Information theory and its application to optical communication

Information Theory and its Application to Optical Communication

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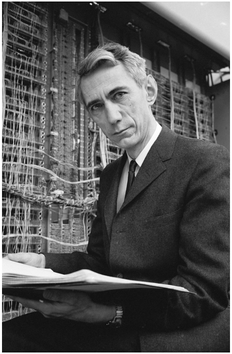
Introduction

- Optical communication links carry most of the data that is transmitted around the world. Home connections are being replaced by optical links.
- We like to achieve the **highest possible data rates for the smallest cost**. **Replacing links should be delayed** as long as possible.
- Therefore **advance transmission protocols** (equalisation, modulation, coding) are required.
- **INFORMATION THEORY tells us what the ultimate performances are (e.g. capacity), and what the techniques are that achieve ultimate performance.**
- Wireless communication is characterized by major developments (coding, mimo, cooperative communications, etc.), often boosted by information theoretical methods.
- **Optical Communication is going through a similar innovation cycle now. Information theory can also be useful here.**

Claude Shannon (1916-2001)



- **1948: "A Mathematical Theory of Communication," Bell Syst. Tech. J.:** Shannon combined the noise power spectral density $N_0/2$, the channel bandwidth W, and the transmit power P, into a single parameter C, which he called the **channel capacity**. More precisely
$$C = W \log_2(1 + \frac{P}{N_0 W})$$
represents the maximum number of bits that can be sent per second **reliably** from transmitter to receiver. **Codes** can be used to achieve capacity.
- **1938:** Shannon also applied **Boole's algebra to switching circuits** (MSc thesis, MIT).
- **WW2:** Shannon developed **cryptographic equipment** for transoceanic conferences (Roosevelt-Churchill). His ideas can be found in **"Communication Theory of Secrecy Systems"**, a confidential report from 1945, published in 1949.
- **1949:** Shannon introduced the **sampling theorem** to the engineering community.

Entropy

Let X be a **discrete random variable** with alphabet $\mathcal{X}$ and **probability mass function** $p(x) = \Pr\{X = x\}$ for $x \in \mathcal{X}$.

Definition

The **entropy** $H(X)$ of discrete random variable X is defined as

$$H(X) = \sum_{x \in \mathcal{X}} p(x) \log_2 \frac{1}{p(x)} \quad \text{[bit]}.$$

Example

Binary random variable X with alphabet $\mathcal{X} = \{0, 1\}$. Let

$$X = \begin{cases} 0 \text{ with probability } 1 - p, \\ 1 \text{ with probability } p. \end{cases}$$

Then the entropy of X is

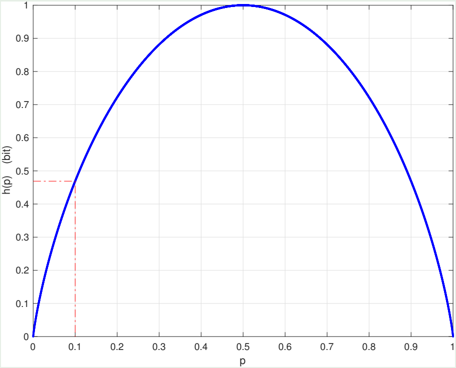
$$H(X) = h(p),$$

where

$$h(p) \stackrel{\Delta}{=} (1-p)\log_2 \frac{1}{1-p} + p \log_2 \frac{1}{p} \quad \text{[bit]}.$$

Entropy, Binary Entropy Function

Example



Observe that $h(p) = h(1-p)$ and that e.g. $h(0.1) = 0.4690$.

- Think of $H(X)$ as the **uncertainty** in X.
- It can be shown that $0 \leq H(X) \leq \log_2 |\mathcal{X}|$.

Conditional Entropy

Let X and y be **discrete random variables** with alphabets $\mathcal{X}$ and $\mathcal{Y}$ respectively and **joint probability mass function** $p(x,y) = \Pr\{X = x, Y = y\}$ for $x \in \mathcal{X}$ and $x \in \mathcal{Y}$.
Note that $p(y) = \sum_{x\in\mathcal{X}} p(x,y)$ and $p(x|y) = p(x,y)/p(y)$.

Definition

The **conditional entropy** $H(X|Y)$ of discrete random variable X given Y is defined as

$$H(X|Y) = \sum_{y\in\mathcal{Y}} p(y) \sum_{x\in\mathcal{X}} p(x|y) \log_2 \frac{1}{p(x|y)} \quad \text{[bit]}.$$

- Think of $H(X|Y)$ as the **uncertainty** in X when Y is given.
- It can be shown that $0 \leq H(X|Y) \leq H(X)$. Conditioning can only reduce entropy.
- Note also that

$$H(X|Y) = \sum_{y\in\mathcal{Y}} p(y) H(X|Y=y),$$

where

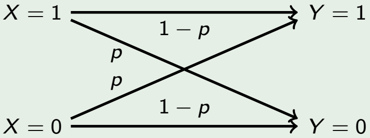
$$H(X|Y=y) = \sum_{x\in\mathcal{X}} p(x|y) \log_2 \frac{1}{p(x|y)}.$$

Conditional entropy $H(X|Y)$ is the expected value of entropies $H(X|Y=y)$ w.r.t. $p(y)$.

Example: Binary Symmetric Channel (BSC)

Example

- **Transition probabilities** $p(Y=1|X=0)=p(Y=0|X=1)=p$.



- For **uniform** $p(X=0)=p(X=1)=1/2$ we obtain that the **entropy**

$$H(X)=h(1/2)=1.$$

- Moreover:

$$\begin{aligned} p(Y=1) &= p(X=0)\cdot p + p(X=1)\cdot(1-p) = 1/2, \\ p(X=1|Y=0) &= \frac{p(X=1,Y=0)}{p(Y=0)} = \frac{p(X=1)\cdot p}{p(Y=0)} = p, \\ p(X=1|Y=1) &= \frac{p(X=1,Y=1)}{p(Y=1)} = \frac{p(X=1)\cdot(1-p)}{p(Y=1)} = 1-p, \end{aligned}$$

and the **conditional entropy**

$$H(X|Y)=p(Y=0)h(p)+p(Y=1)h(1-p)=h(p).$$

Mutual Information

Definition

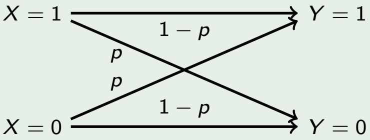
The **mutual information** $I(X;Y)$ between the discrete random variable X and Y is defined as

$$I(X;Y) = H(X) - H(X|Y) \quad \text{[bit]}.$$

- Think of $I(X;Y)$ as the **decrease in uncertainty** about X when Y is released. Equivalently it is the information that Y contains about X.
- It can be shown that always $0 \leq I(X;Y) \leq H(X)$.
- $I(X;Y)$ is also the **decrease in uncertainty** about Y when X is released.

Example

- Binary symmetric channel (BSC) with transition probability p:



- For **uniform** $p(X=0) = p(X=1) = 1/2$, we obtain that

$$I(X;Y) = H(X) - H(X|Y) = 1 - h(p).$$

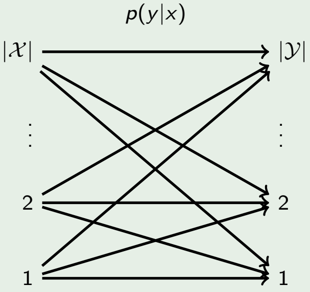
For $p = 0.1$ we obtain $I(X;Y) = 1 - 0.4690 = 0.5310$ bit.

Discrete Memoryless Channel

Definition

Channel **input** alphabet $\mathcal{X}$, channel **output** alphabet $\mathcal{Y}$.
For each $x \in \mathcal{X}$ the **transition probabilities** $\Pr\{Y = y|X = x\}$ for $y \in \mathcal{Y}$ are denoted by $p(y|x)$, where $\sum_{y\in\mathcal{Y}} p(y|x) = 1$.

Example



Here $\mathcal{X} \triangleq \{1, 2, \cdots, |\mathcal{X}|\}$ and $\mathcal{Y} \triangleq \{1, 2, \cdots, |\mathcal{Y}|\}$.

Channel Capacity, Definition

Observe that:

- the **channel input distribution** $\{p(x), x \in \mathcal{X}\}$ determines the joint distribution

$$p(x,y) = p(x)p(y|x), \text{ for all } x \in \mathcal{X}, y \in \mathcal{Y},$$

- and therefore the **mutual information**

$$I(X;Y) = H(X) - H(X|Y).$$

- The maximum value of $I(X;Y)$ is called the **channel capacity** C. Hence

Definition

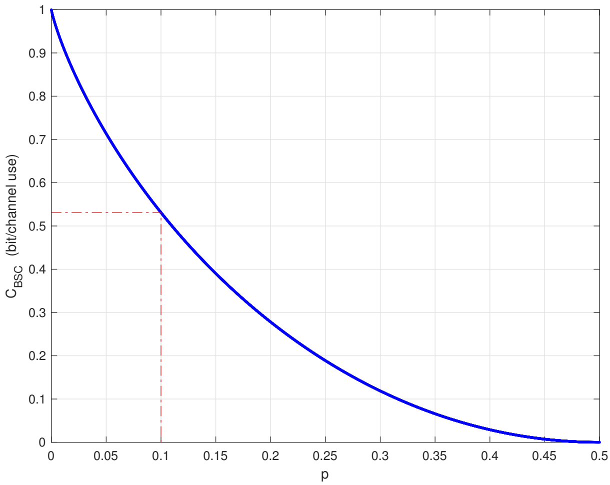
$$C_{\text{DMC}} = \max_{p(x)} I(X;Y) \quad \text{[bit/channel use]}.$$

Example

For a **BSC with crossover probability** p a **uniform input** $p(X=0) = p(X=1) = 1/2$ achieves maximum mutual information $1 - h(p)$. The **channel capacity** of the BSC is therefore

$$C_{\text{BSC}}(p) = 1 - h(p).$$

Capacity of the BSC



Channel Capacity, Operational Meaning

- **Message index** m assumes values in $\{1, 2, \cdots, |M|\}$, uniformly.
- There is a **codeword** $x_1x_2\cdots x_N$ of length X for each message index w.
- The codeword is **transmitted over the DMC** with transition probabilities $\{p(y|x), x \in \mathcal{X}, y \in \mathcal{Y}\}$.
- The receiver makes an **estimate** $\widehat{m}$ of the transmitted index m from the channel output $y_1y_2, \cdots y_N$.
- **Transmission rate**

$$\frac{1}{N}\log_2|M|.$$

- **Error probability**

$$P_e = \Pr\{\widehat{M} \neq M\}.$$

Theorem (Shannon, 1948)

- *Rate R is said to be* **achievable** *if, for any $\varepsilon > 0$, for all large enough N, there exist codes with operational rate $\frac{1}{N}\log_2|M| \geq R - \varepsilon$ and error probability $P_e \leq \varepsilon$.*
- *Rates R not exceeding C are achievable. Rates R larger than C are not achievable.*

Channel Capacity, Proof

1. **RANDOM CODING ARGUMENT**
   Shannon showed that if the $|M|$ codewords are generated **at random according to the capacity-achieving input distribution** $\{p(x), x \in \mathcal{X}\}$ the error probability averaged over the ensemble of codes

$$\overline{P_e} = \sum_{\text{all codes}} P(\text{code}) P_e(\text{code})$$

   can be made arbitrarily small for $\frac{1}{N}\log_2|M| = C - \varepsilon$ and $N \to \infty$, for any $\varepsilon > 0$.
   There **exist** codes with arbitrarily small P_e therefore.

