Extremum-seeking control for steady-state performance optimization of nonlinear plants with time-varying steady-state outputs

Technical report DC2018.028
Extremum-seeking control for steady-state performance optimization of nonlinear plants with time-varying steady-state outputs

L. Hazeleger
Eindhoven University of Technology
Department of Mechanical Engineering
Dynamics and Control group
The Netherlands

February 26, 2018

Contents

1 Introduction

In this technical report, the proof of Theorem 1 can be found. Sections 2-4 are copies of the Sections 2-4 of the American Control Conference paper. Section 5 presents the proof of Theorem 1.

2 Extremum-seeking control problem for time-varying outputs

Consider the following multi-input-multi-output nonlinear plant:

$$\Sigma_p : \begin{cases} \dot{\boldsymbol{x}}(t) &= \boldsymbol{f}(\boldsymbol{x}(t), \boldsymbol{u}(t), \boldsymbol{w}(t)) \\ \boldsymbol{e}(t) &= \boldsymbol{g}(\boldsymbol{x}(t), \boldsymbol{u}(t), \boldsymbol{w}(t)), \end{cases} \tag{1}$$

where $\boldsymbol{x} \in \mathbb{R}^{n_{\boldsymbol{x}}}$ is the state of the plant, $\boldsymbol{u} \in \mathbb{R}^{n_{\boldsymbol{u}}}$ is the input of the plant, $\boldsymbol{e} \in \mathbb{R}^{n_{\boldsymbol{e}}}$ is the output of the plant, $\boldsymbol{w} \in \mathbb{R}^{n_{\boldsymbol{w}}}$ are disturbances, and $t \in \mathbb{R}$ is time. In the context of extremum-seeking control, the input $\boldsymbol{u}$ is a vector of tunable plant parameters, the output $\boldsymbol{e}$ is a vector of measured performance variables, and $\boldsymbol{w}$ are (time-varying) disturbances, for which we adopt the following assumption.

Assumption 1 *The disturbances $\boldsymbol{w}(t)$ are piecewise continuous, defined and bounded on $t \in \mathbb{R}$. Moreover, there exists a constant $\rho_{\boldsymbol{w}} \in \mathbb{R}_{>0}$ such that $\boldsymbol{w}(t) \in \mathcal{W}$ for all $t \in \mathbb{R}$, with $\mathcal{W} = \{\boldsymbol{w} \in \mathbb{R}^{n_{\boldsymbol{w}}} : \|\boldsymbol{w}\| \leq \rho_{\boldsymbol{w}}\}$.*

In addition, we adopt the following assumption on the plant.

Assumption 2 *The plant Σ_p in* (1) *is globally exponentially convergent*[1] *for all constant inputs $\boldsymbol{u} \in \mathcal{U}$, where $\mathcal{U} \subset \mathbb{R}^{n_{\boldsymbol{u}}}$ is a compact set.*

Remark 1 *Assumption 2 guarantees that, for any constant $\boldsymbol{u} \in \mathcal{U}$ and any $\boldsymbol{w}(t) \in \mathcal{W}$, there exists a unique globally exponentially stable (time-varying) steady-state solution. This assumption is the time-varying analogue of the common assumption in extremum-seeking literature on the plant exhibiting globally asymptotically stable equilibria. In many (nonlinear) control problems, for example tracking, synchronization, observer design and output regulation problems, the convergent system property that all solutions of a closed-loop system converges to some steady-state solution and thus "forget" their initial condition plays an important role. Moreover, this property is immediate for asymptotically stable linear time-invariant systems with inputs.*

From Assumptions 1 and 2, it follows that for all constant inputs $\boldsymbol{u} \in \mathcal{U}$ and all disturbances $\boldsymbol{w}(t) \in \mathcal{W}$ there exists a unique steady-state solution of the plant Σ_p, which is defined and bounded on $t \in \mathbb{R}$ and globally exponentially stable (GES). The steady-state solution is denoted by $\bar{\boldsymbol{x}}_{\boldsymbol{w}}(t, \boldsymbol{u})$, emphasizing the dependency on time-varying disturbances $\boldsymbol{w}(t)$ and constant inputs $\boldsymbol{u}$, and satisfies

$$\dot{\bar{\boldsymbol{x}}}_{\boldsymbol{w}}(t, \boldsymbol{u}) = \boldsymbol{f}(\bar{\boldsymbol{x}}_{\boldsymbol{w}}(t, \boldsymbol{u}), \boldsymbol{u}, \boldsymbol{w}(t)). \tag{2}$$

In addition, we adopt the following assumption.

Assumption 3 *The steady-state solution $\bar{\boldsymbol{x}}_{\boldsymbol{w}}(t, \boldsymbol{u})$ is twice continuously differentiable in $\boldsymbol{u}$ and satisfies*

$$\left\| \frac{\partial \bar{\boldsymbol{x}}_{\boldsymbol{w}}}{\partial \boldsymbol{u}}(t, \boldsymbol{u}) \right\| \leq L_{\boldsymbol{xu}}, \tag{3}$$

for all $t \in \mathbb{R}$, all $\boldsymbol{u} \in \mathcal{U}$, and some constant $L_{\boldsymbol{xu}} \in \mathbb{R}_{>0}$.

Furthermore, it follows from Assumption 2 that there exists a unique steady-state output of the plant Σ_p in (1), denoted by $\bar{\mathbf{e}}_{\boldsymbol{w}}(t, \boldsymbol{u})$, which is given by $\bar{\mathbf{e}}_{\boldsymbol{w}}(t, \boldsymbol{u}) = \boldsymbol{g}(\bar{\boldsymbol{x}}_{\boldsymbol{w}}(t, \boldsymbol{u}), \boldsymbol{u}, \boldsymbol{w}(t))$. It is the task of the designer to define a bounded cost function, denoted by Z, that quantifies the performance of interest for the plant under study. Then, the corresponding measured plant performance is given by

$$y(t) = Z(\boldsymbol{e}(t), \boldsymbol{u}(t)), \quad y \in \mathbb{R}. \tag{4}$$

For all constant inputs $\boldsymbol{u} \in \mathcal{U}$ and all (time-varying) disturbances $\boldsymbol{w}(t) \in \mathcal{W}$, the steady-state plant performance $\bar{y}_{\boldsymbol{w}}(t, \boldsymbol{u})$ is given by $\bar{y}_{\boldsymbol{w}}(t, \boldsymbol{u}) = Z(\boldsymbol{g}(\bar{\boldsymbol{x}}_{\boldsymbol{w}}(t, \boldsymbol{u}), \boldsymbol{u}, \boldsymbol{w}(t)), \boldsymbol{u})$. Our aim is to find the constant input values $\boldsymbol{u}$ that minimize the measured steady-state plant performance $\bar{y}_{\boldsymbol{w}}$, yielding

[1] For definitions of convergent systems the reader is referred to Section 2.2 in [19].

the optimization of the steady-state plant output $\bar{\boldsymbol{e}}_{\boldsymbol{w}}$. In the context of extremum-seeking control, ideally the measured steady-state plant performance $\bar{y}_{\boldsymbol{w}}$ is constant for constant inputs $\boldsymbol{u}$; this forms one of the basic assumptions in the extremum-seeking literature [2], [7]. However, due to the time-varying nature of the disturbances $\boldsymbol{w}(t)$ in (1), in general, the measured steady-state plant performance $\bar{y}_{\boldsymbol{w}}$ is time-varying in nature (also for constant $\boldsymbol{u}$).

To deal with time-varying plant outputs, consider the series connection of the plant Σ_p as in (1), the cost function Z as in (4), and additionally a filter, denoted by Σ_f, which reads

$$\Sigma_f : \begin{cases} \dot{\boldsymbol{z}}(t) &= \alpha_{\boldsymbol{z}} \boldsymbol{h}(\boldsymbol{z}(t), y(t)) \\ l(t) &= k(\boldsymbol{z}(t)), \end{cases} \tag{5}$$

where $\alpha_{\boldsymbol{z}} \in \mathbb{R}_{>0}$ is a tuning parameter, $\boldsymbol{z} \in \mathbb{R}^{n_{\boldsymbol{z}}}$ is the state of the filter, $y \in \mathbb{R}$ is the input of the filter defined by (4), and $l \in \mathbb{R}$ is the output of the filter. Intuitively, the filter Σ_f acts as an averaging operator on $y(t)$, utilized to quantify performance of the plant similar to the use of exponentially weighting filters [9], [17]. Basically, if we tune α_z small, the solution of $\boldsymbol{z}(t)$ will vary "slowly" in time, i.e., the output of the filter $l(t)$ will be quasi-constant and determined predominantly by the average of $y(t)$.

The series connection of the cost function Z in (4) and the filter Σ_f in (5), we call the *dynamic cost function*. We adopt the following assumption on the dynamic cost function.

Assumption 4 *The dynamic cost function consisting of the cascade of Z and Σ_f, given by (4) and (5), respectively, is exponentially input-to-state convergent[2] for all constant inputs $\boldsymbol{u} \in \mathcal{U}$ and all $\alpha_{\boldsymbol{z}} \in \mathbb{R}_{>0}$.*

The series connection of the nonlinear plant Σ_p in (1), the user-defined cost function Z in (4), and the to-be-designed filter Σ_f in (5) is referred to as the extended plant Σ and is schematically depicted in Fig. 1. The dynamics of the extended plant is given by

$$\Sigma : \begin{cases} \dot{\boldsymbol{x}}(t) &= \boldsymbol{f}(\boldsymbol{x}(t), \boldsymbol{u}(t), \boldsymbol{w}(t)) \\ \dot{\boldsymbol{z}}(t) &= \alpha_{\boldsymbol{z}} \boldsymbol{h}(\boldsymbol{z}(t), Z(\boldsymbol{g}(\boldsymbol{x}(t), \boldsymbol{u}(t), \boldsymbol{w}(t)), \boldsymbol{u}(t))) \\ l(t) &= k(\boldsymbol{z}(t)). \end{cases} \tag{6}$$

We adopt the following assumption on the extended plant regarding the smoothness of functions.

Assumption 5 *Functions $\boldsymbol{f}$ and $\boldsymbol{g}$ in (1) are twice continuously differentiable in $\boldsymbol{x}$ and $\boldsymbol{u}$ and continuous in $\boldsymbol{w}$. Function Z in (4) is twice continuously differentiable with respect to both arguments. Functions $\boldsymbol{h}$ and k in (5) are twice continuously differentiable with respect to all arguments.*

Remark 2 *The smoothness of the functions $\boldsymbol{f}$ and $\boldsymbol{g}$ in Assumption 5 is a common assumption in the extremum-seeking literature, see, e.g., [2], [7]. The smoothness of the functions Z, $\boldsymbol{h}$, and k can easily be satisfied by design.*

By similar arguments as in the proof of Property 2.27 in [19], we can conclude from Assumptions 2 and 4 that the extended plant Σ in (6) is globally exponentially convergent for all constant inputs $\boldsymbol{u} \in \mathcal{U}$ and disturbances $\boldsymbol{w}(t) \in \mathcal{W}$. As such, there exists a unique steady-state solution of Σ_f, induced by the extended plant, which is defined and bounded on $t \in \mathbb{R}$ and GES. This steady-state solution is denoted by $\bar{\boldsymbol{z}}_{\boldsymbol{w}}(t, \boldsymbol{u}, \alpha_{\boldsymbol{z}})$, emphasizing the dependency on time-varying disturbances $\boldsymbol{w}(t)$, constant inputs $\boldsymbol{u}$, and the tunable parameter $\alpha_{\boldsymbol{z}}$, and satisfies

$$\dot{\bar{\boldsymbol{z}}}_{\boldsymbol{w}}(t, \boldsymbol{u}, \alpha_{\boldsymbol{z}}) = \alpha_{\boldsymbol{z}} \boldsymbol{h}(\bar{\boldsymbol{z}}_{\boldsymbol{w}}(t, \boldsymbol{u}, \alpha_{\boldsymbol{z}}), \bar{y}_{\boldsymbol{w}}(t, \boldsymbol{u})). \tag{7}$$

In addition, we adopt the following assumption.

Assumption 6 *There exists a twice continuously differentiable function $\boldsymbol{q}_{\boldsymbol{w}} : \mathbb{R}^{n_{\boldsymbol{u}}} \to \mathbb{R}^{n_{\boldsymbol{z}}}$, referred to as the constant performance cost, such that*

$$\lim_{\alpha_{\boldsymbol{z}} \to 0} \bar{\boldsymbol{z}}_{\boldsymbol{w}}(t, \boldsymbol{u}, \alpha_{\boldsymbol{z}}) = \boldsymbol{q}_{\boldsymbol{w}}(\boldsymbol{u}), \tag{8}$$

[2]For the definition of input-to-state convergent the reader is referred to Definition 2.18 in [19].

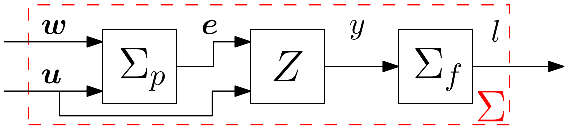

Figure 1: The extended plant Σ, i.e., series connection of the nonlinear plant Σ_p, the user-defined cost function Z, and the to-be-designed filter Σ_f.

for all $t \in \mathbb{R}$ *and all* $\boldsymbol{u} \in \mathcal{U}$ *and* $\boldsymbol{w}(t) \in \mathcal{W}$. *Moreover, there exist constants* $\delta_{\boldsymbol{w}} \in \mathbb{R}_{\geq 0}$, *related to the disturbances* $\boldsymbol{w}(t)$ *and the extended plant* Σ, *and* $L_{\boldsymbol{z}1} \in \mathbb{R}_{>0}$, *such that*

$$\|\bar{\boldsymbol{z}}_{\boldsymbol{w}}(t, \boldsymbol{u}, \alpha_{\boldsymbol{z}}) - \boldsymbol{q}_{\boldsymbol{w}}(\boldsymbol{u})\| \leq \alpha_{\boldsymbol{z}} \delta_{\boldsymbol{w}}, \tag{9}$$

and

$$\left\| \frac{\partial \bar{\boldsymbol{z}}_{\boldsymbol{w}}}{\partial \boldsymbol{u}}(t, \boldsymbol{u}, \alpha_{\boldsymbol{z}}) - \frac{d\boldsymbol{q}_{\boldsymbol{w}}}{d\boldsymbol{u}}(\boldsymbol{u}) \right\| \leq \alpha_{\boldsymbol{z}} L_{\boldsymbol{z}1}, \tag{10}$$

for all $t \in \mathbb{R}$, *all* $\boldsymbol{u} \in \mathcal{U}$ *and all* $0 < \alpha_{\boldsymbol{z}} \leq \epsilon_{\boldsymbol{z}}$ *for some* $\epsilon_{\boldsymbol{z}} \in \mathbb{R}_{>0}$.

