

## Tighter enumeration of matroids of fixed rank

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## TIGHTER ENUMERATION OF MATROIDS OF FIXED RANK

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ABSTRACT. We prove asymptotic upper bounds on the number of paving matroids of fixed rank, using a mixture of entropy counting, sparse encoding, and the probabilistic method.

**Keywords:** Matroid,  $d$ -partition, design, Steiner triple system, entropy

## 1. INTRODUCTION

Let  $m(n, r)$  denote the number of matroids of rank  $r$  on a fixed ground set  $E$  of cardinality  $n$ , and let  $p(n, r)$  denote the corresponding number of paving matroids. The following is Theorem 3 of [PvdP17].

**Theorem 1.1** (Pendavingh, van der Pol (2017)). *For each fixed rank  $r$ ,*

$$\ln p(n, r) \leq \ln m(n, r) \leq \left(1 + \frac{r + o(1)}{n - r + 1}\right) \ln p(n, r) \quad \text{as } n \rightarrow \infty.$$

Theorem 1.1 motivates the main thrust of this paper: to establish tight bounds on  $p(n, r)$ .

A matroid  $M$  on ground set  $E$  and of rank  $r$  is *paving* if all subsets  $F$  of  $E$  with  $|F| < r$  are independent sets of  $M$ . Compared to matroids in general, paving matroids are relatively straightforward objects. A set  $\mathcal{H}$  of subsets of  $E$  is the set of hyperplanes of a paving matroid of rank  $r = d + 1$  if and only if  $\mathcal{H}$  is a  *$d$ -partition* of  $E$  in the sense of Hartmanis [Har59]: each  $H \in \mathcal{H}$  has  $|H| \geq d$ , and for each  $I \subseteq E$  of cardinality  $d$  there a unique  $H \in \mathcal{H}$  such that  $I \subseteq H$ . Thus paving matroids on  $E$  of rank  $r$  correspond one-to-one to  $d$ -partitions of  $E$ , and to determine  $p(n, r)$  is to count the number of  $d$ -partitions of a fixed set  $E$  of cardinality  $n$ .

The following observation is central to our methods for establishing upper bounds on  $p(n, d + 1)$ . Given any set  $\mathcal{V}$  of  $(d + 1)$ -subsets of  $E$  (i.e. subsets  $V \subseteq E$  with  $|V| = d + 1$ ), there is a unique  $d$ -partition  $\mathcal{H}$  such that for each  $V \in \mathcal{V}$  there is an  $H \in \mathcal{H}$  with  $V \subseteq H$ , such that  $|\mathcal{H}|$  is as large as possible. In turn, given any  $d$ -partition  $\mathcal{H}$ , it is not difficult to find some set of  $\mathcal{V}$  of  $(d + 1)$ -subsets of  $E$  which points to  $\mathcal{H}$  in this manner. Thus we may encode  $d$ -partitions by sets of  $(d + 1)$ -subsets of  $E$ , which encodings may even be assumed to be of a special form. To bound the number of  $d$ -partitions of  $E$ , it will then suffice to bound the number of sets  $\mathcal{V}$  of  $(d + 1)$ -subsets of  $E$  of this special form.

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Part of the results presented here were obtained while JvdP was a PhD candidate at Eindhoven University of Technology. A preliminary version of some of the results presented here is contained in his PhD thesis [vdP17].

A matroid  $M$  on ground set  $E$  and of rank  $r$  is *sparse paving* if it is paving and all hyperplanes of  $M$  have cardinality  $r - 1$  or  $r$ . Sparse paving matroids have a special position in the above encoding scheme, since if  $\mathcal{H}$  is the set of hyperplanes of a sparse paving matroid, then the unique set  $\mathcal{V}$  which encodes  $\mathcal{H}$  is  $\mathcal{V} := \{V \in \mathcal{H} : |V| = r\}$ . As a first step towards bounding  $p(n, r)$ , we obtain the following asymptotic estimate the number of sparse paving matroids of rank  $r$  on a fixed ground set of cardinality  $n$ , denoted  $s(n, r)$ .

**Theorem 1.2.** *For each  $r \geq 3$ ,*

$$\ln s(n, r) = \frac{1}{n - r + 1} \binom{n}{r} (\ln(n - r + 1) + 1 - r + o(1)) \quad \text{as } n \rightarrow \infty.$$

We use an entropy counting method for obtaining this bound on  $s(n, r)$ , which was inspired by a method for counting Steiner triple systems due to Linal and Luria [LL13]. More generally, we derive an upper bound on  $s_k(n, r)$ , the number of paving matroids of rank  $r$  such that each dependent hyperplane has cardinality  $r + k$ . The upper bound on  $s(n, r) = s_0(n, r)$  of Theorem 1.2 is the special case where  $k = 0$ .

**Theorem 1.3.** *For each  $r \geq 3$  and  $k \geq 0$ ,*

$$\ln s_k(n, r) \leq Q^{-1} \binom{n}{r - 1} (\ln N + 1 - Q + o(1)) \quad \text{as } n \rightarrow \infty,$$

where  $N = \binom{n - r + 1}{k + 1}$  and  $Q = \binom{r + k}{k + 1}$ .

To bound the number of paving matroids  $p(n, r)$  of rank  $r \geq 4$ , will argue that each paving matroid  $M$  with hyperplanes  $\mathcal{H}$  is encoded by a set of  $r$ -sets  $\mathcal{V}$  which is the disjoint union of  $r$ -sets  $\mathcal{V}^0$  and  $r$ -sets  $\mathcal{V}^1$ , such that  $\mathcal{V}^0$  encodes the hyperplanes of a sparse paving matroid. By exploiting a tradeoff between the cardinalities of  $\mathcal{V}^0$  and  $\mathcal{V}^1$ , we will be able to bound the number of paving matroids very close to the number of sparse paving matroids.

**Theorem 1.4.** *For each  $r \geq 4$ ,*

$$\ln p(n, r) = \frac{1}{n - r + 1} \binom{n}{r} (\ln(n - r + 1) + 1 - r + o(1)) \quad \text{as } n \rightarrow \infty.$$

In rank  $r = 3$ , the observed tradeoff between the cardinalities of  $\mathcal{V}^0$  and  $\mathcal{V}^1$  is not as significant as in higher ranks, and we resort to a different method. As noted, the sets  $\mathcal{V}$  which we use to encode the hyperplanes  $\mathcal{H}$  have a special form. We will derive bounds on the probability that a random set of triples from an  $n$ -set is *good* in this sense, and then bound  $p(n, r)$  as the total number of sets of triples times this probability.

**Theorem 1.5.** *For  $r = 3$ ,*

$$\begin{aligned} \frac{1}{n - r + 1} \binom{n}{r} (\ln(n - r + 1) - 2 + o(1)) &\leq \ln p(n, r) \\ &\leq \frac{1}{n - r + 1} \binom{n}{r} (\ln(n - r + 1) + 0.35) \quad \text{as } n \rightarrow \infty. \end{aligned}$$

After giving preliminaries in the next section, the paper is subdivided according to the methods used. Section 3 uses entropy methods to bound the number of

partial designs and sparse paving matroids. Section 4 describes the encoding of the hyperplanes of a paving matroid which was outlined above to establish the bounds in rank  $r \geq 4$ . This section uses elementary combinatorial counting arguments. Section 5 uses probabilistic arguments and continuous optimization, and settles the bounds in rank  $r = 3$ . In the final section, we speculate on the remaining gap between the upper and lower bounds in the rank-3 case.

## 2. PRELIMINARIES

Throughout this paper, we use  $\mathbb{P}(n, r)$  and  $\mathbb{S}(n, r)$  for the sets of paving and sparse paving matroids, respectively, of rank  $r$  on ground set  $[n]$ . In addition, we use  $p(n, r) = |\mathbb{P}(n, r)|$  and  $s(n, r) = |\mathbb{S}(n, r)|$ .

If  $E$  is a finite set, and  $0 \leq r \leq |E|$ , then we write

$$\binom{E}{r} := \{X \subseteq E : |X| = r\}.$$

The following bounds, which are valid for all integers  $k \geq 1$ , are a form of Stirling's approximation:

$$(1) \quad \sqrt{2\pi k} \left(\frac{k}{e}\right)^k \leq k! \leq e\sqrt{k} \left(\frac{k}{e}\right)^k.$$

We freely use the standard bound on sums of binomial coefficients

$$(2) \quad \sum_{i=0}^m \binom{n}{k} \leq \left(\frac{en}{m}\right)^m.$$

The following lemma provides a bound in the other direction. It essentially shows that the constant  $e$  that appears in the upper bound cannot be dispensed with.

