

Designing fractal line pied-de-poules

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January 2020

Designing Fractal Line Pied-de-poules: A Case Study in Algorithmic Design Mediating between Culture and Fractal Mathematics

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Cover Page Footnote

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Designing Fractal Line Pied-de-poules: A Case Study in Algorithmic Design Mediating between Culture and Fractal Mathematics

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Abstract

Millions of people own and wear pied-de-poule (houndstooth) garments. The pattern has an intriguing basic figure and a typical set of symmetries. The origin of the pattern lies in a specific type of weaving. In this article, I apply computational techniques to modernize this ancient decorative pattern. In particular I describe a way to enrich pied-de-poule with a fractal structure.

Although a first fractal line pied-de-poule was shown at Bridges 2015, a number of fundamental questions still remained. The following questions are addressed in this article: Does the original pied-de-poule appear as a limiting case when the fractal structure is increasingly refined? Can we prove that the pattern is regular in the sense that one formula describes all patterns? What is special about pied-de-poule when it comes to making these fractals? Can the technique be generalised?

The results and techniques in this article anticipate the future of fashion where decorative patterns, including pied-de-poule, will be part of our global culture, as they are now, but rendered in more refined ways, using new technologies. These new technologies include digital manufacturing technologies such as laser-cutting and 3D printing, but also computational and mathematical tools such as Lindenmayer rules (originally devised to describe the algorithmic beauty of plants).

1. Introduction

Regular and symmetric ornamental patterns are among the oldest forms of design. Neolithic societies, such as the (Neolithic) Linear Pottery culture in my own region and the iron-age Vikings in Scandinavia considered it worthwhile to combine functionalities, such as pottery and garments with the art of decorative patterns. The pottery of Figure 1 (a) was found in Stein, The Netherlands [21]. The (fragment of) a little statue of Figure 1 (b) was excavated in Sittard, my home town, and is dated about 5000 BC. It probably was part of a little statue with a textile decorative pattern [19].



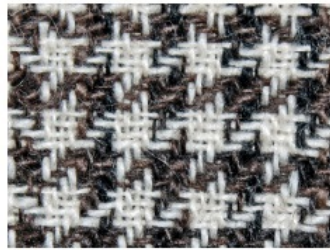
Figure 1: Ancient pottery with decorative patterns found in Stein, The Netherlands (left). Neolithic decorative pattern found in Sittard, The Netherlands (right). Sources: [21] and [19].

The Gerum cloak (Sweden), has has been radiocarbon dated to 360-100 BC, the pre-Roman iron age [14]. Figure 2 shows a fragment of the garment and a modern reconstruction of the weaving pattern. The pattern is what we nowadays call pied-de-poule or houndstooth (more about pied-de-poule in Section 2).

As technology became increasingly sophisticated, the decorative patterns found application in other functional artefacts such as architecture (frieze patterns), woven baskets, paintings, etc. Decorative patterns are among the oldest components of human culture and are deserving of our continued attention. In my view, continued attention means not only studying and preserving old patterns but also looking in-depth and applying contemporary technologies. In this context, mathematics and computer programming are considered technologies, just like materials and production techniques.



Detalj av Gerumsmanteln som visar ljusa och mörka trådar vävda i grupper om fyra.
Fotograf: Gabriel Hildebrand, RAÄ



Adapted from:
Hundtandsmönster i Gerumsmanteln
ursprungliga färgställning.
Tyg från Vävmagasinet 2/06.
Foto: Andreas Hamrin, SHM

Figure 2: Fragment of the Gerum cloak (360–100 BC) and a modern reconstruction. Source: historiska.se/upptack-historien-start/gerumsmanteln/

In past centuries, technology has evolved enormously. In the domain of decorative patterns, we have powerful tools such as printing, Jacquard, wallpaper theory, group theory, tessellation theory and much more. Correspondingly, mathematics and computation have merged into an even more powerful technology, creating fresh new tools such as laser cutters, 3D printers, computer-controlled embroidery and computer Jacquard machines. This work fits in the intersection of arts, math *and* technology. My collaborators and I presented several works aimed at revitalizing one specific decorative pattern, viz. the pied-de-poule, also called houndstooth, already shown in Figure 2.

The work presented in this paper aligns with the goal of a project series to revitalize and refresh pied-de-poule. Specifically, I want to add more depth to the fractal line pied-de-poule presented in [9]. Although the fractal pattern [9], its textile implementation and the garments based on it were well-received at the Bridges Mathematical Art Exhibition held in Baltimore in 2015, it was unclear how the construction works, and why. In this paper I will dive deeper into the formal rules behind the zigzagged pied-de-poule, and find out what is so special about it. The purpose of this work is to discover why and how it works and how it can (or cannot) be generalised. In the following section, I explain more about pied-de-poule, ending in an overview of the plan of the work presented in this paper.

2. Pied-de-poule

Pied-de-poule, also called houndstooth, is a textile pattern which appears from a specific form of weaving with black and white threads. Pied-de-poule has an extensive history, with the oldest known occurrence being the Gerum cloak (Sweden), which has been radiocarbon dated to 360–100 BC, the pre-Roman Iron Age [14]. Pied-de-poule was introduced in fashion by Prince of Wales (Edward the VII) in the 1930s and in haute couture by Dior in the 1950s.

Currently, pied-de-poule is frequently used in haute couture, prêt-à-porter and mass-produced fashion. Although the classic pattern is old, the same design is recycled repeatedly in different contexts, cuts and combinations. Pied-de-poule is very much alive. As part of our research on generative design, pied-de-poule is a recurring theme.

There is a family of pied-de-poule patterns [5], one for each integer $N > 0$. The case $N = 1$ is ambiguous in the sense that it is both a block pattern and a pied-de-poule pattern. Moreover there is a pattern which arises as a limit case when $N \rightarrow \infty$, although this cannot be woven; it can be printed or laser-cut [5], however. In Figure 3 we show the successive pied-de-poule patterns for $N = 1, 2, 3$ and 4. For more details of the computer programs used to generate Figure 3 we refer to [5]. In essence they are grid based, counting row and column numbers.

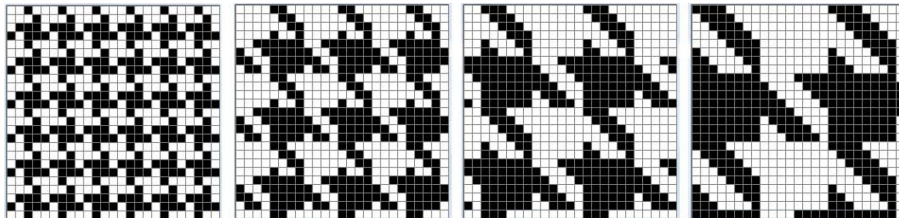


Figure 3: Successive pied-de-poule patterns ($N = 1, 2, 3, 4$).