2. **CONVERSE**
   Using **Fano's inequality**[1] it can be shown that for $\frac{1}{N}\log_2|M| > C + \delta$ the error probability P_e can not be made arbitrarily small for large N, for any $\delta > 0$.

3. **LINEAR CODES** Hamming introduced Hamming codes. Elias [1955] demonstrated that **for the BSC also parity check codes achieve capacity.**

[1] $H(W|\widehat{W}) \leq h(P_e) + P_e \log_2(|M| - 1)$

Linear Error-Correcting Code, Syndrome

A **linear code** is defined by its **generator-matrix** G or by the corresponding **parity-check matrix** H.

Codewords are **linear combinations of the rows** of generator matrix G. There are 2^K codewords. With the parity-check matrix H it can be **checked** whether $\underline{x} = (x_1, x_2, \cdots, x_N)$ is a codeword or not.

N is the length of the codewords and K the number of rows in G. Now $N - K$ is the number of rows in H, which is the number of parity-check equations.

Example

Hamming code, $N = 7$, $K = 4$, can correct a **single error**.

$$G = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}, H = \begin{pmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

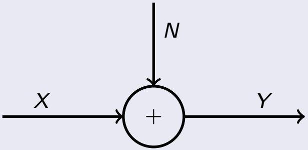
If $\underline{x}H^T = \underline{0}$ then $\underline{x}$ must be a codeword in our code. For non-codewords the so-called **syndrome** of $\underline{x}$

$$\underline{s} = \underline{x}H^T \neq \underline{0}.$$

This syndrome $\underline{s} = (s_1, s_2, s_3)$ can assume eight different values, $(0,0,0)$ when $\underline{x}$ is a codeword, $(1,1,1)$, when there is an error at position 1, $(1,1,0)$, when there is an error at position 2, etc.

Additive Gaussian Noise (AGN) Channel, Definition, Capacity

Definition (AGN channel)



- Input variable X satisfies $E[X^2] \leq E_x$. Here E_x is input symbol enery.
- Noise variable N is zero-mean Gaussian, variance σ^2, hence

$$p(n) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\frac{n^2}{2\sigma^2})$$

- Output $Y = X + N$.

- **MODEL** for transmission links where thermal noise is dominant.
- For the **capacity of the AGN channel we find that**

$$C_{\mathrm{AGN}} = \frac{1}{2}\log_2(1 + \frac{E_x}{\sigma^2}) \quad \text{[bit/channel use]}.$$

AGN Channel, Capacity Derivation

- Derivation

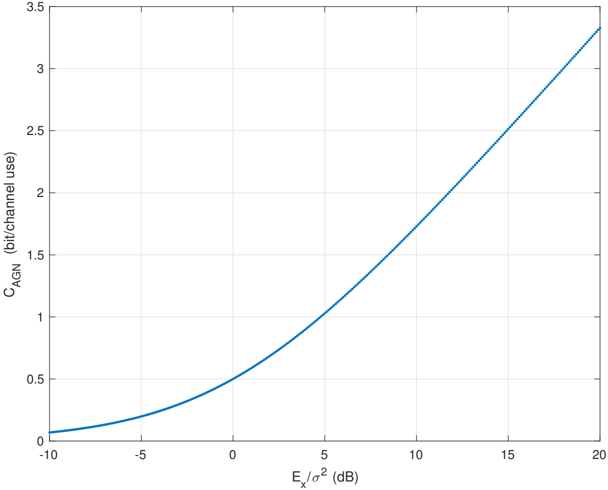
$$
\begin{aligned}
C_{\text{AGN}} &\overset{(a)}{=} \max_{X:E[X^2]\leq E_x} I(X;Y) \\
&\overset{(b)}{=} \max_{X:E[X^2]\leq E_x} h(Y) - h(Y|X) \\
&= \max_{X:E[X^2]\leq E_x} h(Y) - h(N) \\
&\overset{(c)}{\leq} \max_{X:E[X^2]\leq E_x} \frac{1}{2}\log_2 2\pi e(E_x+\sigma^2) - \frac{1}{2}\log_2 2\pi e\sigma^2 \\
&= \frac{1}{2}\log_2(1+\frac{E_x}{\sigma^2}).
\end{aligned}
$$

  Note that (a) is power constrained optimization, (b) splits $I(X;Y)$ into differential entropies, (c) is based on upper bound on entropy given variance.

- Observe that **equality (capacity)is obtained only if X is Gaussian**.
- **Signal-to-noise ratio** definition:

$$\text{SNR} \triangleq \frac{E_x}{\sigma^2}.$$

AGN Channel, Capacity Plot



AGN Channel, Questions

- Q1: What is the **minimal signal-to-noise ratio** E_x/σ^2 for rate R? From

$$R \leq C_{\text{AGN}} = \frac{1}{2}\log_2(1+\frac{E_x}{\sigma^2})$$

  we obtain that

$$(\frac{E_x}{\sigma^2})_{\text{min}} = 2^{2R}-1,$$

  This is called **Shannon limit**.

- Q2: What is the **minimal transmit energy** per transmitted bit?

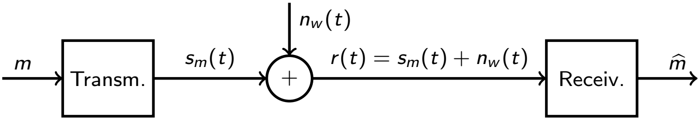
$$(\frac{E_x}{R})_{\text{min}} = \sigma^2 \cdot \frac{2^{2R}-1}{R}.$$

  Since

$$\lim_{R\downarrow 0} \frac{2^{2R}-1}{R} = \lim_{R\downarrow 0} \frac{\exp(2R\ln 2)-1}{R} = 2\ln 2.$$

  the **minimal energy per bit** is equal to $2\sigma^2 \ln 2$, achieved only if $R \downarrow 0$.

Waveform Channel Model



- **Message index** m assumes values in $\{1, 2, \cdots, |M|\}$, uniformly.
- There is a **waveform** $s_m(t)$ for each message index m.
- The waveform is **transmitted over the channel that adds white noise** $n_w(t)$ to it. The channel output-waveform is

  $$r(t) = s_m(t) + n_w(t).$$

  The **Gaussian stationary noise process** $N_w(t)$ is **zero-mean**, hence $E[N_w(t)] = 0$ for all t, and its **autocorrelation function**

  $$E[N_w(t)N_w(s)] = \frac{N_0}{2}\delta(t - s).$$

- **Transmission rate**

  $$\frac{1}{N}\log_2 |M|.$$

- **Error probability**

  $$P_e = \Pr\{\widehat{W} \neq W\}.$$

Transmit Power and Bandwidth Constraint

- If the (effective) time-duration of the signals is Δ then the energy of signal $s_m(t), m \in \{1, 2, \cdots, |M|\}$ should satisfy

$$E_{s_m} = \int_{-\infty}^{\infty} s_m^2(t)dt \leq P\Delta.$$

  This inequality is called the **power constraint**, power is P.

- Moreover the signals $s_m(t), m \in \{1, 2, \cdots, |M|\}$, should satisfy the **bandwidth constraint**

$$S_m(f) = \int_{-\infty}^{\infty} s_m(t) \exp(-j2\pi ft)dt = 0 \text{ for } |f| > W,$$

  where W is the bandwidth.

The sinc-pulse

$$p(t) = \frac{1}{\sqrt{T}} \frac{\sin(\pi t/T)}{\pi t/T}$$

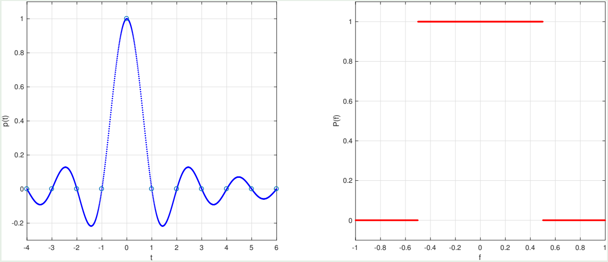
has Fourier spectrum

$$P(f) = \begin{cases} \sqrt{T} & \text{for } |f| < 1/(2T) \\ 0 & \text{for } |f| > 1/(2T). \end{cases}$$

Therefore this sinc-pulse satisfies the bandwidth constraint for $T = \frac{1}{2W}$.

Example

The pulse $p(t)$ and its spectrum $P(f)$ for $T = 1$.



