

Platoon forming algorithms for intelligent street intersections

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Platoon Forming Algorithms for Intelligent Street Intersections

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Abstract

We study intersection access control for autonomous vehicles. Platoon forming algorithms, which aim to organize individual vehicles in platoons, are very promising. To create those platoons, we slow down vehicles before the actual arrival at the intersection in such a way that each vehicle can traverse the intersection at high speed. This increases the capacity of the intersection significantly, offering huge potential savings with respect to travel time in nowadays traffic.

We propose several new platoon forming algorithms and provide an approximate mean delay analysis for our algorithms. A comparison between the current day practice at intersections (through a case study in SUMO) and our proposed algorithms is provided. Simulation results for fairness are obtained as well, showing that platoon forming algorithms with a low mean delay sometimes are relatively unfair, indicating a potential need for balancing mean delay and fairness.

KEYWORDS

Platoon forming algorithms, speed control algorithms, autonomous vehicles, queueing theory, polling models.

1. Introduction

Congestion is commonplace at intersections in urban traffic, but intersections are inevitable to divide capacity among vehicles from conflicting flows. To do so in a fair and efficient manner, intersections are typically managed by some kind of switching process that alternately gives access to batches of vehicles, imposing a constraint on the maximal batch size that can pass the intersection.

The traditional way of regulating the switching process is by installing traffic lights with static signalling, using timers, see e.g. Darroch (1964) and van Leeuwen (2006), or dynamic signalling with sensor data of currently existing traffic flows, see e.g. Papageorgiou et al. (2003). Anticipating the emergence of self-driving vehicles, efficient and fair algorithms for intersection access should be designed. Platoon Forming Algorithms (PFAs) provide such alternatives for self-driving vehicles, no longer letting the traffic light dictate the switching process and hence batch forming, but letting the vehicles organize themselves in batches, well in advance of arriving at the intersection as in Miculescu and Karaman (2014, 2016) and Tachet et al. (2016). This way platoons of vehicles are formed that can pass the intersection collectively.

There is a natural tension between capacity and fairness. One of the fairest switching rules is to let vehicles pass the intersection in order of arrival (on an intersection wide

basis). This rapidly becomes unsustainable, because each switch requires an additional clearance time, which decreases the capacity of the intersection. In near-saturation conditions, when the flows together impose a high volume-to-capacity ratio, the loss of capacity due to switching will have a dramatic effect on delays.

In PFAs, vehicles arriving at the intersection arrange themselves in platoons, not adapting their relative position to other vehicles on the same lane but adapting their speed. The key feature is that cars, while approaching the intersection, adjust their speeds and upon arrival at the intersection are at high speed, accessing the conflict area of the intersection for a minimum period of time. In this way, time bans to give way to other traffic flows still exist, but the platoons are processed in the quickest possible way, because the size and speed of the platoons, of all directions, are organized by the PFA.

PFAs are one particular example of the ‘slower is faster’ effect, which is also observed in e.g. Helbing, Farkas, and Vicsek (2000) and Helbing and Mazloumian (2009), where, perhaps counter-intuitively, slowing down early results in less delay on average in the future. Moreover, this phenomenon results in environmental advantages as less braking-and-pulling-up-again is needed and cars reach their destination more quickly.

The importance of intersection access algorithms has been recognized for several years. Examples of PFAs can be found in Tachet et al. (2016), which uses a batch formation algorithm based on arrival times of vehicles and a maximum batch size and in Miculescu and Karaman (2014, 2016), which use queueing models, and to be more specific polling models. The key link between the PFAs and polling models is a speed control algorithm introduced in Miculescu and Karaman (2014), which we will use in our models as well. However, many more models and control techniques are investigated like reservation based control algorithms, see e.g. Dresner and Stone (2008), and controls based on fuzzy logic, see e.g. Milanés et al. (2010). For an overview we refer to Rios-Torres and Malikopoulos (2017), yet we focus on PFAs in this paper.

The area of application of PFAs is not restricted to intersections. There are numerous cases where PFAs could be used to achieve a good performance. An example in traffic would be the merging of different streams of vehicles (discussed in e.g. Rios-Torres and Malikopoulos 2017). Another possible application can be found in automated guided vehicles (AGVs) systems, where AGVs cross each other or have to merge, see e.g. Kockelkoren (2018). In Kockelkoren (2018), ideas stemming from speed profile algorithms are used and so PFAs can be used in similar types of AGV systems.

Main Contributions

We present several new PFAs, based on enhanced polling policies, that perform well regarding mean delay, unifying and extending ideas from Miculescu and Karaman (2016) and Tachet et al. (2016).

A key building block are speed profile algorithms such as the MotionSynthesize procedure, introduced in Miculescu and Karaman (2014), revealing a relation between PFAs and polling models. We have developed an alternative to the MotionSynthesize procedure exhibiting desirable properties and we have found closed-form solutions for the MotionSynthesize procedure and this alternative, alleviating the need for linear optimization solvers.

Using such speed profile algorithms, a link between polling models and PFAs is established, making it possible to conduct a performance analysis on e.g. mean delay,

which is the main performance characteristic considered in the literature for algorithms like PFAs.

Another important characteristic is the fairness of a PFA. Fairness in queueing models (and therefore PFAs) is important in the perception of customers, see e.g. Rafaeli, Barron, and Haber (2002). We use the quantification of fairness as defined in Shapira and Levy (2016) for polling models, in order to assess the fairness of the various PFAs.

Furthermore, we provide a comparison between the performance of traditional traffic technologies and PFAs through simulations in SUMO and show that intersections in the future can be used much more efficiently, reducing congestion.

Organization of the paper

This paper is organized in the following way. We start with a description of the various ingredients of the model and provide an extensive description of the new PFAs that we introduce in Section 2. Section 3 is devoted to speed profile algorithms. Afterwards, we introduce polling models and give the analytical results that we need for the analysis of mean delay and fairness of PFAs in Section 4. Subsequently, Section 5 provides a comparison between the traditional traffic light (represented by simulations in SUMO) and our PFAs, focusing on mean delay and we wrap up in Section 6.

2. Model Formulation

We will consider models in which autonomous vehicles are crossing an intersection. We assume the existence of a control region around the intersection with at the center a centralized controller communicating with all vehicles within the control region, setting the access times of each of the arriving vehicles to the intersection. This central controller creates platoons of vehicles by scheduling the crossing times of the vehicles according to some policy (the PFA) in such a way that every vehicle is able to cross the intersection at its designated time. We assume that we can control the speed of a vehicle and do so in such a way that the intersection is used efficiently. We make sure that vehicles drive at maximum speed at the moment they are starting to cross the intersection, using ideas introduced in Miculescu and Karaman (2014). Instead of stopping at the stop line and still having to accelerate when crossing the intersection, a vehicle is already slowed down before it reaches the intersection and starts accelerating again before it reaches the intersection. This amongst others implies that the time to cross the intersection is the same for each vehicle. Each vehicle is namely simply driving at full speed through the intersection. The last assumption discussed here, is that we assume that the central controller can look ‘ahead’ for the same amount of time for each of the lanes, to ease the notation and algorithms.

An advantage of the control region, besides the ability to control the speed of arriving vehicles, is that we can adjust the scheduling of the vehicles based on the arriving vehicles that are not yet at the intersection. This specific anticipation is key to the forming of platoons and is up to the central controller at the intersection and results in a specific PFA. There are many PFAs, yet we will specifically focus on PFAs that find their origin in so-called polling models, because they are efficient, well understood and have proven their value in other application areas, such as communication systems and production lines.

2.1. Platoon Forming Algorithms

We present our new PFAs as standalone algorithms, based on service disciplines for polling models, which are described in a way fit for PFAs. We also briefly discuss the Batch Algorithm, originating from Tachet et al. (2016), which serves as a benchmark for our PFAs. The PFAs we discuss, are all derived from so-called branching-type disciplines, which find their origin in the polling literature, see e.g. Resing (1993). Branching-type service disciplines include the exhaustive and the gated discipline, which all allow for many exact results.