As an important example of an innovative pied-de-poule in fashion we included an image (Figure 4), which shows a jacket designed by Dior in 2012. This was very innovative: near the shoulder, the pattern appears as a classic pied-de-poule, but as it descends, the pattern gradually changes and the individual “tiles” become separated.



Figure 4: Pied-de-poule pattern in design by Dior in 2012 (with kind permission of Dior Paris).

In [6], a first fractal pied-de-poule was described, which was a kind of Cantor-set approach, recursively leaving out blocks from the classic pattern. Then in [9] another fractal was described, a line fractal based on recursive zigzagging and a specific idea about pen-up and pen-down inspired by turtle graphics. The design by Dior (Figure 4) inspired us for this zigzagging, as we noted that each basic figure was, in fact, a kind of zigzag line. We already knew that inside a classic pied de poule pattern, the black basic tiles are connected and thus form chains. Zooming out, such a chain could be considered a kind of line. The zigzags could be chained, and at the same time, the line drawing could recursively be done zigzag-wise. As described in [9]:

If we would be allowed to use pen-up and pen-down turtle graphics commands, then we could draw all the essential diagonals and connect them by special line segments and arcs.

The special line segments and arcs would be outside of the classic figure, but we could draw them with pen-up and thus they would not be harmful. Or perhaps we could draw them with a very thin pen, and they would be “almost” harmless.¹

The elaboration of this idea is in Figure 5 for $N = 1, 2$, and $N = 3$. The bold red lines are “pen-down”, the black segments and the blue arcs are “pen-up”. The same can be done for any $N > 0$. Instead of working with pen-down and pen-up commands, we choose to let the drawing function work either recursively (writing pied-de-poules all along), or draw non-recursive lines (thin lines).

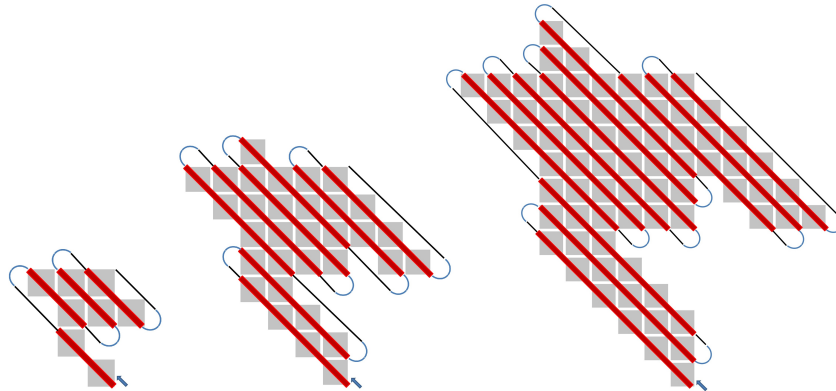


Figure 5: Drawing the diagonals of a classic pied-de-poule with outer loops drawn with a thinner pen for $N = 1$ (left), $N = 2$ (center) and $N = 3$ (right).

To make sure the figures tessellate correctly, we have to do two diagonal pied-de-poule figures in each cell. So they shrink by a factor of $\frac{1}{8}\sqrt{2}$ (for $N = 1$), by $\frac{1}{16}\sqrt{2}$ (for $N = 2$) and by $\frac{1}{24}\sqrt{2}$ (for $N = 3$). In general they shrink by a factor $\frac{1}{8N}\sqrt{2}$. The effect of recursion is demonstrated in Figure 6 for $N = 3$.

The fractal pattern described in [9] was claimed to be a line fractal satisfying the following requirements: pied-de-poule-like, recursively tessellated, parameterised (the figure for recursion level n is a tessellation of figures for type $n - 1$), generic (the same for all n and N) and continuous (no jumps).

¹I make the notion of “almost harmless” more precise in Section 5. WHERE??

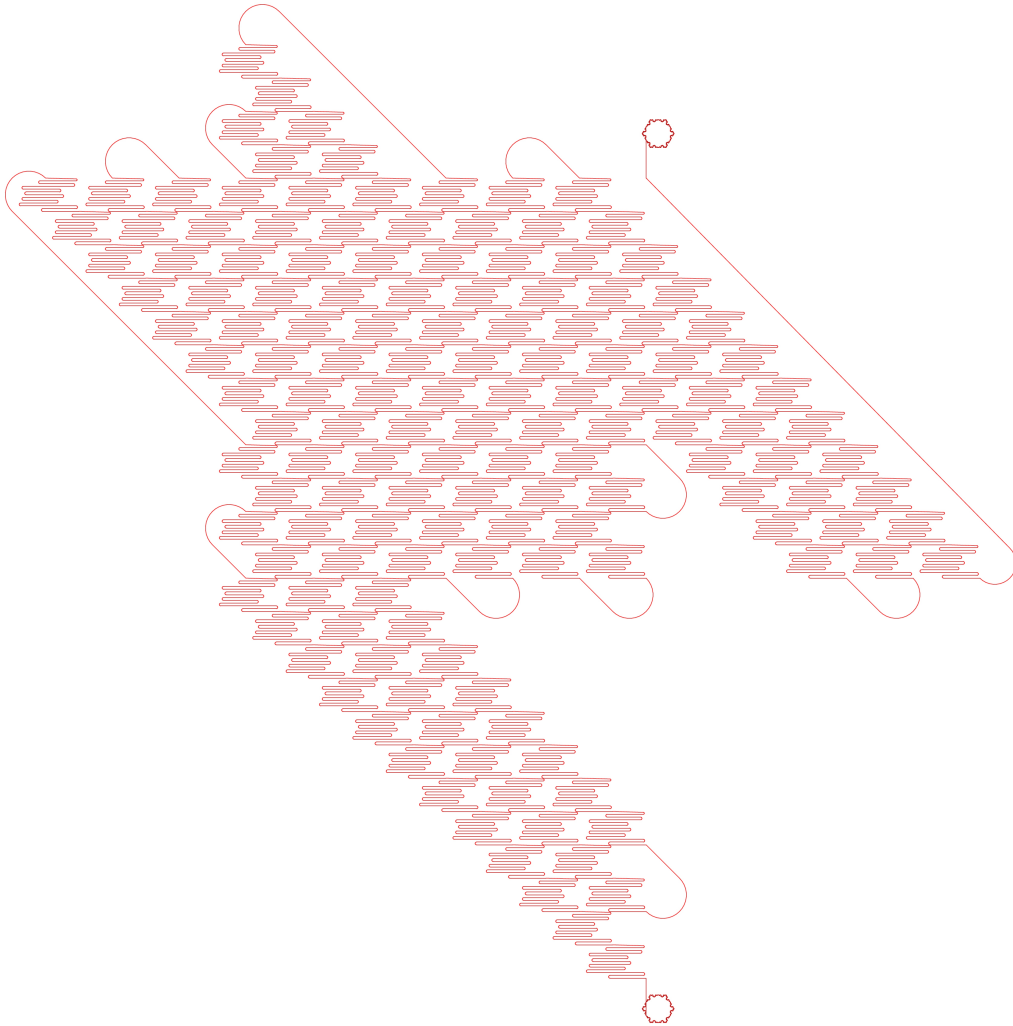


Figure 6: Fractal line pied-de-poule approximation: solution for pied-de-poule type $N = 3$ and recursion level $n = 2$.