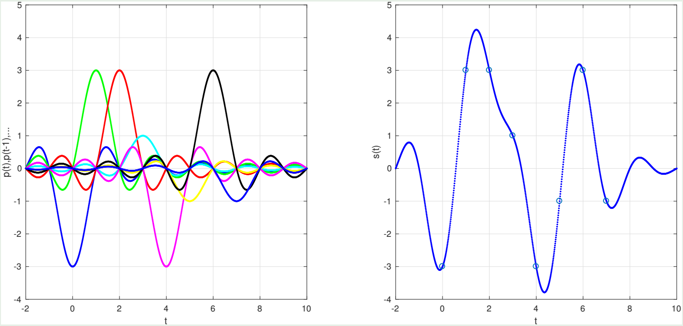
Pulse-Amplitude Modulation (PAM)

We can now transmit a sinc-pulse every T seconds. If we give the pulse $p(t-kT)$ amplitude $x_k \in \mathcal{X}$, and add these scaled pulses, we get the waveform

$$s(t) = \sum_{k=0,K-1} x_k p(t-kT).$$

Example

Let $\mathcal{X} = \{-3,-1,+1,+3\}$ and take $K=8$. Now let $x_1,\cdots,x_8 = (-3,+3,+3,+1,-3,-1,+3,-1)$ then the scaled pulses ($T=1$) and their sum are:



Orthonormality of the Pulses

Definition

The sinc-pulses $p(t-kT)$ for $k=0,1,\cdots,K-1$ are **orthonormal**, i.e.

$$\int_{-\infty}^{\infty} p(t-kT)p(t-k'T)dt = \begin{cases} 1 & \text{for } k'=k, \\ 0 & \text{for } k' \neq k. \end{cases}$$

For the **correlation** y_k of the output waveform $r(t)$ with the pulse $p(t-kT)$ for $k=0,1,\cdots,K-1$ we can write

$$\begin{aligned} y_k &= \int_{-\infty}^{\infty} r(t)p(t-kT)dt \\ &= \int_{-\infty}^{\infty} \left(\sum_{k'} x_{k'} p(t-k'T) + n_w(t) \right) p(t-kT)dt = x_k + n_k \end{aligned}$$

with

$$n_k = \int_{-\infty}^{\infty} n_w(t)p(t-kT)dt.$$

Note that this correlation **yields the desired signal amplitude** x_k to which noise term n_k is added.

Statistic of the Noise Variables

- The noise variables N_k for $k = 0, 1, \cdots, K-1$ are **jointly Gaussian.**
- For their **expectation** we get

$$\begin{aligned} E[N_k] &= E\left[\int_{-\infty}^{\infty} N_w(t)p(t-kT)dt\right] \\ &= \int_{-\infty}^{\infty} E[N_w(t)]p(t-kT)dt = 0. \end{aligned}$$

- Moreover their **correlation**

$$\begin{aligned} E[N_k N_{k'}] &= E\left[\int\int N_w(t)p(t-kT)N_w(t')p(t'-k'T)dtdt'\right] \\ &= \int\int E[N_w(t)N_w(t')]p(t-kT)p(t'-k'T)dtdt' \\ &= \int\int \frac{N_0}{2}\delta(t-t')p(t-kT)p(t'-k'T)dtdt' \\ &= \int \frac{N_0}{2} p(t-kT)p(t-k'T)dt = \begin{cases} \frac{N_0}{2} & \text{for } k'=k, \\ 0 & \text{for } k' \neq k. \end{cases} \end{aligned}$$

Hence the **noise variables (a) are Gaussian, (b) have zero mean, (c) are independent of each other, and (d) have variance $\frac{N_0}{2}$.**

Transmit Power Constraint

- The **effective time-duration** of the signals

$$s_m(t) = \sum_{k=0,K-1} x_{mk} p(t-kT) \text{ for } m \in \{1,2,\cdots,|M|\},$$

  is KT.

- The **energy** of the signal $s_m(t), m \in \{1,2,\cdots,|M|\}$, should therefore satisfy

$$\begin{aligned} E_{s_m} = \int_{-\infty}^{\infty} s_m^2(t)dt &= \int_{-\infty}^{\infty} \sum_k \sum_{k'} x_{mk} p(t-kT) x_{mk'} p(t-k'T)dt \\ &= \sum_k \sum_{k'} x_{mk} x_{mk'} \int_{-\infty}^{\infty} p(t-kT)p(t-k'T)dt \\ &= \sum_{k=0,K-1} x_{mk}^2 \\ &\leq PKT. \end{aligned}$$

Vector Channel

- We have **transformed** the W-bandlimited waveform channel into a **vector channel**, i.e.

$$\begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_{K-1} \end{pmatrix} = \begin{pmatrix} x_{m,0} \\ x_{m,1} \\ \vdots \\ x_{m,K-1} \end{pmatrix} + \begin{pmatrix} n_0 \\ n_1 \\ \vdots \\ n_{K-1} \end{pmatrix},$$

  where transmission of each component (dimension) **requires** $T = \frac{1}{2W}$ **seconds**.

- The **noise vector** consists of K independent Gaussian zero-mean components, each having variance $\frac{N_0}{2}$.
- The **code vectors** $(x_{m,0}, \cdots, x_{m,K-1})$ should satisfy the power constraint

$$\frac{1}{K} \sum_{k=0,K-1} x_{mk}^2 \leq TP = \frac{P}{2W} \text{ for all } m \in \{1, 2, \cdots, |M|\}.$$

- Moreover the actually transmitted waveforms $s_m(t)$ **satisfy the bandwidth constraint**

$$S_m(f) = 0 \text{ for } |f| > W \text{ for all } m \in \{1, 2, \cdots, |M|\}.$$

Bandwidth Optimality

- The W-bandwidth constraint imposes a restriction on the **number of vector components (dimensions) that are available per second**[2]. It can be shown that this number is **at most** $2W$.
- Our signaling method **achieves the optimum** since $1/T = 2W$.

[2]Wozencraft and Jacobs [1965], Dimensionality Theorem.

Waveform Channel, Capacity per Second

- We have seen before that the **capacity of the AGN channel** is

$$C_{\text{AGN}} = \frac{1}{2}\log_2(1+\frac{E_x}{\sigma^2}) \quad \text{[bits/channel use]}.$$

- Note that there are $2W$ **channel uses (dimensions) per second**.
- Moreover the **energy per channel use** $E_x = PT = \frac{P}{2W}$.
- **Noise variance** $\sigma^2 = \frac{N_0}{2}$.
- Therefore:

Theorem (Shannon, 1948)

The **capacity (in bits per second) of the W-bandlimited waveform channel** *when the transmit power is P, is*

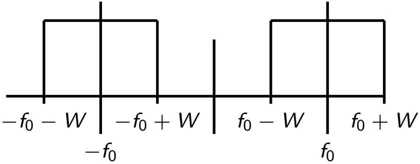
$$C_{W\text{-bandlim.ch.}} = W\log_2(1+\frac{P}{N_0 W}) \quad \textit{[bits/second]}.$$

Bandpass Channel, Model

- **Bandpass constraint.** For all signals

$$S_m(f) = \int_{-\infty}^{\infty} s_m(t) \exp(-j2\pi f t) dt = 0 \text{ except for } |f \pm f_0| < W,$$

where W is the **bandwidth**, and f_0 the **center frequency**.



- **Power constraint.** If the (effective) time-duration of the signals is Δ then the energy of all the signal $s_m(t)$ should satisfy

$$E_{s_m} = \int_{-\infty}^{\infty} s_m^2(t) dt \leq P\Delta.$$

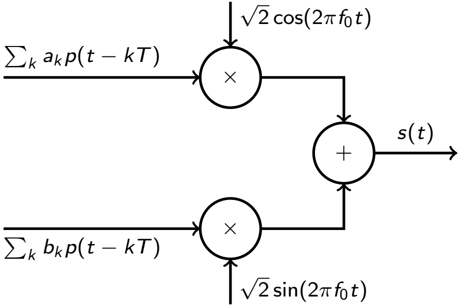
Bandpass Channel, Quadrature Amplitude Modulation

- **Carrier transmission**, use frequency f_0.
- Take $T = \frac{1}{2W}$, let $a_k \in \mathcal{X}$, $b_k \in \mathcal{X}$ for $k = 0, 1, \cdots, K-1$, then let

$$s(t) = \sum_{k=0,1,\cdots,K-1} a_k p(t-kT)\sqrt{2}\cos(2\pi f_0 t) + b_k p(t-kT)\sqrt{2}\sin(2\pi f_0 t).$$

This leads to $2 \cdot 2W = 4W$ orthonormal components per second.
Observe that the total bandwidth is now $4W$ however.

- **Schematic:**



Bandpass Channel, Capacity

- Again note that the **capacity of the AGN channel** is

$$C_{\mathrm{AGN}} = \frac{1}{2}\log_2(1+\frac{E_x}{\sigma^2}) \quad \text{[bits/channel use]}.$$

- Now there are $4W$ **channel uses (dimensions) per second**.
- Therefore the **energy per channel use** $E_x = PT = \frac{P}{4W}$.
- **Noise variance** $\sigma^2 = \frac{N_0}{2}$.
- Therefore:

Theorem

The **capacity (in bits per second) of the W-bandpass waveform channel** *when the transmit power is P, is*

$$C_{W\text{-}bandpass.ch.} = 2W\log_2(1+\frac{P}{2N_0W}) \quad \textit{[bits/second]}.$$

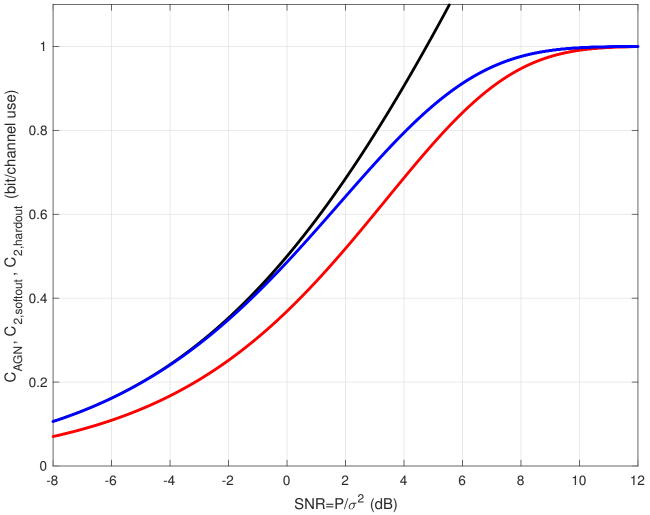
- **Spectral Efficiency (capacity per Hz):**

$$\log_2(1+\frac{E_c}{N_0}) \quad \text{[bit/second/Hz]},$$

where E_c is the energy per (a,b) pair.

