Periodic homogenization of a pseudo-parabolic equation via a spatial-temporal decomposition

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Arthur. J. Vromans and Fons van de Ven and Adrian Muntean

Abstract Pseudo-parabolic equations have been used to model unsaturated fluid flow in porous media. In this paper it is shown how a pseudo-parabolic equation can be upscaled when using a spatio-temporal decomposition employed in the Peszyńska-Showalter-Yi paper [8]. The spatial-temporal decomposition transforms the pseudo-parabolic equation into a system containing an elliptic partial differential equation and a temporal ordinary differential equation. To strengthen our argument, the pseudo-parabolic equation has been given advection/convection/drift terms. The upscaling is done with the technique of periodic homogenization via two-scale convergence. The well-posedness of the extended pseudo-parabolic equation is shown as well. Moreover, we argue that under certain conditions, a non-local-in-time term arises from the elimination of an unknown.

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1 Introduction

Groundwater recharge and pollution prediction for aquifers need models for describing unsaturated fluid flow in porous media. Pseudo-parabolic equations were found to be adequate models, see eqn. 25 in [3]. In [8] a spatial-temporal decomposition

A.J. Vromans and A.A.F. van de Ven
Centre for Analysis, Scientific computing and Applications (CASA), Eindhoven University of Technology, Den Dolech 2, 5612AZ, Eindhoven, The Netherlands, e-mail: a.j.vromans@tue.nl; a.a.f.v.d.ven@tue.nl

A. Muntean
Department of Mathematics and Computer Science, Karlstad University, Universitetsgatan 2, 651 88, Karlstad, Sweden, e-mail: adrian.muntean@kau.se

of a pseudo-parabolic system was introduced. It was shown that this decomposition made upscaling of this system rather straightforward in several classical situations such as vanishing time-delay and double-porosity systems. In [8] a toy pseudo-parabolic model was derived from a balance equation describing flow through a partially saturated porous medium. In our framework, a convective term that was dropped in [8], is retained in order to show that this term yields no additional problems for upscaling with the spatial-temporal decomposition. We want to convey the message that this decomposition can be applied not only to the physical system in [8] but also to other physical systems with pseudo-parabolic equations, such as the concrete corrosion reaction model introduced in [9]. Both these pseudo-parabolic systems are physical systems on a spatial micro scale with an intrinsic microscopic periodicity of size $\varepsilon \ll 1$. Similar intrinsic microscopic periodic behaviors are found in highly active research fields using composite structures or nano-structures.

In this paper, we use this spatial-temporal decomposition to upscale our pseudo-parabolic equation by using the concept of periodic homogenization via two-scale convergence, which leads to a homogenized system that retains the spatial-temporal decomposition. We start in Section 2 with formulating our pseudo-parabolic system $(\mathbf{Q}^\varepsilon)$, the decomposition system $(\mathbf{P}^\varepsilon)$ and stating our assumptions. In Section 3, an existence and uniqueness result for weak solutions to our problem $(\mathbf{P}^\varepsilon)$ is derived. In Section 4, we apply the idea of two-scale convergence to a weak version of problem $(\mathbf{P}^\varepsilon)$, denoted $(\mathbf{P}^\varepsilon_w)$, that contains the microscopic information at the ε-level. Furthermore in this section, an upscaled system $(\mathbf{P}^0_w)$ of the weak system $(\mathbf{P}^\varepsilon_w)$ is derived in the limit $\varepsilon \downarrow 0$, and, under certain conditions, an upscaled strong system $(\mathbf{P}^0_s)$ is obtained after eliminating several variables. This upscaled strong system contains a non-local-in-time term, but the system has lost the partial differential equation framework as a consequence. Contrary, the upscaled weak system $(\mathbf{P}^0_w)$ keeps the partial differential equation framework due to the spatial-temporal decomposition.