The fourth claim, *generic*, means that the figures can be described by a generic recipe with a minimum of ad-hoc tricks and which works the same for each N and each n . Although the idea shown in Figure 5 appears to be effective and generic, my collaborators and I were not able to present a formal description of the fractal as a single formula. In this article, I fill part of that gap, showing how to develop a *generic* description for the pattern.

To describe the pattern, we need a *language*; to this end we will deploy the language of Lindenmayer systems [20], an elegant formalism which has been used for describing fractals; both fractals as found in biology, and designed fractals. In Section 3, I introduce these Lindenmayer systems. Then in Section 4, I describe the recursive tessellation of Section 2 as a Lindenmayer system. In Section 5, I present the formal properties of the fractal. Section 6 gives practical implementation details, not only about coding the fractal in the language of the computer but also about practical aspects of contemporary production machines. In the last section, Section 7, I explore whether the technique, first applied to pied-de-poule, can be generalised to other tessellations. By way of example, I try this for a tessellation by the great master of tessellation, the famous Dutch graphic artist Maurits Escher (1898-1972). I summarise my findings in Section 8.

I should like to inform the reader that Sections 3–7 are relatively technical. This is for two reasons. The first reason is to explain the concepts and statements very precisely. When I say that the “zigzagging of pied-de-poule is a generic recipe”, I want this to be a precise statement, not a vague claim. The second reason is that I can envisage a future in which computational rules, math, new technologies and art come together. This is very promising, but demanding in terms of digital skills and algorithmic technicalities; Sections 3–7 give a preview of this aspect of the envisaged future.

3. Lindenmayer Systems

Lindenmayer systems [20] are often used to describe the growth of fractal plants. In the core of this formalism, there is a substitution approach, for example, a forward move F can be replaced by $F-F++F-F$. As a formal rule we can write: $F \rightarrow F-F++F-F$. The idea is to apply the rule repeatedly (to all F simultaneously). Starting from F we get $F-F++F-F$, then $F-F++F-F-F-F++F-F++F-F++F-F-F-F++F-F$, and so on. Interpreting the symbols as turtle graphics commands, one may for example assign F the meaning of drawing forward, $+$ to turn right 60° , and $-$ to turn left 60° . Then this Lindenmayer system describes the Koch fractal [17] shown in Figure 7.

As another illustration of a Lindenmayer system in action let us look at a fractal inspired by warp-knitting, yet having plant-like qualities (Figure 8). Warp-knitting is a special type of knitting which is well-suited for machine-

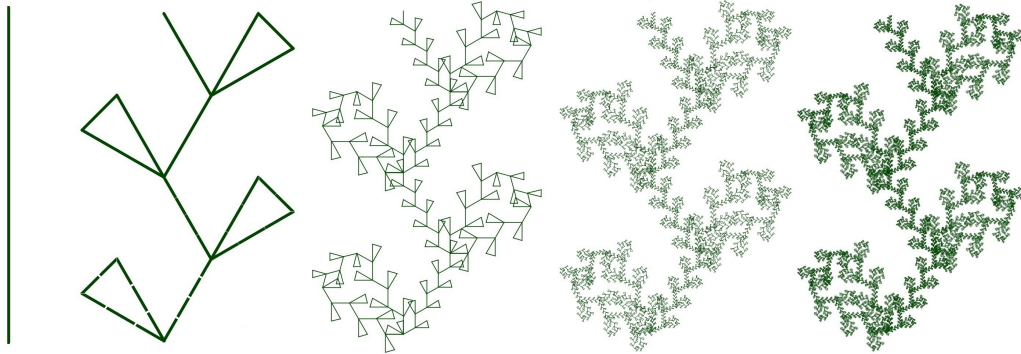


Figure 8: Approximations of the warp-knitting fractal for $N = 0, 1, 2, 3$ and 4.

enough for the complexity at hand. One particular task was to describe the half-circles, which could be done, although the approach is atypical (usually, one would not take a Lindenmayer approach for circles). The next challenge was to describe the phenomenon that each tile has to be described twice: once when travelling up with the turtle, and once when travelling down (in reversed manner). As a third challenge, we found that we had to divide the pied-de-poule tile in four components (“body parts”), each of which had to be formalised in a slightly different manner.

First, how to make the half-circles? We found it useful to adopt special versions of the $+$ and $-$ signals, giving them an extra angle parameter denoted as φ . In particular we interpret $+\varphi$ as turn right over φ and $-\varphi$ as turn left over φ . If the $+$ or $-$ have no subscript, we take by default φ to be $\pi/4$, that is, 45 degrees. Now we can define R_d , which describes a clockwise half-circle with diameter d and L_d , a counter-clockwise half-circle with diameter d . These half turns can be approximated using k steps as follows:

$$\begin{aligned} R_d &\rightarrow (+\varphi F_{d'} +\varphi)^k \\ L_d &\rightarrow (-\varphi F_{d'} -\varphi)^k \end{aligned}$$

where d' is $d \times \sin(\pi/2k)$, and φ is $\pi/2k$. For example, taking $k = 10$ each $+\varphi$ means turning 9 degrees. By increasing k this R_d becomes a very good approximation of a half-circle. Note that these R and L are *not* the basic turtle graphics right and left turn commands; the latter are denoted by $+$ and $-$ respectively.

The symbol F is interpreted as moving forward over a certain distance, say L . We need two distinct symbols for forward, F being interpreted as the usual forward command of turtle graphics, and \mathbf{F} , being a formal symbol during the Lindenmayer substitutions, yet interpreted as F when taking an approximating snapshot after a certain number of simultaneous substitutions (in practice we make recursive Processing programs and then this number is the recursion depth n). As before, we use exponentiation notation for repeated symbols, for example \mathbf{F}^4 means \mathbf{FFFF} .