Before we start with the descriptions of the PFAs we introduce some concepts and notation. The PFA determines the crossing time of each of the vehicles in the control region that have not yet crossed the intersection. We represent this schedule by an entity we call ‘vehicles’. A vehicle V has three properties: a lane d_V , an earliest crossing time a_V and the currently scheduled crossing time c_V . We assume that at every point in time we have such a list of vehicles, ordered on basis of the c_V ’s. The PFA updates (part of) the crossing time of the vehicles upon arrival and departure epochs of vehicles in the control region. The latter is dealt with in an easy way: if the current time is $c_V + B$, where B denotes the difference in crossing times between two vehicles on the same lane, then vehicle V is removed from the ordering. Turning towards arrivals of vehicles within the control region, we need to consider the crossing times of all vehicles already scheduled in order to schedule V . There are several ways to schedule those vehicles and the first we discuss is the exhaustive discipline, as described in Algorithm 1. This discipline is known for its low mean delay, which is the main reason to consider this discipline. We also introduce one more constant, S , that represents the time between the start of crossing of two vehicles on different lanes (similar to clearance times at intersections nowadays).

An intuitive explanation of the exhaustive discipline is the following: if a vehicle that arrives in the control region is able to get within B seconds of the vehicle in front of it on the same lane (which might occur if the vehicle is delayed by its predecessor), it is allowed to join the same platoon as its predecessor. This would imply that all vehicles on different lanes have to wait an additional B seconds. If a vehicle cannot join the platoon in front of it, it will form a new platoon. If no vehicle (on the current lane) is able to join the platoon currently crossing the intersection, the next platoon of vehicles at the next lane may cross the intersection. As a result we have a cyclic structure of departures of platoons.

Although the exhaustive PFA will have very good delay characteristics, we will consider the gated PFA (discussed below) as well. An advantage of the gated discipline is that there is less variation in the size of platoons and, hence, cycle lengths are less variable as well. It may result in longer delays though, as we will see in the numerical examples in Section 4.

For the gated PFA, we need to keep track of one additional variable for each vehicle, compared to the exhaustive PFA. In this gated discipline we are ‘putting gates’ which can be seen as ‘fixing the vehicles of a platoon’, meaning that future arrivals in the same lane cannot join the currently formed platoon (i.e. they are ‘behind the gate’). We use an indicator, g_V , for vehicle V . If $g_V = 1$, it denotes that the vehicle is allowed in the same, currently departing, platoon as its predecessor, whereas a 0 denotes that the vehicle is behind the gate and is not part of the currently departing platoon.

The intuitive explanation of the gated algorithm is quite close to that of the exhaustive discipline, with one exception. It is not always allowed to join a platoon, even if a vehicle is able to get within B seconds from its predecessor on the same lane. This

Algorithm 1 exhaustive algorithm.

```
1: Input: current ordering of vehicles, denoted  $(V_1, V_2, \dots, V_k)$ , ordered on basis of  $c_V$ ;
   the last vehicle that started to cross the intersection,  $V_{last}$ ; and a to be scheduled
   vehicle  $V_0$  with earliest arrival time at the intersection  $a_{V_0}$  in lane  $d_{V_0}$ .
2: if  $c_{V_k} + B < a_{V_0}$  then ▷ The ordering is empty
3:   if  $d_{V_0} = d_{V_{last}}$  then
4:     Put  $c_{V_0} = \max\{a_{V_0}, c_{V_{last}} + B\}$ .
5:   else
6:     Put  $c_{V_0} = \max\{a_{V_0}, c_{V_{last}} + S\}$ .
7:   end if
8: else ▷ The ordering is nonempty
9:   if there exists an  $i, i = 1, 2, \dots, k$ , with  $d_{V_i} = d_{V_0}$  then
10:    Find the vehicle  $V_j$  in ordering with  $d_{V_j} = d_{V_0}$  and  $c_{V_j}$  is maximal.
11:    Put  $c_{V_0} = c_{V_j} + B$ .
12:    For each vehicle  $V$  with  $c_V > c_{V_j}$ , put  $c_V = c_V + B$ .
13:   else ▷ Search for the predecessor of  $V_0$  in the schedule
14:     for  $l$  in  $(d_{V_0} - 1, d_{V_0} - 2, \dots, 1, n, n - 1, \dots, d_{V_0} + 1)$  do
15:       if there is a vehicle  $V$  with  $d_V = l$  then
16:         break
17:       end if
18:     end for
19:     Find the vehicle  $V_j$  with  $d_{V_j} = l$  and  $c_{V_j}$  is maximal.
20:     if  $c_{V_j} + S > a_{V_0}$  then
21:       Put  $c_{V_0} = c_{V_j} + S$ .
22:       For each vehicle  $V$  with  $c_V > c_{V_j}$ , put  $c_V = c_V + S$ .
23:     else
24:       Put  $c_{V_0} = c_{V_k} + S$ .
25:     end if
26:   end if
27: end if
28: Add vehicle  $V_0$  to the ordering.
29: Output: the new ordering  $(V_1, V_2, \dots, V_0, \dots, V_k)$ , where the vehicles  $V$  are ordered
   on basis of  $c_V$ .
```

is only allowed if the lane is *not* the lane from which vehicles are currently departing (the platoon is not yet fixed). Therefore, the variables g_V are governed by the following process: as soon as a vehicle V' starts to cross the intersection and $g_{V'} = 0$, then we put $g_{V'} = 1$ and for all vehicles V that have a scheduled time within B from its predecessor (i.e. the vehicles that are forming a platoon at that specific moment), the variable g_V is put to 1 (i.e. they are in front of the gate and the platoon is finalized). Departures of vehicles are dealt with in the same way as in the exhaustive discipline. We again have the cyclic structure as in the exhaustive discipline. The gated algorithm can then be described as in Algorithm 2.

As a reference to algorithms so far established in the literature, we also consider the Batch Algorithm from Tachet et al. (2016). For the full description we refer to (Tachet et al. 2016, Supplementary Information, Section 1.5). The Batch Algorithm has some ingredients of a gated PFA (also in the Batch Algorithm ‘gates’ are put), together with a maximum number of vehicles dealt with in one cycle.

Algorithm 2 gated algorithm.

```
1: Input: current ordering of vehicles, denoted  $(V_1, V_2, \dots, V_k)$ , ordered on basis of  $c_V$ ;
   the last vehicle that started to cross the intersection,  $V_{last}$ ; and a to be scheduled
   vehicle  $V_0$  with earliest arrival time at the intersection  $a_{V_0}$  in lane  $d_{V_0}$ .
2: if  $c_{V_K} + B < a_{V_0}$  then ▷ The ordering is empty
3:   if  $d_{V_0} = d_{V_{last}}$  then
4:     Put  $c_{V_0} = \max\{a_{V_0}, c_{V_{last}} + B\}$ .
5:   else
6:     Put  $c_{V_0} = \max\{a_{V_0}, c_{V_{last}} + S\}$ .
7:   end if
8: else ▷ The ordering is nonempty
9:   if there exists an  $i$  with  $d_{V_i} = d_{V_0}$  then
10:    if all vehicles  $V'$  with lane  $d_{V'} = d_{V_0}$  satisfy  $g_{V'} = 1$  then
11:      if  $d_{V_K} = d_{V_0}$  then
12:        Put  $c_{V_0} = c_{V_K} + B$ .
13:      else
14:        Put  $c_{V_0} = c_{V_K} + S$ .
15:      end if
16:    else
17:      Find the vehicle  $V_j$  with  $d_{V_j} = d_{V_0}$  and  $c_{V_j}$  is maximal.
18:      Put  $c_{V_0} = c_{V_j} + B$ .
19:      For each vehicle  $V$  with  $c_V > c_{V_j}$ , put  $c_V = c_V + B$ .
20:    end if
21:    else ▷ Search for the predecessor of  $V_0$  in the schedule
22:      for  $l$  in  $(d_{V_0} - 1, d_{V_0} - 2, \dots, 1, n, n - 1, \dots, d_{V_0} + 1)$  do
23:        if there is a vehicle  $V$  with  $d_V = l$  then
24:          break
25:        end if
26:      end for
27:      if all vehicles  $V'$  with lane  $d_{V'} = i$  satisfy  $g_{V'} = 0$  then
28:        Find the  $j$  with  $d_{V_j} = i$  and  $c_{V_j}$  is maximal.
29:        Put  $c_{V_0} = c_{V_j} + s$ .
30:        For each vehicle  $V$  with  $c_V > c_{V_j}$ , put  $c_V = c_V + S$ .
31:      else
32:        Find the  $j$  with  $d_{V_j} = i$ ,  $g_{V_j} = 1$  and  $c_{V_j}$  is maximal.
33:        Put  $c_{V_0} = c_{V_j} + S$ .
34:      if all vehicles  $V$  satisfy  $d_V = i$  then
35:        For each vehicle  $V$  with  $c_V > c_{V_j}$ , put  $c_V = c_V + 2S - B$ .
36:      else
37:        For each vehicle  $V$  with  $c_V > c_{V_j}$ , put  $c_V = c_V + S$ .
38:      end if
39:    end if
40:  end if
41: end if
42: Add vehicle  $V_0$  to the ordering.
43: Output: the new ordering  $(V_1, V_2, \dots, V_0, \dots, V_K)$ , where the vehicles  $V$  are ordered
   on basis of  $c_V$ .
```