Hence, by Assumption 6, under steady-state conditions of the plant Σ_p, the cost function Z, the filter Σ_f, the limit $\alpha_{\boldsymbol{z}} \to 0$, and for constant inputs $\boldsymbol{u} \in \mathcal{U}$, we have that the parameter-to-steady-state performance cost of the plant can be characterized by the static input-to-output map

$$F_{\boldsymbol{w}}(\boldsymbol{u}) := k(\boldsymbol{q}_{\boldsymbol{w}}(\boldsymbol{u})), \quad \forall\ \boldsymbol{u} \in \mathcal{U}. \tag{11}$$

We refer to the map $F_{\boldsymbol{w}}$ as the objective function. To minimize the steady-state plant performance $\bar{y}_{\boldsymbol{w}}$, we aim to find the plant parameter values for which the objective function in (11) is minimal. We further assume that the dynamic cost function $Z + \Sigma_f$ is designed such that there exists a unique minimum of the objective function $F_{\boldsymbol{w}}$ on the compact set $\mathcal{U}$ for any (time-varying) disturbance $\boldsymbol{w}(t) \in \mathcal{W}$ satisfying Assumption 1, where the minimum of the map $F_{\boldsymbol{w}}$ corresponds to the optimal plant performance. This assumption is formulated as follows.

Assumption 7 *The objective function* $F_{\boldsymbol{w}} : \mathbb{R}^{n_{\boldsymbol{u}}} \to \mathbb{R}$ *in* (11) *is twice continuously differentiable and exhibits a unique minimum in the interior of the compact set* $\mathcal{U}$. *Let the corresponding optimal input* $\boldsymbol{u}^*$ *be defined as*

$$\boldsymbol{u}^* = \arg\min_{u \in \mathcal{U}} F_{\boldsymbol{w}}(\boldsymbol{u}). \tag{12}$$

Furthermore, there exists a constant $L_{F1} \in \mathbb{R}_{>0}$ *such that*

$$\frac{dF_{\boldsymbol{w}}}{d\boldsymbol{u}}(\boldsymbol{u})(\boldsymbol{u} - \boldsymbol{u}^*) \geq L_{F1}\|\boldsymbol{u} - \boldsymbol{u}^*\|^2, \quad \forall\ \boldsymbol{u} \in \mathcal{U}. \tag{13}$$

From Assumption 7, it follows that the vector of tunable plant parameters $\boldsymbol{u}$ will converge to optimal input $\boldsymbol{u}^*$ if we are able to design a controller that drives the tunable plant parameters $\boldsymbol{u}$ in opposite direction of the gradient of the objective function in (11). However, since the steady-state solutions of the plant in (1) and the filter in (5) and the objective function $F_{\boldsymbol{w}}$ are unknown, we typically cannot design such a gradient-descent controller. Information of the objective function can only be obtained through measured outputs l of the extended plant in (6). The measured output differs from the objective function $F_{\boldsymbol{w}}$ in two ways; i) due to the dynamics of the plant in (1) and the filter in (5) not being in steady-state, and ii) due to the presence of (time-varying) disturbances $\boldsymbol{w}(t)$ and the design parameter $\alpha_{\boldsymbol{z}}$ which, in the presence of time-varying disturbances $\boldsymbol{w}(t)$, is typically designed to be small, but still non-zero and positive. Nevertheless, we aim to steer the inputs $\boldsymbol{u}$ to their performance optimizing values $\boldsymbol{u}^*$ by using the measured extended plant output $l(t)$ as feedback to an extremum-seeking controller that is introduced in the next section.

3 Extremum-seeking controller

The controller design proposed here follows from the one in [12, Ch. 2]. In Section 3.1, a dither signal design is presented, in Section 3.2, a model of the input-to-output behavior of the plant is presented to be used as a basis for gradient estimation, in Section 3.3, a least-squares observer to estimate the state of that model (and therewith the gradient) and a normalized optimizer to steer the plant parameters $\boldsymbol{u}$ to the minimizer $\boldsymbol{u}^*$ are presented, and, in Section 3.4, tuning guidelines are provided for the closed-loop system composed of the extended plant Σ in (6) and the extremum-seeking controller.

3.1 Dither signal

To estimate the gradient of the objective function and use this estimated gradient to drive $\boldsymbol{u}$ towards $\boldsymbol{u}^*$ by an optimizer, we define the following input signal:

$$\boldsymbol{u}(t) = \hat{\boldsymbol{u}}(t) + \alpha_{\boldsymbol{\omega}}\boldsymbol{\omega}(t), \tag{14}$$

where $\alpha_{\boldsymbol{\omega}}\boldsymbol{\omega}$ is a vector of perturbation signals with amplitude $\alpha_{\boldsymbol{\omega}} \in \mathbb{R}_{>0}$, and $\hat{\boldsymbol{u}}$ is referred to as the nominal input to be generated by the extremum-seeking controller. The vector $\boldsymbol{\omega}$ is defined by $\boldsymbol{\omega}(t) = [\omega_1(t), \omega_2(t), ..., \omega_{n_{\boldsymbol{u}}}(t)]^\top$, with

$$\omega_i(t) = \begin{cases} \sin\left(\frac{i+1}{2}\eta_{\boldsymbol{\omega}} t\right), & \text{if } i \text{ is odd}, \\ \cos\left(\frac{i}{2}\eta_{\boldsymbol{\omega}} t\right), & \text{if } i \text{ is even}, \end{cases} \tag{15}$$

for $i = \{1, 2, ..., n_{\boldsymbol{u}}\}$, where $\eta_{\boldsymbol{\omega}} \in \mathbb{R}_{>0}$ is a tuning parameter. The purpose of the perturbation signal is to provide sufficient excitation to accurately estimate the gradient of the objective function. The nominal plant parameters $\hat{\boldsymbol{u}}$ can be regarded as an estimate of the minimizer $\boldsymbol{u}^*$.

3.2 Model of input-to-output behavior of the extended plant

To obtain an estimate of the gradient of the objective function, we model the input-to-output behavior of the extended plant in (6), that is, from the nominal input $\hat{\boldsymbol{u}}$ to the measured output of the extended plant l, in a general form. Let the state of the model be given by

$$\boldsymbol{m}(t) = [F_{\boldsymbol{w}}(\hat{\boldsymbol{u}}(t)) \quad \alpha_{\boldsymbol{\omega}} \tfrac{dF_{\boldsymbol{w}}}{d\boldsymbol{u}}(\hat{\boldsymbol{u}}(t))]^\top. \tag{16}$$

From Taylor's Theorem and (14), $F_{\boldsymbol{w}}$ can be written as

$$\begin{aligned} F_{\boldsymbol{w}}(\boldsymbol{u}(t)) &= F_{\boldsymbol{w}}(\hat{\boldsymbol{u}}(t)) + \alpha_{\boldsymbol{\omega}} \frac{dF_{\boldsymbol{w}}}{d\boldsymbol{u}}(\hat{\boldsymbol{u}}(t))\boldsymbol{\omega}(t) \\ &+ \alpha_{\boldsymbol{\omega}}^2 \boldsymbol{\omega}^\top(t) \int_0^1 (1-\sigma) \frac{d^2 F_{\boldsymbol{w}}}{d\boldsymbol{u} d\boldsymbol{u}^\top}(\hat{\boldsymbol{u}}(t) + \sigma\alpha_{\boldsymbol{\omega}}\boldsymbol{\omega}(t)) d\sigma \boldsymbol{\omega}(t). \end{aligned} \tag{17}$$

The dynamics of the state in (16) is governed by

$$\begin{aligned} \dot{\boldsymbol{m}}(t) &= \boldsymbol{A}(t)\boldsymbol{m}(t) + \alpha_{\boldsymbol{\omega}}^2 \boldsymbol{B}\boldsymbol{s}(t) \\ l(t) &= \boldsymbol{C}(t)\boldsymbol{m}(t) + \alpha_{\boldsymbol{\omega}}^2 v(t) + r(t) + d(t), \end{aligned} \tag{18}$$

with the matrices $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ defined as

$$\boldsymbol{A}(t) = \begin{bmatrix} 0 & \frac{\dot{\hat{\boldsymbol{u}}}^\top(t)}{\alpha_{\boldsymbol{\omega}}} \\ 0 & 0 \end{bmatrix}, \; \boldsymbol{B} = \begin{bmatrix} 0 \\ \boldsymbol{I} \end{bmatrix}, \; \boldsymbol{C}(t) = \begin{bmatrix} 1 & \boldsymbol{\omega}^\top(t) \end{bmatrix}, \tag{19}$$

and the signals $\boldsymbol{s}$, v, r, and d defined as

$$\begin{aligned} \boldsymbol{s}(t) &:= \frac{d^2 F_{\boldsymbol{w}}}{d\boldsymbol{u} d\boldsymbol{u}^\top}(\hat{\boldsymbol{u}}(t)) \frac{\dot{\hat{\boldsymbol{u}}}(t)}{\alpha_{\boldsymbol{\omega}}}, \\ v(t) &:= \boldsymbol{\omega}^\top(t) \int_0^1 (1-\sigma) \frac{d^2 F_{\boldsymbol{w}}}{d\boldsymbol{u} d\boldsymbol{u}^\top}(\hat{\boldsymbol{u}}(t) + \sigma\alpha_{\boldsymbol{\omega}}\boldsymbol{\omega}(t)) d\sigma \boldsymbol{\omega}(t), \\ r(t) &:= k(\boldsymbol{z}(t)) - k(\bar{\boldsymbol{z}}_{\boldsymbol{w}}(t, \boldsymbol{u}, \alpha_{\boldsymbol{z}})), \\ d(t) &:= k(\bar{\boldsymbol{z}}_{\boldsymbol{w}}(t, \boldsymbol{u}, \alpha_{\boldsymbol{z}})) - k(\boldsymbol{q}_{\boldsymbol{w}}(\boldsymbol{u}(t))). \end{aligned} \tag{20}$$

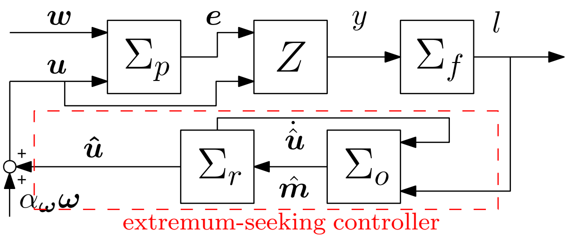

Figure 2: The closed-loop system composed of the extended plant Σ, the observer Σ_o, the optimizer Σ_r, and the dither signal $\alpha_{\boldsymbol{\omega}}\boldsymbol{\omega}$.

The signals $\boldsymbol{s}$, v, r and d can be interpreted as unknown disturbances to the model. The influences of $\boldsymbol{s}$, v, r and d on the state and output of the model in (18) are small if i) $\hat{\boldsymbol{u}}$ is slowly time varying, if ii) $\alpha_{\boldsymbol{\omega}}$ is small, if iii) the states $\boldsymbol{x}$ of the plant in (1) and the states $\boldsymbol{z}$ of the filter in (5) are close to their steady-state values, and if iv) $\alpha_{\boldsymbol{z}}$ is small.

The state $\boldsymbol{m}$ in (16) contains an estimate of the gradient of the objective function, scaled by the perturbation amplitude $\alpha_{\boldsymbol{\omega}}$. Hence, an estimate of the gradient of the objective function can be obtained from an estimate of the state $\boldsymbol{m}$. Based on this gradient estimate, an optimizer can steer the plant parameters $\boldsymbol{u}$ to the minimizer $\boldsymbol{u}^*$. In the next section, a least-squares observer and an optimizer for this purpose are proposed.

3.3 Controller design

We introduce an extremum-seeking controller that is composed of a dither signal as in (14), a least-squares observer to estimate the state $\boldsymbol{m}$ of the model in (18), and an optimizer that uses the estimate of the state $\boldsymbol{m}$ of the observer, denoted by $\hat{\boldsymbol{m}}$, to steer the nominal plant inputs $\hat{\boldsymbol{u}}$ to their performance optimal values $\boldsymbol{u}^*$.

The least-squares observer, denoted by Σ_o, is given by

$$\Sigma_o : \begin{cases} \dot{\hat{\boldsymbol{m}}}(t) = \left(\boldsymbol{A}(t) - \eta_{\boldsymbol{m}}\sigma_r \boldsymbol{Q}(t)\boldsymbol{D}^\top \boldsymbol{D}\right)\hat{\boldsymbol{m}}(t) \\ \qquad\qquad +\eta_{\boldsymbol{m}}\boldsymbol{Q}(t)\boldsymbol{C}^\top(t)(l(t) - \boldsymbol{C}(t)\hat{\boldsymbol{m}}(t)) \\ \dot{\boldsymbol{Q}}(t) = \eta_{\boldsymbol{m}}\boldsymbol{Q}(t) + \boldsymbol{A}(t)\boldsymbol{Q}(t) + \boldsymbol{Q}(t)\boldsymbol{A}^\top(t) \\ \qquad\qquad -\eta_{\boldsymbol{m}}\boldsymbol{Q}(t)(\boldsymbol{C}^\top(t)\boldsymbol{C}(t) + \sigma_r\boldsymbol{D}^\top\boldsymbol{D})\boldsymbol{Q}(t), \end{cases} \tag{21}$$

where $\boldsymbol{D} = [\boldsymbol{0} \quad \boldsymbol{I}]$, and $\eta_{\boldsymbol{m}}$, $\sigma_r \in \mathbb{R}_{>0}$ are tuning parameters related to the observer, referred to as a forgetting factor and a regularization constant.

The optimizer, denoted by Σ_r, is given by

$$\Sigma_r : \qquad \dot{\hat{\boldsymbol{u}}}(t) = -\lambda_{\boldsymbol{u}}\frac{\eta_{\boldsymbol{u}}\boldsymbol{D}\hat{\boldsymbol{m}}(t)}{\eta_{\boldsymbol{u}} + \lambda_{\boldsymbol{u}}\|\boldsymbol{D}\hat{\boldsymbol{m}}(t)\|}, \tag{22}$$

with $\lambda_{\boldsymbol{u}}$, $\eta_{\boldsymbol{u}} \in \mathbb{R}_{>0}$ being tuning parameters related to the optimizer. Normalization of the adaptation gain in (22) is done to prevent solutions of the closed-loop system of the extended plant and the extremum-seeking controller from having a finite escape time if the state estimate $\hat{\boldsymbol{m}}$ is inaccurate [12, Ch. 2]. The closed-loop system, composed of the extended plant Σ in (6), the observer Σ_o in (21), and the optimizer Σ_r in (22), is depicted in Fig. 2.