We define what it means to execute a path in reverse manner, using negative exponent notation: $(c_1 \cdots c_k)^{-1}$ means $c_k^{-1} \cdots c_1^{-1}$, $(-\varphi)^{-1}$ is $+\varphi$ and $(+\varphi)^{-1}$ is $-\varphi$. Similarly L_d^{-1} means R_d , R_d^{-1} means L_d and finally \mathbf{F}^{-1} is just F . For each Lindenmayer rule $\mathbf{F} \rightarrow c_1 \cdots c_k$ tacitly add $\mathbf{F}^{-1} \rightarrow (c_1 \cdots c_k)^{-1}$ (treating \mathbf{F}^{-1} as a symbol).

First we focus on the leftmost zigzag of Figure 5, which is the case $N = 1$. When using recursion to make a single pied de poule figure so that the distance between the begin and end points equals L , we need an equation: $\mathbf{F}_L = -\mathbf{F}_s^4 \mathbf{F}_s^3 \mathbf{R}_s \mathbf{F}_s^{-4} \mathbf{F}_s \mathbf{L}_s \mathbf{F}_s^4 \mathbf{R}_s \mathbf{F}_s \mathbf{F}_s^{-4} \mathbf{L}_s \mathbf{F}_s^3 +$ where $s = L\sqrt{2}/8$ (the general rule is $s = L\sqrt{2}/8N$). In the Lindenmayer approach we omit the size parameters and just let the term expand by n -fold rule-application (recursion level n). It is implicitly understood that the half-circles have a diameter which equals the stepsize of the adjacent F steps. We give the formulas for $N = 1$, $N = 2$ and $N = 3$ now.

$$(N = 1) \quad \mathbf{F} \rightarrow -\mathbf{F}^4 \mathbf{F}^3 \mathbf{R} \mathbf{F}^{-4} \mathbf{F} \mathbf{L} \mathbf{F}^4 \mathbf{R} \mathbf{F} \mathbf{F}^{-4} \mathbf{L} \mathbf{F}^3 +$$

$$(N = 2) \quad \mathbf{F} \rightarrow -\mathbf{F}^8 \mathbf{F} \mathbf{R} \mathbf{F}^{-8} \mathbf{L} \mathbf{F}^5 \mathbf{F}^8 \mathbf{R} \mathbf{F} \mathbf{F}^{-8} \mathbf{F} \mathbf{L} \mathbf{F}^8 \mathbf{F} \mathbf{R} \mathbf{F}^{-8} \mathbf{F}^3 \mathbf{L} \mathbf{F}^8 \mathbf{R} \mathbf{F}_s \mathbf{F}^{-8} \mathbf{L} \mathbf{F}^7 +$$

$$(N = 3) \quad \mathbf{F} \rightarrow -\mathbf{F}^{12} \mathbf{F} \mathbf{R} \mathbf{F}^{-12} \mathbf{L} \mathbf{F} \mathbf{F}^{12} \mathbf{F}^7 \mathbf{R} \mathbf{F}^{-12} \mathbf{F} \mathbf{L} \mathbf{F}^{12} \mathbf{R} \mathbf{F} \mathbf{F}^{-12} \mathbf{F} \mathbf{L} \mathbf{F}^{12} \mathbf{F} \mathbf{R} \mathbf{F}^{-12} \mathbf{L} \mathbf{F} \mathbf{F}^{12} \mathbf{R} \mathbf{F}^5 \mathbf{F}^{-12} \mathbf{F} \mathbf{L} \mathbf{F}^{12} \mathbf{R} \mathbf{F} \mathbf{F}^{-12} \mathbf{L} \mathbf{F}^{11} +$$

Formulas like these are easy to read as zigzags. Each \mathbf{F}^4 is a “zig”, and each \mathbf{F}^{-4} is a zag. For the orientation adopted throughout all the drawings such as Figure 9, a *zig* goes up; a *zag* goes down. Everything else builds the outer loops that connect the zigs and the zags.

Now we sketch the main tasks at hand when developing a formula for arbitrary N . First, the main skeleton of the formula will be a “—” followed by $4N$

copies of \mathbf{F}^{4N} or \mathbf{F}^{-4N} , alternating, followed by one final “+”. Between the zigs and the zags we need extra turtle commands of the non-recursive type, the details being slightly different for each of the transitions, for example when temporarily leaving one body part, or when moving between adjacent body parts. The main statement here is that the generic formula exists. The details are outside the scope of the present paper (they are tedious, but not really difficult). They can be found in [13].

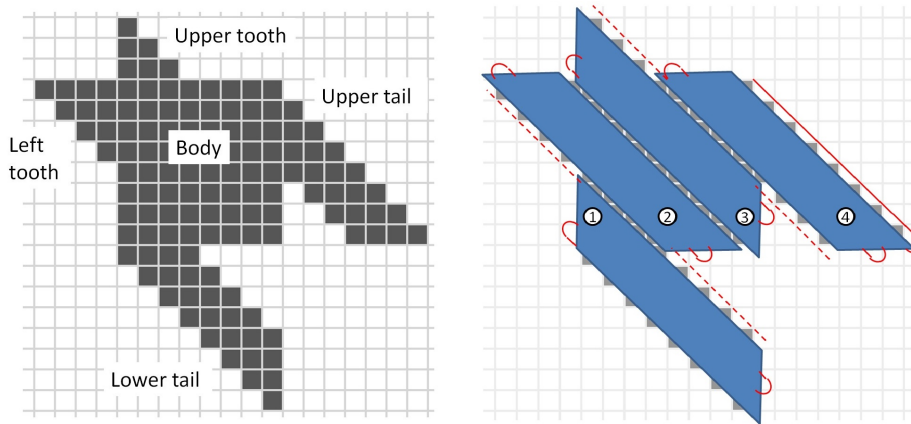


Figure 9: Main geometric parts of the classic pied-de-poule basic tile.

5. Formal Properties

Intuitively we can say that the outer loops are a minor thing, but can we prove it in a formal sense? We shall present two theorems doing precisely that. Certain technicalities are outside the scope of this journal, but can be inspected at Github, see [13]. First we need some preliminaries. We write PDP as an abbreviation of “classic pied-de-poule”. We say that a set $\mathcal{P} \subseteq \mathbb{R}^2$ is a PDP of type N if it has been constructed according to the methods mentioned in Section 1 and detailed in [5]. Such a \mathcal{P} is the union of $8N^2$ non-overlapping square regions of width d for some $d \in \mathbb{R}$. We call d the *grid size*. We say that the *size* of a PDP \mathcal{P} , denoted by $\text{size}(\mathcal{P})$ is the width of the smallest square box which is aligned with the grid and which encloses \mathcal{P} . It equals $5N - 1$. We write fPDP for “fractal line pied-de-poule approximation” and we say that a set $\mathcal{F} \subseteq \mathbb{R}^2$ is an fPDP of type N and recursion level n if it has been constructed according to the methods explained in Sections 2 and 4.