3. Speed Profile Algorithms

Now that we know how to schedule the crossing times of vehicles at the intersection, we turn to the other key ingredient of our model, which is the speed profiling of arriving vehicles. We start with some requirements that the PFAs have to satisfy before we can control the speed of the arriving vehicles in a proper and safe way. The main condition a PFA has to satisfy is *regularity*.

Definition 3.1 (Miculescu and Karaman 2014, 2016). A polling policy is *regular* if an arrival in a queue does not change the *order* of service of all currently present vehicles. I.e. the new arrival is inserted somewhere in the order of service of all waiting vehicles.

A regular polling policy, together with assuming a sufficiently big control region, ensures that the intersection coordination algorithm in Miculescu and Karaman (2014, 2016) and the speed profile algorithms that we will introduce are solvable. These assumptions are necessary with respect to the (possibility of) rescheduling of vehicles. As can be seen in Algorithms 1 and 2, the access time of (some of the) vehicles to the intersection might be increased, upon which trajectories have to be rescheduled. The above assumptions ensure that we can find feasible and safe trajectories for every vehicle, also in case of rescheduling, cf. Miculescu and Karaman (2014, 2016).

Besides these two assumptions on regularity and the size of the control region, we also need to make sure that there are not too many vehicles in the control region at the same time: if there are too many vehicles present in the control region, it might be the case that a newly arriving vehicle cannot decelerate to a complete stop in the distance between entering the control region and the stopping position of its predecessor. This phenomenon is called overcrowding, see Miculescu and Karaman (2016). A way to deal with this is proposed as well: we assume that a vehicle that cannot enter the control region safely, does not enter the control region at all.

All PFAs that we discussed are regular in the sense of Definition 3.1. For the Batch Algorithm of Tachet et al. (2016) we postulate that this condition is also satisfied.

3.1. Optimization based Speed Profile Algorithms

In this subsection, we discuss two algorithms that, satisfying above conditions, result in an efficient use of the intersection which is our main purpose. To this end, we require that vehicles drive at maximum speed while crossing the intersection, so we need to control the speed of arriving vehicles while they are in the control region. A relatively simple optimization algorithm can then be formulated that does the trick, as is shown in Miculescu and Karaman (2014, 2016) (the MotionSynthesize procedure). In order to solve this minimization problem, time is discretized. The MotionSynthesize procedure is then reduced to a linear optimization problem, for which efficient solvers exist.

The optimization procedure has several nice properties, among which is that the algorithm is provably safe (Miculescu and Karaman 2016). Another property is that the distance between vehicle and intersection is minimized across the whole time period a vehicle is in the control region. This is equivalent with the minimization of the area under the distance-time diagram, where the distance is defined as the distance between vehicle and intersection. The physical length of the queue of vehicles is thus also minimized. This is favorable in a network setting, minimizing the amount of spillback to other intersections. Yet, this specific property of minimizing the distance between

vehicle and intersection has a high energy consumption and may not be very pleasant for passengers. Below, in Algorithm 3, we discuss a slightly different formulation of the problem and we minimize the total amount of the absolute value of the *acceleration* instead of the *distance* between vehicle and intersection. Instead of minimizing the area under the distance-time graph, we now minimize the area under the ‘absolute value of the acceleration-time’ graph. Note that t_f is set by the PFA, which corresponds to the time c_V in Algorithms 1 and 2. We use the notation t_f for consistency with the notation in Miculescu and Karaman (2016). Assuming regularity of the PFA and a sufficiently big control region is not sufficient to ensure a feasible optimization problem as it is for the MotionSynthesize procedure. We formulate a mild additional constraint to guarantee feasibility of the optimization problem, which is that one needs to be sure that when the preceding vehicle is done decelerating, the next vehicle is able to decelerate to that same speed as well before the preceding vehicle is decelerating further (due to rescheduling for example). As will turn out, a vehicle starts decelerating immediately after entering the control region (see e.g. Figure 1). As a consequence, if a vehicle is entering the control region, it needs to be sure that it is able to decelerate to the speed of its predecessor while maintaining a certain distance to its predecessor at the same time, showing that we need the additional assumption.

Before we turn to the algorithm, we introduce some notation. Each vehicle has a trajectory that is computed along the lines of the algorithm, given the current time, t_0 , and the scheduled crossing time t_f . The algorithm will compute $x(t)$, the place of the vehicle at time t , $t_0 \leq t \leq t_f$, the speed $v(t)$ at time t and the acceleration $a(t)$ at time t . Furthermore, $y(t)$ denotes the trajectory of the predecessor (if any); l denotes the minimal distance between the front part of two successive vehicles; a_m denotes the maximum acceleration; $-a_m$ denotes the maximum deceleration; and v_m denotes the maximum speed. Algorithm 3 can be discretized in order to obtain a linear optimization problem, just as the MotionSynthesize procedure.

Algorithm 3 MotionSynthesize procedure with a minimal acceleration

- 1: Input: $x_0(t_0)$, $v_0(t_0)$, t_0 , t_f , y .
- 2: Compute

$$\text{MotionSynthesizeAcc}(x_0(t_0), v_0(t_0), t_0, t_f, y) := \arg \min_{x: [t'_0, t'_f] \rightarrow \mathbb{R}} \int_{t'_0}^{t'_f} |a(t)| dt$$

subject to

$$\begin{aligned} x''(t) &= a(t), \text{ for all } t \in [t'_0, t'_f]; \\ 0 \leq x'(t) &\leq v_m, \text{ for all } t \in [t'_0, t'_f]; \\ |a(t)| &\leq a_m, \text{ for all } t \in [t'_0, t'_f]; \\ |x(t) - y(t)| &\geq l, \text{ for all } t \in [t'_0, t'_f]; \\ x(t_0) &= x_0(t_0); x'(t'_0) = v_0(t_0); \\ x(t_f) &= 0; x'(t_f) = v_m. \end{aligned}$$

- 3: Output: $x(t)$.
-

Algorithm 3 is solvable under the set of conditions formulated above, i.e. regularity of the PFA, a sufficiently big control region and the assumption on decelerating of a predecessor of a vehicle. The main difference is that instead of minimizing the distance

from vehicle to intersection, we minimize the (absolute value of the) acceleration applied by the vehicle while being in the control region. This obviously has consequences for the amount of energy consumption (it will be lower than in the MotionSynthesize procedure). Disadvantages include that the physical length of the queue grows and that vehicles cannot enter the control region as close to each other (as vehicles slow down immediately when entering the control region).

We have found analytical solutions to the MotionSynthesize procedure and Algorithm 3. So instead of the need to solve a linear optimization problem each time, we have a simple set of calculations that we can perform to find the trajectory of a vehicle, which is optimal with respect to minimizing the distance or acceleration. These analytical solutions are discussed in the next subsection and are similar in spirit as the results in e.g. Lawitzky, Wollherr, and Buss (2013) and Dib et al. (2014).

3.2. Closed-form Speed Profile Algorithms

We start with the observation that the optimization problem formulated in the MotionSynthesize procedure always leads to piece-wise constant acceleration, where at most four changes in the acceleration occur. Below we will see why.