2 Basic system and assumptions

Our pseudo-parabolic system $(\mathbf{Q}^\varepsilon)$ consists of a family of N partial differential equations for the variable vector $\mathbf{U}^\varepsilon(t,\mathbf{x},\mathbf{x}/\varepsilon) = (U_1^\varepsilon,\dots,U_\alpha^\varepsilon,\dots,U_N^\varepsilon)$ with $t>0$ and $\mathbf{x} = (x_1,\dots,x_i,\dots,x_d) \in \Omega \subset \mathbf{R}^d$. For $\varepsilon \in (0,\varepsilon_0)$ with $\varepsilon_0 > 0$, system $(\mathbf{Q}^\varepsilon)$ is formulated as

$$(\mathbf{Q}^\varepsilon) \quad \begin{cases} \mathsf{M}^\varepsilon \mathsf{G}^{-1}\partial_t \mathbf{U}^\varepsilon - \nabla\cdot\big((\mathsf{E}^\varepsilon\cdot\nabla + \mathsf{D}^\varepsilon)\mathsf{G}^{-1}(\partial_t\mathbf{U}^\varepsilon + \mathsf{L}\mathbf{U}^\varepsilon)\big) & \\ \qquad = \mathbf{H}^\varepsilon + (\mathsf{K}^\varepsilon - \mathsf{M}^\varepsilon\mathsf{G}^{-1}\mathsf{L})\mathbf{U}^\varepsilon + \mathsf{J}^\varepsilon\cdot\nabla\mathbf{U}^\varepsilon & \text{on } \mathbf{R}_+\times\Omega, \\ \mathbf{U}^\varepsilon = \mathbf{U}_* & \text{on } \{0\}\times\Omega, \\ \partial_t\mathbf{U}^\varepsilon + \mathsf{L}\mathbf{U}^\varepsilon = \mathbf{0} & \text{on } \mathbf{R}_+\times\partial\Omega. \end{cases}$$

The vectors $\mathbf{V}^\varepsilon$ and $\mathbf{U}^\varepsilon$ are both functions of the time coordinate t, the global or macro position coordinate $\mathbf{x}$, and also periodic functions of the micro (or nano) coordinate $\mathbf{y} \in Y$, where $\mathbf{y} = \mathbf{x}/\varepsilon$, where the size of the micro domain Y is $\mathscr{O}(\varepsilon)$ of the size of the macro domain Ω.
Our dimensionless decomposition system $(\mathbf{P}^\varepsilon)$ consists of a family of N partial differential equations (PDEs) and a family of N ordinary differential equations (ODEs) for the two variable vectors $\mathbf{V}^\varepsilon(t,\mathbf{x},\mathbf{x}/\varepsilon) = (V_1^\varepsilon,\dots,V_\alpha^\varepsilon,\dots,V_N^\varepsilon)$ and $\mathbf{U}^\varepsilon(t,\mathbf{x},\mathbf{x}/\varepsilon)$. For $\varepsilon \in (0,\varepsilon_0)$ with $\varepsilon_0 > 0$, it is formulated as

$$(\mathbf{P}^\varepsilon) \quad \begin{cases} \mathsf{M}^\varepsilon \mathbf{V}^\varepsilon - \nabla\cdot(\mathsf{E}^\varepsilon \cdot \nabla \mathbf{V}^\varepsilon + \mathsf{D}^\varepsilon \mathbf{V}^\varepsilon) = \mathbf{H}^\varepsilon + \mathsf{K}^\varepsilon \mathbf{U}^\varepsilon + \mathsf{J}^\varepsilon \cdot \nabla \mathbf{U}^\varepsilon & \text{on } \mathbf{R}_+ \times \Omega, \\ \partial_t \mathbf{U}^\varepsilon + \mathsf{L}\mathbf{U}^\varepsilon = \mathsf{G}\mathbf{V}^\varepsilon & \text{on } \mathbf{R}_+ \times \Omega, \\ \mathbf{U}^\varepsilon = \mathbf{U}_* & \text{on } \{0\} \times \Omega, \\ \mathbf{V}^\varepsilon = \mathbf{0} & \text{on } \mathbf{R}_+ \times \partial\Omega. \end{cases}$$

Above, the ε-dependent notation $c^\varepsilon(t,\mathbf{x}) = c(t,\mathbf{x},\mathbf{x}/\varepsilon)$ is used for the ε-independent 1-,2- and 3-tensors of assumption (A1).