**Lemma 2.1.** *For all  $1 \leq k \leq n$ ,  $\binom{n}{k} \geq \left(\frac{e^{1-\varepsilon}n}{k}\right)^k$ , where  $\varepsilon \equiv \varepsilon_{k,n} = \frac{1}{k} \ln \frac{e\sqrt{k}}{\prod_{i=0}^{k-1} (1-i/n)}$ .*

*Proof.* Since  $\binom{n}{k} = \frac{n \cdots (n-k+1)}{k!}$ , it follows from (1) that

$$\binom{n}{k} \geq \left(\frac{en}{k}\right)^k \frac{\prod_{i=0}^{k-1} \left(1 - \frac{i}{n}\right)}{e\sqrt{k}}. \quad \square$$

## 3. SPARSE PAVING MATROIDS

In this section, we prove Theorem 1.2. The lower bound was proved in [PvdP17]:

**Proposition 3.1** ([PvdP17, Theorem 10]). *For each  $r \geq 3$ ,*

$$\ln s(n, r) \geq \frac{1}{n-r+1} \binom{n}{r} (\ln(n-r+1) + 1 - r + o(1))$$

as  $n \rightarrow \infty$ .

It thus remains to prove the corresponding upper bound.

**3.1. Entropy.** The upper bound on  $s(n, r)$  is proved using information-theoretic techniques. We review some of the notation and terminology that we require; for a more thorough introduction, we refer the reader to [AS08, Section 15.7].

In what follows, bold-faced symbols, such as  $\mathbf{X}$ , are random variables that take their values in some finite set  $\mathcal{X}$ . The entropy  $\mathcal{H}(\mathbf{X})$  of  $\mathbf{X}$  is defined as

$$\mathcal{H}(\mathbf{X}) := - \sum_{x \in \mathcal{X}} \mathbf{P}(\mathbf{X} = x) \ln \mathbf{P}(\mathbf{X} = x),$$

where for convenience we use  $0 \ln 0 = 0$ .

It is always true that  $\mathcal{H}(\mathbf{X}) \leq \ln |\mathcal{X}|$ . The upper bound is attained if (and only if)  $\mathbf{X}$  has the uniform distribution on  $\mathcal{X}$ . This observation makes entropy useful for enumeration purposes: questions about the cardinality of  $\mathcal{X}$  immediately translate to questions about the entropy of random variables with uniform distribution on  $\mathcal{X}$ .

For a pair of random variables  $(\mathbf{X}, \mathbf{Y})$ , the conditional entropy of  $\mathbf{X}$  given  $\mathbf{Y}$  is

$$\mathcal{H}(\mathbf{X} | \mathbf{Y}) = - \sum_y \mathbf{P}(\mathbf{Y} = y) \sum_x \mathbf{P}(\mathbf{X} = x | \mathbf{Y} = y) \ln \mathbf{P}(\mathbf{X} = x | \mathbf{Y} = y),$$

which can be written as  $\mathcal{H}(\mathbf{X} | \mathbf{Y}) = \mathcal{H}(\mathbf{X}, \mathbf{Y}) - \mathcal{H}(\mathbf{Y})$ . More generally, if  $\mathbf{X} = (\mathbf{X}_1, \dots, \mathbf{X}_n)$  be a sequence of random variables, then the chain rule for entropy states that

$$\mathcal{H}(\mathbf{X}) = \mathcal{H}(\mathbf{X}_1) + \mathcal{H}(\mathbf{X}_2 | \mathbf{X}_1) + \dots + \mathcal{H}(\mathbf{X}_n | \mathbf{X}_1, \dots, \mathbf{X}_{n-1}).$$

**3.2. Upper bound.** Given a collection of sets  $\mathcal{X} \subseteq 2^E$ , the  $s$ -shadow  $\partial_s \mathcal{X}$  of  $\mathcal{X}$  is

$$\partial_s \mathcal{X} := \left\{ Y \in \binom{E}{s} : \text{there exists } X \in \mathcal{X} \text{ such that } Y \subseteq X \right\}.$$

Let  $\mathbb{S}_k(n, r) \subseteq \mathbb{P}(n, r)$  be the collection of paving matroids all of whose hyperplanes have cardinality  $r - 1$  or  $r + k$  (the hyperplanes of cardinality  $r + k$  of such a matroid form a partial Steiner system on  $n$  points, in which each block has cardinality  $r + k$  and each  $(r - 1)$ -set is contained in at most one block). Note that  $\mathbb{S}(n, r) = \mathbb{S}_0(n, r)$ . Partition  $\mathbb{S}_k(n, r)$  according to the  $(r - 1)$ -shadows of hyperplanes. For a matroid  $M$ , let  $\mathcal{H}_k(M)$  be the collection of its hyperplanes of cardinality  $r + k$ . For  $\mathcal{A} \in \binom{[n]}{r-1}$ , write

$$\mathbb{S}_k(n, r, \mathcal{A}) = \{M \in \mathbb{S}_k(n, r) : \partial_{r-1} \mathcal{H}_k(M) = \mathcal{A}\},$$

and let  $s_k(n, r, \mathcal{A}) = |\mathbb{S}_k(n, r, \mathcal{A})|$ . (Note that  $\mathbb{S}_k(n, r, \mathcal{A})$  may be empty for some choices of  $\mathcal{A}$ , but this is immaterial to our argument.)

The following lemma is a generalisation to partial Steiner systems of a result of Linial and Luria [LL13] for Steiner triple systems (their result was generalised to arbitrary designs by Keevash in [Kee15, Theorem 6.1]).

**Lemma 3.2.** *For each  $r \geq 3$ , and  $k \geq 0$ , there exists a function  $f_{r,k}^{(3,2)}(n)$  with the property that  $f_{r,k}^{(3,2)}(n) \rightarrow 0$  as  $n \rightarrow \infty$ , such that*

$$\ln s_k(n, r, \mathcal{A}) \leq \frac{|\mathcal{A}|}{Q} (\ln N + 1 - Q + f_{r,k}^{(3,2)}(n))$$

for all  $\mathcal{A} \subseteq \binom{[n]}{r-1}$ , where  $Q = \binom{r+k}{k+1}$  and  $N = \binom{n-r+1}{k+1}$ . In particular,

$$\ln s(n, r, \mathcal{A}) \leq \frac{|\mathcal{A}|}{r} (\ln(n - r + 1) + 1 - r + f_{r,0}^{(3,2)}(n))$$

for all  $\mathcal{A} \subseteq \binom{[n]}{r-1}$ .

*Proof.* Fix  $\mathcal{A}$  and let  $\mathbf{X}$  be a matroid chosen uniformly at random from  $\mathbb{S}_k(n, r, \mathcal{A})$ . As  $\ln s_k(n, r, \mathcal{A}) = \mathcal{H}(\mathbf{X})$ , it suffices to bound  $\mathcal{H}(\mathbf{X})$ .

Consider the collection of random variables  $\{\mathbf{X}_A : A \in \mathcal{A}\}$ , where  $\mathbf{X}_A$  is the closure (in  $\mathbf{X}$ ) of  $A$ , and note that  $\mathcal{H}(\mathbf{X}) = \mathcal{H}(\mathbf{X}_A : A \in \mathcal{A})$ . Order the collection  $\mathcal{A}$ . This is conveniently done by introducing an injective function  $\lambda : \mathcal{A} \rightarrow [0, 1]$  and ordering  $\mathcal{A}$  by decreasing  $\lambda$ -values. Write  $\mathbf{X}_A^\lambda := (\mathbf{X}_{A'} : \lambda(A') > \lambda(A))$ . By the chain rule for entropy,

$$\mathcal{H}(\mathbf{X}) = \sum_{A \in \mathcal{A}} \mathcal{H}(\mathbf{X}_A \mid \mathbf{X}_A^\lambda).$$

For  $A \in \mathcal{A}$ , let

$$\mathcal{X}_A := \left\{ X \in \binom{[n]}{r+k} : A \subseteq X, \text{ and } A' \in \mathcal{A} \text{ for all } A' \in \binom{X}{r-1} \right\}.$$

Clearly,  $\mathcal{X}_A$  depends only on  $A$  and  $\mathcal{A}$ . Note that  $\mathbf{X}_A \in \mathcal{X}_A$  and  $1 \leq |\mathcal{X}_A| \leq \binom{n-r+1}{k+1}$ . We further restrict the number of possible values for  $\mathbf{X}_A$  conditional on  $\mathbf{X}_A^\lambda$ . If  $A \subseteq X$  for some  $X \in \mathbf{X}_A^\lambda$ , then we must have  $\mathbf{X}_A = X$ . On the other hand if  $A$  is not contained in any member of  $\mathbf{X}_A^\lambda$ , then in order for  $H \in \mathcal{X}_A$  to be available for  $\mathbf{X}_A$ , we cannot have  $\mathbf{X}_{A''} \in \mathbf{X}_A^\lambda$  for any  $A'' \in \binom{X}{r-1}$ , where  $A' \in \binom{H}{r-1} \setminus \{A\}$ . We make this precise by introducing the random variable  $\mathbf{N}_A \equiv \mathbf{N}_A(\lambda, \mathbf{X}_A^\lambda)$ ,

$$(3) \quad \mathbf{N}_A := \begin{cases} \sum_{H \in \mathcal{X}_A} \prod_{A' \in \binom{H}{r-1} \setminus \{A\}} \prod_{A'' \in \binom{A'}{r-1}} \mathbf{1}_{\{\forall X \in \mathbf{X}_A^\lambda : A'' \not\subseteq X\}} & \text{if } \forall X \in \mathbf{X}_A^\lambda : A \not\subseteq X, \\ 1 & \text{otherwise.} \end{cases}$$