3.4 Tuning guidelines

For the closed-loop system to operate properly, we have design guidelines that guarantee time-scale separation:

1) The convergence of the solutions of the plant dynamics in (1) to its steady-state operation is assumed to be *fast*,

2) The tuning parameter $\alpha_{\boldsymbol{z}}$ of the filter in (5) is chosen small such that the difference between the time-varying steady-state solution of the extended plant Σ and the performance cost is small (see Assumption 6), however sufficiently large such that convergence of solutions of the filter dynamics is on a *medium-to-fast* time scale,

3) The dither frequencies parameterized by $\eta_{\boldsymbol{\omega}}$ are chosen slower than the filter dynamics to provide sufficient excitation, admitting a *medium* time-scale,

4) The observer should use a sufficiently long time history of the perturbation signals and measurement signal to be able to accurately extract the state of the model [12, Ch. 2]; the observer dynamics and its design parameter $\eta_{\boldsymbol{m}}$ should be associated with a *medium-to-slow* time scale compared to the dither signal,

5) The nominal plant parameters $\hat{\boldsymbol{u}}$, induced by the optimizer, should be slowly time varying with respect to the observer by proper design of the design parameters $\lambda_{\boldsymbol{u}}$ and $\eta_{\boldsymbol{u}}$, admitting a *slow* (optimizer) time-scale.

4 Stability analysis

In this section, we will provide a stability result for the closed-loop system described in the previous sections. Due to the perturbation of the tunable parameter $\boldsymbol{u}$, the optimizer state $\hat{\boldsymbol{u}}$ will in general converge to a region of the performance-optimal value $\boldsymbol{u}^*$. The next result states conditions on tuning parameters and initial conditions under which the extremum-seeking scheme guarantees that $\hat{\boldsymbol{u}}$ converge to an arbitrarily small set around the optimum $\boldsymbol{u}^*$.

Theorem 1 *Under Assumptions 1-7, there exist (sufficiently small) constants* $\epsilon_1, ..., \epsilon_6 \in \mathbb{R}_{>0}$, *and initial conditions* $\boldsymbol{x}(0) \in \mathcal{X}_0$, *symmetric and positive-definite* $\boldsymbol{Q}(0) \in \mathcal{Q}_0$, $\hat{\boldsymbol{u}}(0) \in \mathcal{U}_0$, $\boldsymbol{z}(0) \in \mathcal{Z}_0$, *and* $\hat{\boldsymbol{m}}(0) \in \mathcal{M}_0$, *where* $\mathcal{X}_0 \subset \mathbb{R}^{n_{\boldsymbol{x}}}$, $\mathcal{U}_0 \subset \mathbb{R}^{n_{\boldsymbol{u}}}$, $\mathcal{Q}_0 \subset \mathbb{R}^{n_{\boldsymbol{u}}+1 \times n_{\boldsymbol{u}}+1}$, $\mathcal{Z}_0 \subset \mathbb{R}^{n_{\boldsymbol{z}}}$, $\mathcal{M}_0 \subset \mathbb{R}^{n_{\boldsymbol{u}}+1}$ *are compact sets, such that the solutions of the closed-loop system consisting of the extended plant in* (6) *and the extremum-seeking controller (consisting of the dither signal in* (14)*, the observer* Σ_o *in* (21)*, and the optimizer* Σ_r *in* (22)*) are uniformly bounded for all* $\alpha_{\boldsymbol{z}}, \alpha_{\boldsymbol{\omega}}, \eta_{\boldsymbol{u}}, \lambda_{\boldsymbol{u}}, \eta_{\boldsymbol{m}}, \eta_{\boldsymbol{\omega}} \in \mathbb{R}_{>0}$ *and all* $\sigma_r \in \mathbb{R}_{\geq 0}$ *that satisfy* $\alpha_{\boldsymbol{z}} \leq \epsilon_1$, $\eta_{\boldsymbol{\omega}} \leq \alpha_{\boldsymbol{z}}\epsilon_2$, $\eta_{\boldsymbol{m}} \leq \eta_{\boldsymbol{\omega}}\epsilon_3$, $\alpha_{\boldsymbol{\omega}}\lambda_{\boldsymbol{u}} \leq \eta_{\boldsymbol{m}}\epsilon_4$, $\eta_{\boldsymbol{u}} \leq \alpha_{\boldsymbol{\omega}}\eta_{\boldsymbol{m}}\epsilon_5$, *and* $\sigma_r \leq \epsilon_6$. *Moreover, the solutions* $\hat{\mathbf{u}}(t)$ *satisfy*

$$\limsup_{t \to \infty} \|\tilde{\boldsymbol{u}}(t)\| \leq \max\left\{\alpha_{\boldsymbol{\omega}} c_1, \frac{\eta_{\boldsymbol{\omega}}}{\alpha_{\boldsymbol{z}}} c_2, \frac{\alpha_{\boldsymbol{z}} \delta_{\boldsymbol{w}}}{\alpha_{\boldsymbol{\omega}}} c_3\right\}, \tag{23}$$

for some constants $c_1, ..., c_3 \in \mathbb{R}_{>0}$, *with* $\tilde{\mathbf{u}}(t) = \hat{\mathbf{u}}(t) - \boldsymbol{u}^*$.

Proof of Theorem 1 *The proof can be found in Section 5.* □

Remark 3 *Tuning guidelines. Under the conditions of Theorem 1, it follows that, if we are dealing with constant (or no) disturbances* $\boldsymbol{w}(t)$, *i.e.,* $\delta_{\boldsymbol{w}} = 0$, *the optimizer state* $\hat{\mathbf{u}}$ *converges to an arbitrarily small region of the performance-optimal value* $\boldsymbol{u}^*$ *if the dither parameters* $\alpha_{\boldsymbol{\omega}}$ *and* $\eta_{\boldsymbol{\omega}}$ *are chosen sufficiently small for an arbitrary bounded* $\alpha_{\boldsymbol{z}}$. *To make the region to which* $\hat{\mathbf{u}}$ *converges arbitrarily small in case we are dealing with time-varying disturbances* $\boldsymbol{w}(t)$, *i.e.,* $\delta_{\boldsymbol{w}} > 0$, *see* (23), *we subsequently tune* $\alpha_{\boldsymbol{\omega}}$ *small to make the first term in the right-hand side of* (23) *arbitrarily small, tune* $\alpha_{\boldsymbol{z}}$ *small to make the third term in the right-hand side of* (23) *arbitrarily small, and finally tune* $\eta_{\boldsymbol{\omega}}$ *small to make the second term in the right-hand side of* (23) *arbitrarily small.*

5 Proof of Theorem 1

The proof of Theorem 1 is partially inspired by the one in [12, Ch. 2]. To prove Theorem 1, we introduce the following coordinate transformation:

$$\begin{aligned}\tilde{\boldsymbol{x}}(t) &= \boldsymbol{x}(t) - \bar{\boldsymbol{x}}_{\boldsymbol{w}}(t, \boldsymbol{u}(t)),\\ \tilde{\boldsymbol{z}}(t) &= \boldsymbol{z}(t) - \bar{\boldsymbol{z}}_{\boldsymbol{w}}(t, \boldsymbol{u}(t), \alpha_{\boldsymbol{z}}),\\ \tilde{\boldsymbol{m}}(t) &= \hat{\boldsymbol{m}}(t) - \boldsymbol{m}(t),\\ \tilde{\boldsymbol{Q}}(t) &= \boldsymbol{Q}^{-1}(t) - \Xi^{-1} - \tfrac{\eta_{\boldsymbol{m}}}{\eta_\omega}\boldsymbol{n}(t),\\ \tilde{\boldsymbol{u}}(t) &= \hat{\boldsymbol{u}}(t) - \boldsymbol{u}^*,\end{aligned} \tag{24}$$

with

$$\boldsymbol{n}(t) = \int_0^t \eta_\omega \begin{bmatrix} 0 & \boldsymbol{\omega}^T(\tau) \\ \boldsymbol{\omega}(\tau) & \boldsymbol{\omega}(\tau)\boldsymbol{\omega}^T(\tau) - \frac{1}{2}\boldsymbol{I}\end{bmatrix} d\tau \tag{25}$$

and

$$\Xi = \begin{bmatrix} 1 & \boldsymbol{0} \\ \boldsymbol{0} & \frac{2}{1+2\sigma_r}\boldsymbol{I}\end{bmatrix}. \tag{26}$$

Let us define the following vector fields:

$$\begin{aligned}\tilde{\boldsymbol{f}}(t, \tilde{\boldsymbol{x}}, \boldsymbol{u}) &:= \boldsymbol{f}(\tilde{\boldsymbol{x}} + \bar{\boldsymbol{x}}_{\boldsymbol{w}}(t, \boldsymbol{u}), \boldsymbol{u}, \boldsymbol{w}(t)) - \boldsymbol{f}(\bar{\boldsymbol{x}}_{\boldsymbol{w}}(t, \boldsymbol{u}), \boldsymbol{u}, \boldsymbol{w}(t)),\\ \tilde{\boldsymbol{h}}(t, \tilde{\boldsymbol{z}}, \boldsymbol{u}, \alpha_{\boldsymbol{z}}) &:= \boldsymbol{h}(\tilde{\boldsymbol{z}} + \bar{\boldsymbol{z}}_{\boldsymbol{w}}(t, \boldsymbol{u}, \alpha_{\boldsymbol{z}}), \bar{y}_{\boldsymbol{w}}(t, \boldsymbol{u})) - \boldsymbol{h}(\bar{\boldsymbol{z}}_{\boldsymbol{w}}(t, \boldsymbol{u}, \alpha_{\boldsymbol{z}}), \bar{y}_{\boldsymbol{w}}(t, \boldsymbol{u})).\end{aligned} \tag{27}$$

Furthermore, let us consider all variables on compact sets, i.e., let us define positive constants $L_{\boldsymbol{x}}, L_{\boldsymbol{u}}, \rho_{\boldsymbol{x}}, \rho_{\boldsymbol{Q}}, L_{\boldsymbol{z}}, \rho_{\boldsymbol{z}}, \rho_{\boldsymbol{u}}, \rho_{\boldsymbol{m}} \in \mathbb{R}_{>0}$ and the following compact sets:

$$\begin{aligned}\mathcal{X} &= \{\boldsymbol{x} \in \mathbb{R}^{n_{\boldsymbol{x}}} : \|\tilde{\boldsymbol{x}}\| \le L_{\boldsymbol{x}}\},\\ \mathcal{U} &= \{\boldsymbol{u} \in \mathbb{R}^{n_{\boldsymbol{u}}} : \|\boldsymbol{u} - \boldsymbol{u}^*\| \le L_{\boldsymbol{u}}\},\\ \mathcal{X}_0 &= \{\boldsymbol{x} \in \mathbb{R}^{n_{\boldsymbol{x}}} : \|\tilde{\boldsymbol{x}}\| \le \rho_{\boldsymbol{x}}\},\\ \mathcal{Q}_0 &= \{\boldsymbol{Q} \in \mathbb{R}^{n_{\boldsymbol{u}}+1 \times n_{\boldsymbol{u}}+1} : \|\tilde{\boldsymbol{Q}}\| \le \rho_{\boldsymbol{Q}}\},\\ \mathcal{Z} &= \{\boldsymbol{z} \in \mathbb{R}^{n_{\boldsymbol{z}}} : \|\tilde{\boldsymbol{z}}\| \le L_{\boldsymbol{z}}\},\\ \mathcal{Z}_0 &= \{\boldsymbol{z} \in \mathbb{R}^{n_{\boldsymbol{z}}} : \|\tilde{\boldsymbol{z}}\| \le \rho_{\boldsymbol{z}}\},\\ \mathcal{U}_0 &= \{\hat{\boldsymbol{u}} \in \mathbb{R}^{n_{\boldsymbol{u}}} : \|\hat{\boldsymbol{u}} - \boldsymbol{u}^*\| \le \rho_{\boldsymbol{u}}\},\\ \mathcal{M}_0 &= \{\hat{\boldsymbol{m}} \in \mathbb{R}^{n_{\boldsymbol{u}}+1} : \|\tilde{\boldsymbol{m}}\| \le \rho_{\boldsymbol{m}}\}\end{aligned} \tag{28}$$

Loosely speaking, the analysis of the stability properties of the closed-loop system can be divided into three temporal stages, where we defined some finite time instances t_1 and t_2:

- for $0 \le t < t_1$ the solutions $\tilde{\boldsymbol{x}}$ and $\tilde{\boldsymbol{Q}}$ converge to a neighborhood of the origin and remain there, the solution $\tilde{\boldsymbol{z}}$ converges but may still be away from a neighborhood of the origin, while the solutions $\tilde{\boldsymbol{m}}$ and $\tilde{\boldsymbol{u}}$ may drift, but remain bounded.

- for $t_1 \le t \le t_2$, the solutions $\tilde{\boldsymbol{x}}$ and $\tilde{\boldsymbol{Q}}$ have already converged to a neighborhood of the origin, the solution $\tilde{\boldsymbol{z}}$ converges to a neighborhood of the origin, while the solutions $\tilde{\boldsymbol{m}}$ and $\tilde{\boldsymbol{u}}$ may drift, but remain bounded.

- for $t \ge t_2$, the solutions $\tilde{\boldsymbol{m}}$ and $\tilde{\boldsymbol{u}}$ also converge to a neighborhood of the origin.

We first derive bounds on each of the variables in (24) corresponding to these three temporal stages of convergence.