Let $X \subseteq \mathbb{R}^2$ be an arbitrary set then we define the ϵ -fattening of X , denoted as $[X]^\epsilon$ to be a set like X , but with a band of size ϵ added all around it. For each PDP \mathcal{P} of type N , let $\mathcal{F}_n(\mathcal{P})$ be the fPDP which runs through the diagonals of \mathcal{P} and has recursion level n .

Theorem Let $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \dots$ be a sequence of PDPs of type $1, 2, 3, \dots$ and fixed $\text{size}(\mathcal{P}_N) = s$ for all integer $N > 0$. The \mathcal{P}_N thus have shrinking grid sizes $s/4, s/9, s/14, \dots$. Then for each recursion level $n \in \mathbb{N}$ there is a sequence of corresponding fPDPs $\mathcal{F}_n(\mathcal{P}_1), \mathcal{F}_n(\mathcal{P}_2), \mathcal{F}_n(\mathcal{P}_3), \dots$ and a sequence of real numbers $\epsilon_1, \epsilon_2, \epsilon_3, \dots$ such that for all $N > 0$ we have $\mathcal{F}_n(\mathcal{P}_N) \subseteq [\mathcal{P}_N]^{\epsilon_N}$ and

$$\lim_{N \rightarrow \infty} \frac{\epsilon_N}{\text{size}(\mathcal{P}_N)} = 0$$

Proof. See [13].

The idea of the theorem is presented in Figure 10. The outer loops get closer and closer to the edge of the tile; in the visual Gestalt of the tile they become almost invisible, they can be neglected.

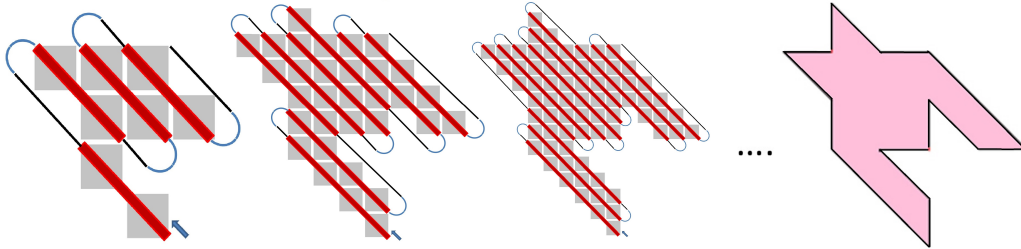


Figure 10: Illustration of the first theorem. By taking successive pied-de-poule types, that is, successive tiles from the pied-de-poule family, the zigzagging becomes more refined so that the auxiliary outer loops protrude less and less.

So we can neglect the outer loops for large pied-de-poule type N , but what about fixed N ? The next theorem says that the outer loops are neglectable anyhow as they have infinitesimal thickness (by which we mean that in the limit case, the thickness tends to zero). If we make a practical drawing of an fPDP \mathcal{F} , the mathematical line has to become visible. Whenever we use a pen of a certain stroke width, we are in fact making an ϵ -fattening $[\mathcal{F}]^\epsilon$ where ϵ is half the pen stroke width. How wide such a pen stroke can be is

naturally limited by the condition that adjacent strokes still have sufficient white space in between – even inside the smallest recursive figures embedded in \mathcal{F} . Without loss of generality we interpret ‘sufficient white space’ to mean that the white space is equally wide as the pen strokes themselves.

Theorem Let $\mathcal{F}^0, \mathcal{F}^1, \mathcal{F}^2, \dots$ be a sequence of fPDPs of increasing recursion level $n = 0, 1, 2, \dots$ and such that the \mathcal{F}^n all run through the diagonals of a single PDP \mathcal{P} of given type N and given size. Let $\epsilon_0, \epsilon_1, \epsilon_2, \dots$ be the sequence of values in \mathbb{R} (half stroke widths) such that for each $n \in \mathbb{N}$ the band of points between the adjacent parallel diagonal strokes of the smallest recursive figures embedded in \mathcal{F}^n is equally wide as the diagonal strokes $[\mathcal{F}^n]^\epsilon$ themselves, viz. $2\epsilon_n$. Then

$$\lim_{n \rightarrow \infty} \frac{\epsilon_n}{\text{size}(\mathcal{P})} = 0$$

Proof. See [13].

The idea of the second theorem is presented in Figure 11.

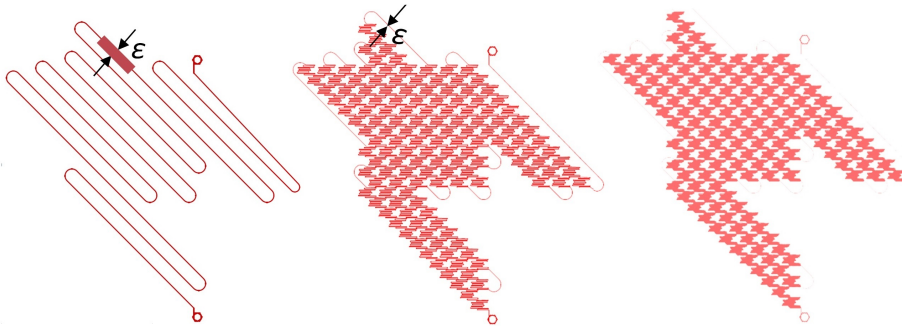


Figure 11: Illustration of the second theorem. The pied-de-poule type does not change, but the recursion level increases. Then the stroke-width ϵ becomes arbitrarily small; in the limit case, it goes to 0.

