) + 2xyi$.

Rollett factor in thermo-acoustic systems is

$$K = \frac{1-|z|^2}{|1-z^2|} \text{ where } z = \frac{T_{21}}{T_{11}}$$

Hence, $K = \frac{1-|z|^2}{|1-z^2|} = \frac{1-(x^2+y^2)}{|(1-x^2+y^2)-2xyi|}$ so,

$$K = \frac{1-x^2+y^2-2y^2}{\sqrt{(1-x^2+y^2)^2+(-2xy)^2}}$$

Also, it is clear that $\frac{1-x^2+y^2}{\sqrt{(1-x^2+y^2)^2+(-2xy)^2}} \leq 1$ and

$$\frac{-2y^2}{\sqrt{(1-x^2+y^2)^2+(-2xy)^2}} \leq 0. \text{ Therefore } K \leq 1.$$

A.2) Edwards-Sinsky factor for Thermo-Acoustic

$$\mu = \frac{1-|S_{11}|^2}{|S_{22}-S_{11}^*\Delta+|S_{12}S_{21}|} = \frac{1-|\frac{T_{21}}{T_{11}}|^2}{|\frac{T_{21}}{T_{11}}+\frac{T_{21}^*}{T_{11}}(-1)|+|1-(\frac{T_{21}}{T_{11}})^2|}$$

$$\mu = \frac{1-|\frac{T_{21}}{T_{11}}|^2}{|2Im(\frac{T_{21}}{T_{11}})|+|1-(\frac{T_{21}}{T_{11}})^2|}$$

As mentioned before $\frac{1-|z|^2}{|1-z^2|} \leq 1$ so, $\frac{1-|z|^2}{|2Im(z)|+|1-z^2|} \leq 1$ as well.

B.1) Quadratic equation for Thermo-Acoustic systems

We know $R_{in} = \frac{S_{11}-\Delta R_2}{1-S_{22}R_2}$, $|R_{up}R_{in}|^2 < 1$,

$$2Re(z) = z + \bar{z}, \quad Re(z_1z_2) \leq |z_1||z_2|,$$

and $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2Re(z_1\bar{z}_2)$. So,

$$|R_{up}R_{in}|^2 < 1 \rightarrow |R_{up}(S_{11} - \Delta R_{dn})|^2 < |1 - S_{22}R_{dn}|^2.$$

Expanding the power results in

$$|R_{up}|^2 (|S_{11}|^2 + |\Delta R_{dn}|^2 - 2Re(S_{11}\overline{\Delta R_{dn}})) < 1 + |S_{22}|^2 |R_{dn}|^2 - 2Re(\overline{S_{22}R_{dn}}).$$

Applying $2Re(z) = z + \bar{z}$, one writes

$$|R_{up}|^2 |S_{11}|^2 + |R_{up}|^2 |\Delta|^2 |R_{dn}|^2 - [|R_{up}|^2 S_{11}\overline{\Delta R_{dn}} - |R_{up}|^2 \overline{S_{11}\Delta R_{dn}}] - 1 - |S_{22}|^2 |R_{dn}|^2 + [S_{22}R_{dn} + \overline{S_{22}R_{dn}}] < 0.$$

Factorizing $|R_{dn}|^2$, R_{dn} , and $\overline{R_{dn}}$ concludes

$$\begin{aligned} & (|R_{up}|^2 |\Delta|^2 - |S_{22}|^2) |R_{dn}|^2 + [R_{dn} (S_{22} - |R_{up}|^2 \overline{S_{11}\Delta}) \\ & + \overline{R_{dn}} (\overline{S_{22}} - |R_{up}|^2 S_{11}\overline{\Delta})] \\ & + |R_{up}|^2 |S_{11}|^2 - 1 < 0. \end{aligned}$$

Employing $z + \bar{z} = 2Re(z)$, one finds

$$\begin{aligned} & (|R_{up}|^2 |\Delta|^2 - |S_{22}|^2) |R_{dn}|^2 \\ & + 2Re(R_{dn} [S_{22} - |R_{up}|^2 \overline{S_{11}\Delta}]) \\ & + |R_{up}|^2 |S_{11}|^2 - 1 < 0. \end{aligned}$$