From the plant Σ_p in (1) and its steady-state solution that satisfies (2), the coordinate transformation in (24), and the vector field defined by (27), it follows that the dynamics of $\tilde{\boldsymbol{x}}$ for *constant* inputs $\boldsymbol{u}$ is governed by

$$\begin{aligned}\dot{\tilde{\boldsymbol{x}}}(t) &= \dot{\boldsymbol{x}}(t) - \dot{\bar{\boldsymbol{x}}}_{\boldsymbol{w}}(t,\boldsymbol{u}) \\ &= \boldsymbol{f}(\boldsymbol{x},\boldsymbol{u},\boldsymbol{w}(t)) - \boldsymbol{f}(\bar{\boldsymbol{x}}_{\boldsymbol{w}}(t,\boldsymbol{u}),\boldsymbol{u},\boldsymbol{w}(t)) \\ &= \boldsymbol{f}(\tilde{\boldsymbol{x}} + \bar{\boldsymbol{x}}_{\boldsymbol{w}}(t,\boldsymbol{u}),\boldsymbol{u},\boldsymbol{w}(t)) - \boldsymbol{f}(\bar{\boldsymbol{x}}_{\boldsymbol{w}}(t,\boldsymbol{u}),\boldsymbol{u},\boldsymbol{w}(t)) \\ &= \tilde{\boldsymbol{f}}(t,\tilde{\boldsymbol{x}},\boldsymbol{u}).\end{aligned} \tag{29}$$

To derive a bound on the solutions of $\tilde{\boldsymbol{x}}(t)$ for time-varying inputs $\boldsymbol{u}(t)$, a preliminary result is presented in Lemma 1 on the existence of a Lyapunov function for the $\tilde{\boldsymbol{x}}$-dynamics, on a compact set, for constant inputs $\boldsymbol{u} \in \mathcal{U}$ in (29), and satisfying Assumptions 1-3 and 5.

Lemma 1 *Under Assumptions 1-3 and 5, there exists a function* $V_{\boldsymbol{x}} : \mathbb{R} \times \mathbb{R}^{n_{\boldsymbol{x}}} \times \mathbb{R}^{n_{\boldsymbol{u}}} \to \mathbb{R}$*, and constants* $\gamma_{\boldsymbol{x}1}, \gamma_{\boldsymbol{x}2}, ..., \gamma_{\boldsymbol{x}5} \in \mathbb{R}_{>0}$*, such that the inequalities*

$$\gamma_{\boldsymbol{x}1}\|\tilde{\boldsymbol{x}}\|^2 \leq V_{\boldsymbol{x}}(t,\tilde{\boldsymbol{x}},\boldsymbol{u}) \leq \gamma_{\boldsymbol{x}2}\|\tilde{\boldsymbol{x}}\|^2, \tag{30}$$

$$\frac{\partial V_{\boldsymbol{x}}}{\partial t}(t,\tilde{\boldsymbol{x}},\boldsymbol{u}) + \frac{\partial V_{\boldsymbol{x}}}{\partial \tilde{\boldsymbol{x}}}(t,\tilde{\boldsymbol{x}},\boldsymbol{u})\tilde{\boldsymbol{f}}(t,\tilde{\boldsymbol{x}},\boldsymbol{u}) \leq -\gamma_{\boldsymbol{x}3}\|\tilde{\boldsymbol{x}}\|^2, \tag{31}$$

and

$$\left\|\frac{\partial V_{\boldsymbol{x}}}{\partial \tilde{\boldsymbol{x}}}(t,\tilde{\boldsymbol{x}},\boldsymbol{u})\right\| \leq \gamma_{\boldsymbol{x}4}\|\tilde{\boldsymbol{x}}\|, \qquad \left\|\frac{\partial V_{\boldsymbol{x}}}{\partial \boldsymbol{u}}(t,\tilde{\boldsymbol{x}},\boldsymbol{u})\right\| \leq \gamma_{\boldsymbol{x}5}\|\tilde{\boldsymbol{x}}\|, \tag{32}$$

are satisfied for all $t \in \mathbb{R}$*, all* $\boldsymbol{x} \in \mathcal{X}$*, all constant* $\boldsymbol{u} \in \mathcal{U}$*, and each (time-varying) disturbance* $\boldsymbol{w}(t) \in \mathcal{W}$*.*

Proof. See Section 6.1. □

From the plant Σ_p in (1) and its steady-state solution that satisfies (2), the coordinate transformation in (24), and the vector field defined by (27), it follows that the state equation for $\tilde{\boldsymbol{x}}$ for *time-varying* inputs $\boldsymbol{u}(t)$ is given by

$$\begin{aligned}\dot{\tilde{\boldsymbol{x}}}(t) &= \dot{\boldsymbol{x}}(t) - \dot{\bar{\boldsymbol{x}}}_{\boldsymbol{w}}(t,\boldsymbol{u}(t)) \\ &= \dot{\boldsymbol{x}}(t) - \frac{\partial \bar{\boldsymbol{x}}_{\boldsymbol{w}}}{\partial t}(t,\boldsymbol{u}(t)) - \frac{\partial \bar{\boldsymbol{x}}_{\boldsymbol{w}}}{\partial \boldsymbol{u}}(t,\boldsymbol{u}(t))\dot{\boldsymbol{u}}(t) \\ &= \boldsymbol{f}(\boldsymbol{x}(t),\boldsymbol{u}(t),\boldsymbol{w}(t)) - \boldsymbol{f}(\bar{\boldsymbol{x}}_{\boldsymbol{w}}(t,\boldsymbol{u}(t)),\boldsymbol{u}(t),\boldsymbol{w}(t)) - \frac{\partial \bar{\boldsymbol{x}}_{\boldsymbol{w}}}{\partial \boldsymbol{u}}(t,\boldsymbol{u}(t))\dot{\boldsymbol{u}}(t) \\ &= \boldsymbol{f}(\tilde{\boldsymbol{x}}(t) + \bar{\boldsymbol{x}}_{\boldsymbol{w}}(t,\boldsymbol{u}(t)),\boldsymbol{u}(t),\boldsymbol{w}(t)) - \boldsymbol{f}(\bar{\boldsymbol{x}}_{\boldsymbol{w}}(t,\boldsymbol{u}(t)),\boldsymbol{u}(t),\boldsymbol{w}(t)) - \frac{\partial \bar{\boldsymbol{x}}_{\boldsymbol{w}}}{\partial \boldsymbol{u}}(t,\boldsymbol{u}(t))\dot{\boldsymbol{u}}(t) \\ &= \tilde{\boldsymbol{f}}(t,\tilde{\boldsymbol{x}}(t),\boldsymbol{u}(t)) - \frac{\partial \bar{\boldsymbol{x}}_{\boldsymbol{w}}}{\partial \boldsymbol{u}}(t,\boldsymbol{u}(t))\dot{\boldsymbol{u}}(t).\end{aligned} \tag{33}$$

A bound on the solutions of the $\tilde{\boldsymbol{x}}$-dynamics for time-varying inputs $\boldsymbol{u}(t)$ in (33) is presented in Lemma 2.

Lemma 2 *Under the conditions of Theorem 1, there exist constants* $c_{\boldsymbol{x}1}$*,* $c_{\boldsymbol{x}2}$*,* $\beta_{\boldsymbol{x}} \in \mathbb{R}_{>0}$ *such that the solutions of* $\tilde{\boldsymbol{x}}$ *satisfy*

$$\|\tilde{\boldsymbol{x}}(t)\| \leq \max\left\{c_{\boldsymbol{x}1}\|\tilde{\boldsymbol{x}}(0)\|e^{-\beta_{\boldsymbol{x}}t}, \alpha_{\boldsymbol{\omega}}\eta_{\boldsymbol{\omega}}c_{\boldsymbol{x}2}\right\}, \tag{34}$$

for all $t \geq 0$*, all* $\boldsymbol{x}(0) \in \mathcal{X}_0$*, and all time-varying* $\boldsymbol{u}(t) \in \mathcal{U}$*.*

Proof. See Section 6.2. □

From the observer in (21), the coordinate transformation in (24), and the model of the input-output behavior in (18) we obtain that the state equation for $\tilde{\boldsymbol{Q}}$ is given by

$$\begin{aligned}\dot{\tilde{\boldsymbol{Q}}}(t) = &-\eta_{\boldsymbol{m}}\tilde{\boldsymbol{Q}}(t) - \tilde{\boldsymbol{Q}}(t)\boldsymbol{A}(t) - \boldsymbol{A}^T(t)\tilde{\boldsymbol{Q}}(t) - \left(\Xi^{-1} + \frac{\eta_{\boldsymbol{m}}}{\eta_{\boldsymbol{\omega}}}\boldsymbol{n}(t)\right)\boldsymbol{A}(t) \\ &- \boldsymbol{A}^T(t)\left(\Xi^{-1} + \frac{\eta_{\boldsymbol{m}}}{\eta_{\boldsymbol{\omega}}}\boldsymbol{n}(\boldsymbol{t})\right) - \eta_{\boldsymbol{m}}\frac{\eta_{\boldsymbol{m}}}{\eta_{\boldsymbol{\omega}}}\boldsymbol{n}(t).\end{aligned} \tag{35}$$

A bound on the solutions of $\tilde{\boldsymbol{Q}}(t)$ is presented in Lemma 3.

Lemma 3 *Under the conditions of Theorem 1, there exist constants* $c_{\boldsymbol{Q}}$*,* $\beta_{\boldsymbol{Q}} \in \mathbb{R}_{>0}$ *such that the solutions of* $\tilde{\boldsymbol{Q}}$ *satisfy*

$$\|\tilde{\boldsymbol{Q}}(t)\| \leq \max\left\{c_{\boldsymbol{Q}}\|\tilde{\boldsymbol{Q}}(0)\|e^{-\eta_m \beta_Q t}, \frac{1}{8}\right\}, \tag{36}$$

for all $t \geq 0$*, all* $\boldsymbol{Q}(0) \in \mathcal{Q}_0$ *for which* $\boldsymbol{Q}(0)$ *is symmetric and positive definite, and all time-varying* $\boldsymbol{u}(t) \in \mathcal{U}$*.*

Proof. See [12, Ch. 2]. □

From Lemma 2 and Lemma 3, we conclude that there exists a finite time $t_1 \geq 0$ such that $\|\tilde{\boldsymbol{x}}(t)\| \leq \alpha_{\boldsymbol{\omega}}\eta_{\boldsymbol{\omega}}c_{\boldsymbol{x}2}$ and $\|\tilde{\boldsymbol{Q}}(t)\| \leq \frac{1}{8}$ for all $t \geq t_1$. These bounds on $\tilde{\boldsymbol{x}}(t)$ and $\tilde{\boldsymbol{Q}}(t)$ are utilized to obtain bounds on the solutions of $\tilde{\boldsymbol{z}}(t)$, $\tilde{\boldsymbol{u}}(t)$, and $\tilde{\boldsymbol{m}}(t)$ in Lemmas 5, 6, and 7, respectively.

From the filter Σ_f in (5) and its steady-state solution that satisfies (7), the coordinate transformation in (24), and the vector field defined by (27), it follows that the state equation for $\tilde{\boldsymbol{z}}$ for *constant* inputs $\boldsymbol{u}$ is given by

$$\begin{aligned}
\dot{\tilde{\boldsymbol{z}}}(t) &= \dot{\boldsymbol{z}}(t) - \dot{\bar{\boldsymbol{z}}}_{\boldsymbol{w}}(t, \boldsymbol{u}, \alpha_{\boldsymbol{z}}) \\
&= \alpha_{\boldsymbol{z}}\boldsymbol{h}(\boldsymbol{z}, y) - \alpha_{\boldsymbol{z}}\boldsymbol{h}(\bar{\boldsymbol{z}}_{\boldsymbol{w}}(t, \boldsymbol{u}, \alpha_{\boldsymbol{z}}), \bar{y}_{\boldsymbol{w}}(t, \boldsymbol{u})) \\
&= \alpha_{\boldsymbol{z}}\big(\boldsymbol{h}(\boldsymbol{z}, \bar{y}_{\boldsymbol{w}}(t, \boldsymbol{u})) - \boldsymbol{h}(\bar{\boldsymbol{z}}_{\boldsymbol{w}}(t, \boldsymbol{u}, \alpha_{\boldsymbol{\omega}}), \bar{y}_{\boldsymbol{w}}(t, \boldsymbol{u}))\big) + \alpha_{\boldsymbol{z}}\big(\boldsymbol{h}(\boldsymbol{z}, y) - \boldsymbol{h}(\boldsymbol{z}, \bar{y}_{\boldsymbol{w}}(t, \boldsymbol{u}))\big) \\
&= \alpha_{\boldsymbol{z}}\tilde{\boldsymbol{h}}(t, \tilde{\boldsymbol{z}}(t), \boldsymbol{u}, \alpha_{\boldsymbol{z}}) + \alpha_{\boldsymbol{z}}\big(\boldsymbol{h}(\boldsymbol{z}, y) - \boldsymbol{h}(\boldsymbol{z}, \bar{y}_{\boldsymbol{w}}(t, \boldsymbol{u}))\big).
\end{aligned} \tag{37}$$

To derive a bound on the solutions of $\tilde{\boldsymbol{z}}(t)$ for time-varying inputs $\boldsymbol{u}(t)$, a preliminary result is presented in Lemma 4 on the existence of a Lyapunov function for the $\tilde{\boldsymbol{z}}$-dynamics, on a compact set, for constant inputs $\boldsymbol{u} \in \mathcal{U}$ in (37), and satisfying Assumptions 4 and 5.