This observation implies that if we can find the points at which the acceleration changes, we are able to determine the trajectory analytically and in closed-form. We give some intuition behind the main ideas of Algorithm 4. We have to plan the trajectory from t_0 until t_f , the crossing time set by the PFA. It is sufficient to give the acceleration for any time t for which $t_0 \leq t \leq t_f$. As said, the acceleration is piece-wise constant and there are at most four changes in the acceleration. We shortly describe those five parts of the arriving trajectory.

- No acceleration or deceleration from t_0 until t_{dec} ;
- Deceleration at maximum rate from t_{dec} until t_{stop} ;
- A stop from t_{stop} until t_{acc} ;
- Acceleration at maximum rate from t_{acc} until t_{full} ;
- No acceleration or deceleration from t_{full} until t_f .

All that remains is that we have to find t_{dec} , t_{stop} , t_{acc} and t_{full} in such a way that we minimize the distance between vehicle and intersection. The four times are found using the following observations. We continue as long as possible at maximum speed, decreasing the distance between intersection and vehicle as quickly as possible (this corresponds to time t_{dec}). After some time we know that we have to decelerate, because if we do not, we either are too early at the intersection or not at a maximum speed. This implies that the remainder of the trajectory is fixed: we decelerate at maximum rate (until t_{stop}), possibly stop for some time (until t_{acc}), and accelerate at maximum rate (until t_{full}). Then the vehicle might drive at full speed for some time until t_f .

The closed-form solution of the MotionSynthesize procedure is formulated in Algorithm 4, where we assume that $t_0 = 0$ to ease the notation and that $v_0(0) = v_m$. We can allow for general $v_0(0)$, but the algorithm would become more involved and in the interest of space and clarity we focus on the case $v_0(0) = v_m$. The input consists of the current distance between vehicle and intersection, $x_0(0)$, the scheduled crossing time of the vehicle, t_f , and the trajectory of the predecessor of the vehicle for which we are currently planning the trajectory, y .

We prove that the MotionSynthesize procedure and Algorithm 4 are equivalent, which is the subject of the next lemma.

Algorithm 4 closed-form solution to the MotionSynthesize procedure.

- 1: Input: $x_0(0)$, t_f and y .
- 2: Define the arrival time of the vehicle associated with y as $t_{f,y}$.
- 3: **if** $t_f - t_{f,y} = B$ **then**
- 4: Consider trajectory y and determine the time at which the vehicle continues at full speed. Call this time t_{full} .
- 5: **else**
- 6: Put $t_{full} = t_f$.
- 7: **end if**
- 8: Put

$$L = v_m(t_f - \frac{v_m}{a_m}). \quad (1)$$

▷ L represents the distance covered if a vehicle stops for 0 seconds

- 9: **if** $L \geq x_0$ **then** ▷ The vehicle has to stop
- 10: Put $t_{acc} = t_{full} - v_m/a_m$.
- 11: Put $t_{stop} = t_{acc} - (t_f - v_m/a_m - x_0/v_m)$.
- 12: Put $t_{dec} = t_{stop} - v_m/a_m$.
- 13: **else** ▷ The vehicle does not have to stop
- 14: Define

$$\tilde{t} = \sqrt{\frac{-x_0 + t_f v_m}{a_m}}. \quad (2)$$

▷ \tilde{t} is the deceleration time

- 15: Put $t_{acc} = t_{full} - \tilde{t}$.
- 16: Put $t_{stop} = t_{acc}$.
- 17: Put $t_{dec} = t_{acc} - \tilde{t}$.
- 18: **end if**
- 19: Then

$$a(t) = x''(t) = \begin{cases} 0 & \text{if } 0 \leq t < t_{dec}, \\ -a_m & \text{if } t_{dec} \leq t < t_{stop}, \\ 0 & \text{if } t_{stop} \leq t < t_{acc}, \\ a_m & \text{if } t_{acc} \leq t < t_{full}, \\ 0 & \text{if } t_{full} \leq t < t_f. \end{cases} \quad (3)$$

- 20: Knowing $a(t)$, we can compute $x(t)$ by integrating twice and using the conditions $x(0) = x_0$ and the velocity at time 0 being v_m .
 - 21: Output: $x(t)$.
-

Lemma 3.2. *The MotionSynthesize procedure and Algorithm 4 are equivalent in the sense that both minimize the distance between vehicle and intersection across the time period t_0 to t_f .*

Proof. We split the proof in two parts. First we prove that the obtained *form* of the trajectory is optimal (i.e. the five parts of the trajectory) and then we prove that the times t_{dec} , t_{stop} , t_{acc} and t_{full} in Algorithm 4 indeed result in the trajectory having

the minimal area under the distance-time graph.

Segment 1. We first argue that the trajectory consists of at most five parts. The last part, from t_{full} until t_f , has non-zero length if (and only if) the vehicle V of which we are currently planning the trajectory is delayed by its predecessor, of which the trajectory is denoted with y . This means that at some point in time, the constraint that there should be a minimal distance between the two vehicles (i.e. the constraint $|x(t) - y(t)| \geq l$ in the MotionSynthesize procedure) is active. Then we observe that: the time between the crossing of V and its predecessor is B ; the minimum distance between two vehicles is fixed for the whole time period; and that at some point the predecessor of V is driving at full speed, which we denote with $t_{f,y}$. So, from the first moment that the distance between the two vehicles is the minimum distance, the distance remains the same. This means that we need the vehicle V to drive at full speed from $t_{f,y}$, as well, so $t_{full} = t_{f,y}$.

We describe the first four parts of the trajectory, which are split in the following way: driving at full speed (until t_{dec}), decelerating (until t_{stop}), stop (until t_{acc}) and accelerating (until t_{full}), where the last three periods may have zero length. V continues as long as possible at full speed (decreasing the distance between vehicle and intersection as quickly as possible). This implies that if V decelerates, it has to decelerate at maximum rate (otherwise we could have continued at full speed longer). Depending on the amount of delay, the vehicle might be stopped for some time. It stops at the place where V is just able to reach the maximum speed, when accelerating at maximum rate (for if V does not, it could stop closer to the intersection). The described trajectory minimizes the distance between vehicle and intersection, because we continue as long as possible at full speed from the start.

Segment 2. As argued in Part 1, the time t_{full} is determined by the trajectory y of the predecessor of V and is fixed. If the crossing times differ a time B , then the time at which the predecessor starts driving at full speed, $t_{f,y}$, should equal t_{full} , and otherwise it is simply t_f , which is the way we choose t_{full} in lines 3-7.

Then we need to check whether or not the vehicle has to stop. We do this by calculating the distance that V would cover when it stops for 0 seconds. This is done in (1). Define $v_m/a_m = t_1$, then indeed

$$L = (t_f - 2t_1)v_m - \frac{1}{2}a_m t_1^2 + v_m t_1 + \frac{1}{2}a_m t_1^2 = (t_f - t_1)v_m.$$

If $L \geq x_0$, it means that V would cover a larger distance than or the same distance as x_0 if it stops for 0 seconds. But then it potentially crosses the intersection before t_f , so we need to stop V . We do this for a time $L/v_m - x_0/v_m = t_f - v_m/a_m - x_0/v_m$, as this is the time at which the vehicle would start crossing the intersection ‘too early’. So, building backwards from t_{full} , we see that $t_{acc} = t_{full} - v_m/a_m$ (as the vehicle has to accelerate from 0 to v_m), $t_{stop} = t_{acc} - (t_f - v_m/a_m - x_0/v_m)$ and $t_{dec} = t_{stop} - v_m/a_m$ (as the vehicle has to decelerate from v_m to 0) as in lines 10-12.

If $L < x_0$, then the vehicle does not have to stop. This means that, if t represents the time V decelerates (and also accelerates), we traverse a distance

$$v_m(t_f - t) + (v_m - a_m t)t,$$

which has to equal x_0 . Solving for t yields one positive solution, which is given by \tilde{t} in (2). So, in this case we can choose $t_{acc} = t_{full} - \tilde{t}$, $t_{stop} = t_{acc}$ and $t_{dec} = t_{stop} - \tilde{t}$, as

in lines 15-17. Note that t_{stop} does not correspond to a stop of the vehicle if $L < x_0$, but to the time we start accelerating (which is why we choose $t_{stop} = t_{acc}$).