Generally speaking, if $A|R_{dn}|^2 + 2|z_1||z_2| + C$ is smaller than zero then for sure $A|R_{dn}|^2 + 2Re(z_1z_2) + C$ would be less than zero because of $Re(z_1z_2) \leq |z_1||z_2|$. Therefore, one allows to consider below inequality instead of the last inequality.

$$\begin{aligned} & (|R_{up}|^2 |\Delta|^2 - |S_{22}|^2) |R_{dn}|^2 + 2|R_{dn}| [S_{22} - |R_{up}|^2 \overline{S_{11}\Delta}] \\ & + |R_{up}|^2 |S_{11}|^2 - 1 < 0. \end{aligned}$$

So, if

$$A|R_{dn}|^2 + B|R_{dn}| + C < 0$$

where $A = (|R_{up}|^2 |\Delta|^2 - |S_{22}|^2)$;

$$B = 2|S_{22} - |R_{up}|^2 \overline{S_{11}\Delta}|; C = |R_{up}|^2 |S_{11}|^2 - 1$$

then for sure $|R_{up}R_{in}|^2 < 1$.

B.2)

$$B^2 - 4AC = 4|S_{22} - |R_{up}|^2 \overline{S_{11}\Delta}|^2 - 4(|R_{up}|^2 |\Delta|^2 - |S_{22}|^2)(|R_{up}|^2 |S_{11}|^2 - 1).$$

Expanding the equation and applying $2Re(z) = z + \bar{z}$, one writes

$$4[|S_{22}|^2 + |R_{up}|^4 |\overline{S_{11}\Delta}|^2 - S_{22}|R_{up}|^2 S_{11}\overline{\Delta} - \overline{S_{22}}|R_{up}|^2 S_{11}\Delta -$$

$$|R_{up}|^4 |\Delta|^2 |S_{11}|^2 - |R_{up}|^2 |\Delta|^2 - |S_{22}|^2 |R_{up}|^2 |S_{11}|^2 - |S_{22}|^2].$$

Simplifying the equation leads to

$$B^2 - 4AC = 4|R_{up}|^2 |S_{11}S_{22} - \Delta|^2 \rightarrow$$

$$B^2 - 4AC = 4|R_{up}|^2 |S_{12}S_{21}|^2 \geq 0.$$

B.3)

$$\lambda_1 = \frac{-B + \sqrt{B^2 - 4AC}}{2A}$$

Substituting A, B, C defined in Appendix B.1 leads to

$$\lambda_1 = \frac{-2|S_{22} - |R_{up}|^2 \overline{S_{11}\Delta}| + 2|R_{up}||S_{12}S_{21}|}{2(|R_{up}|^2 |\Delta|^2 - |S_{22}|^2)}.$$

Also, the same procedure can be used to prove Eq. (7.b) from $|R_{dn}R_{out}|^2 < 1$.

C.1) A new proof of Balsi et al. criterion

Balsi et al. showed that necessary and sufficient conditions for conditional stability can be ascertained by means of a single parameter. Their theorem was,

'Provided that the S -parameters defined for at least one pair of positive constant reference impedances have no RHP poles, the necessary and sufficient condition for a linear active two-port to be stable is

$$\frac{1-|S_{11}|^2 R_{up}^2}{|S_{22}-\overline{S_{11}}\Delta R_{up}^2| |R_{dn}| + |S_{12}S_{21}| R_{up} R_{dn}} > 1.$$

Proof. As shown in Edwards-Sinsky paper, one can readily show that (Edwards & Sinsky, 1992)

$$|S_{22}|^2 - |R_{up}|^2 |\Delta|^2 = \frac{|S_{22}-|R_{up}|^2 \overline{S_{11}}\Delta|^2 - |R_{up}|^2 |S_{11}S_{22}-\Delta|^2}{1-|R_{up}|^2 |S_{11}|^2}.$$

By substituting the term into the denominator of Eq. (7.c) and simplify it, one can derive

$$|R_{dn}| < \frac{1-|R_{up}|^2 |S_{11}|^2}{|S_{22}-|R_{up}|^2 \overline{S_{11}}\Delta| + |R_{up}| |S_{12}S_{21}|}.$$

By moving $|R_{dn}|$ to the right side of the earlier equation, the conditional stability criterion, provided by Balsi et al., is derived.