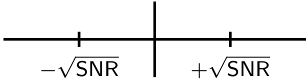
AGN Capacity at Low SNR

For signal-to-noise ratio SNR $= P/\sigma^2$ horizontally in dB the AGN capacity C_{AGN} in bits/chan use is depicted in **BLACK** in the figure below.



Binary Signalling, Soft or Hard Decision

- Instead of using a Gaussian inputs we can use **equally likely binary inputs** $-\sqrt{\text{SNR}}$ and $+\sqrt{\text{SNR}}$, assuming that $\sigma^2 = 1$.



- The capacity of such a **binary-in, soft-out** channel is

$$C_{2,\text{soft}} = \frac{\text{SNR} - E[\ln(\cosh(\text{SNR} + \sqrt{\text{SNR}}N)]}{\ln(2)}.$$

  This capacity is depicted in the plot in **BLUE**. Note that for SNR $\downarrow 0$ this capacity approaches C_{AGN}.

- The receiver can make a **hard-decision based on the channel output, with a threshold at 0.** The resulting channel is a BSC with cross-over probability

$$p = Q(\sqrt{\text{SNR}}) = \int_{\sqrt{\text{SNR}}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp(-\frac{\alpha^2}{2}) d\alpha.$$

  The capacity

$$C_{2,\text{hard}} = 1 - h(p)$$

  of this **binary-in, hard-out** channel is depicted in **RED** in the figure. For SNR $\downarrow 0$ **hard decision results in an SNR-loss of roughly 2 dB**.

Signalling at Larger SNR's

For larger SNR's we need more signal points. Assume that we use **equidistant** and **equiprobable** points. This leads to the following constellations:

- $M = 2$ (2-PAM):

1/2	1/2
-1	$+1$

- $M = 4$ (4-PAM):

1/4	1/4	1/4	1/4
-3	-1	$+1$	$+3$

- $M + 8$ (8-PAM):

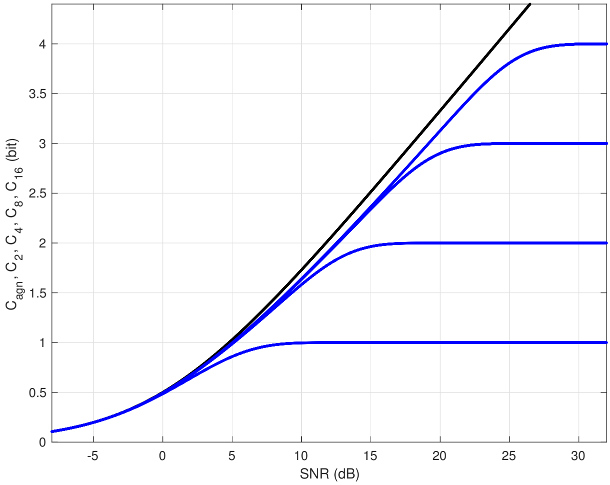
1/8	1/8	1/8	1/8	1/8	1/8	1/8	1/8
-7	-5	-3	-1	$+1$	$+3$	$+5$	$+7$

Note that the **average energy** of a signal set is

$$E_{\text{PAM}} = \frac{M^2 - 1}{3}.$$

Large SNR Capacity and PAM-Capacities

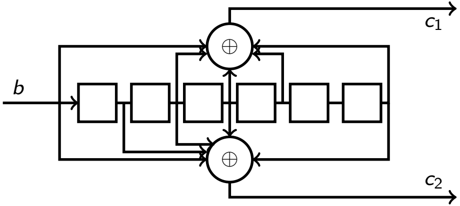
For signal-to-noise ratio SNR $= E_{\text{PAM}}/\sigma^2$ horizontally in dB the **PAM "capacities" in blue** and the **AGN capacity C_{AGN} in black** in bits/channel use are depicted in the figure below.



Observe that for 2^K-PAM can at most reach a capacity of K bit.

A Convolutional Code (NASA Code)

- Elias [1955]
- Binary input digits $b_1, b_2, \cdots, b_K$ are independent and uniform.
- Digits are encoded using a **64-state convolutional encoder**. Schematic:



- Description:

$$
\begin{aligned}
c_1(k) &= b(k) \oplus b(k-2) \oplus b(k-3) \oplus b(k-4) \oplus b(k-6), \\
c_2(k) &= b(k) \oplus b(k-1) \oplus b(k-2) \oplus b(k-3) \oplus b(k-6).
\end{aligned}
$$

- One input digit produces two output digits, **code rate** is 1/2.
- **Free Hamming distance** $d_H = 10$ of the code is 10, hence up to 4 errors can be corrected, and 5 detected.
- When used e.g. for the AGN channel, **soft-decision decoding can be realized by the Viterbi algorithm [1967]**.
- **NASA code (WiFi)**.

NASA Code, Performance

Our $R = 1/2$ code has **constraint length** $\nu = 7$. **Coding gain** at bit-error probability 10^{-5} is roughly 6 dB. **Gap to the Shannon bound** is 3.8 dB. (Clark and Cain [1981])

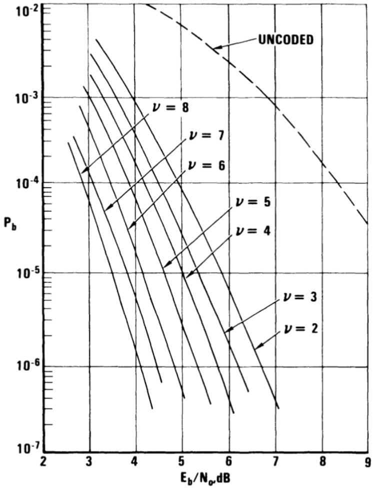
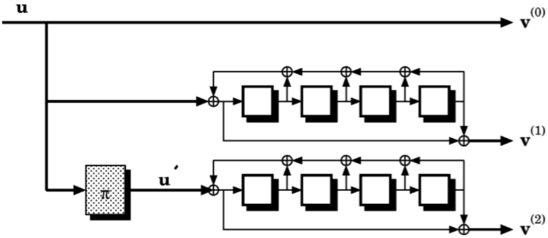


Figure 6-13. Bit error probability for $R = 1/2$ codes with Viterbi decoding (infinite quantization) and PSK modulation.

Turbo Codes

- **Turbo Codes** (Berrou, Glavieux, and Thitimajhshima [1993])

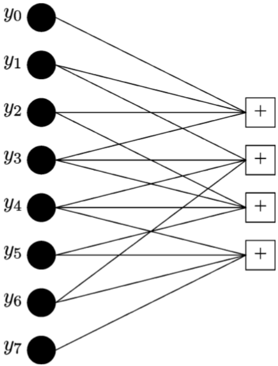


  Based on systematic recursive convolutional codes connected by an interleaver.
  Bahl, Cocke, Jelinek, Raviv (BCJR) algorithm [1974] used for iterative decoding.

- Near-Shannon-limit performance (with a dB).

LDPC Codes

- **Low Density Parity Check (LDPC)-codes** (Gallager [1963], rediscovered in 1993):

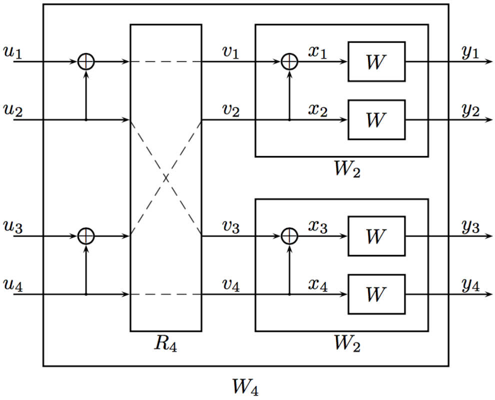


  Code is specified by its **parity-check matrix**. The symbol nodes on the left are checked by equation nodes on the right. Low density of the matrix makes **message-passing algorithms** possible.

- Near-Shannon-limit performance (within tenths of a dB).

Polar Codes

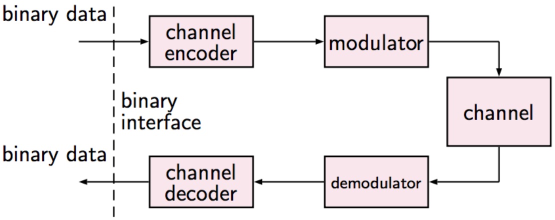
- **Polar codes** Arikan [2006], Arikan and Telatar [2007].



- **IDEA (Polarization):** Two identical channels can be transformed into a channel that is **better** and a channel that is **worse** than the original ones. The sum of the capacities remains constant however.
- **Shannon-limit performance.** Decoding complexity $|M| \log_2 |M|$.

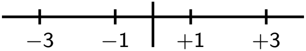
Coding and Modulation

- **Modulation** maps **binary digits onto signals for the AGN channel**, e.g. three binary digits map onto an 8-PAM signal.
- **Coding** is used to map **message (data) sequences onto a set of binary codewords.** These codewords are input to the modulator. The codewords are chosen such that the corresponding signal sequences are e.g. far apart (large Euclidean distance), or e.g. maximize mutual information between AGN channel input and output.
- Block diagram:

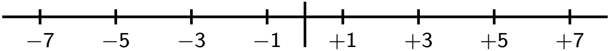


Trellis Coded Modulation

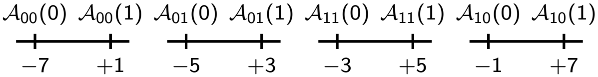
- **COMBINE CODING and MODULATION, Ungerboeck [1982]**
- **BASELINE:** Uncoded 4-PAM. Now average energy $E_{\text{PAM}}(\text{unc}) = 5$ and squared Euclidean distance $d_E^2(\text{unc}) = 4$.