Lemma 4 *Under 4 and 5, there exists a function* $V_{\boldsymbol{z}} : \mathbb{R} \times \mathbb{R}^{n_{\boldsymbol{z}}} \times \mathbb{R}^{n_{\boldsymbol{u}}} \times \mathbb{R} \to \mathbb{R}$*, and constants* $\gamma_{\boldsymbol{z}1}, \gamma_{\boldsymbol{z}2}, ..., \gamma_{\boldsymbol{z}5} \in \mathbb{R}_{>0}$*, such that the inequalities*

$$\gamma_{\boldsymbol{z}1}\|\tilde{\boldsymbol{z}}\|^2 \leq V_{\boldsymbol{z}}(t, \tilde{\boldsymbol{z}}, \boldsymbol{u}, \alpha_{\boldsymbol{z}}) \leq \gamma_{\boldsymbol{z}2}\|\tilde{\boldsymbol{z}}\|^2, \tag{38}$$

$$\frac{\partial V_{\boldsymbol{z}}}{\partial t}(t, \tilde{\boldsymbol{z}}, \boldsymbol{u}, \alpha_{\boldsymbol{z}}) + \alpha_{\boldsymbol{z}}\frac{\partial V_{\boldsymbol{z}}}{\partial \tilde{\boldsymbol{z}}}(t, \tilde{\boldsymbol{z}}, \boldsymbol{u}, \alpha_{\boldsymbol{z}})\tilde{\boldsymbol{h}}(t, \tilde{\boldsymbol{z}}, \boldsymbol{u}, \alpha_{\boldsymbol{z}}) \leq -\alpha_{\boldsymbol{z}}\gamma_{\boldsymbol{z}3}\|\tilde{\boldsymbol{z}}\|^2, \tag{39}$$

and

$$\left\|\frac{\partial V_{\boldsymbol{z}}}{\partial \tilde{\boldsymbol{z}}}(t, \tilde{\boldsymbol{z}}, \boldsymbol{u}, \alpha_{\boldsymbol{z}})\right\| \leq \gamma_{\boldsymbol{z}4}\|\tilde{\boldsymbol{z}}\|, \qquad \left\|\frac{\partial V_{\boldsymbol{z}}}{\partial \boldsymbol{u}}(t, \tilde{\boldsymbol{z}}, \boldsymbol{u}, \alpha_{\boldsymbol{z}})\right\| \leq \gamma_{\boldsymbol{z}5}\|\tilde{\boldsymbol{z}}\|, \tag{40}$$

are satisfied for all $t \in \mathbb{R}$*, all* $\boldsymbol{z} \in \mathcal{Z}$*, all* $\alpha_{\boldsymbol{z}} \in \mathbb{R}_{>0}$*, all constant* $\boldsymbol{u} \in \mathcal{U}$*, and each (time-varying) disturbance* $\boldsymbol{w}(t) \in \mathcal{W}$*.*

Proof. The proof of Lemma 4 follows similar arguments as the proof of Lemma 1. □

From the filter Σ_f in (5) and its steady-state solution that satisfies (7), the coordinate transformation in (24), and the vector field defined by (27), it follows that the state equation for $\tilde{\boldsymbol{z}}$ for *time-varying* inputs $\boldsymbol{u}(t)$ is given by

$$\begin{aligned}
\dot{\tilde{\boldsymbol{z}}}(t) &= \dot{\boldsymbol{z}}(t) - \dot{\bar{\boldsymbol{z}}}_{\boldsymbol{w}}(t, \boldsymbol{u}(t), \alpha_{\boldsymbol{z}}) \\
&= \dot{\boldsymbol{z}}(t) - \frac{\partial \bar{\boldsymbol{z}}_{\boldsymbol{w}}}{\partial t}(t, \boldsymbol{u}(t), \alpha_{\boldsymbol{z}}) - \frac{\partial \bar{\boldsymbol{z}}_{\boldsymbol{w}}}{\partial \boldsymbol{u}}(t, \boldsymbol{u}(t), \alpha_{\boldsymbol{z}})\dot{\boldsymbol{u}}(t) \\
&= \alpha_{\boldsymbol{z}}\boldsymbol{h}(\boldsymbol{z}(t), y(t)) - \alpha_{\boldsymbol{z}}\boldsymbol{h}(\bar{\boldsymbol{z}}_{\boldsymbol{w}}(t, \boldsymbol{u}(t), \alpha_{\boldsymbol{z}}), \bar{y}_{\boldsymbol{w}}(t, \boldsymbol{u}(t))) - \frac{\partial \bar{\boldsymbol{z}}_{\boldsymbol{w}}}{\partial \boldsymbol{u}}(t, \boldsymbol{u}(t), \alpha_{\boldsymbol{z}})\dot{\boldsymbol{u}}(t) \\
&= \alpha_{\boldsymbol{z}}\big(\boldsymbol{h}(\boldsymbol{z}(t), \bar{y}_{\boldsymbol{w}}(t, \boldsymbol{u}(t))) - \boldsymbol{h}(\bar{\boldsymbol{z}}_{\boldsymbol{w}}(t, \boldsymbol{u}(t), \alpha_{\boldsymbol{\omega}}), \bar{y}_{\boldsymbol{w}}(t, \boldsymbol{u}(t)))\big) \\
&\quad + \alpha_{\boldsymbol{z}}\big(\boldsymbol{h}(\boldsymbol{z}(t), y(t)) - \boldsymbol{h}(\boldsymbol{z}(t), \bar{y}_{\boldsymbol{w}}(t, \boldsymbol{u}(t)))\big) - \frac{\partial \bar{\boldsymbol{z}}_{\boldsymbol{w}}}{\partial \boldsymbol{u}}(t, \boldsymbol{u}(t), \alpha_{\boldsymbol{z}})\dot{\boldsymbol{u}}(t) \\
&= \alpha_{\boldsymbol{z}}\tilde{\boldsymbol{h}}(t, \tilde{\boldsymbol{z}}(t), \boldsymbol{u}(t), \alpha_{\boldsymbol{z}}) + \alpha_{\boldsymbol{z}}\big(\boldsymbol{h}(\boldsymbol{z}(t), y(t)) - \boldsymbol{h}(\boldsymbol{z}(t), \bar{y}_{\boldsymbol{w}}(t, \boldsymbol{u}(t)))\big) - \frac{\partial \bar{\boldsymbol{z}}_{\boldsymbol{w}}}{\partial \boldsymbol{u}}(t, \boldsymbol{u}(t), \alpha_{\boldsymbol{z}})\dot{\boldsymbol{u}}(t).
\end{aligned} \tag{41}$$

A bound on the solutions of the $\tilde{\boldsymbol{z}}$-dynamics for time-varying inputs $\boldsymbol{u}(t)$ in (41) is presented in Lemma 5.

Lemma 5 *Under the conditions of Theorem 1, for any finite time $t_1 \geq 0$, the solutions of $\tilde{\boldsymbol{z}}$ are bounded for all $0 \leq t \leq t_1$, all $\boldsymbol{z}(0) \in \mathcal{Z}_0$, and all time-varying $\boldsymbol{u}(t) \in \mathcal{U}$. In addition, there exist constants $c_{\boldsymbol{z}1}, c_{\boldsymbol{z}2}, c_{\boldsymbol{z}3}, \beta_{\boldsymbol{z}} \in \mathbb{R}_{>0}$ such that the solutions of $\tilde{\boldsymbol{z}}$ satisfy*

$$\|\tilde{\boldsymbol{z}}(t)\| \leq \max\{c_{\boldsymbol{z}_1}\|\tilde{\boldsymbol{z}}(t_1)\|e^{-\alpha_{\boldsymbol{z}}\beta_{\boldsymbol{z}}(t-t_1)}, \alpha_{\boldsymbol{\omega}}^2\eta_{\boldsymbol{\omega}}\alpha_{\boldsymbol{z}}c_{\boldsymbol{z}2}, \frac{\alpha_{\boldsymbol{\omega}}\eta_{\boldsymbol{\omega}}}{\alpha_{\boldsymbol{z}}}c_{\boldsymbol{z}3}\} \tag{42}$$

for all $t \geq t_1$, all $\boldsymbol{z}(0) \in \mathcal{Z}_0$, all $\boldsymbol{x}(0) \in \mathcal{X}_0$, and all time-varying $\boldsymbol{u}(t) \in \mathcal{U}$.

Proof. See Section 6.3. □

From Lemma 5 we conclude that there exists a finite time $t_2 \geq t_1 \geq 0$ such that

$$\|\tilde{\boldsymbol{z}}(t)\| \leq \max\{\alpha_{\boldsymbol{\omega}}^2\eta_{\boldsymbol{\omega}}\alpha_{\boldsymbol{z}}c_{\boldsymbol{z}2}, \frac{\alpha_{\boldsymbol{\omega}}\eta_{\boldsymbol{\omega}}}{\alpha_{\boldsymbol{z}}}c_{\boldsymbol{z}3}\}, \tag{43}$$

for all $t \geq t_2 \geq t_1$. Moreover, from Lemmas 2 and 3 it follows that $\|\tilde{\boldsymbol{x}}(t)\| \leq \alpha_{\boldsymbol{\omega}}\eta_{\boldsymbol{\omega}}c_{\boldsymbol{x}2}$ and $\|\tilde{\boldsymbol{Q}}(t)\| \leq \frac{1}{8}$ for all $t \geq t_2 \geq t_1$. These bounds on $\tilde{\boldsymbol{x}}(t)$, $\tilde{\boldsymbol{z}}(t)$, and $\tilde{\boldsymbol{Q}}(t)$ are utilized to obtain bounds on the solutions of $\tilde{\boldsymbol{u}}(t)$ and $\tilde{\boldsymbol{m}}(t)$.

Firstly, consider the $\tilde{\boldsymbol{u}}$-dynamics. From the optimizer in (22) and the coordinate transformation in (24) it follows that the state equation for $\tilde{\boldsymbol{u}}$ for *time-varying* inputs $\boldsymbol{u}(t)$ is given by

$$\begin{aligned} \dot{\tilde{\boldsymbol{u}}}(t) = \dot{\hat{\boldsymbol{u}}}(t) &= -\lambda_{\boldsymbol{u}}\frac{\eta_{\boldsymbol{u}}\boldsymbol{D}\hat{\boldsymbol{m}}(t)}{\eta_{\boldsymbol{u}} + \lambda_{\boldsymbol{u}}\|\boldsymbol{D}\hat{\boldsymbol{m}}(t)\|} \\ &= -\lambda_{\boldsymbol{u}}\frac{\eta_{\boldsymbol{u}}\boldsymbol{D}\left(\tilde{\boldsymbol{m}}(t) + \boldsymbol{m}(t)\right)}{\eta_{\boldsymbol{u}} + \lambda_{\boldsymbol{u}}\|\boldsymbol{D}\left(\tilde{\boldsymbol{m}}(t) + \boldsymbol{m}(t)\right)\|}. \end{aligned} \tag{44}$$

A bound on the solutions of $\tilde{\boldsymbol{u}}(t)$ for time-varying inputs $\boldsymbol{u}(t)$ in (44) is presented in Lemma 6.

Lemma 6 *Under the conditions of Theorem 1, for any finite time $t_2 \geq 0$, the solutions of $\tilde{\boldsymbol{u}}$ are bounded for all $0 \leq t \leq t_2$, and all $\hat{\boldsymbol{u}}(0) \in \mathcal{U}_0$. In addition, there exist a constant $c_{\boldsymbol{u}1} \in \mathbb{R}_{>0}$ such that the solutions $\tilde{\boldsymbol{u}}$ satisfy*

$$\sup_{t\geq t_2}\|\tilde{\boldsymbol{u}}(t)\| \leq \max\left\{\|\tilde{\boldsymbol{u}}(t_2)\|, \frac{1}{\alpha_{\boldsymbol{\omega}}}c_{\boldsymbol{u}1}\sup_{t\geq t_2}\|\tilde{\boldsymbol{m}}(t)\|\right\} \tag{45}$$

and

$$\limsup_{t\to\infty}\|\tilde{\boldsymbol{u}}(t)\| \leq \frac{1}{\alpha_{\boldsymbol{\omega}}}c_{\boldsymbol{u}1}\limsup_{t\to\infty}\|\tilde{\boldsymbol{m}}(t)\|. \tag{46}$$

Proof. See [12, Ch. 2]. □

Secondly, consider the $\tilde{\boldsymbol{m}}$-dynamics. From the observer in (21), the coordinate transformation in (24), the model of the input-output behavior in (18), and the state definition in (16) we obtain that the state equation for $\tilde{\boldsymbol{m}}$ is given by

$$\begin{aligned} \dot{\tilde{\boldsymbol{m}}}(t) &= \dot{\hat{\boldsymbol{m}}}(t) - \dot{\boldsymbol{m}}(t), \\ &= \left(\boldsymbol{A}(t) - \eta_{\boldsymbol{m}}\boldsymbol{Q}(t)\left(\boldsymbol{C}^T(t)\boldsymbol{C}(t) + \sigma_r\boldsymbol{D}^T\boldsymbol{D}\right)\right)\tilde{\boldsymbol{m}}(t) + \alpha_{\boldsymbol{\omega}}^2\boldsymbol{B}\left(\hat{\boldsymbol{s}}(t) - \boldsymbol{s}(t)\right) \\ &\quad - \eta_{\boldsymbol{m}}\boldsymbol{Q}(t)\boldsymbol{C}^T(t)(\alpha_{\boldsymbol{\omega}}^2\left(\hat{v}(t) - v(t)\right) - r(t) - d(t)) - \eta_{\boldsymbol{m}}\sigma_r\alpha_{\boldsymbol{\omega}}\boldsymbol{Q}(t)\boldsymbol{D}^T\frac{dF_{\boldsymbol{w}}}{d\boldsymbol{u}^T}(\hat{\boldsymbol{u}}(t)). \end{aligned} \tag{47}$$

A bound on the solutions of $\tilde{\boldsymbol{m}}(t)$ for time-varying inputs $\boldsymbol{u}(t)$ in (47) is presented in Lemma 7.

Lemma 7 *Under the conditions of Theorem 1, for any finite time $t_2 \geq 0$, the solutions of $\tilde{\boldsymbol{m}}$ are bounded for all $0 \leq t \leq t_2$, all $\boldsymbol{m}(0) \in \mathcal{M}_0$, and all time-varying $\boldsymbol{u}(t) \in \mathcal{U}$. In addition, there exist constants $c_{\boldsymbol{m}1}, ..., c_{\boldsymbol{m}7} \in \mathbb{R}_{>0}$ such that the solutions of $\tilde{\boldsymbol{m}}$ satisfy*

$$\begin{aligned} \sup_{t\geq t_2}\|\tilde{\boldsymbol{m}}(t)\| \leq \sup_{t\geq t_2}\max\Bigg\{&c_{\boldsymbol{m}1}\|\tilde{\boldsymbol{m}}(t_2)\|, c_{\boldsymbol{m}2}\frac{\alpha_{\boldsymbol{\omega}}^2\lambda_{\boldsymbol{u}}}{\eta_{\boldsymbol{m}}}\|\tilde{\boldsymbol{u}}(t)\|, c_{\boldsymbol{m}3}\alpha_{\boldsymbol{\omega}}^2, c_{\boldsymbol{m}4}\alpha_{\boldsymbol{\omega}}^2\eta_{\boldsymbol{\omega}}\alpha_{\boldsymbol{z}}, \\ &c_{\boldsymbol{m}5}\frac{\alpha_{\boldsymbol{\omega}}\eta_{\boldsymbol{\omega}}}{\alpha_{\boldsymbol{z}}}, c_{\boldsymbol{m}6}\alpha_{\boldsymbol{z}}\delta_{\boldsymbol{w}}, c_{\boldsymbol{m}7}\sqrt{\sigma_r}\alpha_{\boldsymbol{\omega}}\|\tilde{\boldsymbol{u}}(t)\|\Bigg\}, \end{aligned} \tag{48}$$

and

$$\limsup_{t\to\infty} \|\tilde{\boldsymbol{m}}(t)\| \leq \limsup_{t\to\infty} \max\left\{ c_{\boldsymbol{m}2}\frac{\alpha_{\boldsymbol{\omega}}^2\lambda_{\boldsymbol{u}}}{\eta_{\boldsymbol{m}}}\|\tilde{\boldsymbol{u}}(t)\|, c_{\boldsymbol{m}3}\alpha_{\boldsymbol{\omega}}^2, c_{\boldsymbol{m}4}\alpha_{\boldsymbol{\omega}}^2\eta_{\boldsymbol{\omega}}\alpha_{\boldsymbol{z}}, c_{\boldsymbol{m}5}\frac{\alpha_{\boldsymbol{\omega}}\eta_{\boldsymbol{\omega}}}{\alpha_{\boldsymbol{z}}}, c_{\boldsymbol{m}6}\alpha_{\boldsymbol{z}}\delta_{\boldsymbol{w}}, c_{\boldsymbol{m}7}\sqrt{\sigma_r}\alpha_{\boldsymbol{\omega}}\|\tilde{\boldsymbol{u}}(t)\| \right\}, \tag{49}$$