(A1) For all $\alpha,\beta \in \{1,\dots,N\}$ and for all $i,j \in \{1,\dots,d\}$, we have

$$\begin{aligned} \mathsf{M}_{\alpha\beta}, \mathsf{E}_{ij}, \mathsf{D}_{i\alpha\beta}, \mathbf{H}_\alpha, \mathsf{K}_{\alpha\beta}, \mathsf{J}_{i\alpha\beta} &\in L^\infty(\mathbf{R}_+ \times \Omega; C_\#(Y)), \\ \mathsf{L}_{\alpha\beta}, \mathsf{G}_{\alpha\beta} &\in L^\infty(\mathbf{R}_+; W^{1,\infty}(\Omega)), \\ \mathbf{U}_* &\in C^1(\Omega)^N, \end{aligned}$$

with G invertible.
(A2) Let the tensors M^ε and E^ε be in diagonal form[1] with elements $m_\alpha^\varepsilon > 0$ and $e_i^\varepsilon > 0$, respectively, satisfying $1/m_\alpha^\varepsilon, 1/e_i^\varepsilon \in L^\infty(\mathbf{R}_+ \times \Omega; C_\#(Y))$.
(A3) The inequality

$$\|\mathsf{D}_{i\beta\alpha}^\varepsilon\|^2_{L^\infty(\mathbf{R}_+\times\Omega^\varepsilon;C_\#(Y))} < \frac{4}{dN^2 \|1/m_\alpha^\varepsilon\|_{L^\infty(\mathbf{R}_+\times\Omega^\varepsilon;C_\#(Y))} \|1/e_i^\varepsilon\|_{L^\infty(\mathbf{R}_+\times\Omega^\varepsilon;C_\#(Y))}}$$

holds for all $\alpha,\beta \in \{1,\dots,N\}$, for all $i \in \{1,\dots,n\}$, and for all $\varepsilon \in (0,\varepsilon_0)$.

Remark, inequality (2) implies that automatically (2) holds for the Y-averaged functions $\overline{\mathsf{D}_{i\beta\alpha}^\varepsilon}$, $\overline{\mathsf{M}_{\beta\alpha}^\varepsilon}$, and $\overline{\mathsf{E}_{ij}^\varepsilon}$ in $L^\infty(\mathbf{R}_+ \times \Omega)$, using $|Y|\overline{f}(t,\mathbf{x}) = \int_Y f(t,\mathbf{x},\mathbf{y})\mathrm{d}\mathbf{y}$.

3 Existence and uniqueness of weak solutions to $(\mathbf{P}_w^\varepsilon)$

In this section, we show the existence and uniqueness of a weak solution $(\mathbf{U},\mathbf{V})$ to $(\mathbf{P}^\varepsilon)$. We define a weak solution to $(\mathbf{P}^\varepsilon)$ for $\varepsilon \in (0,\varepsilon_0)$ and $T \in \mathbf{R}_+$ as a pair of

[1] Due to the Theorem of Jacobi about quadratic forms (cf. [4]) in combination with the coercivity of both M^ε and E^ε, we are allowed to assume diagonal forms of M^ε and E^ε as the orthogonal transformations, necessary to put their quadratic forms in diagonal form, modify the domain Ω^ε and the coefficients of D^ε, $\mathbf{H}^\varepsilon$, K^ε and J^ε without changing their regularity.

sequences $(\mathbf{U}^\varepsilon, \mathbf{V}^\varepsilon) \in H^1((0,T)\times\Omega)^N \times L^\infty((0,T), H^1_0(\Omega))^N$ satisfying

$$(\mathbf{P}^\varepsilon_w) \begin{cases} \int_\Omega \phi^\top [\mathsf{M}^\varepsilon \mathbf{V}^\varepsilon - \mathbf{H}^\varepsilon - \mathsf{K}^\varepsilon \mathbf{U}^\varepsilon - \mathsf{J}^\varepsilon \cdot \nabla \mathbf{U}^\varepsilon] + (\nabla\phi)^\top \cdot (\mathsf{E}^\varepsilon \cdot \nabla \mathbf{V}^\varepsilon + \mathsf{D}^\varepsilon \mathbf{V}^\varepsilon)\, \mathrm{d}\mathbf{x} = 0, \\ \int_\Omega \psi^\top [\partial_t \mathbf{U}^\varepsilon + \mathsf{L}^\varepsilon \mathbf{U}^\varepsilon - \mathsf{G}^\varepsilon \mathbf{V}^\varepsilon]\, \mathrm{d}\mathbf{x} = 0, \\ \mathbf{U}^\varepsilon(0,\mathbf{x}) = \mathbf{U}_*(\mathbf{x}) \text{ for all } \mathbf{x} \in \Omega, \end{cases}$$

for a.e. $t \in (0,T)$, for all test-functions $\phi \in H^1_0(\Omega)^N$ and $\psi \in L^2(\Omega)^N$.
The existence and uniqueness can only hold when the first equation of $(\mathbf{P}^\varepsilon_w)$ satisfies all the conditions of Lax-Milgram. The next lemma provides the coercivity condition, while the continuity condition is trivially satisfied.