- Coding starts by **expanding the signal constellation** to 8-PAM. Now $E_{\text{PAM}}(\text{cod}) = 21$.



- **Partition the 8-PAM signal set** into 4 subsets $\mathcal{A}_{00}$, $\mathcal{A}_{01}$, $\mathcal{A}_{11}$, and $\mathcal{A}_{10}$, each containing 2 signals for which the distance is 8.

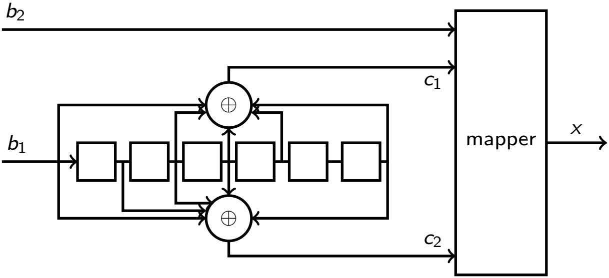


- The distance between **signals in different subsets** can be as small as 2.
- Points in **subsets with complementary labels** have a distance 4 or more.

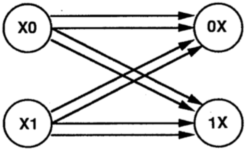
Trellis Coded Modulation

- **Use the NASA code.** For each channel use let the **coded** binary digits c_1 and c_2 determine the subset $\mathcal{A}_{c_1 c_2}$ and the uncoded bit b_2 determine the symbol within this subset. Hence the mapper realizes

$$x = \mathcal{A}_{c_1 c_2}(b_2).$$



- Decoding done with **Viterbi algorithm**. Trellis structure:



Trellis Coded Modulation

- **Distance analysis:** An error event starts and ends with $c_1 = c_2$. This leads to a starting and an ending subset with complementary labels and squared distance 4^2.
  For the Euclidean distance between the 8-PAM sequeces we obtain, using $d_H = 10$ of the NASA code that
  $$d_E^2(\text{cod}) = \min(8^2, (d_H - 4)2^2 + 2 \cdot 4^2) = 56.$$
- What we have gained is now
  $$G = \frac{d_E^2(\text{cod})/E_{av}(\text{cod})}{d_E^2(\text{unc})/E_{av}(\text{unc})} = \frac{56/21}{4/5} = 10/3 = 5.2 \text{ dB}.$$
- The gain G is **asymptotic coding gain**. This implies that at large SNR TCM achieves the same error probability as uncoded transmission with 5.2 dB less SNR.
- We followed here the **Pragmatic approach to trellis coded modulation** (Viterbi, Wolf, Zehavi, and Padovani [1989]).
- Ungerboeck received the **Shannon Award** recently from the IEEE Information Theory Society.

Multi-Level Coded Modulation, Set Partition Mapping

- Consider a one-to-one mapping from three binary digits to an 8-PAM symbol. We consider here a **set partition mapping**.

b_1	b_2	b_3	x
0	0	0	-7
1	0	0	-5
0	1	0	-3
1	1	0	-1
0	0	1	$+1$
1	0	1	$+3$
0	1	1	$+5$
1	1	1	$+7$

- Another representation of this mapping

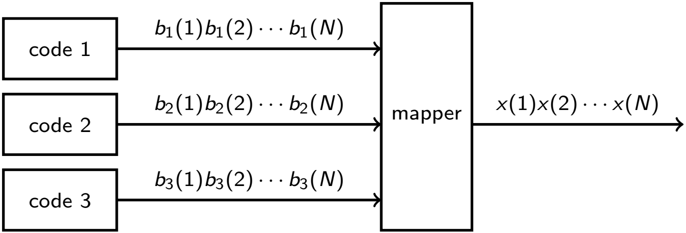
000	100	010	110	001	101	011	111
-7	-5	-3	-1	$+1$	$+3$	$+5$	$+7$

  Note that the **distance increases by a factor of 2** after b_1 is exposed. If in adittion b_2 is exposed the distance is again increased by a factor of 2.

- Ungerboeck.

Multi-Level Coding: Use Binary Code for Every Label

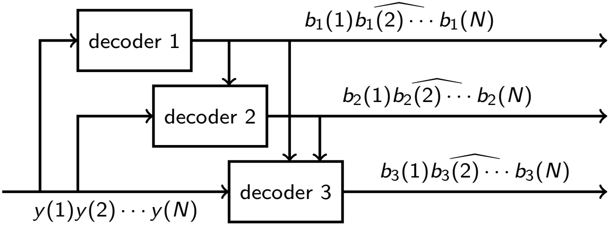
- Use a first (strong) code for bit labels b_1, a second code for bit labels b_2, and a third code for bit labels b_3. All codewords have length N.



- Bit labels B_1, B_2, and B_3 are uniform and independent of each other.
- The mapper combines the three codewords into a 8-PAM sequence of length N that is transmitted.

Decoding the Binary Codes for All Labels Sequentially Using Already Decoded Results

- The first decoder decodes the first bit-label sequence $\widehat{b_1(1)b_1(2)\cdots b_1(N)}$.
- Then the second decoder decodes the second bit-label sequence $\widehat{b_2(1)b_2(2)\cdots b_2(N)}$, using the decoded first bit-label sequence.
- Finally the third decoder decodes the third bit-label sequence $\widehat{b_3(1)b_3(2)\cdots b_3(N)}$, using the decoded first bit-label sequence and the decoded second bit-label sequence.
- Block diagram:



- Decoding must be performed **sequentially**.

Multi-Level Capacity Analysis

- The first bit channel has as input B_1 and as output Y.
- The second bit channel has as input B_2 and as output (YB_1).
- The third bit channel has as input B_3 and the output is (YB_1B_2).
- Therefore the **mutual information (MI)** is:

$$
\begin{aligned}
&I(B_1;Y)+I(B_2;YB_1)+I(B_3;YB_1B_2)\\
&\quad= I(B_1;Y)+I(B_1;B_2)+I(B_2;Y|B_1)+I(B_3;B_1B_2)+I(B_3;Y|B_1B_2)\\
&\quad= I(B_1;Y)+I(B_2;Y|B_1)+I(B_3;Y|B_1B_2)\\
&\quad= I(B_1B_2B_3;Y)\\
&\quad= I(X;Y)
\end{aligned}
$$

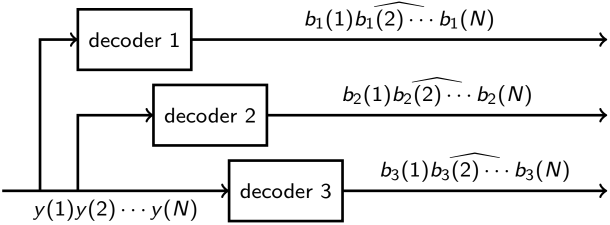
This implies that there is no loss!

- Imai and Hirakawa [1977].

Decoding Not Using Already Decoded Results

Bit-Interleaved Coded Modulation (Zehavi [1991], also Caire, Taricco, and Biglieri [1998]):

- The first decoder decodes the first bit-label sequence $\widehat{b_1(1)b_1(2)\cdots b_1(N)}$.
- The second decoder decodes the second bit-label sequence $\widehat{b_2(1)b_2(2)\cdots b_2(N)}$.
- The third decoder decodes the third bit-label sequence $\widehat{b_3(1)b_3(2)\cdots b_3(N)}$.
- Block diagram:



- Decoding can be performed **in parallel**.

Bit-Interleaved Capacity Analysis

- The first bit channel has as input B_1 and as output Y.
- The second bit channel has as input B_2 and as output Y.
- The third bit channel has as input B_3 and the output is Y.
- Therefore now the **generalised mutual information (GMI)** is:

$$
\begin{aligned}
&I(B_1;Y) + I(B_2;Y) + I(B_3;Y) \\
&\quad \le \quad I(B_1;Y) + I(B_2;Y,B_1) + I(B_3;Y,B_1,B_2) \\
&\quad = \quad I(B_1;Y) + I(B_2;Y|B_1) + I(B_3;Y|B_1,B_2) \\
&\quad = \quad I(B_1,B_2,B_3;Y) \\
&\quad = \quad I(X;Y)
\end{aligned}
$$

This implies that **now there could be loss!**

Bit-Interleaved Coded Modulation, Gray Mapping

- Consider a one-to-one mapping from three binary digits to an 8-PAM symbol. We consider now a **Gray mapping**[3].

b_1	b_2	b_3	x
0	0	0	-7
0	0	1	-5
0	1	1	-3
0	1	0	-1
1	1	0	$+1$
1	1	1	$+3$
1	0	1	$+5$
1	0	0	$+7$

- Another representation of this mapping

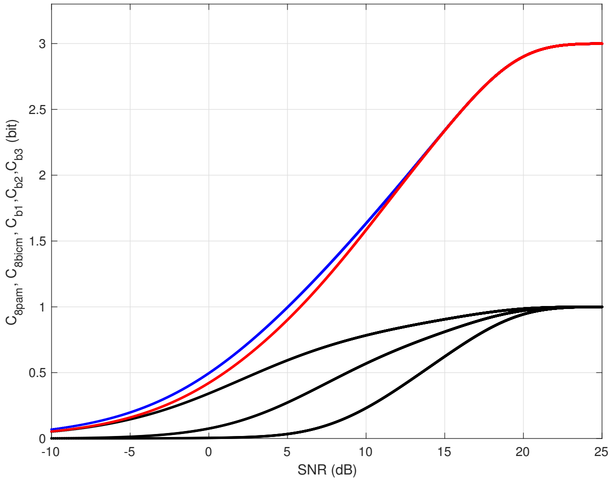
000	001	011	010	110	111	101	100
-7	-5	-3	-1	$+1$	$+3$	$+5$	$+7$

Note that **only one digit changes if we go from a signal point to its neighbour**.

[3] Binary Reflected Gray Mapping.