By the above discussion,  $\mathcal{H}(\mathbf{X}_A \mid \mathbf{X}_A^\lambda) \leq \mathbf{E}_{\mathbf{X}}[\ln \mathbf{N}_A]$ . The inequality holds for any injection  $\lambda$ , so it remains true after randomising  $\lambda$  and taking the expected value. Such a random  $\lambda$  can be constructed by choosing  $\lambda(A)$  uniformly at random from the interval  $[0, 1]$ , independently of all other choices and  $\mathbf{X}$ . (Note that almost surely no two  $\lambda$ -values are the same.) We obtain

$$\mathbf{E}_{\mathbf{X}}[\ln \mathbf{N}_A] = \mathbf{E}_{\lambda(A)}[\mathbf{E}_{\mathbf{X}}[\mathbf{E}_{\lambda}[\ln \mathbf{N}_A \mid \lambda(A)]]].$$

Let  $\mathcal{F}_A$  be the event that  $\lambda(A) > \lambda(A')$  for all  $A' \in \binom{\mathbf{X}_A}{r-1} \setminus \{A\}$ , i.e. that  $A$  comes first (in the  $\lambda$ -ordering) among all  $(r-1)$ -subsets of  $\mathbf{X}_A$ . Using that  $\mathbf{N}_A = 1$  on  $\overline{\mathcal{F}_A}$ , we obtain

$$\begin{aligned} \mathbf{E}_{\lambda}[\ln \mathbf{N}_A \mid \lambda(A)] &= (\lambda(A))^{Q-1} \mathbf{E}_{\lambda}[\ln \mathbf{N}_A \mid \lambda(A), \mathcal{F}_A] \\ &\leq (\lambda(A))^{Q-1} \ln \mathbf{E}_{\lambda}[\mathbf{N}_A \mid \lambda(A), \mathcal{F}_A], \end{aligned}$$

where the inequality follows from Jensen's inequality. We claim that

$$(4) \quad \mathbf{E}_{\lambda}[\mathbf{N}_A \mid \lambda(A), \mathcal{F}_A] \leq 1 + (Q-1)(\lambda(A))^{Q-1}.$$

To prove (4), first note that on the event  $\mathcal{F}_A$  the term in (3) corresponding to  $\mathbf{X}_A \in \mathcal{F}_A$  evaluates to 1. For each of the remaining terms, note that the event

$$\bigcap_{A' \in \binom{H}{r-1} \setminus \{A\}} \bigcap_{A'' \in \binom{A'}{r-1}} \{\forall X \in \mathbf{X}_A^\lambda : A'' \not\subseteq X\}$$

happens precisely when  $A$  precedes (in the  $\lambda$ -ordering) all  $Q(Q-1)$  of the  $(r-1)$ -sets contained in a set of the form  $\mathbf{X}_{A'}$  with  $A' \in \binom{H}{r-1} \setminus \{A\}$ ; as these events

are mutually independent, and each happens with probability  $\lambda(A)$ , (4) follows by linearity of expectation.

We conclude that

$$\begin{aligned} \mathcal{H}(\mathbf{X}) &= \sum_{A \in \mathcal{A}} \int_0^1 \lambda^{Q-1} \ln \left( 1 + (N-1) \lambda^{Q(Q-1)} \right) d\lambda \\ &= \frac{|\mathcal{A}|}{Q} \int_0^1 \ln \left( 1 + (N-1) u^{Q-1} \right) du \\ &\leq \frac{|\mathcal{A}|}{Q} \left[ \ln N + \int_0^1 \ln \left( u^{Q-1} + \frac{1}{N} \right) du \right]. \end{aligned}$$

The integral on the right-hand side is at most  $1 - Q + 3QN^{-Q}$ ; this proves the first claim, with  $f_{r,k}^{(3.2)}(n) = 3QN^{-Q}$ .

The second claim follows from the first, since  $s(n, r, \mathcal{A}) = s_0(n, r, \mathcal{A})$ .  $\square$

The following lemma bounds the number of partial designs with given parameters. In particular, it proves Theorem 1.2, as  $s(n, r) = s_0(n, r)$ .

**Lemma 3.3.** *For each  $r \geq 3$  and  $k \geq 0$ , there exists a function  $f_{r,k}^{(3.3)}(n)$  with the property that  $f_{r,k}^{(3.3)}(n) \rightarrow 0$  as  $n \rightarrow \infty$ , such that*

$$\ln s_k(n, r) \leq Q^{-1} \binom{n}{r-1} \left( \ln N + 1 - Q + f_{r,k}^{(3.3)}(n) \right),$$

where  $N = \binom{n-r+1}{k+1}$  and  $Q = \binom{r+k}{k+1}$ .

*Proof.* Define  $f_{r,k}^{(3.3)}(n) = f_{r,k}^{(3.2)}(n) + Q \ln \left( 1 + \frac{1}{N e^{1-Q+f_{r,k}^{(3.2)}(n)}} \right)$ . A straightforward argument shows that  $f_{r,k}^{(3.3)}(n) \rightarrow 0$  as  $n \rightarrow \infty$ . As  $s_k(n, r) = \sum_{\mathcal{A}} s_k(n, r, \mathcal{A})$ , where the sum is over all subsets  $\mathcal{A} \subseteq \binom{[n]}{r-1}$ , it follows from Lemma 3.2 that

$$\begin{aligned} s_k(n, r) &\leq \sum_{a=0}^{\binom{n}{r-1}} \binom{\binom{n}{r-1}}{a} \left( e^{1-Q+f_{r,k}^{(3.2)}(n)} N \right)^{Q^{-1}a} \\ &= \left( 1 + \left( e^{1-Q+f_{r,k}^{(3.2)}(n)} N \right)^{Q^{-1}} \right)^{\binom{n}{r-1}} = \left( e^{1-Q+f_{r,k}^{(3.3)}(n)} N \right)^{Q^{-1} \binom{n}{r-1}}, \end{aligned}$$

as required.  $\square$

#### 4. PAVING MATROIDS OF RANK AT LEAST 4

**4.1. An encoding of paving matroids.** We describe an encoding of paving matroids that was used in [PvdP17] to prove a weaker bound on the number of paving matroids.

Let  $E$  be a finite set and assume that it is linearly ordered. A paving matroid  $M$  of rank  $r$  on  $E$  can be reconstructed from the collection

$$\mathcal{V}(M) := \bigcup_{H \in \mathcal{H}(M)} \mathcal{V}(H),$$

where for each hyperplane  $H$ , the elements of  $\mathcal{V}(H)$  are exactly the consecutive  $r$ -subsets of  $H$ :

$$\mathcal{V}(H) := \left\{ V \in \binom{H}{r} : \text{there are no } v, v' \in V \text{ and } h \in H \setminus V \text{ so that } v < h < v' \right\}.$$

If  $M$  is a sparse paving matroid, then  $\mathcal{V}(M)$  is the collection of circuit-hyperplanes of  $M$ , and hence a stable set in the Johnson graph  $J(E, r)$ . In general, this is not the case, as  $\mathcal{V}(M)$  may contain distinct sets  $V, V'$  so that  $|V \cap V'| = r - 1$ . The occurrence of such sets, however, is relatively restricted; this is the content of the following two lemmas, whose proofs are given in [PvdP17, Section 4.2] as well as [vdP17, Lemmas 5.5.3 and 5.5.5].

**Lemma 4.1.** *Let  $M$  be a paving matroid of rank  $r$  and let  $H, H'$  be distinct hyperplanes of  $M$ . If  $V \in \mathcal{V}(H)$  and  $V' \in \mathcal{V}(H')$ , then  $|V \cap V'| < r - 1$ .*

**Lemma 4.2.** *If  $M$  is a paving matroid of rank  $r$  on a ground set of cardinality  $n$ , then  $|\mathcal{V}(M)| \leq \frac{1}{n-r+1} \binom{n}{r}$ .*

For each hyperplane  $H$ , we may write  $\mathcal{V}(H) = \{V_H^0, \dots, V_H^k\}$  so that  $|V_H^i \cap V_H^j| = r - 1$  if and only if  $i = j \pm 1$ . Consider the partition  $\mathcal{V}(H) = \mathcal{V}^0(H) \dot{\cup} \mathcal{V}^1(H)$  where

$$\mathcal{V}^0(H) = \{V_H^i : i \text{ even}\} \quad \text{and} \quad \mathcal{V}^1(H) = \{V_H^i : i \text{ odd}\}.$$

The collections  $\mathcal{V}^0(H)$  and  $\mathcal{V}^1(H)$  are both stable sets of  $J(E, r)$ . Writing

$$\mathcal{V}^0(M) := \bigcup_{H \in \mathcal{H}(M)} \mathcal{V}^0(H) \quad \text{and} \quad \mathcal{V}^1(M) := \bigcup_{H \in \mathcal{H}(M)} \mathcal{V}^1(H),$$

we evidently have  $\mathcal{V}(M) = \mathcal{V}^0(M) \dot{\cup} \mathcal{V}^1(M)$ , and by Lemma 4.1 both  $\mathcal{V}^0(M)$  and  $\mathcal{V}^1(M)$  are stable sets of  $J(E, r)$ .