Proof. See Section 6.4. □

The dynamics of $\tilde{\boldsymbol{u}}$ and $\tilde{\boldsymbol{m}}$ can be seen as feedback-interconnected subsystems for which the solutions satisfy the bounds in Lemmas 6 and 7, respectively. To verify that this feedback-interconnected system exhibits uniformly bounded solutions, the cyclic-small-gain criterion in [22] is employed. The cyclic-small-gain criterion for each simple cycle follows from Lemmas 6 and 7, and are given by

$$\begin{aligned} c_{\boldsymbol{u}1}c_{\boldsymbol{m}2}\frac{\alpha_{\boldsymbol{\omega}}\lambda_{\boldsymbol{u}}}{\eta_{\boldsymbol{m}}} &< 1, \\ c_{\boldsymbol{u}1}c_{\boldsymbol{m}7}\sqrt{\sigma_r} &< 1. \end{aligned} \tag{50}$$

For (sufficiently small) constants $\epsilon_1, ..., \epsilon_6 \in \mathbb{R}_{>0}$, the cyclic-small-gain criterion for each simple cycle is satisfied for all $\alpha_{\boldsymbol{\omega}}\lambda_{\boldsymbol{u}} \leq \eta_{\boldsymbol{m}}\epsilon_4$, and all $\sigma_r \leq \epsilon_6$, rendering the closed-loop system of the extended plant and the extremum-seeking controller ISS with respect to the dither signal $\alpha_{\boldsymbol{\omega}}\boldsymbol{\omega}(t)$. Therefore, from (45) and (48), we obtain that

$$\sup_{t\geq t_2} \|\tilde{\boldsymbol{u}}(t)\| \leq \max\left\{ \|\tilde{\boldsymbol{u}}(t_2)\|, \frac{1}{\alpha_{\boldsymbol{\omega}}}c_{\boldsymbol{u}1}c_{\boldsymbol{m}1}\|\tilde{\boldsymbol{m}}(t_2)\|, \alpha_{\boldsymbol{\omega}}c_{\boldsymbol{u}1}c_{\boldsymbol{m}3}, \alpha_{\boldsymbol{\omega}}\eta_{\boldsymbol{\omega}}\alpha_{\boldsymbol{z}}c_{\boldsymbol{u}1}c_{\boldsymbol{m}4}, \frac{\eta_{\boldsymbol{\omega}}}{\alpha_{\boldsymbol{z}}}c_{\boldsymbol{u}1}c_{\boldsymbol{m}5}, \frac{\alpha_{\boldsymbol{z}}\delta_{\boldsymbol{w}}}{\alpha_{\boldsymbol{\omega}}}c_{\boldsymbol{u}1}c_{\boldsymbol{m}6} \right\} \tag{51}$$

and

$$\sup_{t\geq t_2} \|\tilde{\boldsymbol{m}}(t)\| \leq \max\left\{ c_{\boldsymbol{m}1}\|\tilde{\boldsymbol{m}}(t_2)\|, \frac{\alpha_{\boldsymbol{\omega}}^2\lambda_{\boldsymbol{u}}}{\eta_{\boldsymbol{m}}}c_{\boldsymbol{m}2}\|\tilde{\boldsymbol{u}}(t_2)\|, c_{\boldsymbol{m}3}\alpha_{\boldsymbol{\omega}}^2, c_{\boldsymbol{m}4}\alpha_{\boldsymbol{\omega}}^2\eta_{\boldsymbol{\omega}}\alpha_{\boldsymbol{z}}, \frac{\alpha_{\boldsymbol{\omega}}\eta_{\boldsymbol{\omega}}}{\alpha_{\boldsymbol{z}}}c_{\boldsymbol{m}5}, \alpha_{\boldsymbol{z}}\delta_{\boldsymbol{w}}c_{\boldsymbol{m}6}, \alpha_{\boldsymbol{\omega}}\sqrt{\sigma_r}c_{\boldsymbol{m}7}\|\tilde{\boldsymbol{u}}(t_2)\| \right\}, \tag{52}$$

Similarly, from (46) and (49), we have that

$$\limsup_{t\to\infty} \|\tilde{\boldsymbol{u}}(t)\| \leq \max\left\{ \alpha_{\boldsymbol{\omega}}c_{\boldsymbol{u}1}c_{\boldsymbol{m}3}, \alpha_{\boldsymbol{\omega}}\eta_{\boldsymbol{\omega}}\alpha_{\boldsymbol{z}}c_{\boldsymbol{u}1}c_{\boldsymbol{m}4}, \frac{\eta_{\boldsymbol{\omega}}}{\alpha_{\boldsymbol{z}}}c_{\boldsymbol{u}1}c_{\boldsymbol{m}5}, \frac{\alpha_{\boldsymbol{z}}\delta_{\boldsymbol{w}}}{\alpha_{\boldsymbol{\omega}}}c_{\boldsymbol{u}1}c_{\boldsymbol{m}6} \right\} \tag{53}$$

and

$$\limsup_{t\to\infty} \|\tilde{\boldsymbol{m}}(t)\| \leq \max\left\{ c_{\boldsymbol{m}3}\alpha_{\boldsymbol{\omega}}^2, c_{\boldsymbol{m}4}\alpha_{\boldsymbol{\omega}}^2\eta_{\boldsymbol{\omega}}\alpha_{\boldsymbol{z}}, \frac{\alpha_{\boldsymbol{\omega}}\eta_{\boldsymbol{\omega}}}{\alpha_{\boldsymbol{z}}}c_{\boldsymbol{m}5}, \alpha_{\boldsymbol{z}}\delta_{\boldsymbol{w}}c_{\boldsymbol{m}6} \right\}, \tag{54}$$

The boundedness of the solutions of the closed-loop system in Theorem 1 follows from Lemmas 2, 3, 5, 6, and 7, the bounds in (51), (52), and the coordinate transformation in (24). The bound in (23) of Theorem 1 directly follows from (53) and the coordinate transformation in (24).

A final remark has to be made about considering the inputs $\boldsymbol{u}$ on the compact set $\mathcal{U}$. Throughout Lemmas 1-7 we have considered $\boldsymbol{u}(t) \in \mathcal{U}$, where $\mathcal{U}$ is a compact set as defined in (28). From the result of Theorem 1 can be concluded that by subsequently tuning $\alpha_{\boldsymbol{\omega}}, \alpha_{\boldsymbol{z}}$, and $\eta_{\boldsymbol{\omega}}$ small, we can tune the region to which $\hat{\boldsymbol{u}}$ converges arbitrarily small, such that in the limit for $t \to \infty$ we have $\boldsymbol{u}(t) \in \mathcal{U}$. However, $\boldsymbol{u}$ needs to stay in the compact set $\mathcal{U}$ for all time, not only in the limit for $t \to \infty$.

The condition $\boldsymbol{u}(t) \in \mathcal{U}$ as used throughout Lemmas 1-7 requires that $\|\boldsymbol{u}(t) - \boldsymbol{u}^*\| \leq L_{\boldsymbol{u}}$ for all $t \geq 0$, with some constant $L_{\boldsymbol{u}} \in \mathbb{R}_{>0}$. From the definition of $\boldsymbol{\omega}$ in (15) it follows that there exists a positive constant $L_{\boldsymbol{\omega}2} \in \mathbb{R}_{>0}$ such that $\|\boldsymbol{\omega}\| \leq L_{\boldsymbol{\omega}2}$. As such, from (14) and (24) we have that

$$\begin{aligned} \|\boldsymbol{u}(t) - \boldsymbol{u}^*\| &= \|\hat{\boldsymbol{u}}(t) - \boldsymbol{u}^* + \alpha_{\boldsymbol{\omega}}\boldsymbol{\omega}(t)\|, \\ &\leq \|\hat{\boldsymbol{u}}(t) - \boldsymbol{u}^*\| + \alpha_{\boldsymbol{\omega}}\|\boldsymbol{\omega}(t)\|, \\ &\leq \|\hat{\boldsymbol{u}}(t) - \boldsymbol{u}^*\| + \alpha_{\boldsymbol{\omega}}L_{\boldsymbol{\omega}2}, \\ &\leq \|\tilde{\boldsymbol{u}}(t)\| + \alpha_{\boldsymbol{\omega}}L_{\boldsymbol{\omega}2}, \end{aligned} \tag{55}$$

Basically, if we show that $\|\tilde{\boldsymbol{u}}(t)\| \leq L_{\boldsymbol{u}} - \alpha_{\boldsymbol{\omega}} L_{\boldsymbol{\omega}2}$ for all $t \geq 0$, then $\boldsymbol{u}(t) \in \mathcal{U}$ for all $t \geq 0$.

First, we consider $0 \leq t \leq t_2$. From the optimizer in (22) we have that

$$\|\dot{\tilde{\boldsymbol{u}}}(t)\| = \|\dot{\hat{\boldsymbol{u}}}(t)\| \leq \eta_{\boldsymbol{u}}, \tag{56}$$

which yields

$$\|\tilde{\boldsymbol{u}}(t)\| \leq \|\tilde{\boldsymbol{u}}(0)\| + \eta_{\boldsymbol{u}} t \qquad \forall\, t \geq 0. \tag{57}$$

This implies that the bound on $\tilde{\boldsymbol{u}}(t)$ in (57) grows with time on the interval $0 \geq t \geq t_2$. Nevertheless, for any initial condition $\hat{\boldsymbol{u}}(0) \in \mathcal{U}_0 \subset \mathcal{U}$ and for any finite time $t_2 \geq 0$, from (57) it follows that we can tune $\eta_{\boldsymbol{u}}$ sufficiently small such that $\|\tilde{\boldsymbol{u}}(t)\| \leq L_{\boldsymbol{u}} - \alpha_{\boldsymbol{\omega}} L_{\boldsymbol{\omega}2}$ for all $0 \leq t \leq t_2$. In particular, for all $\hat{\boldsymbol{u}}(0) \in \mathcal{U}_0$, with $\mathcal{U}_0$ as in (28), we should tune $\eta_{\boldsymbol{u}} \leq \frac{L_{\boldsymbol{u}} - \alpha_{\boldsymbol{\omega}} L_{\boldsymbol{\omega}2} - \rho_{\boldsymbol{u}}}{t_2}$, such that $\boldsymbol{u}(t) \in \mathcal{U}$ for $0 \leq t \leq t_2$.

Second, consider $t \geq t_2$. From (51), it follows that we can subsequently choose $\alpha_{\boldsymbol{\omega}}$, $\alpha_{\boldsymbol{z}}$, $\eta_{\boldsymbol{\omega}}$, and $\eta_{\boldsymbol{u}}$ sufficiently small such that the last four terms and the first term in the right-hand side of (51) are smaller than $L_{\boldsymbol{u}} - \alpha_{\boldsymbol{\omega}} L_{\boldsymbol{\omega}2}$. In particular:

- From the third term in (51) it follows that $\boldsymbol{u}(t) \in \mathcal{U}$ for $t \geq t_2$ if $\alpha_{\boldsymbol{\omega}} \leq \frac{L_{\boldsymbol{u}}}{(c_{\boldsymbol{u}1} c_{\boldsymbol{m}3} + L_{\boldsymbol{\omega}2})}$;
- From the sixth term in (51) it follows that $\boldsymbol{u}(t) \in \mathcal{U}$ for $t \geq t_2$ if $\alpha_{\boldsymbol{z}} \leq \frac{\alpha_{\boldsymbol{\omega}}(L_{\boldsymbol{u}} - \alpha_{\boldsymbol{\omega}} L_{\boldsymbol{\omega}2})}{\delta_{\boldsymbol{w}} c_{\boldsymbol{u}1} c_{\boldsymbol{m}6}}$;
- From the fourth and fifth term in (51) it follows that $\boldsymbol{u}(t) \in \mathcal{U}$ for $t \geq t_2$ if $\eta_{\boldsymbol{\omega}} \leq (L_{\boldsymbol{u}} - \alpha_{\boldsymbol{\omega}} L_{\boldsymbol{\omega}2}) \min\{\frac{1}{\alpha_{\boldsymbol{\omega}} \alpha_{\boldsymbol{z}} c_{\boldsymbol{u}1} c_{\boldsymbol{m}4}}, \frac{\alpha_{\boldsymbol{z}}}{c_{\boldsymbol{u}1} c_{\boldsymbol{m}5}}\}$;
- From the first term in (51) it follows that $\boldsymbol{u}(t) \in \mathcal{U}$ for $t \geq t_2$ if $\eta_{\boldsymbol{u}} \leq \frac{L_{\boldsymbol{u}} - \alpha_{\boldsymbol{\omega}} L_{\boldsymbol{\omega}2} - \rho_{\boldsymbol{u}}}{t_2}$;