Bit-Interleaved Coded Modulation, Capacity Plot

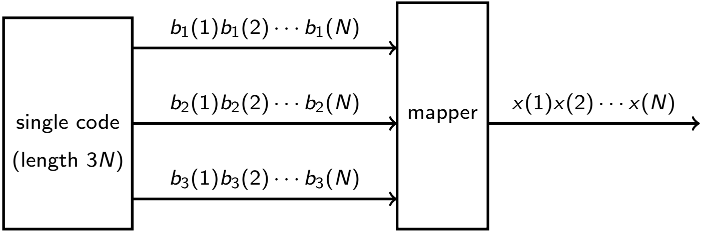
The figure contains the capacities of the sub-channels and the total bit-interleaved capacity for Gray coding.



Loss is acceptable for the Gray mapping!

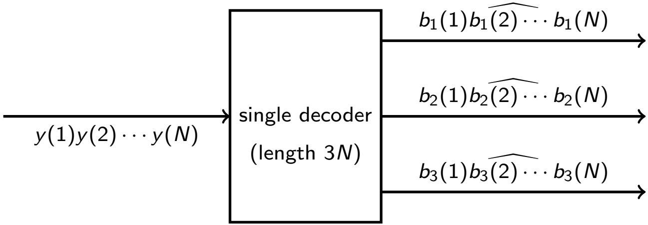
Bit-Interleaved Coded Modulation, Use a SINGLE Binary Code

- Transmitter:



Bit-Interleaved Coded Modulaton: Use a SINGLE Binary Code

- Receiver:



- **Log-Likelihood Ratio calculation:**

$$\begin{aligned} LLR_i &= \frac{\sum_{b_1 b_2 b_3 : b_i = 0} p(y|x(b_1, b_2, b_3))}{\sum_{b_1 b_2 b_3 : b_i = 1} p(y|x(b_1, b_2, b_3))} \\ &\approx \frac{\max_{b_1 b_2 b_3 : b_i = 0} p(y|x(b_1, b_2, b_3))}{\max_{b_1 b_2 b_3 : b_i = 1} p(y|x(b_1, b_2, b_3))}. \end{aligned}$$

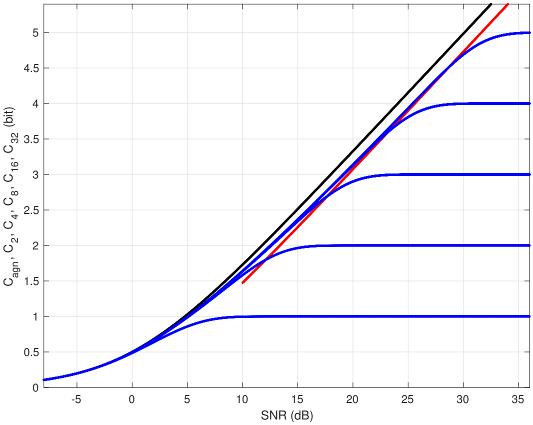
- Used everywhere.

Remarks

- **Trellis Coded Modulation (Ungerboeck)** focussed on obtaining **distance gain**.
- **Multi-Level Coding** and **Bit-Interleaved Coded Modulation** is based on **mutual information considerations**.

Gap to Channel Capacity

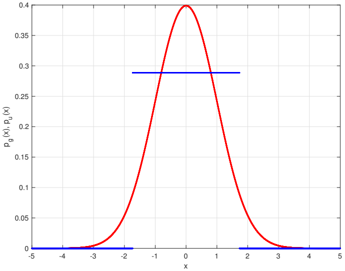
For signal-to-noise ratio SNR horizontally in dB the AGN capacity C_{AGN} in bits/channel use is depicted in **black** in the figure below.



The curves for uniform 2-PAM, 4-PAM, 8-PAM, 16-PAM and 32-PAM are depicted in **blue**. A gap to AGN-capacity appears since the PAM inputs are not **Gaussian**, but **Uniform**.

Gap in bit and in SNR loss

- Assumptions: (a) M-PAM where $M \to \infty$ and (b) SNR $\to \infty$ or equivalently that $\sigma_n^2 \to 0$.

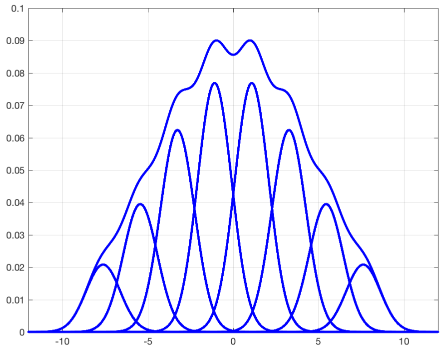


- Consider difference of the **capacity** $I(X_g; Y_g)$, where X_g is the **Gaussian** channel input and Y_g the corresponding output, and the **mutual information** $I(X_u; Y_u)$, where X_u is a **uniform** channel input and Y_u the corresponding output:

$$
\begin{aligned}
&I(X_g; Y_g) - I(X_u; Y_u) \\
&\quad = h(Y_g) - h(Y_g|X_g) - h(Y_u) + h(Y_u|X_u) \\
&\quad = h(Y_g) - h(Y_u) \\
&\quad = \frac{1}{2}\log_2(2\pi e \sigma_x^2) - \log_2 12\sigma_x^2 = \frac{1}{2}\log_2\frac{\pi e}{6} = 0.2546 \text{ bit},
\end{aligned}
$$

or equivalently 1.53 dB in SNR.

Probabilistic Shaping: Equidistant Signal Points



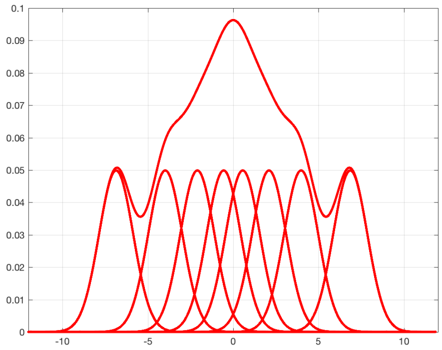
Eight equidistant signals $x \in \{-7\gamma, -5\gamma, \cdots, +7\gamma\}$ for $\gamma = 1.089$.
Non-uniform probability distribution $P(x)$: $\{0.0521, 0.0989, 0.1562, 0.1927, 0.1927, 0.1562, 0.0989, 0.0521\}$.
In the plot:

$$p(x,y) = P(x)p(y|x) \text{ for } x \in \{-7\gamma, -5\gamma, \cdots, +7\gamma\}$$

$$p(y) = \sum_x P(x)p(y|x) \text{ where } p(y|x) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{(y-x)^2}{2}).$$

Moreover $I(X;Y) = 2.001$ bit at SNR $= 11.96$ dB.

Geometric Shaping: Equiprobable Signals

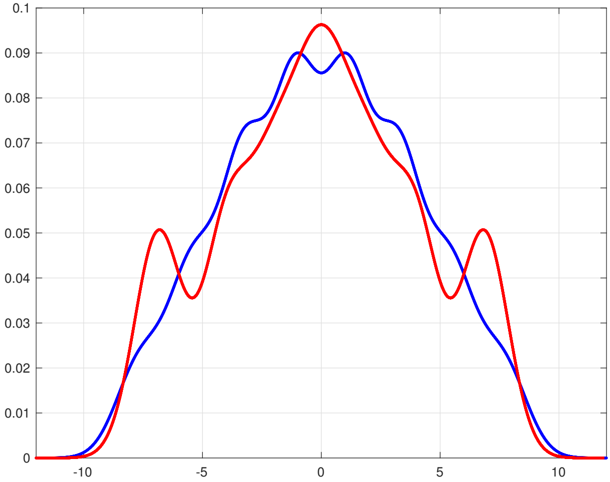


Uniform probability distribution $P(x) : \{1/8, 1/8, \cdots, 1/8\}$ for all possible x.
Eight non-equidistant signals: $x \in \{-6.86, -3.98, -2.10, -0.56, +0.56, +2.10, +3.98, +6.86\}$.
In the plot

$$\begin{aligned} p(x,y) &= P(x)p(y|x) \text{ for } x \in \{-6.86, -3.98, \cdots, +6.86\} \\ p(y) &= \sum_x P(x)p(y|x) \text{ where } p(y|x) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{(y-x)^2}{2}). \end{aligned}$$

Moreover $I(X;Y) = 2.001$ bit at SNR $= 12.28$ dB.

Probabilistic Shaping and Geometric Shaping



In the plot the **output densities** $p(y)$. **Blue:** Probabilistic Shaping (SNR $= 11.96$ dB). **Red:** Geometric Shaping (SNR $= 12.28$ dB). $I(X;Y) = 2.001$ bit in both cases.

Probabilistic Shaping, Distribution Matching

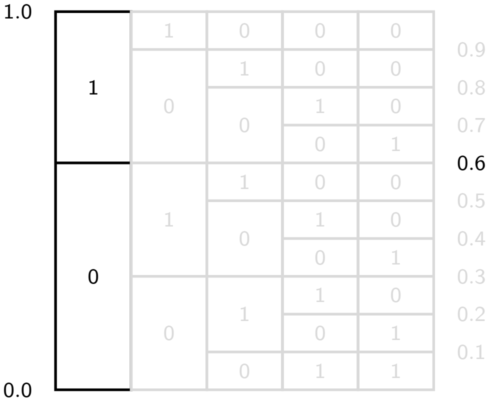
Q: How can we generate sequences with a given composition?

We can use **Distribution Matching** (Boecherer [2014], Schulte and Boecherer [2016]).

Distribution matching is inspired by **arithmetic data compression techniques** (e.g. Langdon and Rissanen [1979], Witten, Neal, and Cleary [1987]). In their methods **sequences are represented by intervals**.

Sequences and Intervals, Example

We want to generate binary sequences of length 5 **containing** 2 **ones. There are** $\binom{5}{2} = 10$ **such sequences.**
Each of these sequences corresponds to a subinterval of length $1/10$ of the $[0, 1)$ interval.
This interval can be computed **sequentially** from the sequence. The first digit of the sequence splits the interval in fractions $3/5$ and $2/5$. After a first 0 the interval $[0, 3/5)$ is split according to $2/4$ and $2/4$, etc.
Note that the sequences and their intervals are now in a **lexicographical order**.