Lemma 1. *Assume assumptions (A1) - (A3) hold, then there exist positive constants $\tilde{m}_\alpha$, $\tilde{e}_i$, $\tilde{H}$, $\tilde{K}_\alpha$, $\tilde{J}_{i\alpha}$ for $\alpha \in \{1,\ldots,N\}$ and $i \in \{1,\ldots,d\}$ such that the following a-priori estimate holds for a.e. $t \in (0,T)$.*

$$\sum_{\alpha=1}^N \tilde{m}_\alpha \|\mathbf{V}^\varepsilon_\alpha\|^2_{L^2(\Omega)} + \sum_{i=1}^d \sum_{\alpha=1}^N \tilde{e}_i \|\partial_{x_i} \mathbf{V}^\varepsilon_\alpha\|^2_{L^2(\Omega)} \leq \tilde{H} + \sum_{\alpha=1}^N \tilde{K}_\alpha \|\mathbf{U}^\varepsilon_\alpha\|^2_{L^2(\Omega)} + \sum_{i=1}^d \sum_{\alpha=1}^N \tilde{J}_{i\alpha} \|\partial_{x_i} \mathbf{U}^\varepsilon_\alpha\|^2_{L^2(\Omega)} \quad (1)$$

Proof. See pages 92, 93 in [9] for proof and relation with parameters of $(\mathbf{P}^\varepsilon_w)$. □

Theorem 1. *Assume assumptions (A1) - (A3) hold, then there exists a unique pair $(\mathbf{U}^\varepsilon, \mathbf{V}^\varepsilon) \in H^1((0,T)\times\Omega)^N \times L^\infty((0,T), H^1_0(\Omega))^N$ such that $(\mathbf{U}^\varepsilon, \mathbf{V}^\varepsilon)$ is a weak solution to $(\mathbf{P}^\varepsilon_w)$.*

Proof. Use $\phi = \mathbf{V}^\varepsilon$ and apply Lemma 1. Then use $\psi \in \{\mathbf{U}^\varepsilon, \partial_t \mathbf{U}^\varepsilon\}$. Moreover, apply a gradient to the second equation of $(\mathbf{P}^\varepsilon)$ and test that equation with $\nabla \mathbf{U}^\varepsilon$ and $\partial_t \nabla \mathbf{U}^\varepsilon$. Application of Young's inequality, use of (1) and application of Gronwall's inequality, see [2, Thm. 1], yields the existence for $\mathbf{U}^\varepsilon$. Then Lax-Milgram yields the existence for $\mathbf{V}^\varepsilon$. Uniqueness follows from the bilinearity of $(\mathbf{P}^\varepsilon_w)$. For more details, see pages 93 and 94 in [9]. □

4 Upscaling the system $(\mathbf{P}^\varepsilon_w)$ via two-scale convergence

Based on two-scale convergence, see [1], [5], [7] for details, we obtain the following Lemma ensuring that the weak solution to problem $(\mathbf{P}^\varepsilon_w)$ has two-scale limits in the limit $\varepsilon \downarrow 0$.

Lemma 2. *Assume assumptions (A0), (A1), (A2) to hold. For each $\varepsilon \in (0, \varepsilon_0)$, let the pair of sequences $(\mathbf{U}^\varepsilon, \mathbf{V}^\varepsilon) \in H^1((0,T)\times\Omega) \times L^\infty((0,T); H^1_0(\Omega))$ be the unique weak solution to $(\mathbf{P}^\varepsilon_w)$. Then this sequence of weak solutions satisfies the estimate*