C.2) A new proof of Edwards-Sinsky criterion

Substituting $|R_{up}| = 1$ and $|R_{dn}| = 1$ into the aforementioned equation (last equation in C.1), it is easy to realize that the right-hand side of the equation is indeed the Edwards-Sinsky criterion.

$$1 < \frac{1-|S_{11}|^2}{|S_{22}-\overline{S_{11}}\Delta| + |S_{12}S_{21}|} = \mu.$$

Also, one can show the second Edwards-Sinsky parameter, μ^* , is the same as the second root of the quadratic equation, $\lambda_2 = \frac{-B-\sqrt{B^2-4AC}}{2A} = \mu^*$.

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References

- Åbom, M. (1992). A note on the experimental determination of acoustical two-port matrices. *Journal of Sound and Vibration*, 155(1), 185–188. [https://doi.org/10.1016/0022-460X\(92\)90655-H](https://doi.org/10.1016/0022-460X(92)90655-H)
- Balsi, M., Scotti, G., Tommasino, P., & Trifiletti, A. (2006). Discussion and new proofs of the conditional stability criteria for multidevice microwave amplifiers. *IEE Proceedings - Microwaves, Antennas and Propagation*, 153(2), 177. <https://doi.org/10.1049/ip-map:20050062>
- Çakmak, G., Deniz, A., & Koçak, Ş. (2018). A Class of Mobius Iterated Function Systems. , 1–10. <http://arxiv.org/abs/1802.01432>
- Edwards, M. L., & Sinsky, J. H. (1992). A new criterion for linear 2-port stability using a single geometrically derived parameter. *IEEE Transactions on Microwave Theory and Techniques*, 40(12), 2303–2311. <https://doi.org/10.1109/22.179894>
- Emmert, T., Bomberg, S., & Polifke, W. (2015). Intrinsic

thermoacoustic instability of premixed flames. *Combustion and Flame*, 162(1), 75–85. <https://doi.org/10.1016/j.combustflame.2014.06.008>

- Ha, T. T. (1981). *Solid-state microwave amplifier design*. New York: Wiley. <http://lib.ugent.be/catalog/rug01:002156145>
- Hoeijmakers, M. (2014). *Flame-acoustic coupling in combustion instabilities* [Ph.D. Thesis, Technical University of Eindhoven]. <https://doi.org/10.6100/IR762773>
- Hoeijmakers, M., Kornilov, V., Arteaga, I. L., de Goey, L. P. H., & Nijmeijer, H. (2016). Flame dominated thermoacoustic instabilities in a system with high acoustic losses. *Combustion and Flame*, 169(2016), 209–215. <https://doi.org/10.1016/j.combustflame.2016.03.009>
- Hoeijmakers, M., Kornilov, V., Lopez Arteaga, I., de Goey, L. P. H., & Nijmeijer, H. (2014). Intrinsic instability of flame-acoustic coupling. *Combustion and Flame*, 161(11), 2860–2867. <https://doi.org/10.1016/j.combustflame.2014.05.009>
- Keller, J. J. (1995). Thermoacoustic oscillations in combustion chambers of gas turbines. *AIAA Journal*, 33(12), 2280–2287. <https://doi.org/10.2514/3.12980>
- Kojourimanesh, M., Kornilov, V., Arteaga, I. L., & de Goey, L. P. H. (2021). Thermo-acoustic flame instability criteria based on upstream reflection coefficients. *Combustion and Flame*, 225, 435–443. <https://doi.org/10.1016/j.combustflame.2020.11.020>
- Kojourimanesh, M., Kornilov, V., Lopez Arteaga, I., & de Goey, L. P. H. (2021a). Mobius transformation between reflection coefficients at upstream and downstream sides of flame in thermoacoustics systems. *The 27th International Congress on Sound and Vibration*.
- Kojourimanesh, M., Kornilov, V., Lopez Arteaga, I., & de Goey, L. P. H. (2021b). Theoretical and experimental investigation on the linear growth rate of the thermoacoustic combustion instability. *Proceedings of the European Combustion Meeting*.
- Kornilov, V., & de Goey, L. P. H. (2017). Combustion thermoacoustics in context of activity and stability criteria for linear two-ports. *Proceedings of the European combustion meeting* April, 18–21.
- Kornilov, V. N., & de Goey, L. P. H. (2015). Approach to evaluate statistical measures for the thermo-acoustic instability properties of premixed burners. *Proceedings of the European combustion Meeting*.
- Ku, W. H. (1966). Unilateral gain and stability criterion of active two-ports in terms of scattering parameters. *Proceedings of the IEEE*, 54(11), 1617–1618. <https://doi.org/10.1109/PROC.1966.5229>
- Kühnau, R. (1988). Jones, G. A.; Singermann, D., *Complex Functions: An Algebraic and Geometric Viewpoint*. Cambridge etc., Cambridge University Press 1987. 342