Sequences and Intervals, Example

We want to generate binary sequences of length 5 **containing** 2 **ones. There are** $\binom{5}{2} = 10$ **such sequences.**
Each of these sequences corresponds to a subinterval of length $1/10$ of the $[0, 1)$ interval.
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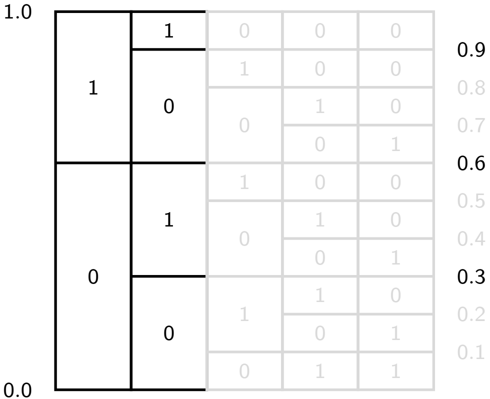

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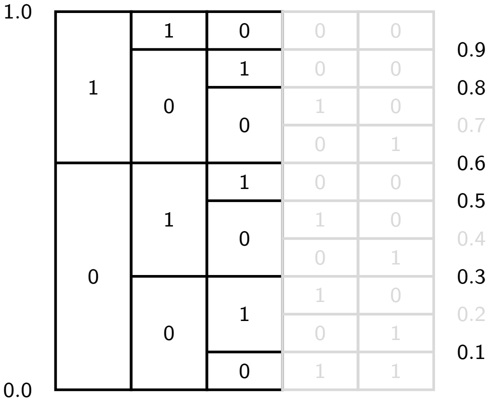

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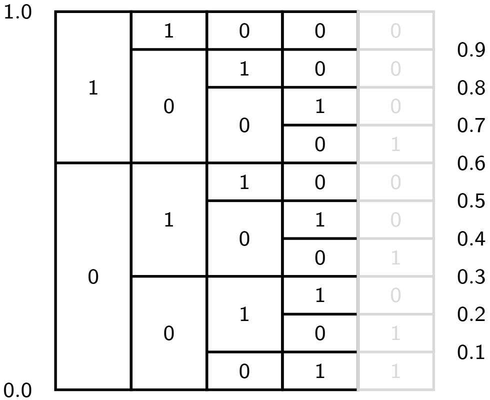

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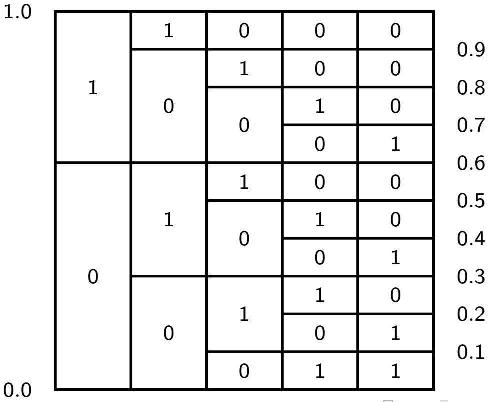
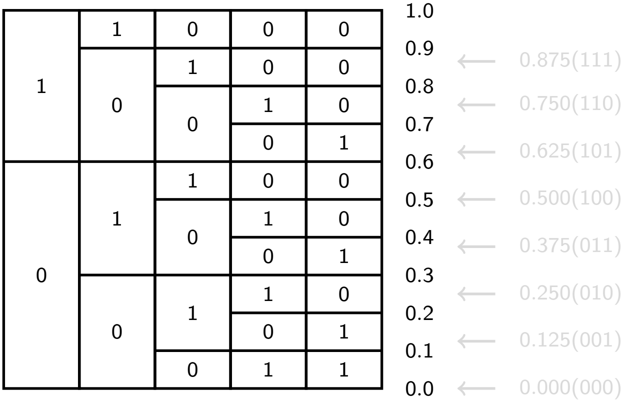


Constant Composition Sequence Intervals, Indices

Consider 3-digit indices $000, 001, \cdots, 111$. Index $b_1 b_2 b_3$ **connects** to a constant composition sequence if $b_1 2^{-1} + b_2 2^{-2} + b_3 2^{-3}$ is in the interval corresponding to the constant composition sequence.



(a) Only one index can connect to a sequence since $2^{-3} \geq 1/10$.
(b) Observe that for not all constant composition sequences there is an index.
(c) For all indices there is a sequence however.

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<table>
<tr><td rowspan="4">1</td><td>1</td><td>0</td><td>0</td><td>0</td></tr>
<tr><td rowspan="3">0</td><td>1</td><td>0</td><td>0</td></tr>
<tr><td rowspan="2">0</td><td>1</td><td>0</td></tr>
<tr><td>0</td><td>1</td></tr>
<tr><td rowspan="6">0</td><td rowspan="3">1</td><td>1</td><td>0</td><td>0</td></tr>
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<tr><td>0</td><td>1</td></tr>
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</table>

1.0
0.9 ⟵ 0.875(111)
0.8 ⟵ 0.750(110)
0.7 ⟵ 0.625(101)
0.6
0.5 ⟵ 0.500(100)
0.4 ⟵ 0.375(011)
0.3 ⟵ 0.250(010)
0.2 ⟵ 0.125(001)
0.1
0.0 ⟵ 0.000(000)

(a) Only one index can connect to a sequence since $2^{-3} \geq 1/10$.
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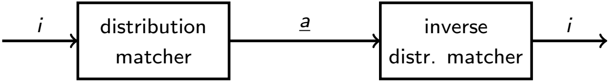
From an index to a const. comp. sequence and back

- Using this method **(distribution matching)** we find a constant composition sequence $\underline{a}$ for all indices i, and from such a constant composition sequence $\underline{a}$ the original index i can be recovered **(inverse distribution matching)**.



- **Use as index the message m that is to be transmitted.** The resulting const. comp. sequence $\underline{a}$ can be used as **amplitude sequence**.
- Now take a **short block length** $N = 96$. The amplitude level composition is

$$96 * (0.1927, 0.1562, 0.0989, 0.0521) * 2 \approx (37, 30, 19, 10).$$

  This leads to

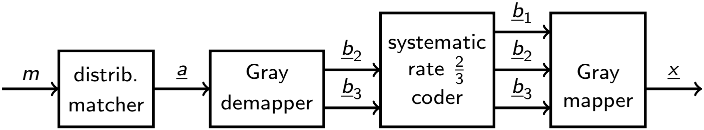
$$\frac{96!}{37!30!19!10!} = 2^{168.72} \text{ const. comp. sequences.}$$

  Rate is $\frac{168}{96} = 1.75$ [bit/symbol], and the **sequence energy**

$$37 * 1 + 30 * 9 + 19 * 25 + 10 * 49 = 1272.$$

Combining Shaping with Coding

- Schematic:



- The distribution matcher converts message m into amplitude sequence $\underline{a}$ of the desired composition.
- The Gray demapper **(sign bit missing!)** converts the amplitude sequence $\underline{a}$ into the two **amplitude** bitstreams $\underline{b}_2$ and $\underline{b}_3$ both of length N.

a	b_2	b_3
1	1	0
3	1	1
5	0	1
7	0	0

- Now **parity is generated** from $\underline{b}_2$ and $\underline{b}_3$, using a systematic code of rate 2/3. This parity is used as bitstream $\underline{b}_1$. This bitstream represents the **sign bits.**

Combining Shaping with Coding

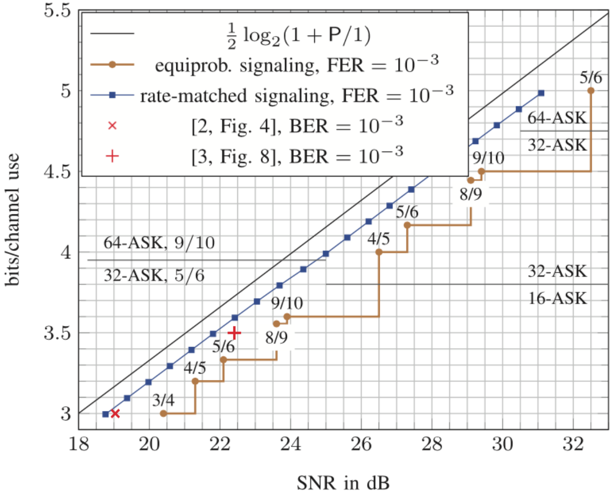
- The three bitstreams are combined into an 8-PAM symbol $\underline{x}$ stream, using Gray mapping.

b_1	b_2	b_3	x
0	0	0	-7
0	0	1	-5
0	1	1	-3
0	1	0	-1
1	1	0	$+1$
1	1	1	$+3$
1	0	1	$+5$
1	0	0	$+7$

- We have described a **Bit-Interleaved Coded Modulation** construction, where only sequences with **constant amplitude composition** are generated.
- Log-Likelihood Ratio calculation now includes **a priori symbol information**.

$$LLR_i = \frac{\sum_{b_1 b_2 b_3 : b_i = 0} p(b_1 b_2 b_3) p(y | x(b_1, b_2, b_3))}{\sum_{b_1 b_2 b_3 : b_i = 1} p(b_1 b_2 b_3) p(y | x(b_1, b_2, b_3))}$$

Boecherer Simulations



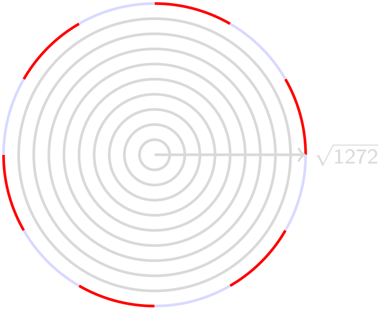
- FER (frame error rate) $= 10^{-3}$, LDPC codes from DVB-S2. Boecherer, Schulte, and Steiner [2016].

Enumerative Shap., from a Partially-Filled Surface to a Sphere

Consider composition $(37, 30, 19, 10)$ that leads to $2^{168.72}$ sequences and sequence energy 1272.

QUESTION: Can we obtain more sequences such that the average sequence energy does not exceed 1272?

- Note first that there are **more compositions** with energy equal to 1272. Add the corresponding sequences. This leads to $2^{172.75}$ sequences.
- Add all the sequences with an **energy smaller than 1272**. Now we obtain $2^{175.04}$ sequences. Moreover the **average energy drops** to 1242.4.