We associate two  $(r - 1)$ -shadows with a paving matroid  $M$ :  $\partial_{r-1} \mathcal{V}^0(M)$  and  $\partial_{r-1} \mathcal{V}^1(M)$ . In the remainder of this section, all shadows will be  $(r - 1)$ -shadows, and we suppress the subscript  $r - 1$  in our notation.

**Lemma 4.3.** *Let  $n \geq r \geq 3$ . For each  $M \in \mathbb{P}(n, r)$ ,*

$$|\partial \mathcal{V}^0(M)| + \frac{r-1}{2} |\partial \mathcal{V}^1(M)| \leq \binom{n}{r-1}.$$

*Proof.* Let  $h_k$  denote the number of hyperplanes of  $M$  that contain exactly  $r + k$  elements. As each  $(r - 1)$ -set from  $E(M)$  is contained in a unique hyperplane, we have

$$(5) \quad \sum_{k=0}^{\infty} h_k \binom{r+k}{r-1} \leq \binom{n}{r-1}.$$

Each hyperplane with  $r + k$  elements contributes  $\lfloor k/2 \rfloor + 1$  elements to  $\mathcal{V}^0(M)$  and  $\lfloor k/2 \rfloor$  elements to  $\mathcal{V}^1(M)$ . Hence, writing  $v_i = |\mathcal{V}^i(M)|$ ,  $i \in \{0, 1\}$ ,

$$v_0 = \sum_{k=0}^{\infty} h_k (\lfloor k/2 \rfloor + 1) \quad \text{and} \quad v_1 = \sum_{k=0}^{\infty} h_k \lfloor k/2 \rfloor.$$



As  $r(\lfloor k/2 \rfloor + 1) + r\frac{r-1}{2}\lfloor k/2 \rfloor \leq \binom{r+k}{r-1}$  for all  $k \in \mathbb{N}$ , it follows from (5) that

$$\begin{aligned} r(v_0 + \frac{r-1}{2}v_1) &\leq \sum_{k=0}^{\infty} h_k \left( r(\lfloor k/2 \rfloor + 1) + r\frac{r-1}{2}\lfloor k/2 \rfloor \right) \\ &\leq \sum_{k=0}^{\infty} h_k \binom{r+k}{r-1} \leq \binom{n}{r-1}. \end{aligned}$$

The lemma follows, since  $|\partial\mathcal{V}^i(M)| = rv_i$  for  $i \in \{0, 1\}$ .  $\square$

**4.2. Upper bound.** We now turn to proving the upper bound in Theorem 1.4. Define

$$\mathbb{P}(n, r, \mathcal{A}, \mathcal{B}) := \{M \in \mathbb{P}(n, r) : \partial\mathcal{V}^0(M) = \mathcal{A}, \partial\mathcal{V}^1(M) = \mathcal{B}\}$$

and  $p(n, r, \mathcal{A}, \mathcal{B}) := |\mathbb{P}(n, r, \mathcal{A}, \mathcal{B})|$ . We consider that

$$p(n, r) = \sum_{a=0}^{\binom{n}{r-1}} \sum_{b=0}^{\binom{n}{r-1}} p(n, r, a, b),$$

where  $p(n, r, a, b)$  denotes the sum of  $p(n, r, \mathcal{A}, \mathcal{B})$  over all  $\mathcal{A}, \mathcal{B} \subseteq \binom{[n]}{r}$  such that  $a = |\mathcal{A}|, b = |\mathcal{B}|$ . Note that for  $p(n, r, a, b) > 0$  to hold, both  $a$  and  $b$  are necessarily multiples of  $r$ . We prove sufficient bounds on  $\ln p(n, r, a, b)$  under two complementary regimes.

**Lemma 4.4.** *Let  $r \geq 4$  and  $n \geq \exp\left(\left(\frac{(r-1)(r+1)}{r-3}\right)^2\right)$ . If  $a \leq \left(1 - \frac{1}{\sqrt{\ln(n)}}\right) \binom{n}{r-1}$ , then*

$$\ln p(n, r, a, b) \leq \frac{1}{r} \binom{n}{r-1} (\ln(n-r+1) + 1 - r).$$

*Proof.* Suppose  $\mathcal{A}, \mathcal{B} \subseteq \binom{[n]}{r-1}$  are such that  $|\mathcal{A}| = a$  and  $|\mathcal{B}| = b$ . Each  $M \in \mathbb{P}(n, r, \mathcal{A}, \mathcal{B})$  is determined uniquely by  $\mathcal{V}(M) = \mathcal{V}^0(M) \cup \mathcal{V}^1(M)$ , and we have  $\partial\mathcal{V}^0(M) = \mathcal{A}$  and  $\partial\mathcal{V}^1(M) = \mathcal{B}$ . Let  $\gamma_r = \frac{r-1}{2}$ . By Lemma 4.3,

$$a + \gamma_r b = |\partial\mathcal{V}^0(M)| + \gamma_r |\partial\mathcal{V}^1(M)| \leq \binom{n}{r-1}.$$

Since  $|\mathcal{V}(M)| = |\mathcal{V}^0(M)| + |\mathcal{V}^1(M)| = (a+b)/r$ , and using the assumed upper bound on  $a$ , we have

$$|\mathcal{V}(M)| \leq \frac{1}{r} \binom{n}{r-1} \left(1 - \frac{\delta_r}{\sqrt{\ln(n)}}\right),$$

where  $\delta_r = 1 - \gamma_r^{-1} > 0$ . As each of the paving matroids  $M$  we are counting is determined uniquely by a set  $\mathcal{V}(M) \subseteq \binom{[n]}{r}$  of this bounded cardinality, we obtain

$$\begin{aligned} \ln p(n, r, a, b) &\leq \frac{1}{r} \binom{n}{r-1} \left(1 - \frac{\delta_r}{\sqrt{\ln(n)}}\right) \ln \left(\frac{e(n-r+1)}{1 - \frac{\delta_r}{\sqrt{\ln(n)}}}\right) \\ &= \ln p(n, r, a, b) \leq \frac{1}{r} \binom{n}{r-1} (\ln(n-r+1) + 1 + u_r(n)). \end{aligned}$$

where  $u_r(n) = -\left(1 - \frac{\delta_r}{\sqrt{\ln(n)}}\right) \ln\left(1 - \frac{\delta_r}{\sqrt{\ln(n)}}\right) - \frac{\delta_r}{\sqrt{\ln(n)}} - \frac{\delta_r}{\sqrt{\ln(n)}} \ln(n-r+1)$ . It is straightforward to verify that  $u_r(n) \leq -r$  whenever  $n \geq \exp((r+1)^2 \delta_r^{-2})$ . The lemma follows.  $\square$

**Lemma 4.5.** *For each  $r \geq 4$ , there exists a function  $h_r(n)$  with the property that  $h_r(n) \rightarrow 0$  as  $n \rightarrow \infty$  such that if  $a \geq \left(1 - \frac{1}{\sqrt{\ln(n)}}\right) \binom{n}{r-1}$ , then*

$$\ln p(n, r, a, b) \leq \frac{1}{r} \binom{n}{r-1} (\ln(n-r+1) + 1 - r + h_r(n)).$$

*Proof.* Each  $M \in \mathbb{P}(n, r, \mathcal{A}, \mathcal{B})$  is determined uniquely by the pair  $(\mathcal{V}^0(M), \mathcal{V}^1(M))$ , which is such that  $\partial\mathcal{V}^0(M) = \mathcal{A}$  and  $\partial\mathcal{V}^1(M) = \mathcal{B}$ . The collection  $\mathcal{V}^0(M)$  is the collection of circuit-hyperplanes of a sparse paving matroid  $N \in \mathbb{S}(n, r, \mathcal{A})$ , and  $\mathcal{V}^1(M)$  is a collection of  $|\mathcal{B}|/r$  elements from  $\binom{[n]}{r}$ , which in turn determines  $\mathcal{B}$ . For fixed  $\mathcal{A}$  and  $b$ , it follows that

$$\sum_{\mathcal{B}:|\mathcal{B}|=b} p(n, r, \mathcal{A}, \mathcal{B}) = \sum_{\mathcal{U}:|\mathcal{U}|=b/r} |\{M \in \mathbb{P}(n, r, \mathcal{A}, \partial\mathcal{U}) : \mathcal{V}^1(M) = \mathcal{U}\}| \leq s(n, r, \mathcal{A}) \binom{\binom{n}{r}}{b/r}.$$

By definition of  $p(n, r, a, b)$ , we have

$$p(n, r, a, b) = \sum_{\mathcal{A}:|\mathcal{A}|=a} \sum_{\mathcal{B}:|\mathcal{B}|=b} p(n, r, \mathcal{A}, \mathcal{B}) \leq \sum_{\mathcal{A}:|\mathcal{A}|=a} s(n, r, \mathcal{A}) \binom{\binom{n}{r}}{b/r}.$$