- pp., £ 12.95 A P/b, \$ 19.95. ISBN 0 521 31366 X. *ZAMM - Journal of Applied Mathematics and Mechanics / Zeitschrift für Angewandte Mathematik und Mechanik*, 68(6), 206. <https://doi.org/10.1002/zamm.19880680605>
- Lavrentjev, J., Åbom, M., & Bodén, H. (1995). A measurement method for determining the source data of acoustic two-port sources. *Journal of Sound and Vibration*, 183(3), 517–531. <https://doi.org/10.1006/jsvi.1995.0268>
- Lombardi, G., & Neri, B. (2019). On the Relationships Between Input and Output Stability in Two-Ports. *IEEE Transactions on Circuits and Systems I: Regular Papers*, 66(7), 2489–2495. <https://doi.org/10.1109/TCSI.2019.2898479>
- Marietti, P., Scotti, G., Trifiletti, A., & Viviani, G. (2006). Stability Criterion for Two-Port Network With Input and Output Terminations Varying in Elliptic Regions. *IEEE Transactions on Microwave Theory and Techniques*, 54(12), 4049–4055. <https://doi.org/10.1109/TMTT.2006.885883>
- Merk, H. J. (1957). An analysis of unstable combustion of premixed gases. *Symposium (International) on Combustion*, 6(1), 500–512. [https://doi.org/10.1016/S0082-0784\(57\)80067-8](https://doi.org/10.1016/S0082-0784(57)80067-8)
- Meys, R. P. (1990). Review and discussion of stability criteria for linear 2-ports. *IEEE Transactions on Circuits and Systems*, 37(11), 1450–1452. <https://doi.org/10.1109/31.62423>
- Munjal, M. L. (1987). *Acoustics of Ducts and Mufflers*. John Wiley & Sons, Inc.
- Olivieri, M., Scotti, G., Tommasino, P., & Trifiletti, A. (2005). Necessary and sufficient conditions for the stability of microwave amplifiers with variable termination impedances. *IEEE Transactions on Microwave Theory and Techniques*, 53(8), 2580–2586. <https://doi.org/10.1109/TMTT.2005.852759>
- Owens, P. J., & Woods, D. M. (1970). *Reappraisal of the unconditional stability criteria for active 2-port networks in terms of S parameters*.
- Özgür, N. Y. (2009). On some mapping properties of Möbius transformations. *Australian Journal of Mathematical Analysis and Applications*, 6(1).
- Paschereit, C. O., Schuermans, B., Polifke, W., & Mattson, O. (2002). Measurement of transfer matrices and source terms of premixed flames. *Journal of Engineering for Gas Turbines and Power*, 124(2), 239–247. <https://doi.org/10.1115/1.1383255>
- Polifke, W., Paschereit, C. O., & Sattelmayer, T. (1997). A Universally Applicable Stability Criterion for Complex Thermo-Acoustic Systems. *Deutsch-Niederländischer Flammentag*, 455–460.
- Rollett, J. M. (1962). Stability and Power-Gain Invariants of Linear Twoports. *IRE Transactions on Circuit Theory*, 9(1), 29–32. <https://doi.org/10.1109/TCT.1962.1086854>
- Schuller, T., Poinso, T., & Candel, S. (2020). Dynamics and control of premixed combustion systems based on flame transfer and describing functions. *Journal of Fluid Mechanics*, 894. <https://doi.org/10.1017/jfm.2020.239>
- Stow, S. R., & Dowling, A. P. (2001). Thermoacoustic oscillations in an annular combustor. *Proceedings of the ASME Turbo Expo*, 2. <https://doi.org/10.1115/2001-GT-0037>
- Yong, K. J., Silva, C. F., & Polifke, W. (2021). A categorization of marginally stable thermoacoustic modes based on phasor diagrams. *Combustion and Flame*, 228, 236–249. <https://doi.org/10.1016/j.combustflame.2021.01.003>