If we are **interested in rate** 1.75 we can decrease the radius to $\sqrt{1120}$. Now we find $2^{168.03}$ sequences with average sequence energy 1096.9. Gain $= \frac{1272}{1096.9} = 0.6431$ dB.

Analysis by Y. Gultekin [2017, TU/e].

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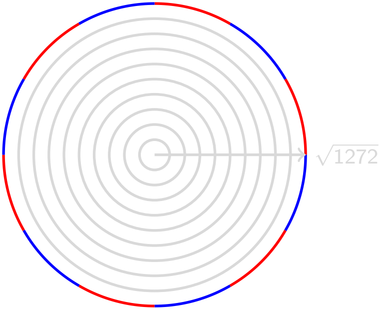

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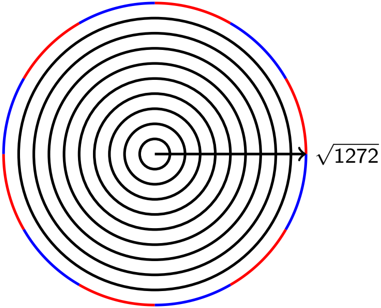

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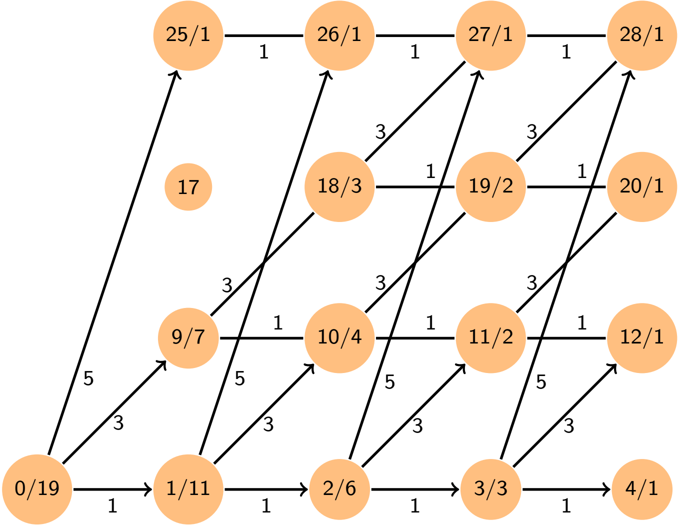
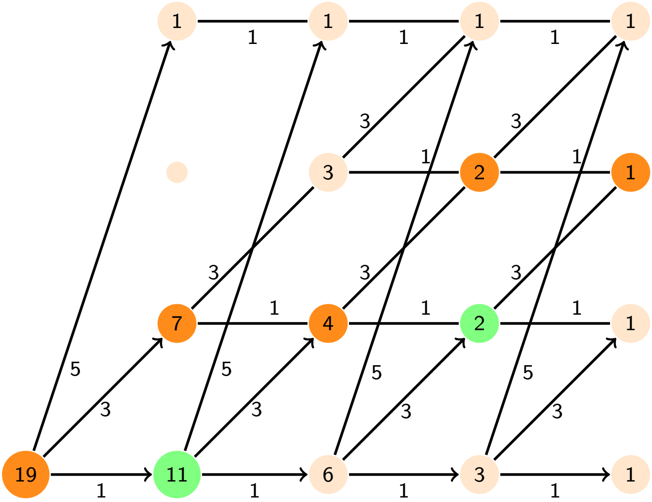
Enumerative Shaping: Bounded Energy Trellis

Wuijts [1991, TU/e], W. and Wuijts [1993]

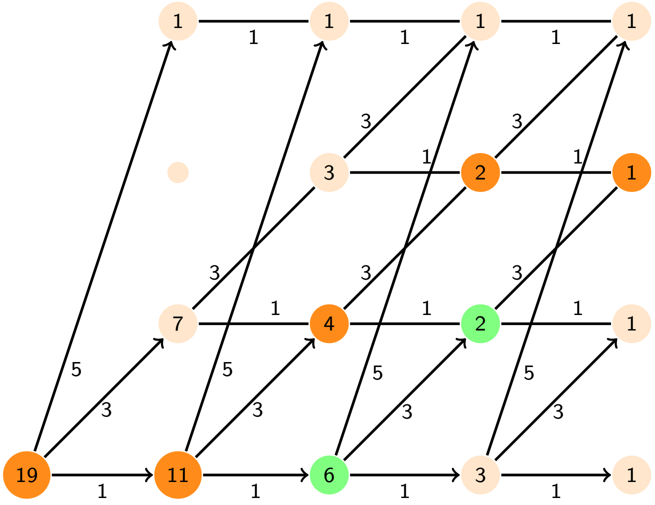
$N = 4$, amplitude alphabet is $\{1, 3, 5, \cdots\}$, $E_{max} = 28$ i.e. sphere radius $\sqrt{28}$.



Lexicographical Ordering. Index of sequence 3131 is 13.



Lexicographical ordering. Sequence with index 8 is 1331.



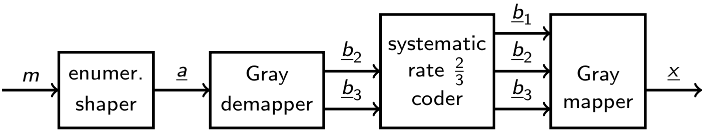
Enumerative Shaping, Analysis

- Maximum energy level $E_{max} = 28$.
- **Adding signs** leads to $16 \cdot 19 = 304$ sequences.
- Total rate
$$R_{tot} = \frac{\log_2 304}{4} = 2.062 \text{ bits/symbol.}$$
- Average energy per symbol $E_{av}/N = 5.211$.
- Gain
$$G = \frac{\frac{2^{2R}-1}{3}}{E_{av}/N} = 0.218 \text{ dB.}$$
- More results for rate $R \approx 2$, where N is sequence-length.

N	E_{max}	E_{av}/N	R_{tot}	G
8	48	5.169	2.102	0.509
16	80	4.638	2.064	0.734
32	136	4.100	2.006	0.901
64	264	4.051	2.019	1.039

Combining Shaping with Coding

- Schematic:



- The enumerative shaper converts a message m into a bounded energy amplitude sequence $\underline{a}$.
- **No difference with Boecherer's approach.**

Probabilistic Shaping versus Enumerative Shaping

Consider a hypersphere in N (even) dimensions with radius ρ, then

$$
\begin{aligned}
h(X) = \frac{h(X_1)+h(X_2)+\cdots+h(X_N)}{N} &\geq \frac{h(X_1,X_2,\cdots,X_N)}{N} \\
&= \frac{1}{N}\log\frac{\pi^{N/2}}{(N/2)!}\rho^N \\
&\geq \frac{1}{N}\log\frac{\pi^{N/2}}{(\frac{N}{2e})^{N/2}\sqrt{\frac{e^2N}{2}}}\rho^N \\
&= \frac{1}{2}\log 2\pi e\frac{\rho^2}{N} - \frac{1}{2N}\log\frac{e^2N}{2}.
\end{aligned}
$$

$$
\begin{aligned}
h(X) &= \frac{h(X_1)+h(X_2)+\cdots+h(X_N)}{N} \\
&\leq \frac{\log 2\pi eE[X_1^2]+\log 2\pi eE[X_2^2]+\cdots+\log 2\pi eE[X_N^2]}{2N} \\
&\leq \frac{1}{2}\log 2\pi e\frac{E[X_1^2]+E[X_2^2]+\cdots+E[X_N^2]}{N} = \frac{1}{2}\log 2\pi e\frac{\rho^2}{N}.
\end{aligned}
$$

HENCE enumerative shaping for large N results in Gaussians components!

Probabilistic Shaping versus Enumerative Shaping

- Both **probabilistic shaping** and **enumerative shaping** lead to Gaussian input distributions, which is required to **achieve capacity.**
- In a **coding environment** the bounded energy constraint could be **more elementary** than the Gaussian input constraint.
- For short blocklengths there is something to gain with enumerative shaping.
- For large blocklengths the gain is negligible (sphere hardening argument).
- Both probabilistic shaping and enumerative shaping are lexicographical indexing methods.
- Complexity of enumerative shaping is larger than that of probabilistic shaping.

Geometric Shaping

Advantage:

- No probabilistic shaper required.

Disadvantages:

- Mapping from symbols to bits causes problems (dependent bits).
- Performance (Steiner and Boecherer [2016]):

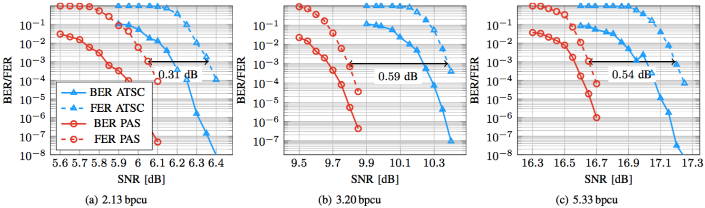


Fig. 4. Comparison of the coded performance of geometrically shaped ATSC 3.0 modcods with one single PAS scheme.

Smart ideas needed!

Shaping References

- **Forney, Gallager, Lang, Longstaff, and Qureshi [1984]:** constellation shaping, coding and shaping can be separated.
- **Kschischang and Pasupathy [1990]:** variable-rate shaping, geometric shaping.
- **Calderbank and Ozarow [1991]:** shaping on regions.
- **Forney [1992]:** trellis shaping (more codewords for same data, choose lowest energy codeword), sign-bit shaping, constellation expansion, peak-to-average power ratio expansion.
- **Sun and van Tilborg [1993]:** geometrical shaping.
- **Laroia, Farvardin, and Tretter [1994]:** enumerative shaping (two schemes).
- **V34 Modem standard [1994]:** shell mapping, enumerative shaping.
- **Fischer [2002]:** overview.

Final Remarks

- Information Theory and Coding Theory is useful in understanding, analysing, and improving communication systems, also optical communication systems.
- Performance indicators as e.g. capacities and mutual informations are very powerful, and well studied (see e.g. work of Szczecinski and Alvarado [2015], Guillen i Fabregas, Martinez, and Caire [2008]).
- Next step. Non-linearities ...