Using the upper bound on  $\ln s(n, r, \mathcal{A})$  from Lemma 3.2, we obtain

$$(6) \quad \ln p(n, r, a, b) \leq \ln \binom{\binom{n}{r-1}}{a} + \frac{a}{r} (\ln(n-r+1) + 1 - r + f_r(n)) + \ln \binom{\binom{n}{r}}{b/r}.$$

By our lower bound on  $a$ , we have

$$(7) \quad \ln \binom{\binom{n}{r-1}}{a} = \ln \left( \frac{\binom{n}{r-1}}{\binom{n}{r-1} - a} \right) \leq \frac{1}{\sqrt{\ln(n)}} \binom{n}{r-1} \ln(e\sqrt{\ln(n)}).$$

If  $p(n, r, a, b) > 0$ , then by Lemma 4.3  $a + \gamma_r b \leq \binom{n}{r-1}$ , where  $\gamma_r = \frac{r-1}{2}$ . By our assumed lower bound on  $a$  it then follows that

$$b \leq \gamma_r^{-1} \left( \binom{n}{r-1} - a \right) \leq \frac{1}{\gamma_r \sqrt{\ln(n)}} \binom{n}{r-1},$$

so that

$$\ln \binom{\binom{n}{r}}{b/r} \leq \frac{1}{\gamma_r r \sqrt{\ln(n)}} \binom{n}{r-1} \ln(e(n-r+1)\sqrt{\ln(n)})$$

With  $\delta := 1 - c_r^{-1}$ , we obtain

$$\ln \binom{\binom{n}{r}}{b/r} \leq \frac{1}{r \sqrt{\ln(n)}} \binom{n}{r-1} (\ln(n-r+1) + 1 + u_r(n)),$$

where  $u_r(n) = \ln(c_r \sqrt{\ln(n)}) - \delta(\ln(n-r+1) + 1 + \ln(c_r \sqrt{\ln(n)}))$ . Since  $\delta$  depends only on  $r$ , there is a constant  $N_r$  depending only on  $r$  so that for all  $n \geq N_r$  we have  $u_r(n) \leq -r$ . Then

$$(8) \quad \ln \binom{\binom{n}{r}}{b/r} \leq \frac{1}{r \sqrt{\ln(n)}} \binom{n}{r-1} (\ln(n-r+1) + 1 - r)$$

for all  $n \geq N_r$ . Combining (6)–(8), we get

$$\ln p(n, r, a, b) \leq \frac{1}{r} \binom{n}{r-1} \left( \ln(n-r+1) + 1 - r + f_r(n) + \frac{r}{\sqrt{\ln(n)}} \ln(e\sqrt{\ln(n)}) \right)$$

for all  $n \geq N_r$ . The term  $\frac{r}{\sqrt{\ln(n)}} \ln(e\sqrt{\ln(n)})$  tends to 0 as  $n \rightarrow \infty$ . Hence any function  $h_r(n)$  such that

$$h_r(n) = f_r(n) + \frac{r}{\sqrt{\ln(n)}} \ln(e\sqrt{\ln(n)})$$

for all  $n \geq N_r$ , and which is sufficiently large for  $n < N_r$  satisfies the requirements of the lemma.  $\square$

We may now complete the proof of Theorem 1.4.

*Proof of Theorem 1.4.* We have

$$p(n, r) = \sum_{a=0}^{\binom{n}{r-1}} \sum_{b=0}^{\binom{n}{r-1}} p(n, r, a, b) \leq \left(1 + \binom{n}{r-1}\right)^2 \max_{a,b} p(n, r, a, b).$$

With the bounds obtained as in Lemmas 4.4 and 4.5,

$$\max\{\ln p(n, r, a, b) : a, b\} \leq \frac{1}{r} \binom{n}{r-1} (\ln(n-r+1) + 1 - r + h_r(n))$$

for all  $n \geq N_r$ , where  $N_r$  the constant of Lemma 4.4 and  $h_r(n)$  is the vanishing function of Lemma 4.5. It follows that

$$\ln p(n, r) \leq 2 \ln \left(1 + \binom{n}{r-1}\right) + \frac{1}{r} \binom{n}{r-1} (\ln(n-r+1) + 1 - r + h_r(n)).$$

As  $r \geq 4$ , the term  $2 \ln \left(1 + \binom{n}{r-1}\right) \leq 2(r-1) \ln(en/(r-1))$  is tiny compared to the upper bound on  $\ln p(n, r, a, b)$ . Theorem 1.4 follows.  $\square$

## 5. PAVING MATROIDS OF RANK 3

**5.1. The result.** In this section, we prove the following upper bound on the number  $p(n, 3)$  of rank-3 paving matroids on a ground set of  $n$  elements.

**Theorem 5.1.** *There exists  $\beta < 0$  such that*

$$\ln p(n, 3) \leq \frac{1}{n-2} \binom{n}{3} \ln(e^{1+\beta} n + o(n)) \quad \text{as } n \rightarrow \infty.$$

Together with the lower bound on  $s(n, 3)$  from Proposition 3.1, Theorem 5.1 implies Theorem 1.5.

We characterise the constant  $\beta$  that appears in the upper bound as the value of a calculus-of-variations problem that we now define. Write  $h(y) = (1-y) \ln(1-y)$  (and  $h(1) = 0$ ). Let  $\Delta := \{(x, y) \in [0, 1]^2 : 0 \leq y \leq \min\{x, 1-x\}\}$ , and define the function  $F: \Delta \rightarrow \mathbb{R}$  by

$$F(x, y) = -2 - 6xh\left(\frac{y}{x}\right) - 6(1-x)h\left(\frac{y}{1-x}\right) - 6y \ln\left(\frac{y}{x(1-x)}\right).$$

Define the functional  $\mathcal{F}[u] = \int_0^1 F(x, u(x)) dx$ . We show that

$$(9) \quad \beta = \sup_{u \in C_{\blacktriangle}^1([0,1])} \mathcal{F}[u],$$

where the supremum is taken over the space  $C_{\blacktriangle}^1([0, 1])$  of all continuously differentiable functions  $u$  on  $[0, 1]$  that satisfy the constraints  $\int_0^1 u(x) dx = 1/6$  and  $0 \leq u(x) \leq \min\{x, 1-x\}$ .

The optimisation problem in (9) can be solved using standard methods from the calculus of variations.

**Lemma 5.2.**  $-0.67 < \beta < -0.65$ .

*Proof.* Maximising  $\mathcal{F}[u]$  subject to the constraint  $\int_0^1 u(x)dx = 1/6$  is a problem of Euler-Lagrange type, and it follows that any extremum must satisfy the Euler-Lagrange equation  $\frac{\partial}{\partial u}F(x, u(x)) = \lambda$ , where  $\lambda$  is a multiplier whose value follows from the constraint. After taking the derivative and rearranging terms, we obtain

$$(10) \quad (x - u)(1 - x - u) = \lambda' u,$$

where  $\lambda' = (\lambda - 6)/6$ . Equation (10) is a quadratic equation with solutions

$$u_{\pm}(x; \lambda') = \frac{1}{2} \left( 1 + \lambda' \pm \sqrt{(1 + \lambda')^2 - 4x(1 - x)} \right).$$

Of the two solutions  $u_{\pm}$ , only  $u_{-}$  satisfies the constraint  $0 \leq u(x) \leq \min\{x, 1 - x\}$ . It remains to find  $\lambda'$  such that  $\int_0^1 u_{-}(x; \lambda')dx = 1/6$ .

The function  $\lambda' \mapsto \int_0^1 u_{-}(x; \lambda')dx$  is continuous and decreasing on  $[0, \infty)$ ; moreover,  $\int_0^1 u_{-}(x; 0)dx = 1/4$ , while  $\lim_{\lambda' \rightarrow \infty} \int_0^1 u_{-}(x; \lambda')dx = 0$ . It follows that there is a unique  $\lambda^*$  for which  $\int_0^1 u_{-}(x; \lambda^*)dx = 1/6$ . Numerical evaluation gives  $0.2 < \lambda^* < 0.21$ .

The function  $\mathcal{F}[u_{-}(\cdot; \lambda')]$  is strictly increasing in  $\lambda'$ , from which it follows that

$$-0.67 < \mathcal{F}[u_{-}(\cdot; 0.2)] < \beta \equiv \mathcal{F}[u_{-}(\cdot; \lambda^*)] < \mathcal{F}[u_{-}(\cdot; 0.21)] < -0.65. \quad \square$$

**5.2. Good sets.** We obtain Theorem 5.1 as a corollary to a stronger result, which we now describe. Call a subset  $\mathcal{X} \subseteq \binom{[n]}{3}$  *good* if

- (i) for any pair of triples  $\{a_1 < a_2 < a_3\}$  and  $\{b_1 < b_2 < b_3\}$  in  $\mathcal{X}$ , if  $a_2 = b_2$ , then  $a_1 \neq b_1$  and  $a_3 \neq b_3$ ; and
- (ii)  $|\mathcal{X}| \leq \frac{1}{n-2} \binom{n}{3}$ .

Let  $g(n)$  be the number of good sets in  $\binom{[n]}{3}$ .