$\|\mathbf{U}^\varepsilon\|_{H^1((0,T)\times\Omega)^N}+\|\mathbf{V}^\varepsilon\|_{L^\infty((0,T),H_0^1(\Omega))^N}\leq C$, *for all* $\varepsilon\in(0,\varepsilon_0)$ *and there exist vector functions*

$$\begin{aligned}&\mathbf{u} \text{ in } H^1((0,T)\times\Omega)^N, &&\mathscr{U} \text{ in } H^1((0,T);L^2(\Omega;H^1_\#(Y)/\mathbf{R}))^N,\\ &\mathbf{v} \text{ in } L^\infty((0,T);H^1_0(\Omega))^N, &&\mathscr{V} \text{ in } L^\infty((0,T)\times\Omega;H^1_\#(Y)/\mathbf{R})^N,\end{aligned}$$

and a subsequence $\varepsilon'\subset\varepsilon$, *for which the following two-scale convergences*

$$\begin{aligned}\mathbf{U}^{\varepsilon'} &\xrightarrow{2} \mathbf{u}(t,\mathbf{x}), & \nabla\mathbf{U}^{\varepsilon'} &\xrightarrow{2} \nabla\mathbf{u}(t,\mathbf{x})+\nabla_\mathbf{y}\mathscr{U}(t,\mathbf{x},\mathbf{y}),\\ \partial_t\mathbf{U}^{\varepsilon'} &\xrightarrow{2} \partial_t\mathbf{u}(t,\mathbf{x}), & \partial_t\nabla\mathbf{U}^{\varepsilon'} &\xrightarrow{2} \partial_t\nabla\mathbf{u}(t,\mathbf{x})+\partial_t\nabla_\mathbf{y}\mathscr{U}(t,\mathbf{x},\mathbf{y}),\\ \mathbf{V}^{\varepsilon'} &\xrightarrow{2} \mathbf{v}(t,\mathbf{x}), & \nabla\mathbf{V}^{\varepsilon'} &\xrightarrow{2} \nabla\mathbf{v}(t,\mathbf{x})+\nabla_\mathbf{y}\mathscr{V}(t,\mathbf{x},\mathbf{y})\end{aligned}$$

hold for a.e. $t\in(0,T)$.

Proof. See pages 95 and 96 of [9]. □

Using Lemma 2, we upscale **($\mathbf{P}_w^\varepsilon$)** to **($\mathbf{P}_w^0$)** via two-scale convergence.

Theorem 2. *Assume the conditions of Lemma 2 are met. Then the two-scale limits* $\mathbf{u}\in H^1((0,T)\times\Omega)^N$, $\mathscr{U}\in H^1((0,T);L^2(\Omega;H^1_\#(Y)/\mathbf{R}))^N$ *and* $\mathbf{v}\in L^\infty((0,T);H^1_0(\Omega))^N$ *introduced in Lemma 2 form the weak solution triple to*

$$(\mathbf{P}_w^0)\qquad\begin{cases}\displaystyle\int_\Omega \phi^\top\left[\overline{\mathsf{M}}\mathbf{v}-\overline{\mathbf{H}}-\overline{\mathsf{K}}\mathbf{u}-\overline{\mathsf{J}}\cdot\nabla\mathbf{u}-\frac{1}{|Y|}\int_Y \mathsf{J}\cdot\nabla_\mathbf{y}\mathscr{U}\,\mathrm{d}\mathbf{y}\right]\\ \qquad\qquad+(\nabla\phi)^\top\cdot(\mathsf{E}^*\cdot\nabla\mathbf{v}+\mathsf{D}^*\mathbf{v})\,\mathrm{d}\mathbf{x}=0,\\ \displaystyle\int_\Omega \psi^\top\left[\partial_t\mathbf{u}+\mathsf{L}\mathbf{u}-\mathsf{G}\mathbf{v}\right]\mathrm{d}\mathbf{x}=0,\\ \displaystyle\int_Y \xi^\top\cdot\nabla_\mathbf{y}\left[\partial_t\mathscr{U}+\mathsf{L}\mathscr{U}-\tilde{\delta}\mathbf{v}-\tilde{\omega}\cdot\nabla\mathbf{v}\right]\mathrm{d}\mathbf{y}=0,\\ \mathbf{u}(0,\mathbf{x})=\mathbf{U}_*(\mathbf{x}) \qquad \text{on } \Omega,\\ \nabla_\mathbf{y}\mathscr{U}(0,\mathbf{x},\mathbf{y})=\mathbf{0} \qquad \text{on } \Omega\times Y,\end{cases}$$