6. Implementation Details

The following Java function shows how we constructed a left-turning half-circle in Oogway. Oogway [4] is a turtle graphics library created by Jun Hu, aimed at creative programming and tessellations. The function is named LARC meaning left-arc (half-circle).

```

void LARC(float diam){
    int steps = max(18,min(1,ceil(sqrt(diam))));
    float phi = 180.0 / steps;
    float segment = diam * sin(radians(phi/2));
    for (int i = 0; i < steps; i++){
        LEFT(phi / 2);
        FORWARD(segment);
        LEFT(phi / 2);
    }
}

```

We can translate the Lindenmayer rules into Oogway commands, which are mixed with regular Processing (=Java) statements. For example, for $N = 3$ the formula would be: $\mathbf{F} \rightarrow -\mathbf{F}^{12}\mathbf{FRF}^{-12}\mathbf{LF}$ etcetera +, which is rendered in Java as follows:

```

void FORPIED(float LEN, int budget){
    int N = 3;
    float grid = LEN / 12;
    float step = grid / sqrt(2);
    if ( budget == 0)
        FORWARD(LEN);
    else {
        LEFT(45);
        for (int i = 0; i < 12; i++)
            FORPIED(step, budget-1);
        FORWARD(step);
        RARC(step);
        for (int i = 0; i < 12; i++)
            FORDIEP(step, budget-1);
        LARC(step);
        FORWARD(step);

        //etcetera
        RIGHT(45);
    }
}

```

The command FORWARD is the basic Oogway turtle graphics command. The function FORPIED codes \mathbf{F} and therefore is the pied-de-poule variation of

going forward (PIED being shorthand for pied-de-poule). And there is a similar function `FORDIEP` which is the reversed version (DIEP being the word PIED in reverse, in Dutch language "IE" is one vowel). One call of the function `FORDIEP` codes \mathbf{F}^{-1} and then of course

```
for (int i = 0; i < 12; i++) FORDIEP(step,budget-1)
codes  $\mathbf{F}^{-12}$ .
```

For the implementation of the garments described in [9] (Figure 13), we did not generate a recursive structure for fixed N , but we made a mixed figure where the pied-de-poule is of the $N = 3$ type at the highest recursion level, $N = 2$ for the smaller pied-de-poules, and $N = 1$ at the smallest level. This is shown in Figure 12.

Also, in Fig 12 it can be seen how we avoided the effect that adjacent figures touch each other. If we would run the turtle graphics commands as obtained by straightforward coding of the Lindenmayer rules, we find that adjacent pied-de-poule figures touch each other. We tweak the turtle path a little bit, so the effect is hardly visible, and then we can claim that the entire path is a single line which does not touch or cross itself. This tweaking is implemented by adding extra statements inside the code of `FORPIED` and `FORDIEP`. Yet the appearance of the total figure is not affected.

Today, there is a significant change happening in the world of fashion manufacturing equipment. This change is one of the reasons why we expect a new wave of innovation in fashion. We chose a production method which is consistent with this development. Novel manufacturing methods are data-driven and are less and less depending on manual machine set-up procedures. Examples include 3D printing, computer-embroidery, laser cutting, Jacquard weaving and computer-printing. These new manufacturing methods will support ultra-personalisation and aesthetic innovation. Examples of aesthetic innovation can be found in [22] and [18]. Jacquard weaving was invented a long time ago and it still expensive, but the digitisation may lead to a renewed interest. In our case, we experimented with computer-embroidery and also with computer-laser engraving; for implementing our fractal pied-de-poule we have chosen laser engraving. The laser produces extremely thin carving lines, which make the fractal appear very subtle and beautiful, both at a short distance and far away. Although our Trotec Speedy300 laser cutter can move at more than 3 m/s, it takes hours to engrave a large fractal

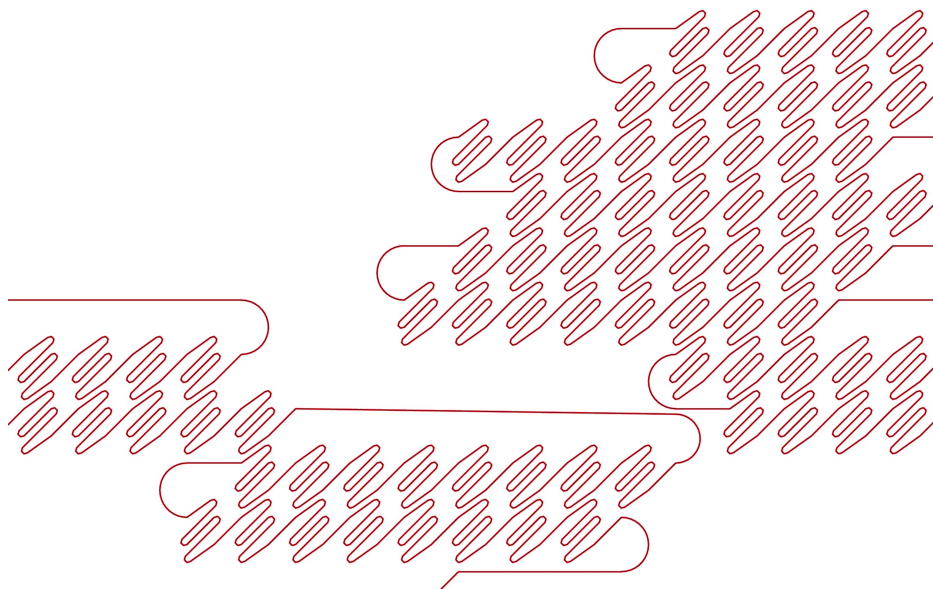


Figure 12: Mixed figure where the pied-de-poule is of the $N = 3$ type at the highest recursion level, $N = 2$ for the smaller pied-de-poules, and $N = 1$ at the lowest level.

(the figure is a very long densely compressed line). Further optimisation is possible, for example, Bézier curves for the semi-circles would be better (as the machine can interpret these faster).

Figure 13 shows one of the attractive fractal line pied-de-poule garments we created and exhibited in Baltimore at the Bridges Mathematical Art Exhibition in 2015 [10].

7. Generalisations

Under certain conditions, the idea of zigzagging a figure can be generalised for more shapes. If the shape is part of a tessellation pattern and if the shape can be zigzagged, then it can be turned into a recursively tessellated line fractal. To zigzag a shape, one needs an entry point and an exit point, which are most conveniently chosen to be a network point of the tessellation. If the shape is convex, then it can always easily be zigzagged, otherwise, additional tricks are needed. One trick is to adjust the angle, which so far was 45° .



Figure 13: Fractal line-pied-de-poule garment by Marina Toeters and Loe Feijs at the Bridges Mathematical Art Exhibition in Baltimore in 2015 (photo by the author).

Worst-case one needs other, less-pure types of zigzagging, such as re-entrant loops. Most interesting artistic tessellations, such as Escher's tessellations are made with non-convex figures indeed. We illustrate this by creating a recursively tessellated line fractal based on one of Escher's birds, E128 (Figure 14).