**Theorem 5.3.**  $\ln g(n) = \frac{1}{n-2} \binom{n}{3} \ln(e^{1+\beta} n + o(n))$  as  $n \rightarrow \infty$ .

If  $M$  is a paving matroid of rank 3 on ground set  $E = [n]$ , then  $\mathcal{V}(M)$  (as defined in Section 4.1) is good: The first property follows from Lemma 4.1 and the fact that  $\mathcal{V}(H)$  consists of consecutive subsets of  $H$  for each hyperplane  $H$ , while the second property is Lemma 4.2. As  $\mathcal{V}(M)$  determines  $M$ , it follows that  $p(n, 3) \leq g(n)$ , and hence Theorem 5.3 implies Theorem 5.1.

**5.3. Good sets: asymptotics.** In this section, we outline a proof of Theorem 5.3, postponing some technical details to the next section. Let  $\mathcal{X}$  be a set of  $t$  triples in  $[n]$ , chosen uniformly at random from among all such  $t$ -sets of triples, and write  $\mathbf{P}_{n,t}$  for its law. Write  $\mathcal{G}$  for the event that  $\mathcal{X}$  is good. Set  $T = \frac{1}{n-2} \binom{n}{3}$ ,  $\mathcal{T} \equiv \mathcal{T}_n = \mathbb{Z} \cap [(1 - 4/\ln n)T, T]$ , and  $\overline{\mathcal{T}} = \mathbb{Z} \cap [0, (1 - 4/\ln n)T)$ . Evidently,

$$g(n) = \sum_{t=0}^T \binom{\binom{n}{3}}{t} \mathbf{P}_{n,t}(\mathcal{G}).$$

Using the trivial bound  $0 \leq \mathbf{P}_{n,t}(\mathcal{G}) \leq 1$ ,

$$0 \leq \sum_{t \in \overline{\mathcal{T}}} \binom{\binom{n}{3}}{t} \mathbf{P}_{n,t}(\mathcal{G}) \leq (e^{-3} n + o(n))^T;$$

thus, in order to proof Theorem 5.3, it remains to show that

$$(11) \quad \sum_{t \in \mathcal{T}_n} \binom{\binom{n}{3}}{t} \mathbf{P}_{n,t}(\mathcal{G}) = (e^{1+\beta} n + o(n))^T \quad \text{as } n \rightarrow \infty.$$

Most of the technical work to prove (11) is done in Lemmas 5.4 and 5.5 below, the proofs of which are deferred to the next section.

In what follows, we write  $a_{n,t} \asymp b_{n,t}$  if

$$\max_{t \in \mathcal{T}_n} \left| \frac{1}{t} \ln a_{n,t} - \frac{1}{t} \ln b_{n,t} \right| = O\left(\frac{\ln n}{n}\right) \quad \text{as } n \rightarrow \infty.$$

Let

$$\mathcal{Z}_{n,t} = \left\{ \vec{z} = (z_2, \dots, z_{n-1}) \in \mathbb{Z}^{n-2} : \begin{array}{l} 0 \leq z_i \leq \min\{i-1, n-i\} \text{ for all } i, \\ \sum_{i=2}^{n-1} z_i = t \end{array} \right\}.$$

Recall that  $h(y) = (1-y) \ln(1-y)$  for  $0 \leq y < 1$  and  $h(1) = 0$ . Define

$$f_{n,t}(\vec{z}) = -2 - \frac{1}{t} \sum_{i=2}^{n-1} \left[ (i-1) h\left(\frac{z_i}{i-1}\right) + (n-i) h\left(\frac{z_i}{n-i}\right) + z_i \ln\left(\frac{z_i/t}{(i-1)(n-i)/N}\right) \right].$$

**Lemma 5.4.**  $\mathbf{P}_{n,t}(\mathcal{G}) \asymp \exp\left(t \max_{\vec{z} \in \mathcal{Z}_{n,t}} f_{n,t}(\vec{z})\right)$ .

**Lemma 5.5.**  $\limsup_{n \rightarrow \infty} \max_{t \in \mathcal{T}_n} \max_{\vec{z} \in \mathcal{Z}_{n,t}} f_{n,t}(\vec{z}) = \liminf_{n \rightarrow \infty} \max_{T-n+2 \leq t \leq T} \max_{\vec{z} \in \mathcal{Z}_{n,t}} f_{n,t}(\vec{z}) = \beta$ .

We are now ready to prove Theorem 5.3 subject to Lemmas 5.4 and 5.5.

*Proof of Theorem 5.3.* As indicated before, it remains to prove (11). Let  $0 < \varepsilon < |\beta|$  be given. We first turn to proving the upper bound. Let  $N \geq \exp(4|\beta|/\varepsilon)$  be so large that  $\mathbf{P}_{n,t}(\mathcal{G}) \leq e^{t(\beta+\varepsilon/2)}$  for all  $t \in \mathcal{T}_n$  whenever  $n \geq N$ ; such an  $N$  exists by Lemmas 5.4 and 5.5. For  $n \geq N$ , we find

$$(12) \quad \sum_{t \in \mathcal{T}_n} \binom{\binom{n}{3}}{t} \leq (e(n-2))^T e^{T(1-4/\ln n)(\beta+\varepsilon/2)} \leq e^{T(1+\beta+\varepsilon+\ln(n-2))}.$$

We turn to proving the lower bound. For  $n = 1, 2, \dots$ , let  $\tau_n$  be a maximiser of  $t \mapsto \max_{\vec{z} \in \mathcal{Z}_{n,t}} f_{n,t}(\vec{z})$  on  $\mathbb{Z} \cap [T-n+2, T]$ . Let  $N'$  be so large that  $\mathbf{P}_{n,\tau_n}(\mathcal{G}) \geq e^{\tau_n(\beta-\varepsilon/3)}$ ,  $(n-2)(1+\ln(n-2))/T \leq \varepsilon/3$ , and  $(T-n+2)^{-1} \ln(T-n+2) < \varepsilon/3$  whenever  $n \geq N'$ . For  $n \geq N'$ , we find, using Lemma 2.1,

$$(13) \quad \begin{aligned} \sum_{t \in \mathcal{T}_n} \binom{\binom{n}{3}}{t} \mathbf{P}_{n,t}(\mathcal{G}) &\geq \binom{\binom{n}{3}}{\tau_n} \mathbf{P}_{n,\tau_n}(\mathcal{G}) \geq \binom{\binom{n}{3}}{\tau_n} e^{\tau_n(\beta-\varepsilon/3)} \\ &\geq e^{(T-n+2)(1+\beta-\frac{2}{3}\varepsilon+\ln(n-2))} \geq e^{T(1+\beta-\varepsilon+\ln(n-2))}. \end{aligned}$$

The theorem now follows as (12)–(13) hold whenever  $n \geq \max\{N, N'\}$ , and  $\varepsilon$  is arbitrarily small.  $\square$

**5.4. Good sets: details.** In this section, we prove Lemmas 5.4 and 5.5, thus finishing the proof of Theorem 5.3.

*Proof of Lemma 5.4.* Recall that  $\mathcal{X}$  is chosen uniformly at random from the collections of  $t$  triples in  $[n]$ , and that  $\mathcal{G}$  denotes the event that  $\mathcal{X}$  is good. For  $i = 2, \dots, n-1$ , let  $Z_i$  denote the number of triples in  $\mathcal{X}$  whose middle element is  $i$ , and write  $\vec{Z} = (Z_2, \dots, Z_{n-1})$ . It is easily verified that if  $\mathcal{X}$  is good, then  $\vec{Z} \in \mathcal{Z}_{n,t}$ .

By conditioning on  $\vec{Z}$ , we obtain

$$\mathbf{P}_{n,t}(\mathcal{G}) = \sum_{\vec{z} \in \mathcal{Z}_{n,t}} \mathbf{P}_{n,t}(\mathcal{G} \mid \vec{Z} = \vec{z}) \mathbf{P}_{n,t}(\vec{Z} = \vec{z}).$$

As  $|\mathcal{Z}_{n,t}| \leq t^{n-2}$  and  $\frac{1}{t} \log t^{n-2} = O\left(\frac{\log n}{n}\right)$  uniformly in  $t \in \mathcal{T}_n$  as  $n \rightarrow \infty$ , it follows that

$$(14) \quad \mathbf{P}_{n,t}(\mathcal{G}) \asymp \max_{\vec{z} \in \mathcal{Z}_{n,t}} \mathbf{P}_{n,t}(\mathcal{G} \mid \vec{Z} = \vec{z}) \mathbf{P}_{n,t}(\vec{Z} = \vec{z}).$$

We start by analysing the second factor. The random variable  $\vec{Z}$  has a multivariate hypergeometric distribution, so that (writing  $k_i = (i-1)(n-i)$ )

$$\mathbf{P}_{n,t}(\vec{Z} = \vec{z}) = \binom{N}{t}^{-1} \prod_{i=2}^{n-1} \binom{k_i}{z_i}, \quad 0 \leq z_i \leq k_i.$$

Using Stirling's approximation (1),

$$\begin{aligned} & \left| \frac{1}{t} \ln \mathbf{P}_{n,t}(\vec{Z} = \vec{z}) + \sum_{i:z_i>0} \frac{z_i}{t} \ln \left( \frac{z_i/t}{k_i/N} \right) \right| \\ & \leq \frac{N-t}{t} \left| \sum_{i:z_i>0} \frac{k_i - z_i}{N-t} \ln \left( \frac{(k_i - z_i)/(N-t)}{k_i/N} \right) \right| \\ & \quad + \frac{1}{2t} \left| \ln \left( \frac{t(N-t)}{N} \right) + \sum_{i:z_i>0} \ln \left( \frac{k_i}{z_i(k_i - z_i)} \right) \right| + C \frac{n-1}{t}, \end{aligned}$$

where  $C = 3 \ln e / \sqrt{2\pi}$ . In particular, there exists a constant  $c > 0$  such that

$$(15) \quad \left| \frac{1}{t} \ln \mathbf{P}_{n,t}(\vec{Z} = \vec{z}) + \sum_{i=2}^{n-1} \frac{z_i}{t} \ln \left( \frac{z_i/t}{k_i/N} \right) \right| \leq \frac{c \ln n}{n},$$

for all  $n$ , and for all  $t \in \mathcal{T}_n$  and  $\vec{z} \in \mathcal{Z}_{n,t}$ .