Then combining the defined times, we see that we obtain (3). With this choice of times, we see that we minimize the area under the distance-time graph. This is exactly the same criterion as we optimize for in the MotionSynthesize procedure, so the two algorithms yield the same trajectory.

□

Remark 1. Algorithm 4 assumes that there is at most *one* period of deceleration, possibly a stop, and acceleration. It is readily seen that this is the case for the exhaustive and gated PFA. However, for other disciplines, like the Batch Algorithm, this might not be the case, and the period from t_0 until t_f might have to be split in more than five different periods. A similar type of speed profile algorithm is still possible, but is more involved and therefore omitted in interest of space and clarity of the algorithm and argumentation.

So, Algorithm 4 has the same desirable properties as the MotionSynthesize procedure, but is computationally much less expensive and also provides intuition on the shape of the trajectories. A visualization of such trajectories can be found in Figure 1.

We can also formulate such an alternative for Algorithm 3, where we, again, put $t_0 = 0$ to ease the notation. We allow for general $v_0(0)$ now. We have the same structure as for Algorithm 4. Also in this case, the acceleration is piece-wise constant, yet there are at most three changes in the acceleration. We shortly describe those four parts of the arriving trajectory.

- Deceleration at maximum rate from t_0 until t_{cruise} ;
- No acceleration or deceleration from t_{cruise} until t_{acc} ;
- Acceleration at maximum rate from t_{acc} until t_{full} ;
- No acceleration or deceleration from t_{full} until t_f .

This is also visible in Figure 1 below. Note that we start decelerating as soon as possible, because we want to cruise at a relatively low speed. If we would not cruise at a low speed, then we would have to decelerate more (as we covered a longer distance at high speed). So we decelerate maximally for some time, continue at a constant speed for some time and then accelerate maximally (taking advantage of the lower cruising speed as long as possible). The resulting algorithm is formulated in Algorithm 5 and equivalence with Algorithm 3 is proven thereafter.

Then we have the following lemma.

Lemma 3.3. *Algorithm 3 and Algorithm 5 are equivalent in the sense that both minimize the absolute value of the applied acceleration across the time period t_0 to t_f .*

Proof. We again split the proof in two parts, the form of the trajectory and then we check the computation of t_{cruise} , t_{acc} and t_{full} in Algorithm 5.

Segment 1. The optimal trajectory consists of at most four parts. The last part, from t_{full} until t_f is determined in the same way as shown in the proof of Lemma 3.2.

The first three parts of the trajectory are split in the following way: decelerating (until t_{cruise}), cruising at a fixed speed (until t_{acc}) and accelerating (until t_{full}), where the first and last period may have zero length. We want to minimize the area under the absolute value of the acceleration-time graph. We decelerate as early as possible

Algorithm 5 closed-form solution to Algorithm 3.

- 1: Input: $x_0(0)$, $v_0(0)$, t_f and y .
- 2: Define the arrival time of the vehicle associated with y as $t_{f,y}$.
- 3: **if** $t_f - t_{f,y} = B$ **then**
- 4: Consider trajectory y and determine the time at which the vehicle continues at full speed. Call this time t_{full} .
- 5: **else**
- 6: Put $t_{full} = t_f$.
- 7: **end if**
- 8: Put

$$t_1 = \frac{a_m t_f + v_0(0) - v_m}{2a_m} - \frac{\sqrt{4a_m x_0 + (a_m t_f - v_0(0))^2 - 2a_m t_f v_m - 4a_m(t_f - t_{full})v_m + 2v_0(0)v_m - v_m^2}}{2a_m} \quad (4)$$

- 9: Put

$$t_2 = \frac{a_m t_f + v_0(0) - v_m}{2a_m} + \frac{\sqrt{4a_m x_0 + (a_m t_f - v_0(0))^2 - 2a_m t_f v_m - 4a_m(t_f - t_{full})v_m + 2v_0(0)v_m - v_m^2}}{2a_m} \quad (5)$$

- 10: Put $t_{cruise} = t_1$ and $t_{acc} = t_2$.
- 11: Then,

$$a(t) = x''(t) = \begin{cases} -a_m & \text{if } 0 \leq t < t_{cruise}, \\ 0 & \text{if } t_{cruise} \leq t < t_{acc}, \\ a_m & \text{if } t_{acc} \leq t < t_{full}, \\ 0 & \text{if } t_{full} \leq t < t_f. \end{cases} \quad (6)$$

- 12: Knowing $a(t)$, we can compute $x(t)$ by integrating twice and using the conditions $x(0) = x_0$ and the velocity at time 0 being $v_0(0)$.
 - 13: Output: $x(t)$.
-

and accelerate as late as possible, and both at the maximum rate. If we would not do one of these three things, it means that we would have to decelerate more as we drive at high speed longer (and as e.g. the average speed is fixed, namely x_0/t_f , we would have to decelerate more to a lower speed). So, indeed the first three parts of a trajectory consist of decelerating at maximum rate, then cruising at a fixed (and relatively low) speed and then accelerating at maximum rate.

Segment 2. As argued in the proof of Lemma 3.2, the time t_{full} is determined by the trajectory y of the predecessor of V and is fixed. So t_{full} is chosen as in lines 3-7.

Knowing this, we can compute the remainder of the trajectory. We can compute the traversed distance if we immediately decelerate for a time t and accelerate as late as possible for a time $t + v_m/a_m - v_0(0)/a_m$ (because it might be that $v_0(0) \neq v_m$),

which is

$$v_0(0)t - \frac{1}{2}a_m t^2 + \left(v_m - a_m\left(t + \frac{v_m}{a_m} - \frac{v_0(0)}{a_m}\right)\right) \left(t + \frac{v_m}{a_m} - \frac{v_0(0)}{a_m}\right) + (t_f - t_{full})v_m + \left(v_m - a_m\left(t + \frac{v_m}{a_m} - \frac{v_0(0)}{a_m}\right)\right) \left(t_f - 2t - \frac{v_m}{a_m} + \frac{v_0(0)}{a_m}\right) + \frac{1}{2}a_m \left(t + \frac{v_m}{a_m} - \frac{v_0(0)}{a_m}\right)^2. \quad (7)$$

Equating (7) with x_0 and solving for t , results in two positive values. The smaller one is given as t_1 in (4) and the larger one as t_2 in (5). So we can put $t_{cruise} = t_1$ and $t_{acc} = t_2$.

Then combining the defined times, we obtain (6). With this choice of times, we see that we minimize the area under the absolute value of the acceleration-time graph. This is exactly the same criterion as we optimize for in Algorithm 3, so the two algorithms yield the same trajectory.

□

A visualization of trajectories generated by Algorithms 4 and 5 is depicted in Figure 1.

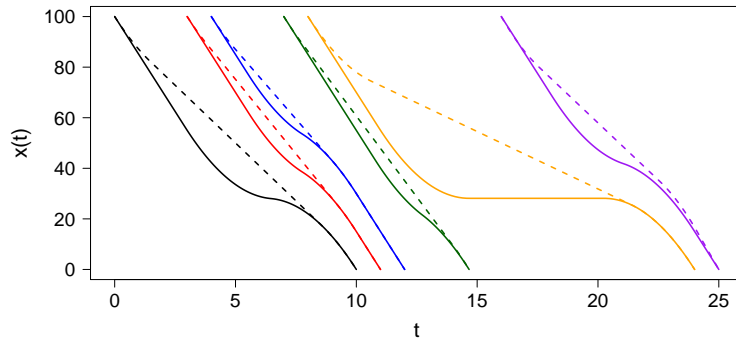


Figure 1. Algorithm 4 (solid lines) and Algorithm 5 (dashed lines) for several vehicles with t (s) on the horizontal axis and $x(t)$ (m) on the vertical axis for several vehicles.

4. Performance Analysis

Having covered the two main ingredients of the model, we turn to the performance analysis. The two measures that we consider are mean delay and fairness. In order to obtain results on mean delay and fairness, we first establish a link between the model we described so far and polling models.