Using the taxonomy of tessellations developed by Heesch and Kienzle in the 1960s, we note that this particular bird configuration has Heesch type TTTT [16], which means that each tile has four edges, pairwise related by

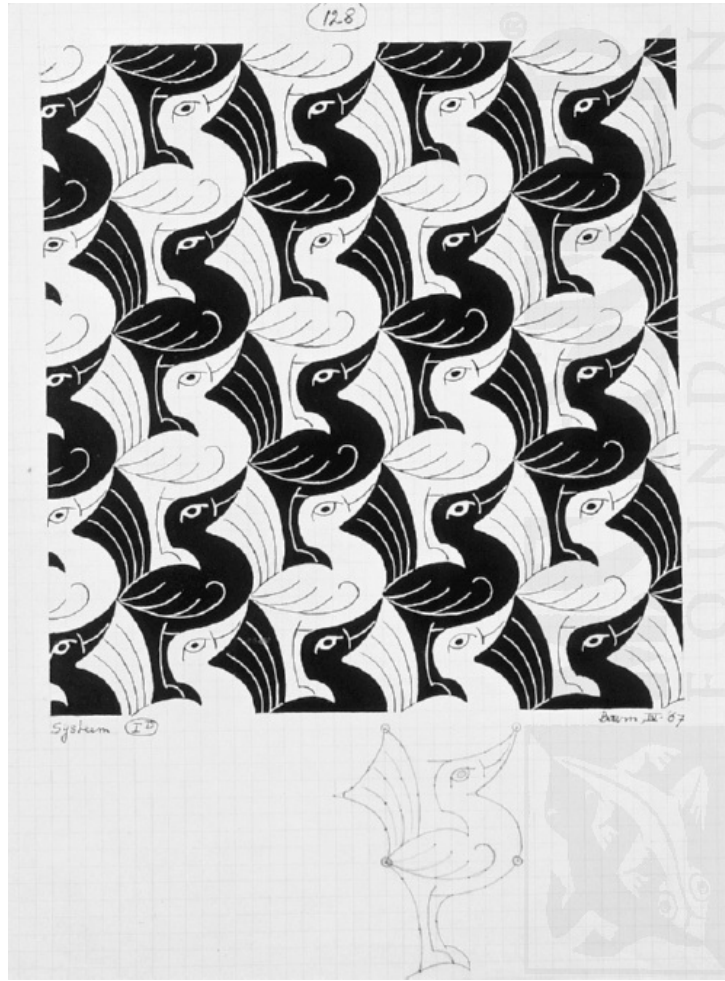


Figure 14: M.C. Escher’s “Symmetry Drawing E128” ©2019 The M.C. Escher Company-The Netherlands. All rights reserved. www.mcescher.com.

translations (unlike pied-de-poule, which has Heesch type TTTTTT). In each network point, four edges come together. Escher’s sketch has explicitly indicated network points, and we choose two of them which are diagonally opposed. The four network points are arranged in a square, but clearly, the bird extends beyond the square (Figure 15).

The bird is not concave, but it (almost) fits in a rectangular box which goes through the entry and exit points, which means that the zigzagging can be done by lines parallel to the edges of the rectangle without missing too

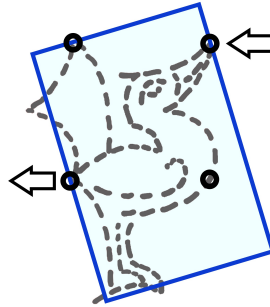


Figure 15: Fitting the bird in a rectangular box. The box is helpful for choosing the main direction of zigzagging. The black dots are the network points. The arrows indicate the entry and exit points.

much of the bird. In this case, we work with an angle of 56° . In Figure 16, the process of interactively choosing a proper zigzag pattern is shown. It is done with a locally made software tool which enables editing a simple Lindenmayer language and simultaneously interpreting it with an arbitrary bitmap as background (here a rotated E128).

A first version of the zigzagging result is shown for three different recursion levels in Figure 17 (the three leftmost birds). It has the following Lindenmayer rule: $\mathbf{F} \rightarrow -\mathbf{FR} \mathbf{F}^{-1}\mathbf{L} \mathbf{F}^2\mathbf{F}^3\mathbf{F}^3\mathbf{FR} \mathbf{F}^{-8}\mathbf{L} \mathbf{F}^9\mathbf{FR} \mathbf{F}^{-9}\mathbf{L} \mathbf{F}^9\mathbf{R} \mathbf{F}^{-4}\mathbf{FF}^{-4}\mathbf{L} \mathbf{F}^5\mathbf{F}^4\mathbf{R} \mathbf{F}^{-13}\mathbf{L} \mathbf{F}^{14}\mathbf{F}^4\mathbf{FR} \mathbf{F}^{-16}\mathbf{L} \mathbf{FF}^7+$. The rightmost bird in Figure 17 shows an additional feature: it has a re-entrant loop in the tail.

After refinement of the details, we obtained the tessellation of Figure 18, which has three recursion levels. The additional feature of making reentrant loops, abandoning pure zigzagging, was used for the bird's tail and foot. It gives more creative freedom, but now the line self-intersects. The generated vector graphics image is extremely detailed, and the challenge to materialize it is still ahead of us (the line of Figure 18 has more than two million "forward" steps). From this generalisation, we conclude that:

- We are given another perspective on the pied-de-poule case. The zigzagging of the pied-de-poule went smooth only because of the mathematical properties and precise rectilinear outline of the basic pied-de-poule figure. We could take advantage of the typicalities of the TTTTTT Heesch type. In fact, the basic pied-de-poule figure turns

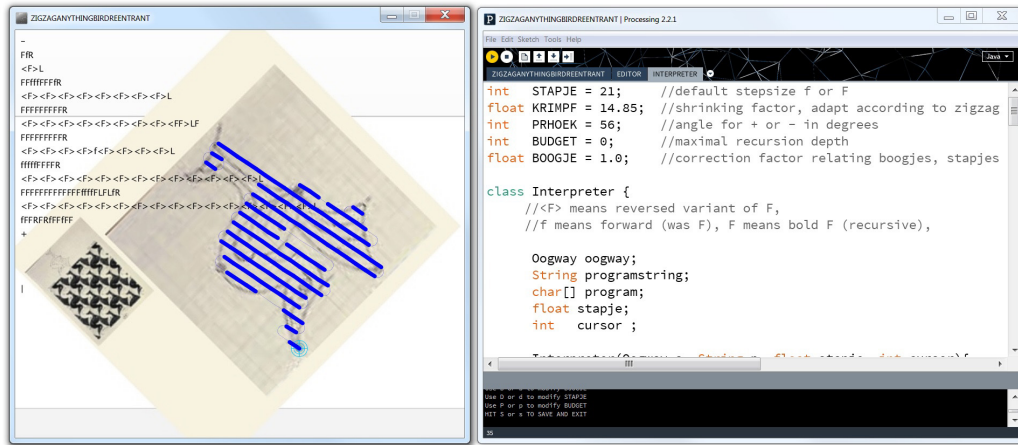


Figure 16: Zigzagging Escher’s bird using a dedicated interactive tool in Processing. Left is the interaction screen which shows both the turtle graphics program and the result, overlaid on a background bitmap. Right is the tool’s code window.

out perfectly fit for 45° zigzagging. The first theorem of Section 5 can only be formulated for pied-de-poules, not for the birds (they do not come as a regular family).