Finally, we show that

$$(16) \quad \left| \ln \mathbf{P}_{n,t}(\mathcal{G} \mid \vec{Z} = \vec{z}) - \sum_{i=2}^{n-1} \left[ -2z_i - (i-1)h \left( \frac{z_i}{i-1} \right) - (n-i)h \left( \frac{z_i}{n-i} \right) \right] \right| \leq 4n \ln(n),$$

which, together with (15) and (14) proves the lemma.

Write  $\mathcal{G}_i$  for the event that the triples with central element  $\leq i$  are good. By the chain rule for probabilities,

$$\mathbf{P}_{n,t}(\mathcal{G} | \vec{\mathcal{Z}} = \vec{z}) = \mathbf{P}_{n,t}\left(\bigcap_{i=2}^{n-1} \mathcal{G}_i \mid \vec{\mathcal{Z}} = \vec{z}\right) = \prod_{i=2}^{n-1} \mathbf{P}_{n,t}\left(\mathcal{G}_i \mid \bigcap_{j<i} \mathcal{G}_j, \vec{\mathcal{Z}} = \vec{z}\right).$$

Fix  $2 \leq i \leq n-1$ . Given  $\mathcal{G}_j$  for all  $j < i$  and  $\vec{\mathcal{Z}} = \vec{z}$ ,  $\mathcal{G}_i$  holds if and only if  $\mathcal{X}_i = \{\{a_1, a_2, a_3\} \in \mathcal{X} : a_2 = i\}$  is good. Each triple in  $\mathcal{X}_i$  is specified by selecting an element that is smaller than  $i$  and an element that is larger than  $i$ , and each of these elements has to be distinct. Thus, there are  $(i-1)_{z_i}(n-i)_{z_i}$  ways of selecting the  $z_i$  triples with central element  $i$ , where we use  $(x)_k = x(x-1)\cdots(x-k+1)$  to denote the falling factorial. It follows that

$$\mathbf{P}_{n,t}\left(\mathcal{G}_i \mid \bigcap_{j<i} \mathcal{G}_j, \vec{\mathcal{Z}} = \vec{z}\right) = \frac{(i-1)_{z_i}(n-i)_{z_i}}{(i-1)^{z_i}(n-i)^{z_i}} = \prod_{k=0}^{z_i-1} \left(1 - \frac{k}{i-1}\right) \left(1 - \frac{k}{n-i}\right).$$

and hence, upon taking logarithms,

$$\ln \mathbf{P}_{n,t}(\mathcal{G} | \vec{\mathcal{Z}} = \vec{z}) = \sum_{i=2}^{n-1} \sum_{k=0}^{z_i-1} \left[ \ln \left(1 - \frac{k}{i-1}\right) + \ln \left(1 - \frac{k}{n-i}\right) \right].$$

Fix  $i$  and  $m \in \{i-1, n-i\}$ . By concavity of the function  $x \mapsto \ln(1-x/m)$ ,

$$\left| \ln \left(1 - \frac{k}{m}\right) - \int_k^{k+1} \ln \left(1 - \frac{x}{m}\right) dx \right| \leq \varepsilon_{m,k},$$

where

$$\varepsilon_{m,k} = \begin{cases} \frac{1}{2} \left[ \ln \left(1 - \frac{k}{m}\right) - \ln \left(1 - \frac{k+1}{m}\right) \right] & \text{if } k = 0, 1, \dots, m-2 \\ 1 & \text{if } k = m-1. \end{cases}$$

Due to the telescoping nature of the  $\varepsilon_{m,k}$ , upon summing over  $k$ , we obtain

$$\left| \sum_{k=0}^{z_i-1} \ln \left(1 - \frac{k}{m}\right) - z_i \int_0^1 \ln \left(1 - \frac{xz_i}{m}\right) dx \right| \leq \sum_{k=0}^{m-1} \varepsilon_{m,k} \leq 1 + \frac{\ln m}{2} \leq 2 \ln n,$$

Using that  $\int_0^1 \ln(1-\alpha x) dx = -1 - \frac{1-\alpha}{\alpha} \ln(1-\alpha)$ , and summing over  $m$ , this proves (16), and hence completes the proof of the lemma.  $\square$

Before proving Lemma 5.5, we require two additional technical results, that relate the discrete optimisation problem of Lemma 5.7 to the continuous optimisation problem (9).

Starting from  $\vec{z} \in \mathcal{Z}_{n,t}$ , define the step function  $z$  associated with  $\vec{z}$  by

$$z(x) = \begin{cases} 0 & \text{if } x \leq \frac{1}{n} \text{ or } x > 1 - \frac{1}{n}, \\ \frac{z_i}{n} & \text{if } \frac{i-1}{n} < x \leq \frac{i}{n}. \end{cases}$$

Writing  $i_n(x) = \lceil xn \rceil$ , it follows that  $z(x) = z_{i_n(x)}/n$  (whenever  $z_{i_n(x)}$  exists).

**Lemma 5.6.** *For all  $\varepsilon > 0$ , there exists  $N^{(5.6)} \equiv N^{(5.6)}(\varepsilon)$  such that for all  $n \geq N^{(5.6)}$ ,  $t \in \mathcal{T}_n$ , and  $\vec{z} \in \mathcal{Z}_{n,t}$ , if  $z$  is the step function associated with  $\vec{z}$ , then  $|f_{n,t}(\vec{z}) - \mathcal{F}[z]| < \varepsilon$ .*

*Proof.* Replacing the sum by an integral, we have

$$f_{n,t}(\vec{z}) = -2 - \frac{n^2}{t} \int_0^1 \frac{i_n(x) - 1}{n} g\left(\frac{z(x)n}{i_n(x) - 1}\right) + \frac{n - i_n(x)}{n} g\left(\frac{z(x)n}{n - i_n(x)}\right) \\ + z(x) \ln \frac{z(x)n(n-2)}{(i_n(x) - 1)(n - i_n(x))} dx.$$

By continuity of the integrand, it follows that, for all  $\vec{z} \in \mathcal{Z}_{n,t}$ ,

$$|f_{n,t}(\vec{z}) - \mathcal{F}[z]| < \varepsilon$$

provided that  $n$  is sufficiently large.  $\square$

The next lemma shows that  $f_{n,t}(\vec{z})$  can be approximated to arbitrary precision by the functional  $\mathcal{F}$ . Recall that  $C_{\blacktriangle}^1([0, 1])$  is the space of all continuously differentiable functions  $u: [0, 1] \rightarrow \mathbb{R}$  that satisfy the constraints  $\int_0^1 u(x) dx = 1/6$  and  $0 \leq u(x) \leq \min\{x, 1 - x\}$ .

**Lemma 5.7.** *For all  $\varepsilon > 0$  there exists  $N^{(5.7)} \equiv N^{(5.7)}(\varepsilon)$  such that for all  $n \geq N^{(5.7)}$ , all  $t \in \mathcal{T}_n$ , and all  $\vec{z} \in \mathcal{Z}_{n,t}$  there exists  $\mathfrak{z} \in C_{\blacktriangle}^1([0, 1])$  such that  $|f_{n,t}(\vec{z}) - \mathcal{F}[\mathfrak{z}]| < \varepsilon$ .*

*Proof.* We construct  $\mathfrak{z}$  in three steps. In the first step, we construct an approximation of  $\vec{z}$  by a step function  $z$ . In the second step, we tweak  $z$  so that its integral evaluates to  $1/6$  which yields another function  $\hat{z}$ . In the third step, we smooth  $\hat{z}$  using convolution to obtain  $\mathfrak{z}$ .