Now it remains to show that the second term in the right-hand side of (51), i.e., $\frac{1}{\alpha_{\boldsymbol{\omega}}} c_{\boldsymbol{u}1} c_{\boldsymbol{m}1} \|\tilde{\boldsymbol{m}}(t_2)\|$, can be upper bounded by $L_{\boldsymbol{u}} - \alpha_{\boldsymbol{\omega}} L_{\boldsymbol{\omega}2}$. From (171) in Lemma 7 and the comparison lemma we can deduce that

$$\begin{aligned} V_{\boldsymbol{m}}(\tilde{\boldsymbol{m}}(t), \boldsymbol{Q}(t)) \leq{}& V_{\boldsymbol{m}}(\tilde{\boldsymbol{m}}(0), \boldsymbol{Q}(0)) e^{-\frac{\eta_{\boldsymbol{m}}}{2} t} + \frac{4\alpha_{\boldsymbol{\omega}}^2}{\eta_{\boldsymbol{m}}^2} L_{F2}^2 \eta_{\boldsymbol{u}}^2 \sup_{t \geq 0} \|\boldsymbol{Q}^{-1}(t)\| + 6 L_k^2 \alpha_{\boldsymbol{z}}^2 \delta_{\boldsymbol{w}}^2 \\ &+ \frac{3}{2} \alpha_{\boldsymbol{\omega}}^4 L_{F2}^2 L_{\boldsymbol{\omega}2}^4 + 6 L_k^2 \sup_{t \geq 0} \|\tilde{\boldsymbol{z}}(t)\|^2 + 2\sigma_r \alpha_{\boldsymbol{\omega}}^2 L_{F2}^2 \sup_{t \geq 0} \|\tilde{\boldsymbol{u}}(t)\|^2, \end{aligned} \tag{58}$$

for $t \geq 0$. Using (159) we obtain

$$\begin{aligned} \|\tilde{\boldsymbol{m}}(t)\|^2 \leq{}& \frac{\lambda_{max}(\boldsymbol{Q}^{-1}(0))}{\lambda_{min}(\boldsymbol{Q}^{-1}(t))} \|\tilde{\boldsymbol{m}}(0)\|^2 e^{-\frac{\eta_{\boldsymbol{m}}}{2} t} + \Big(\frac{4\alpha_{\boldsymbol{\omega}}^2}{\eta_{\boldsymbol{m}}^2} L_{F2}^2 \eta_{\boldsymbol{u}}^2 \sup_{t \geq 0} \|\boldsymbol{Q}^{-1}(t)\| + 6 L_k^2 \alpha_{\boldsymbol{z}}^2 \delta_{\boldsymbol{w}}^2 \\ &+ \frac{3}{2} \alpha_{\boldsymbol{\omega}}^4 L_{F2}^2 L_{\boldsymbol{\omega}2}^4 + 6 L_k^2 \sup_{t \geq 0} \|\tilde{\boldsymbol{z}}(t)\|^2 + 2\sigma_r \alpha_{\boldsymbol{\omega}}^2 L_{F2}^2 \sup_{t \geq 0} \|\tilde{\boldsymbol{u}}(t)\|^2 \Big) \frac{1}{\lambda_{min}(\boldsymbol{Q}^{-1}(t))}, \end{aligned} \tag{59}$$

for $t \geq 0$. From (59) we obtain

$$\begin{aligned} \|\tilde{\boldsymbol{m}}(t)\| \leq \sqrt{\frac{1}{\lambda_{min}(\boldsymbol{Q}^{-1}(t))}} \max \Big\{ & \sqrt{6 \lambda_{max}(\boldsymbol{Q}^{-1}(0))} \|\tilde{\boldsymbol{m}}(0)\| e^{-\frac{\eta_{\boldsymbol{m}}}{4} t}, \\ & 2\sqrt{6} L_{F2} \frac{\alpha_{\boldsymbol{\omega}} \eta_{\boldsymbol{u}}}{\eta_{\boldsymbol{m}}} \sup_{t \geq 0} \|\boldsymbol{Q}^{-1}(t)\|^{\frac{1}{2}}, 6 L_k \alpha_{\boldsymbol{z}} \delta_{\boldsymbol{w}}, \\ & 3\alpha_{\boldsymbol{\omega}}^2 L_{F2} L_{\boldsymbol{\omega}2}^2, 6 L_k \sup_{t \geq 0} \|\tilde{\boldsymbol{z}}(t)\|, 2\sqrt{3\sigma_r} \alpha_{\boldsymbol{\omega}} L_{F2} \sup_{t \geq 0} \|\tilde{\boldsymbol{u}}(t)\| \Big\}, \end{aligned} \tag{60}$$

for $t \geq 0$. From (151) in Lemma 5 we have that

$$\begin{aligned} \sup_{t \geq 0} \|\tilde{\boldsymbol{z}}(t)\|^2 \leq{}& \frac{\gamma_{\boldsymbol{z}2}}{\gamma_{\boldsymbol{z}1}} \|\tilde{\boldsymbol{z}}(0)\|^2 + \frac{3\gamma_{\boldsymbol{z}2} z_1^2}{\gamma_{\boldsymbol{z}1} \gamma_{\boldsymbol{z}3}^2} \sup_{t \geq 0} \|\tilde{\boldsymbol{x}}(t)\|^4 + \frac{3\gamma_{\boldsymbol{z}2} z_2^2}{\gamma_{\boldsymbol{z}1} \gamma_{\boldsymbol{z}3}^2} \sup_{t \geq 0} \|\tilde{\boldsymbol{x}}(t)\|^2 \\ &+ \frac{3\gamma_{\boldsymbol{z}2} z_3^2}{\gamma_{\boldsymbol{z}1} \alpha_{\boldsymbol{z}}^2 \gamma_{\boldsymbol{z}3}^2} \alpha_{\boldsymbol{\omega}}^2 \eta_{\boldsymbol{\omega}}^2 \left(\epsilon_3 \epsilon_5 + L_{\boldsymbol{\omega}1}\right)^2, \end{aligned} \tag{61}$$

which leads to

$$\sup_{t\geq 0}\|\tilde{\boldsymbol{z}}(t)\| \leq 2\sqrt{\frac{\gamma_{\boldsymbol{z}2}}{\gamma_{\boldsymbol{z}1}}}\max\Big\{\|\tilde{\boldsymbol{z}}(0)\|, \sqrt{3}\frac{z_1}{\gamma_{\boldsymbol{z}3}}\sup_{t\geq 0}\|\tilde{\boldsymbol{x}}(t)\|^2, \sqrt{3}\frac{z_2}{\gamma_{\boldsymbol{z}3}}\sup_{t\geq 0}\|\tilde{\boldsymbol{x}}(t)\|, \sqrt{3}\frac{z_3}{\gamma_{\boldsymbol{z}3}}\frac{\alpha_{\boldsymbol{\omega}}\eta_{\boldsymbol{\omega}}}{\alpha_{\boldsymbol{z}}}\left(\epsilon_3\epsilon_5 + L_{\boldsymbol{\omega}1}\right)\Big\}. \tag{62}$$

From (34) in Lemma 2, we have that

$$\sup_{t\geq 0}\|\tilde{\boldsymbol{x}}(t)\|^2 \leq \max\left\{c_{\boldsymbol{x}1}^2\|\tilde{\boldsymbol{x}}(0)\|^2, \alpha_{\boldsymbol{\omega}}^2\eta_{\boldsymbol{\omega}}^2c_{\boldsymbol{x}2}^2\right\}, \tag{63}$$

and

$$\sup_{t\geq 0}\|\tilde{\boldsymbol{x}}(t)\| \leq \max\left\{c_{\boldsymbol{x}1}\|\tilde{\boldsymbol{x}}(0)\|, \alpha_{\boldsymbol{\omega}}\eta_{\boldsymbol{\omega}}c_{\boldsymbol{x}2}\right\}. \tag{64}$$

From the coordinate transformation in (24) we obtain that

$$\sup_{t\geq 0}\|\boldsymbol{Q}^{-1}(t)\| \leq \sup_{t\geq 0}\|\tilde{\boldsymbol{Q}}(t)\| + \|\Xi^{-1}\| + \frac{\eta_{\boldsymbol{m}}}{\eta_{\boldsymbol{\omega}}}\sup_{t\geq 0}\|\boldsymbol{n}(t)\|. \tag{65}$$

From (25) and (26) it follows that there exist constants $N_1, N_2 \in \mathbb{R}_{>0}$ such that $\|\boldsymbol{n}(t)\| \leq N_1 \; \forall \; t \geq 0$ and $\|\Xi^{-1}\| \leq N_2$. Moreover, from the conditions in Theorem 1 follows that $\eta_{\boldsymbol{m}} \leq \eta_{\boldsymbol{\omega}}\epsilon_3$. Using this and (36) in Lemma 3 we obtain

$$\begin{aligned}\sup_{t\geq 0}\|\boldsymbol{Q}^{-1}(t)\| &\leq \sup_{t\geq 0}\|\tilde{\boldsymbol{Q}}(t)\| + N_2 + \frac{\eta_{\boldsymbol{m}}}{\eta_{\boldsymbol{\omega}}}N_1,\\ &\leq \max\left\{c_{\boldsymbol{Q}}\|\tilde{\boldsymbol{Q}}(0)\|, \frac{1}{8}\right\} + N_2 + \epsilon_3 N_1,\end{aligned} \tag{66}$$

From (62)-(64) we obtain

$$\begin{aligned}\sup_{t\geq 0}\|\tilde{\boldsymbol{z}}(t)\| \leq 2\sqrt{\frac{3\gamma_{\boldsymbol{z}2}}{\gamma_{\boldsymbol{z}1}}}\max\Big\{&\frac{1}{\sqrt{3}}\|\tilde{\boldsymbol{z}}(0)\|, \frac{z_1c_{\boldsymbol{x}1}^2}{\gamma_{\boldsymbol{z}3}}\|\tilde{\boldsymbol{x}}(0)\|^2, \frac{z_2c_{\boldsymbol{x}1}}{\gamma_{\boldsymbol{z}3}}\|\tilde{\boldsymbol{x}}(0)\|,\\ &\frac{z_1c_{\boldsymbol{x}2}^2}{\gamma_{\boldsymbol{z}3}}\alpha_{\boldsymbol{\omega}}^2\eta_{\boldsymbol{\omega}}^2, \frac{z_2c_{\boldsymbol{x}2}}{\gamma_{\boldsymbol{z}3}}\alpha_{\boldsymbol{\omega}}\eta_{\boldsymbol{\omega}}, \frac{z_3}{\gamma_{\boldsymbol{z}3}}\frac{\alpha_{\boldsymbol{\omega}}\eta_{\boldsymbol{\omega}}}{\alpha_{\boldsymbol{z}}}\left(\epsilon_3\epsilon_5 + L_{\boldsymbol{\omega}1}\right)\Big\}.\end{aligned} \tag{67}$$

Combining (60), (66), and (67), using the conditions in Theorem 1, i.e., $\eta_{\boldsymbol{u}} \leq \alpha_{\boldsymbol{\omega}}\eta_{\boldsymbol{m}}\epsilon_5$ and $\eta_{\boldsymbol{\omega}} \leq \alpha_{\boldsymbol{z}}\epsilon_2$, and assessing $\|\tilde{\boldsymbol{m}}(t)\|$ at $t = t_2$, the second term in the right-hand side of (51) reads

$$\begin{aligned}\frac{1}{\alpha_{\boldsymbol{\omega}}}c_{\boldsymbol{u}1}c_{\boldsymbol{m}1}\|\tilde{\boldsymbol{m}}(t_2)\| \leq \sqrt{\frac{c_{\boldsymbol{u}1}^2c_{\boldsymbol{m}1}^2}{\lambda_{min}(\boldsymbol{Q}^{-1}(t_2))}}\max\Big\{&\frac{\sqrt{6\lambda_{max}(\boldsymbol{Q}^{-1}(0))}}{\alpha_{\boldsymbol{\omega}}}\|\tilde{\boldsymbol{m}}(0)\|e^{-\frac{\eta_{\boldsymbol{m}}}{4}t_2},\\ &\frac{12L_k}{\alpha_{\boldsymbol{\omega}}}\sqrt{\frac{\gamma_{\boldsymbol{z}2}}{\gamma_{\boldsymbol{z}1}}}\|\tilde{\boldsymbol{z}}(0)\|, \frac{12L_kz_1c_{\boldsymbol{x}1}^2}{\alpha_{\boldsymbol{\omega}}\gamma_{\boldsymbol{z}3}}\sqrt{\frac{3\gamma_{\boldsymbol{z}2}}{\gamma_{\boldsymbol{z}1}}}\|\tilde{\boldsymbol{x}}(0)\|^2, \frac{12L_kz_2c_{\boldsymbol{x}1}}{\alpha_{\boldsymbol{\omega}}\gamma_{\boldsymbol{z}3}}\sqrt{\frac{3\gamma_{\boldsymbol{z}2}}{\gamma_{\boldsymbol{z}1}}}\|\tilde{\boldsymbol{x}}(0)\|,\\ &\alpha_{\boldsymbol{\omega}}2\sqrt{6}L_{F2}\epsilon_5\left(c_{\boldsymbol{Q}}\|\tilde{\boldsymbol{Q}}(0)\| + N_2 + \epsilon_3N_1\right)^{\frac{1}{2}},\\ &\alpha_{\boldsymbol{\omega}}2\sqrt{6}L_{F2}\epsilon_5\left(\frac{1}{8} + N_2 + \epsilon_3N_1\right)^{\frac{1}{2}},\\ &6L_k\frac{\alpha_{\boldsymbol{z}}\delta_{\boldsymbol{w}}}{\alpha_{\boldsymbol{\omega}}}, 3\alpha_{\boldsymbol{\omega}}L_{F2}L_{\boldsymbol{\omega}2}^2, 12L_k\sqrt{\frac{3\gamma_{\boldsymbol{z}2}}{\gamma_{\boldsymbol{z}1}}}\frac{z_1c_{\boldsymbol{x}2}^2}{\gamma_{\boldsymbol{z}3}}\alpha_{\boldsymbol{\omega}}\eta_{\boldsymbol{\omega}}\alpha_{\boldsymbol{z}}\epsilon_2,\\ &12L_k\sqrt{\frac{3\gamma_{\boldsymbol{z}2}}{\gamma_{\boldsymbol{z}1}}}\frac{z_2c_{\boldsymbol{x}2}}{\gamma_{\boldsymbol{z}3}}\eta_{\boldsymbol{\omega}}, 12L_k\sqrt{\frac{3\gamma_{\boldsymbol{z}2}}{\gamma_{\boldsymbol{z}1}}}\frac{z_3}{\gamma_{\boldsymbol{z}3}}\frac{\eta_{\boldsymbol{\omega}}}{\alpha_{\boldsymbol{z}}}\left(\epsilon_3\epsilon_5 + L_{\boldsymbol{\omega}1}\right),\\ &2\sqrt{3\sigma_r}L_{F2}\sup_{t_2\geq t\geq 0}\|\tilde{\boldsymbol{u}}(t)\|\Big\}.\end{aligned} \tag{68}$$

By showing that the right-hand side of (68) is bounded by $L_{\boldsymbol{u}} - \alpha_{\boldsymbol{\omega}}L_{\boldsymbol{\omega}2}$, we have that $\boldsymbol{u}(t) \in \mathcal{U}$ for all $t \geq 0$. In particular:

- From the last term in (68) it follows that, if we have designed $\eta_{\boldsymbol{u}} \leq \frac{L_{\boldsymbol{u}} - \alpha_{\boldsymbol{\omega}} - \rho_{\boldsymbol{u}}}{t_2}$ (as discussed before), and if we design $\sigma_r \leq \frac{1}{3}\left(\frac{1}{2L_{F2}}\right)^2$ we have that the last term is bounded by $L_{\boldsymbol{u}} - \alpha_{\boldsymbol{\omega}}L_{\boldsymbol{\omega}2}$, yielding $\boldsymbol{u}(t) \in \mathcal{U}$ for $t \geq t_2$.