- For different Heesch types, different solutions can be found, as demonstrated by choice of diagonal entry points. Yet the process is somewhat ad-hoc and the complement white-space in Figure 18 creates a bird which is less elegant than Escher’s basic figure. This can be overcome partly by using more zigzag lines and the re-entrant loop feature. Certain media such as high-resolution laser systems are better suited to materialise the result than others (e.g. embroidery).

8. Concluding Remarks

The combination of turtle graphics and the concept of recursive zigzagging is a technology which we can apply to any figure, not just pied-de-poule. But because of the special symmetries embedded in pied-de-poule, it became a fascinating exercise to prepare the program and to analyze the properties. The mathematics of Sections 3–5 are essential to support the precise formulation of the properties of the fractal line pied-de-poule. The negative exponent notation for reversed turtle movements is to the best of our knowledge, new.

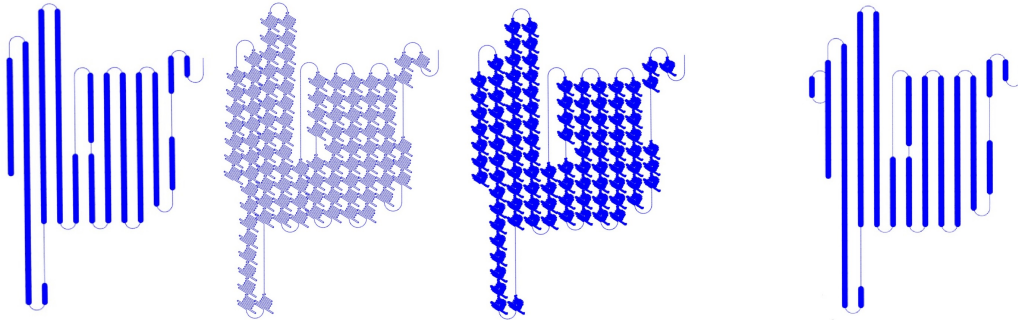


Figure 17: The bird with successive recursion levels adopting a pure zigzagging approach. The rightmost bird shows an additional feature: it has a re-entrant loop in the tail. This solves the problem that the bird would not fit in the rectangle.

We consider it worthwhile to focus serious attention to the pied-de-poule pattern, which is a great asset of European culture (and mediated by fashion now of global culture). In this paper, we used the somewhat technical Lindenmayer rules so that we could see in sufficient detail what was so special about the zigzagged pied-de-poule.

The work was somewhat technical, but that is an essential part of the envisioned fashion future. As limitations of (mass) production machines tend to disappear, a new creative space is opened-up. In this new creative space, computational rules, math, new technologies and art come together. These allow for more personalization and sophistication but is demanding in terms of digital skills and algorithmic technicalities. We would like to finish with a quote from Karl Lagerfeld: “*Fashion is about two things: continuity and the opposite. That’s why you have to keep moving*”. This project contributes to moving towards radically new fractal decorative patterns, with continuity coming from the ancient, almost archetypal pied-de-poule.

Related work: The work is related to ethno modeling (ethnomathematics and ethno computing) as described in, for example [1] and [2]. In ethno modeling, as in this paper, cultural artefacts are dissected using mathematical and then applied creatively in new ways, using computational tools. In the Bridges community, this is a recurring theme, see for example, Gerdes’ descriptions of African Basketry [15]. One of the goals described by Babbitt [1] is to educate and empower young people from under-represented ethnic groups, deploying the mathematics in cultural artefacts. Although I sympathize with the idea,

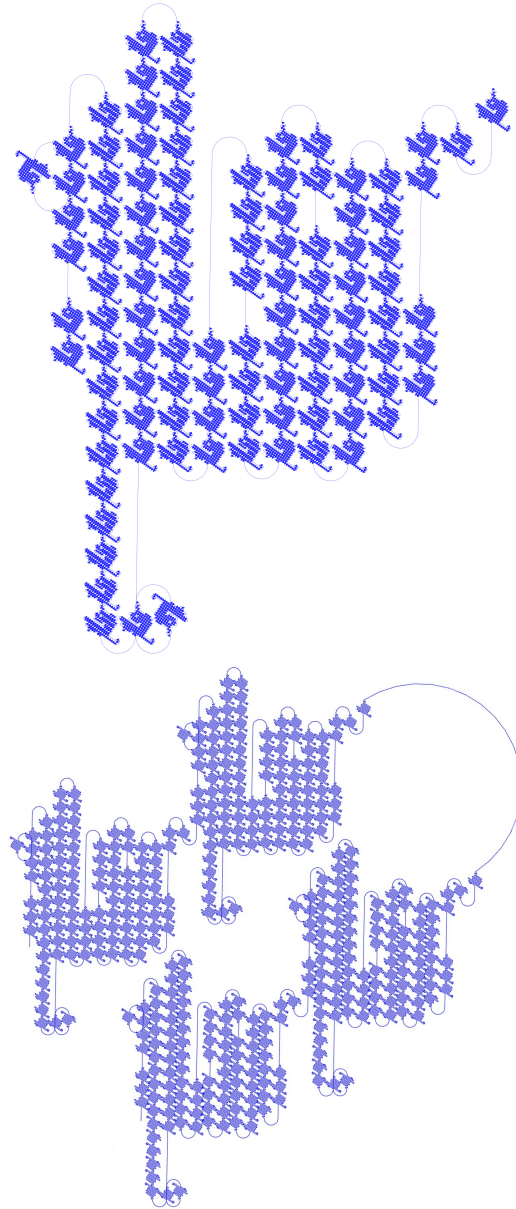


Figure 18: Tessellation obtained by recursively zigzagging the bird with three levels of recursion. It is simultaneously a fractal structure *and* a tessellation structure. It consists of a single continuous zigzag line. Left is the entire bird, right we zoom in to the four sub-birds at the highest point the tail.

in my own work, the ethnic aspect has played a lesser role. Besides pied-de-poule, I worked with cultural themes related to The Netherlands: (Escher-style) tessellations [3, 4] and (Mondrian-style) non-figurative art [7, 12]. Pied-de-poule seems mostly a Western-world theme (Section 2). Tessellations and fractals are used often to raise awareness and pleasure in mathematics for children (see for example www.mathartfun.com). In my teaching, together with colleagues Christoph Bartneck, Jun Hu, and Mathias Funk, we tried to bring mathematical principles to the attention of our (university-level) design students. Many design students like to *make* things, which has determined the pedagogical approach of our course Golden Ratio at TU/e for the past ten years [3, 4].

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