for a.e. $t\in(0,T)$, *for all test-functions* $\phi\in H^1_0(\Omega)^N$, $\psi\in L^2(\Omega)^N$, *and* $\xi\in H^1_\#(Y)^{d\times N}$, *where the* effective *coefficients* E^* *and* D^* *are given by*

$$\mathsf{E}^*=\frac{1}{|Y|}\int_Y \mathsf{E}\cdot(1+\nabla_\mathbf{y}\mathbf{W})\mathrm{d}\mathbf{y},\qquad \mathsf{D}^*=\frac{1}{|Y|}\int_Y \mathsf{D}+\mathsf{E}\cdot\nabla_\mathbf{y}\delta\mathrm{d}\mathbf{y},$$
$$\tilde{\delta}=\nabla_\mathbf{y}(\mathsf{G}\delta),\qquad \tilde{\omega}=\nabla_\mathbf{y}\mathbf{W}\otimes\mathsf{G},$$

and the tensor $\delta_{\alpha\beta}\in L^\infty((0,T)\times\Omega;H^1_\#(Y)/\mathbf{R}))$ *and vector* $\mathbf{W}_i\in L^\infty((0,T)\times\Omega;H^1_\#(Y)/\mathbf{R}))$ *satisfy the cell problems*

$$0=\int_Y \Phi^\top\cdot(\nabla_\mathbf{y}\cdot[\mathsf{E}\cdot(1+\nabla_\mathbf{y}\mathbf{W})])\mathrm{d}\mathbf{y},\qquad 0=\int_Y \Psi^\top(\nabla_\mathbf{y}\cdot[\mathsf{D}+\mathsf{E}\cdot\nabla_\mathbf{y}\delta])\mathrm{d}\mathbf{y}$$

for all $\Phi\in C_\#(Y)^d$, $\Psi\in C_\#(Y)^{N\times N}$.

Proof. In $(\mathbf{P}^{\varepsilon}_{w})$, we choose $\phi = \phi^{\varepsilon} = \Phi\left(t, \mathbf{x}, \frac{\mathbf{x}}{\varepsilon}\right)$ and $\psi = \psi^{\varepsilon} = \Psi(t, \mathbf{x}) + \varepsilon\varphi\left(t, \mathbf{x}, \frac{\mathbf{x}}{\varepsilon}\right)$ for the test-functions $\Phi \in L^2((0,T); \mathscr{D}(\Omega; C^{\infty}_{\#}(Y)))^N$, $\Psi \in L^2((0,T); C^{\infty}_{0}(\Omega))^N$ and for $\varphi \in L^2((0,T); \mathscr{D}(\Omega; C^{\infty}_{\#}(Y)))^N$. Two-scale convergence limits, see [1], [5], [7], and cell-function arguments, see [6], give $(\mathbf{P}^{0}_{w})$. Details in pages 97,98 of [9]. □

We have shown that upscaling system $(\mathbf{P}^{\varepsilon}_{w})$ yields system $(\mathbf{P}^{0}_{w})$. This system contains only PDEs with respect to $(t, \mathbf{x})$. However, an extra variable $\nabla_{\mathbf{y}}\mathscr{U}$ was needed. Removing $\nabla_{\mathbf{y}}\mathscr{U}$ needs the use of continuous semi-group theory, see papers 10 and 14 of [10], for solving the third equation of system $(\mathbf{P}^{0}_{w})$. This leads to a non-local-in-time term as a consequence of removing $\nabla_{\mathbf{y}}\mathscr{U}$.

5 Conclusion

Our main goal of this paper is to show that the spatial-temporal decomposition, as employed in [8], allows for the straighforward upscaling of pseudo-parabolic equations, in specific for system $(\mathbf{Q}^{\varepsilon})$. The upscaling procedure is here performed using the concept of two-scale convergence as reported in Section 4. Moreover, the decomposition is retained in the upscaled limit. A non-local-in-time term arose when an extra variable was eliminated. The spatial-temporal decoupling showed why this non-local term is non-local in time.
In future research we intend to investigate the applicability of the spatial-temporal decomposition of our pseudo-parabolic system to perforated periodic domains, corrector estimates (convergence speed estimate) and high-contrast situations.

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