4.1. Polling Model

Polling models are well-studied mathematical objects representing queueing models with multiple queues sharing a single server. For an overview of applications we refer to Boon, van der Mei, and Winands (2011) and for an overview of commonly used methods we refer to Vishnevskii and Semenova (2006).

A general polling model has n queues, each with a distinct arrival process (usually a Poisson process) with parameter λ_i , which are assumed to be independent from each other. Each queue has its own generally distributed service time from which is sampled independently. A single server is visiting each of the n queues in a certain (possibly random) order to serve customers. After a certain period at a queue the server switches to the next queue. We assume that this switching takes zero time. Instead, we assume that if we switch to a queue that is non-empty, a setup is performed. Otherwise, we do not perform a setup and continue immediately to the next queue (see e.g. Singh and Srinivasan 2002). When all queues were empty before the arrival of a vehicle, we assume that a setup was started at the most recent departure epoch. This is a feature that has not been studied before in the polling literature, but that naturally represents the behaviour of our PFAs.

We will analyze the performance of PFAs regarding delay through polling models. Although we take a vertical queueing approach in those polling models (i.e. the vehicles are all stopped at the stop line at the intersection, occupying no space), the intersection control algorithm provides a one-to-one relation between the vertical queueing model and the PFAs. We visualize this in Figure 2, where the black line represents a self-driving vehicle, and the red dotted line represents the corresponding ‘vehicle’ in the vertical queueing model. Both ‘vehicles’ enter the control region at the same time (so also the earliest possible arrival time at the intersection is the same for both). They also have the same service time, because as soon as the vehicles start to cross the intersection they have the same trajectory. So the delay for both vehicles, the difference between earliest possible crossing time and actual crossing time, is the same, as visualized in Figure 2.

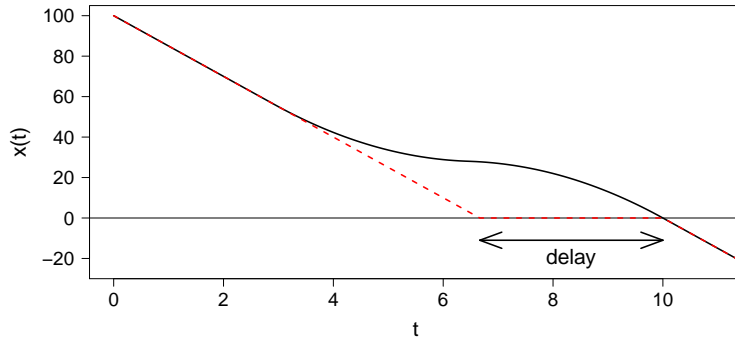


Figure 2. Visualization of the link between the traffic model with PFAs and polling models. The black line represents a self-driving vehicle, and the red dotted line represents the corresponding ‘vehicle’ in the vertical queueing model.

To make the connection between the traffic model and polling models more explicit, we argue how the traffic model translates to a polling model. The time B in between vehicles from the same stream accessing the intersection is the service time in the polling model, whereas the clearance time S is the setup time in the polling model. Which queue or lane is to be served is decided upon by the service discipline, respectively the PFA.

So, our intersection model precisely fits the framework of polling models. We will use the ideas and results already obtained for polling models to give a sound analysis of the traffic model discussed so far. From now on in this section, we will be focusing on the polling model and related results, therefore using queueing terminology.

4.2. Mean Delay

The specific assumptions result in a polling model that does not fall into the standard framework, and a fully analytical solution is difficult (if not impossible) to derive. So, we focus on approximations, being much faster and still accurate, and refrain from providing an analytical solution.

We focus on obtaining approximations for the mean delay that still require some analytical results, but that are easier to derive than the full distribution of the delay. We start with a definition of delay. Delay D_i at lane i is defined as the actual time of a car crossing the intersection minus the free-flow time in which a car could cross the intersection. B_i denotes the service time of queue i , whereas S_i denotes the setup time when we arrive at queue i . We have Poisson arrivals with rate λ_i and define $\rho_i = \lambda_i \mathbb{E}[B_i]$ and $\rho = \sum_i \rho_i$, where ρ is similar to the vehicle-to-capacity ratio. The approximations that we propose for the mean delay are all of the form,

$$\mathbb{E}[D_{i,app}^P] = \frac{K_{0,i}^P + K_{1,i}^P \rho + K_{2,i}^P \rho^2}{1 - \rho}, \quad (8)$$

like in Boon et al. (2011), where $K_{j,i}^P$ are constants that are yet to be determined and P denotes the PFA. The constants, that might depend on P and the arrival distribution (due to space limitations we only consider Poisson arrivals), are derived through requiring (8) to be exact in various limiting cases. These three cases are the following: (8) should match the mean delay for queue i in the light-traffic limit, the derivative of the light-traffic limit and the heavy-traffic limit. Then we have a system of three equations with three unknowns, which we can solve to find the constants $K_{j,i}^P$. These approximations are based on the framework described in Boon et al. (2011), which is in turn based on ideas developed in Reiman and Simon (1988). Note that (8) is only valid for $\rho < 1$, which is the condition for the polling model (and therefore also for our PFAs) to be stable.

We start with deriving the light-traffic limit for general service time and setup time distributions for the mean delay. The light-traffic here corresponds to the case where

$$\mathbb{P}(\text{server not working and not setting up}) \uparrow 1,$$

which means that both $\lambda_i \mathbb{E}[B_i]$ and $\lambda_j \mathbb{E}[S_i]$ should be close to zero. We denote with X_i^{res} the residual or overshoot of the random variable X with mean $\mathbb{E}[X^{res}] = \mathbb{E}[X^2]/(2\mathbb{E}[X])$. Then we have the following lemma.

Lemma 4.1. *The light-traffic limit for the mean delay, up to and including first-order terms, for all discussed PFAs, satisfies*

$$\mathbb{E}[D_i^{LT}] = \rho_i \mathbb{E}[B_i^{res}] + \sum_{j \neq i} \rho_j (\mathbb{E}[B_j^{res}] + \mathbb{E}[S_i]) + \sum_{j \neq i} \lambda_j \mathbb{E}[S_i] \mathbb{E}[S_i^{res}]. \quad (9)$$

Proof. We consider what happens in each phase of the cycle and argue what the waiting time is of a customer arriving at queue i .

We have n different visit periods, numbered $j = 1, \dots, n$. If $j = i$, we only have to wait for a residual service time of the customer that is currently in service (using the PASTA property of Poisson arrivals). This happens with probability $\lambda_i \mathbb{E}[B_i] = \rho_i$. The contribution to the waiting time is thus $\rho_i \mathbb{E}[B_i^{res}]$. If $i \neq j$, we have to wait for the

residual service time of the customer that is in service and for the setup time to our own queue i . This all happens with probability $\lambda_j \mathbb{E}[B_j] = \rho_j$, so the contribution to the waiting time is $\rho_j(\mathbb{E}[B_j^{res}] + \mathbb{E}[S_i])$.

The setup periods: we again have $j = 1, \dots, n$. The case $i = j$ does not occur, as we do not have a setup time in that case (we take the customer immediately into service). The cases $i \neq j$, occur with rate $\lambda_j \mathbb{E}[S_i]$ (which converges to zero) and if we arrive during such a period, we have to wait for a residual setup time. So the contribution is $\lambda_j \mathbb{E}[S_i] \mathbb{E}[S_i^{res}]$.

Cases where we see more than one customer when we arrive in the system are all of order $\mathcal{O}(\rho^2)$ or higher, so we do not consider those terms.