Step 1. Let  $z$  be the step function associated with  $\vec{z}$ . By Lemma 5.6, we can ensure that

$$(17) \quad \text{for all } n \geq N^{(5.6)}(\varepsilon/3): \quad |f_{n,t}(\vec{z}) - \mathcal{F}[z]| < \varepsilon/3.$$

Step 2. Note that  $I_1 := \int_0^1 z(x) dx = t/n^2 < 1/6$ . Let  $I_2 := 1/2 - 1/n^2$ , and let  $\lambda$  be such that  $(1 - \lambda)I_1 + \lambda I_2 = 1/6$ . For large  $n$ ,  $0 \leq \lambda < 5/\ln n \leq 1$ . Define

$$\hat{z}(x) = \begin{cases} 0 & \text{if } x \leq \frac{1}{n} \text{ or } x > 1 - \frac{1}{n}, \\ (1 - \lambda)z(x) + \lambda \min\{x, 1 - x\} & \text{otherwise.} \end{cases}$$

By construction,  $\int_0^1 \hat{z}(x) dx = 1/6$ , while

$$0 \leq z(x) \leq \hat{z}(x) \leq \min\{x, 1 - x\} \quad \text{for all } x \in [0, 1],$$

and the pointwise difference between  $z$  and  $\hat{z}$  satisfies

$$|z(x) - \hat{z}(x)| \leq \lambda < \frac{5}{\ln n} \quad \text{for all } x \in [0, 1].$$

Hence, by uniform continuity, there exists  $N_1^{(5.7)}(\varepsilon)$  such that

$$(18) \quad \text{for all } n \geq N_1^{(5.7)}(\varepsilon): \quad |\mathcal{F}[z] - \mathcal{F}[\hat{z}]| < \varepsilon/3.$$

Step 3. Define

$$K_{\delta}(y) = \begin{cases} \frac{\pi}{4\delta} \cos\left(\frac{\pi}{2\delta}x\right) & \text{if } |x| \leq \delta \\ 0 & \text{otherwise.} \end{cases}$$



Note that  $K_\delta$  is smooth, nonnegative, and has support  $(-\delta, \delta)$ . Define  $\mathfrak{z} = \hat{z} * K_{1/n^2}$ , i.e.

$$\mathfrak{z}(x) = \int_{-\infty}^{\infty} \hat{z}(x-y)K_{1/n^2}(y)dy, \quad x \in [0, 1],$$

where, for convenience, we use  $\hat{z}(x) = 0$  whenever  $x < 0$  or  $x > 1$ . The following properties of  $\mathfrak{z}$  follow from elementary properties of convolutions:

- (a)  $\mathfrak{z}$  is smooth on  $[0, 1]$ , and
- (b)  $\int_0^1 \mathfrak{z}(x)dx = \int_0^1 \hat{z}(x)dx = 1/6$ .

Moreover, since  $0 \leq \hat{z}(x) \leq \min\{x, 1-x\}$  for all  $x \in [0, 1]$  and  $K_\delta(y)$  is symmetric about  $y = 0$ ,

- (c)  $0 \leq \mathfrak{z}(x) \leq \min\{x, 1-x\}$  for all  $x \in [0, 1]$ .

Thus,  $\mathfrak{z} \in C_{\blacktriangle}^1([0, 1])$ .

By construction,  $\mathfrak{z}(x) = \hat{z}(x)$  for all  $x$  except for a set of (Lebesgue) measure at most  $c_2/n^2$ . It follows that there exists  $N_2^{(5.7)}(\varepsilon)$  such that

$$(19) \quad \text{for all } n \geq N_2^{(5.7)}(\varepsilon): \quad |\mathcal{F}[\hat{z}] - \mathcal{F}[\mathfrak{z}]| < \varepsilon/3.$$

The lemma holds with  $N_2^{(5.7)}(\varepsilon) := \max\{N^{(5.6)}(\varepsilon/3), N_1^{(5.7)}(\varepsilon), N_2^{(5.7)}(\varepsilon)\}$ , as (17)–(19) imply that  $|f_{n,t}(\vec{z}) - \mathcal{F}[\mathfrak{z}]| < \varepsilon$  whenever  $n \geq N^{(5.7)}(\varepsilon)$ .  $\square$

We are now ready to prove Lemma 5.5.

*Proof of Lemma 5.5.* Let  $\varepsilon > 0$  be given. By Lemma 5.7, if  $n \geq N^{(5.7)}(\varepsilon)$ , then for all  $t \in \mathcal{T}_n$  and  $\vec{z} \in \mathcal{Z}_{n,t}$ , there exists  $\mathfrak{z} \in C_{\blacktriangle}^1([0, 1])$  such that

$$f_{n,t}(\vec{z}) \leq \mathcal{F}[\mathfrak{z}] + \varepsilon \leq \beta + \varepsilon.$$

As the right-hand side does not depend on  $n$ ,  $t$ , or  $\vec{z}$ , this proves the upper bound in the lemma.

We now turn to proving the corresponding lower bound. Let  $\mathfrak{z}$  be such that  $\mathcal{F}[\mathfrak{z}] > \beta - \varepsilon/3$ . For given  $n \geq 3$ , define the sequence  $\vec{z} = (z_2, \dots, z_n)$  as

$$z_i = \begin{cases} \left\lfloor 6T \int_0^{2/n} \mathfrak{z}(x)dx \right\rfloor & \text{if } i = 2, \\ \left\lfloor 6T \int_{1-2/n}^1 \mathfrak{z}(x)dx \right\rfloor & \text{if } i = n-1, \\ \left\lfloor 6T \int_{(i-1)/n}^{i/n} \mathfrak{z}(x)dx \right\rfloor & \text{otherwise,} \end{cases}$$

and set  $t = \sum_{i=2}^{n-1} z_i$ . It is easily verified that  $T - n + 2 \leq t \leq T$  and that  $\vec{z} \in \mathcal{Z}_{n,t}$ . Let  $z$  be the step function associated with  $\vec{z}$ . By Lemma 5.6,  $|f_{n,t}(\vec{z}) - \mathcal{F}[z]| < \varepsilon/3$  whenever  $n \geq N^{(5.6)}(\varepsilon/3)$ . Since  $\mathfrak{z}$  is continuously differentiable on a compact set, it has bounded derivative; using a Taylor expansion of  $\mathfrak{z}$  around  $x$ , we find that there is a constant  $c > 0$  such that  $|z(x) - \mathfrak{z}(x)| \leq c/n$  for all  $x \in [0, 1]$ . By continuity, there exists  $N^{5.5}(\varepsilon)$  such that  $|\mathcal{F}[z] - \mathcal{F}[\mathfrak{z}]| < \varepsilon/3$  for all  $n \geq N^{5.5}(\varepsilon)$ . Combining the three estimates, we find that

$$|f_{n,t}(\vec{z}) - \beta| \leq |f_{n,t}(\vec{z}) - \mathcal{F}[z]| + |\mathcal{F}[z] - \mathcal{F}[\mathfrak{z}]| + |\mathcal{F}[\mathfrak{z}] - \beta| < \varepsilon.$$

It follows that for  $n \geq \max\{N^{(5.6)}(\varepsilon/3), N^{5.5}(\varepsilon)\}$ , there exist  $t$ ,  $T - n + 2 \leq t \leq T$  and  $\vec{z} \in \mathcal{Z}_{n,t}$  such that  $f_{n,t}(\vec{z}) \geq \beta - \varepsilon$ ; this proves the lower bound.  $\square$

## 6. FINAL REMARKS

We have established tight bounds on the number of paving matroids. With the aid of Theorem 1.1, we may derive upper bounds on the number of matroids  $m(n, r)$ . For fixed rank  $r \geq 4$ , we obtain from Theorem 1.4 that

$$\ln m(n, r) \leq \frac{1}{n-r+1} \binom{n}{r} (\ln(n-r+1) + 1 - r + o(1)) \text{ as } n \rightarrow \infty.$$

Trivially  $\ln m(n, r) \geq \ln p(n, r) \geq \ln s(n, r)$ , and

$$\ln s(n, r) = \frac{1}{n-r+1} \binom{n}{r} (\ln(n-r+1) + 1 - r + o(1)) \text{ as } n \rightarrow \infty,$$

The case where  $r \geq 4$  is therefore settled at this level of precision. For rank  $r = 3$  a greater gap remains, since from Theorem 1.5 we have an upper bound

$$\ln m(n, r) \approx \ln p(n, r) \leq \frac{1}{n-r+1} \binom{n}{r} (\ln(n-r+1) + c) \text{ as } n \rightarrow \infty,$$

where  $c = .35 > -2 = 1 - r$ . We are not entirely convinced that the constant  $c$  in our upper bound is best possible, but we do think that in rank 3 the gap between  $p(n, r)$  and  $s(n, r)$  is more pronounced than in higher rank. Specifically, let  $p_k(n, r)$  denote the number of paving matroids without hyperplanes of cardinality  $> r + k$ . The techniques from Section 4 show that  $p(n, r) \approx p_0(n, r) = s(n, r)$  if  $r > 3$ , but not if  $r = 3$ . We conjecture that this reflects the following underlying truth.

**Conjecture 6.1.** *Let  $r = 3$ . There is a constant  $c > -2$  such that*

$$\ln p(n, r) \approx \ln p_1(n, r) = \frac{1}{n-r+1} \binom{n}{r} (\ln(n-r+1) + c + o(1)) \text{ as } n \rightarrow \infty.$$

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