Summing all possibilities, we arrive at (9). \square

The heavy-traffic limit of the mean delay does depend on the PFA. In heavy traffic, the behaviour of our PFAs and regular polling models is the same. Consequently, the heavy-traffic limits for the exhaustive and gated PFAs are the same as the heavy-traffic limits for the exhaustive and gated disciplines in e.g. Boon (2011), where polling models with switch-over times (rather than setup times) are presented. Indeed, if the lengths of the setups and switch-overs are the same, the polling model with switch-overs (and without setup times) is the same as the polling model with setup times (but no switch-over times), because each setup will be performed in heavy traffic (as all queues are non-empty when the server visits them) and can be seen as an ‘ordinary’ switch-over time. This implies that we can use the results from Boon (2011), so

$$\mathbb{E}[D_i^{HT,P}] = \frac{\omega_i^P}{1-\rho} + o((1-\rho)^{-1}), \quad (10)$$

with P denoting the PFA, where

$$\omega_i^{exh} = \frac{1-\hat{\rho}_i}{2} \left(\frac{\sigma^2}{\sum_{j=1}^n \hat{\rho}_j(1-\hat{\rho}_j)} + \sum_{j=1}^n \mathbb{E}[R_j] \right), \quad (11)$$

for the exhaustive PFA, with $\sigma^2 = \mathbb{E}[B^2]/\mathbb{E}[B]$ (in case of Poisson arrivals) and $\hat{\rho}_i = \rho_i/\rho$ and

$$\omega_i^{gat} = \frac{1+\hat{\rho}_i}{2} \left(\frac{\sigma^2}{\sum_{j=1}^n \hat{\rho}_j(1+\hat{\rho}_j)} + \sum_{j=1}^n \mathbb{E}[R_j] \right) \quad (12)$$

for the gated PFA.

The general approximation in (8) is now ready to be used. We obtain the following theorem.

Theorem 4.2. *The mean delay experienced for PFA P can be approximated with*

Equation (8), where

$$\begin{aligned}
K_{0,i}^P &= 0, \\
K_{1,i}^P &= \hat{\rho}_i \mathbb{E}[B_i^{res}] + \sum_{j \neq i} \hat{\rho}_j (\mathbb{E}[B_j^{res}] + \mathbb{E}[S_i]) + \sum_{j \neq i} \hat{\lambda}_j \mathbb{E}[S_i^{res}] \mathbb{E}[S_i], \\
K_{2,i}^P &= \omega_i^P - K_{1,i}^P,
\end{aligned} \tag{13}$$

with $\hat{\lambda}_i = \hat{\rho}_i / \mathbb{E}[B_i]$.

Proof. As mentioned before, we put three conditions on the constants $K_{j,i}^P$, $j = 0, 1, 2$. These are the following

$$\begin{aligned}
\mathbb{E}[D_{i,app}^P] \Big|_{\rho=0} &= \mathbb{E}[D_i^{LT}] \Big|_{\rho=0}, \\
\frac{d}{d\rho} \mathbb{E}[D_{i,app}^P] \Big|_{\rho=0} &= \frac{d}{d\rho} \mathbb{E}[D_i^{LT}] \Big|_{\rho=0}, \\
(1 - \rho) \mathbb{E}[D_{i,app}^P] \Big|_{\rho \uparrow 1} &= \mathbb{E}[D_i^{HT,P}].
\end{aligned}$$

Using Lemma 4.1 and Equation (10),

$$\begin{aligned}
K_{0,i}^P &= 0, \\
K_{0,i}^P + K_{1,i}^P &= \hat{\rho}_i \mathbb{E}[B_i^{res}] + \sum_{j \neq i} \hat{\rho}_j (\mathbb{E}[B_j^{res}] + \mathbb{E}[S_i]) + \sum_{j \neq i} \hat{\lambda}_j \mathbb{E}[S_i^{res}] \mathbb{E}[S_i], \\
K_{0,i}^P + K_{1,i}^P + K_{2,i}^P &= \mathbb{E}[D_i^{HT,P}] = \omega_i^P.
\end{aligned} \tag{14}$$

It can easily be seen that (14) reduces to (13). \square

Remark 2. The above mentioned results for mean delay can readily be extended to results for the mean number of vehicles in the queue, using Little's law. Together with the speed regulation algorithm, the physical length of the queue can be calculated (for example if we define the last vehicle that has already decelerated to be in the queue). This would give information about e.g. spillback of the intersection to other intersections.

In general the approximations work fine for all discussed PFAs, as can be seen in Figure 3 (comparing the solid lines (the exact results) and the dashed lines (the approximations)). We present examples where we put $v_m = 15$ m/s, $a_m = 4$ m/s², $l = 5$ m and $s = 10$ m and where two lanes cross each other. We consider two cases where the load on both lanes is split differently: one case where $\rho_1 = \rho_2$ (referred to as being symmetric) and one case where $\rho_1 = 3\rho_2$ (referred to as being asymmetric). Following Tachet et al. (2016), we put $B = 1$ s and $S = 2.375$ s. The two discussed PFAs result in the Figure 3, where also, as a benchmark, the Batch Algorithm from Tachet et al. (2016) is considered, with a maximum batch size of 100. The approximations are also good for all other settings we simulated.

We see that the exhaustive PFA performs really well, if we focus on mean delay, compared to the other PFAs. This can also be understood from the heavy-traffic limits (11) and (12). The performance of the Batch Algorithm is similar to that of the gated

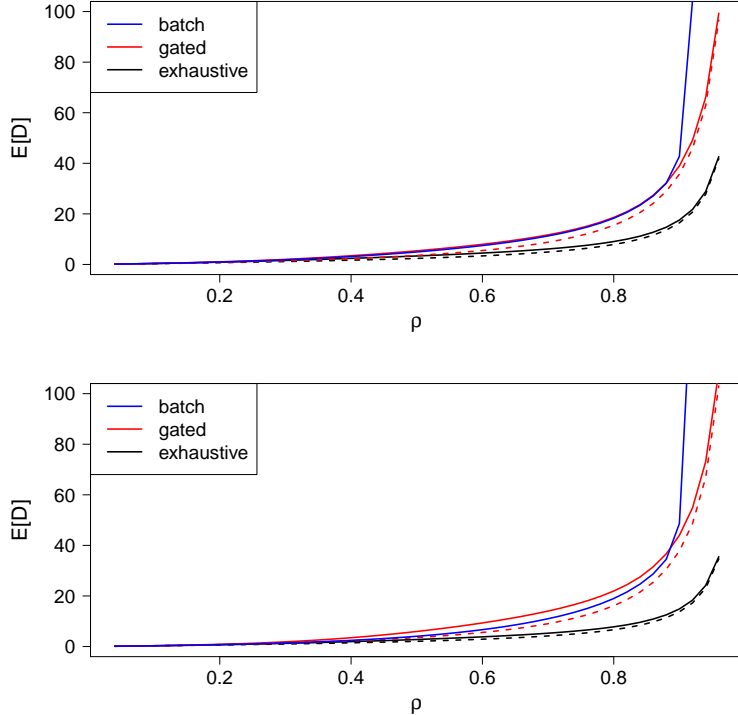


Figure 3. Mean delay experienced by an arbitrary car for the symmetric case (top) and asymmetric case (bottom). The solid lines represent simulation results and the dashed lines approximations.

PFA, except for higher values of ρ , which is due to the maximum batch size of 100. This maximum batch size causes a lower maximum capacity for the Batch Algorithm than for the exhaustive and gated PFA and therefore, the Batch Algorithm has a sharp increase in mean delay earlier than the other two PFAs. We expect the exhaustive PFA to be (very close to) optimal with respect to the mean delay. This optimality was, to some extent, already observed in e.g. Newell (1969), Levy, Sidi, and Boxma (1990) and Wu, Yan, and Abbas-Turki (2013).

4.3. Fairness

In order to show that the exhaustive PFA is not the best for all performance metrics we consider fairness in this subsection. We use the definition of fairness for polling models, denoted with F , as introduced in Shapira and Levy (2016),

$$F = \frac{\mathbb{E}[N_{ahead}]}{\mathbb{E}[N_{total}]},$$

where N_{ahead} denotes the number of cars an arbitrary car sees upon arrival and that are served ahead of it; and where N_{total} denotes the total number of cars an arbitrary car sees upon arrival. In words this means that we quantify the percentage of cars that did not overtake an arbitrary car (on an intersection-wide basis).

We present simulation results for fairness for the same set of examples as for the mean delay.

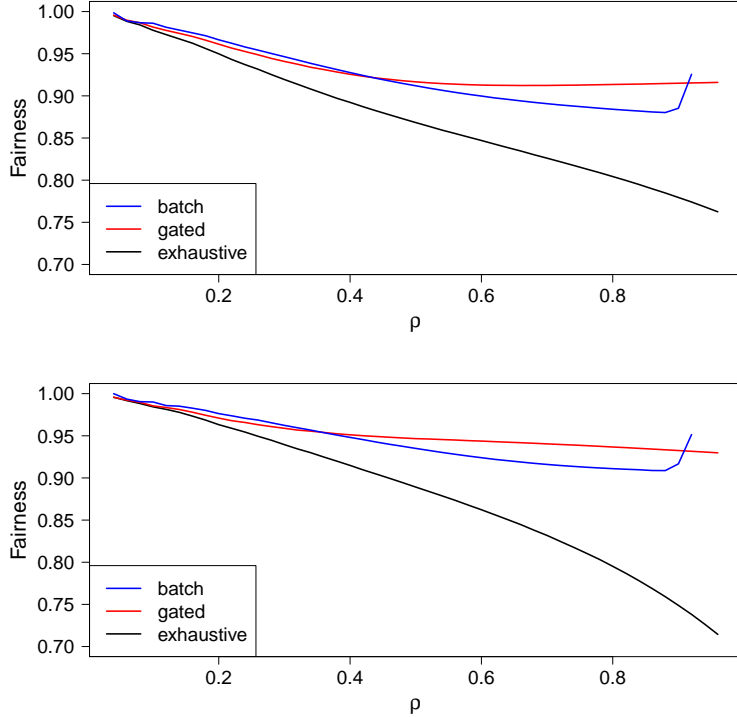


Figure 4. Fairness experienced by an arbitrary car for the symmetric case (top) and asymmetric case (bottom).

Considering fairness, we see once more that the gated PFA is close to the Batch Algorithm for values of ρ that are not too high. The increase of fairness for high values of ρ for the Batch Algorithm is due to the maximum batch size of 100. The exhaustive PFA is worse on fairness, but is still above 75%. It seems that a low mean delay results in a relatively low fairness, showing a potential need to balance the two performance measures, which is to some extent visible in the increase of fairness for the Batch Algorithm and high values of ρ .

5. Comparison traditional Traffic Light and PFAs

The goal of this section is to provide a comparison between traditional traffic lights and PFAs on basis of delay. As a measure for the traditional traffic light we use the traffic simulator SUMO. We will consider two scenarios in SUMO: one with fixed control and one with adaptive control (based on so-called time loss in the SUMO User Documentation). We will compare these two scenarios with the exhaustive PFA.

We again consider two examples where two lanes cross each other and the vehicle to capacity ratio is in the first example the same on both lanes, whereas in the second example the ratio between the loads on the lanes is 1 : 3. For the exhaustive PFA we again put $B = 1$ s and $S = 2.375$ s. For the fixed control simulation in SUMO and the first example we assume a green period for both lanes of 22 s and an amber period of 3 s; for the second example we pick green periods of 11 and 33 s and amber periods of length 3 s. Note that some of the results for the fixed control in Figure 5 could

be slightly improved by adapting the length of the green period. For the adaptive control in SUMO we assume a maximum green period duration of 45 s and an amber period of 3 s for the symmetric example and a maximum green of 22 and 68 s for the asymmetric case. Note that we do not have to define the variable B in SUMO, as the vehicles themselves will decide what B is. The delay in SUMO for the fixed and adaptive control is obtained in the following way: we compute the mean time spent in the system for all vehicles and subtract the mean time vehicles spent in the system under free-flow conditions (i.e. putting the traffic light at green for one lane all the time). We take exactly the same arrivals for all three scenarios.

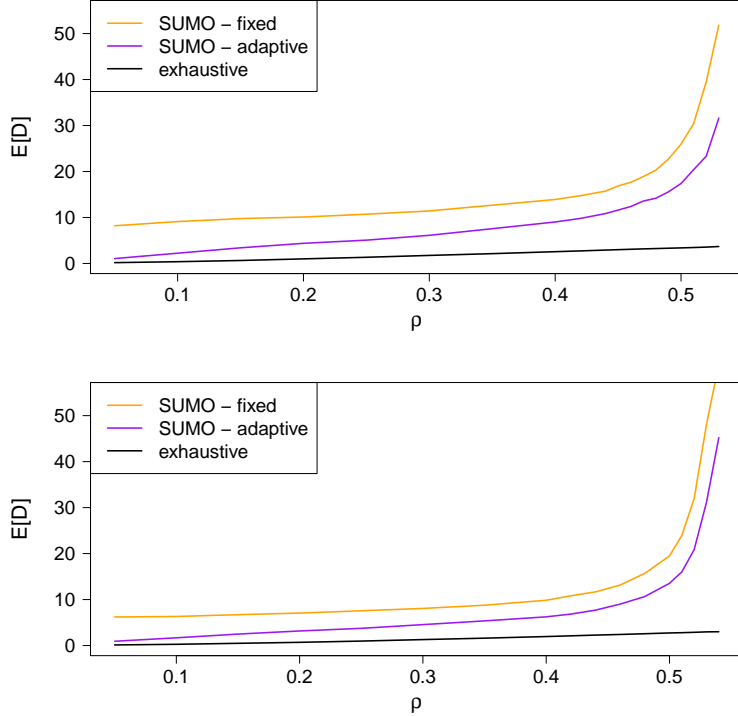


Figure 5. Mean delay for an arbitrary car for traditional traffic lights (represented by SUMO) and the exhaustive PFA for the symmetric example (top) and the asymmetric case (bottom).

We see in Figure 5 that there is quite a difference between either the fixed cycle traffic light or the adaptive traffic light, and the exhaustive PFA. To some extent, this was also observed in Tachet et al. (2016). The capacity of the intersection for the latter case is almost twice as high as for the traditional traffic light, showing a huge potential in resolving congestion. This is mainly due to the speed regulation of vehicles, which increases the speed of vehicles crossing the intersection, but also due to the scheduling strategies of the PFA.

6. Conclusion and Discussion

We have shown that significant gains can be obtained compared to nowadays traffic when speed regulation and PFAs can be employed and have given ways to decrease mean delay on intersections. This has been shown through a connection between

polling models and PFAs.

It seems that the exhaustive PFA is close to optimality with respect to mean delay. However, the exhaustive PFA exhibits relatively poor fairness characteristics. It might be worthwhile to find a balance between mean delay and (e.g.) fairness in order to obtain some kind of optimal setting for the PFA. A possibility hereto might be the so-called k -limited discipline in polling models, where for each lane an upper bound to the platoon size is set. Intuitively, the k -limited discipline is similar to the exhaustive discipline, except for this maximum size of the platoon.

In principle our PFAs could be used in nowadays traffic as well. The only requirement is that it must be known on an intersection wide basis in which order the vehicles arrive. The requirement that we can control the speed of arriving vehicles is not needed to execute the PFAs. This assumption would only play a role in what the variables B and S would look like. But even then, the scheduling part of a PFA might still be used. Using some kind of speed advisory system for conventional vehicles, it might be possible to come close to the performance of the PFAs based on self-driving vehicles.

A future direction of research is to investigate more realistic intersection scenarios, yet we expect similar results. Depending on the extension, our results readily apply, if at most one stream of vehicles is allowed to cross the intersection, or need to be generalized. We also would like to extend our approximations to obtain analytical results for fairness.

We have studied an isolated intersection, where vehicles arrive individually in the control region. In a network of intersections there are several complications. Firstly, the arrival processes of vehicles become dependent. Moreover, the interplay between various intersections is non-trivial. Already for a tandem of fixed cycle traffic light intersections, it is difficult to find a good green wave, see e.g. Oblakova et al. (2017). Our PFAs are much less strict on e.g. the cycle length, imposing an even more difficult task of balancing a whole network of intersections. Once more, the k -limited PFA (having a fixed maximum cycle length) might prove to be an outcome in this respect.

A study on how realistic our proposed models are, might also be relevant. We assume e.g. that each vehicle is able to perfectly match the criteria we set in the speed regulation assumptions. For example, there might be some uncertainty in the control of a self-driving vehicle. A notion like string-stability of a platoon of vehicles (see e.g. Swaroop and Hedrick 1996) might be investigated for our proposed models.

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