- The 5^{th} to 11^{th} term in (68) can be made arbitrarily small by subsequently tuning $\alpha_{\boldsymbol{\omega}}, \alpha_{\boldsymbol{z}}$, and $\eta_{\boldsymbol{\omega}}$ small, such that the last term is bounded by $L_{\boldsymbol{u}} - \alpha_{\boldsymbol{\omega}} L_{\boldsymbol{\omega}2}$ and yielding $\boldsymbol{u}(t) \in \mathcal{U}$ for $t \geq t_2$.

- From the first four terms it follows that we can not choose arbitrarily (large) initial conditions. For example, we require $\alpha_{\boldsymbol{\omega}}$ to be small to make some terms small, while given the first four terms require the initial conditions to be chosen even smaller. Nevertheless, there exist a (small) set of initial conditions such that the first four terms can be bounded by $L_{\boldsymbol{u}} - \alpha_{\boldsymbol{\omega}} L_{\boldsymbol{\omega}2}$, and thus $\boldsymbol{u}(t) \in \mathcal{U}$ for all $t \geq t_2$.

Concluding, there exist (sufficiently small) constants $\epsilon_1, ..., \epsilon_6 \in \mathbb{R}_{>0}$ and initial conditions $\boldsymbol{x}(0) \in \mathcal{X}_0$, $\boldsymbol{Q}(0) \in \mathcal{Q}_0$, $\boldsymbol{z}_0 \in \mathcal{Z}_0$, $\hat{\boldsymbol{u}}(0) \in \mathcal{U}_0$, and $\boldsymbol{m}(0) \in \mathcal{M}_0$, with $\mathcal{X}_0$, $\mathcal{Q}_0$, $\mathcal{Z}_0, \mathcal{U}_0$, and $\mathcal{M}_0$ as in (28), such that i) the solutions of the closed-loop system of the extended plant and the extremum-seeking controller are uniformly bounded for all $\alpha_{\boldsymbol{z}}, \alpha_{\boldsymbol{\omega}}, \eta_{\boldsymbol{u}}, \lambda_{\boldsymbol{u}}, \eta_{\boldsymbol{m}}, \eta_{\boldsymbol{\omega}} \in \mathbb{R}_{>0}$ and all $\sigma_r \in \mathbb{R}_{\geq 0}$ that satisfy $\alpha_{\boldsymbol{z}} \leq \epsilon_1$, $\eta_{\boldsymbol{\omega}} \leq \alpha_{\boldsymbol{z}}\epsilon_2$, $\eta_{\boldsymbol{m}} \leq \eta_{\boldsymbol{\omega}}\epsilon_3$, $\alpha_{\boldsymbol{\omega}}\lambda_{\boldsymbol{u}} \leq \eta_{\boldsymbol{m}}\epsilon_4$, $\eta_{\boldsymbol{u}} \leq \alpha_{\boldsymbol{\omega}}\eta_{\boldsymbol{m}}\epsilon_5$, and $\sigma_r \leq \epsilon_6$, ii) $\boldsymbol{u}(t) \in \mathcal{U}$ for all time, and iii) the region to which $\hat{\boldsymbol{u}}$ converges can be made arbitrarily small. This completes the proof of Theorem 1. □

6 Appendix

6.1 Proof of Lemma 1.

The proof of Lemma 1 follows a similar line of reasoning as Theorem 4.14 and Lemma 9.8 in Khalil (2002). The structure of the proof is as follows. First, it is shown that the inequalities in (30) hold. Second, it is shown that the inequality in (31) holds. Third, it is shown that the inequalities in (32) hold.

Since Assumption 2 implies that the system in (29) is globally exponentially convergent for constant inputs $\boldsymbol{u} \in \mathcal{U}$ and all (time-varying) disturbances $\boldsymbol{w}(t) \in \mathcal{W}$ for all $t \in \mathbb{R}$, satisfying Assumption 1, there exist constants $\mu_{\boldsymbol{x}}, \nu_{\boldsymbol{x}} \in \mathbb{R}_{>0}$ for each pair $\boldsymbol{u}$ and $\boldsymbol{w}(t)$ such that all solution of the dynamics in (29) satisfy

$$\|\tilde{\boldsymbol{x}}(t)\| \leq \bar{\mu}_{\boldsymbol{x}} \|\tilde{\boldsymbol{x}}(t_0)\| e^{-\underline{\nu}_{\boldsymbol{x}}(t-t_0)}, \quad \forall\, \tilde{\boldsymbol{x}}(t_0) \in \mathbb{R}^{n_{\boldsymbol{x}}},\ t \in \mathbb{R}, \tag{69}$$

where $\bar{\mu}_{\boldsymbol{x}}, \underline{\nu}_{\boldsymbol{x}} \in \mathbb{R}_{>0}$ denote the maximum of all $\mu_{\boldsymbol{x}}$ and minimum of all $\nu_{\boldsymbol{x}}$ for each pair $\boldsymbol{u}$ and $\boldsymbol{w}(t)$, respectively.

Let $\boldsymbol{\phi}(\tau; t, \tilde{\boldsymbol{x}}, \boldsymbol{u})$ denote the solution of (29) for constant inputs $\boldsymbol{u}$ that starts at $(t, \tilde{\boldsymbol{x}})$; that is, $\boldsymbol{\phi}(t; t, \tilde{\boldsymbol{x}}, \boldsymbol{u}) = \tilde{\boldsymbol{x}}$. In other words, $\boldsymbol{\phi}$ satisfies the equation

$$\frac{\partial \boldsymbol{\phi}}{\partial \tau}(\tau; t, \tilde{\boldsymbol{x}}, \boldsymbol{u}) = \tilde{\boldsymbol{f}}(\tau, \boldsymbol{\phi}(\tau; t, \tilde{\boldsymbol{x}}, \boldsymbol{u}), \boldsymbol{u}), \qquad \boldsymbol{\phi}(t; t, \tilde{\boldsymbol{x}}, \boldsymbol{u}) = \tilde{\boldsymbol{x}}. \tag{70}$$

The notation $\boldsymbol{\phi}(\tau; t, \tilde{\boldsymbol{x}}, \boldsymbol{u})$ emphasizes the dependence of the solution on the constant input $\boldsymbol{u}$. Moreover, due to the exponentially decaying bound on the trajectories in (69) we can write the following:

$$\|\boldsymbol{\phi}(\tau; t, \tilde{\boldsymbol{x}}, \boldsymbol{u})\| \leq \bar{\mu}_{\boldsymbol{x}} \|\boldsymbol{\phi}(t; t, \tilde{\boldsymbol{x}}, \boldsymbol{u})\| e^{-\underline{\nu}_{\boldsymbol{x}}(\tau-t)}, \quad \forall\, \tau \geq t. \tag{71}$$

Define the function

$$V_{\boldsymbol{x}}(t, \tilde{\boldsymbol{x}}, \boldsymbol{u}) := \int_t^{t+\delta_x} \boldsymbol{\phi}^\top(\tau; t, \tilde{\boldsymbol{x}}, \boldsymbol{u}) \boldsymbol{\phi}(\tau; t, \tilde{\boldsymbol{x}}, \boldsymbol{u}) d\tau, \tag{72}$$

where $\delta_x > 0$ is a positive constant to be chosen. Firstly, we prove that the inequalities in (30) hold. Using (71), we obtain the following upper bound on $V_{\boldsymbol{x}}$:

$$\begin{aligned} V_{\boldsymbol{x}}(t, \tilde{\boldsymbol{x}}, \boldsymbol{u}) &= \int_t^{t+\delta_{\boldsymbol{x}}} \|\boldsymbol{\phi}(\tau; t, \tilde{\boldsymbol{x}}, \boldsymbol{u})\|^2\, d\tau \\ &\leq \int_t^{t+\delta_{\boldsymbol{x}}} \bar{\mu}_{\boldsymbol{x}}^2 e^{-2\underline{\nu}_{\boldsymbol{x}}(\tau-t)} d\tau \|\tilde{\boldsymbol{x}}\|^2 = \frac{\bar{\mu}_{\boldsymbol{x}}^2}{2\underline{\nu}_{\boldsymbol{x}}} \left(1 - e^{-2\underline{\nu}_{\boldsymbol{x}}\delta_{\boldsymbol{x}}}\right) \|\tilde{\boldsymbol{x}}\|^2. \end{aligned} \tag{73}$$

Next, we construct also a lower bound for $V_{\boldsymbol{x}}$. From Assumption 5, it follows that if we consider $\boldsymbol{x}$ and $\boldsymbol{u}$ on compact sets, i.e., $\boldsymbol{x} \in \mathcal{X}$ and $\boldsymbol{u} \in \mathcal{U}$, there exist constants $L_{\boldsymbol{f}\boldsymbol{x}}$, $L_{\boldsymbol{f}\boldsymbol{u}}$, $L_{\boldsymbol{g}\boldsymbol{x}}$, $L_{\boldsymbol{g}\boldsymbol{u}} \in \mathbb{R}_{>0}$ such that

$$\begin{aligned} &\left\|\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{w})\right\| \leq L_{\boldsymbol{f}\boldsymbol{x}}, \qquad \left\|\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{w})\right\| \leq L_{\boldsymbol{f}\boldsymbol{u}}, \\ &\left\|\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{x}}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{w})\right\| \leq L_{\boldsymbol{g}\boldsymbol{x}}, \qquad \left\|\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{u}}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{w})\right\| \leq L_{\boldsymbol{g}\boldsymbol{u}}, \end{aligned} \tag{74}$$

for all $\boldsymbol{x} \in \mathcal{X}$, all $\boldsymbol{u} \in \mathcal{U}$, and all $\boldsymbol{w} \in \mathcal{W}$. In addition, from (29) and the Mean-Value Theorem, we have

$$\begin{aligned} \|\tilde{\boldsymbol{f}}(t, \tilde{\boldsymbol{x}}, \boldsymbol{u})\| &= \|\boldsymbol{f}(\tilde{\boldsymbol{x}} + \bar{\boldsymbol{x}}_{\boldsymbol{w}}(t, \boldsymbol{u}), \boldsymbol{u}, \boldsymbol{w}) - \boldsymbol{f}(\bar{\boldsymbol{x}}_{\boldsymbol{w}}(t, \boldsymbol{u}), \boldsymbol{u}, \boldsymbol{w})\| \\ &= \left\|\int_0^1 \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}(\sigma\tilde{\boldsymbol{x}} + \bar{\boldsymbol{x}}_{\boldsymbol{w}}(t, \boldsymbol{u}), \boldsymbol{u}, \boldsymbol{w}) d\sigma \tilde{\boldsymbol{x}}\right\| \\ &\leq \int_0^1 \left\|\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}(\sigma\tilde{\boldsymbol{x}} + \bar{\boldsymbol{x}}_{\boldsymbol{w}}(t, \boldsymbol{u}), \boldsymbol{u}, \boldsymbol{w})\right\| d\sigma\, \|\tilde{\boldsymbol{x}}\| = L_{\boldsymbol{f}\boldsymbol{x}} \|\tilde{\boldsymbol{x}}\|. \end{aligned} \tag{75}$$

By using (75) and (70) we obtain

$$\begin{aligned}\left|\frac{\partial}{\partial\tau}\left(\|\boldsymbol{\phi}(\tau;t,\tilde{\boldsymbol{x}},\boldsymbol{u})\|^2\right)\right| &= \left|\frac{\partial}{\partial\tau}\left(\boldsymbol{\phi}^\top(\tau;t,\tilde{\boldsymbol{x}},\boldsymbol{u})\boldsymbol{\phi}(\tau;t,\tilde{\boldsymbol{x}},\boldsymbol{u})\right)\right| \\ &= 2\left|\boldsymbol{\phi}^\top(\tau;t,\tilde{\boldsymbol{x}},\boldsymbol{u})\tilde{\boldsymbol{f}}(\tau,\boldsymbol{\phi}(\tau;t,\tilde{\boldsymbol{x}},\boldsymbol{u}),\boldsymbol{u})\right| \\ &\leq 2\|\boldsymbol{\phi}(\tau;t,\tilde{\boldsymbol{x}},\boldsymbol{u})\|\left\|\tilde{\boldsymbol{f}}(\tau,\boldsymbol{\phi}(\tau;t,\tilde{\boldsymbol{x}},\boldsymbol{u}),\boldsymbol{u})\right\|, \\ &\leq 2L_{\boldsymbol{f}\boldsymbol{x}}\|\boldsymbol{\phi}(\tau;t,\tilde{\boldsymbol{x}},\boldsymbol{u})\|^2,\end{aligned} \tag{76}$$

from which we can derive the following bound:

$$\frac{\partial}{\partial\tau}\left(\|\boldsymbol{\phi}(\tau;t,\tilde{\boldsymbol{x}},\boldsymbol{u})\|^2\right) \geq -2L_{\boldsymbol{f}\boldsymbol{x}}\|\boldsymbol{\phi}(\tau;t,\tilde{\boldsymbol{x}},\boldsymbol{u})\|^2, \tag{77}$$

for all $\tau \geq t$. From the inequality in (77) we obtain

$$\frac{\partial}{\partial\tau}\left(\|\boldsymbol{\phi}(\tau;t,\tilde{\boldsymbol{x}},\boldsymbol{u})\|^2 e^{2L_{\boldsymbol{f}\boldsymbol{x}}\tau}\right) \geq 0. \tag{78}$$

By integration of both sides with respect to time over the domain $[t,\tau]$, it follows that

$$\begin{aligned}\int_t^\tau\left(\frac{\partial}{\partial s}\left(\|\boldsymbol{\phi}(s;t,\tilde{\boldsymbol{x}},\boldsymbol{u})\|^2 e^{2L_{\boldsymbol{f}\boldsymbol{x}}s}\right)\right)ds \geq 0 \\ e^{2L_{\boldsymbol{f}\boldsymbol{x}}\tau}\|\boldsymbol{\phi}(\tau;t,\tilde{\boldsymbol{x}},\boldsymbol{u})\|^2 - e^{2L_{\boldsymbol{f}\boldsymbol{x}}t}\|\boldsymbol{\phi}(t;t,\tilde{\boldsymbol{x}},\boldsymbol{u})\|^2 \geq 0 \\ \|\boldsymbol{\phi}(\tau;t,\tilde{\boldsymbol{x}},\boldsymbol{u})\|^2 \geq e^{-2L_{\boldsymbol{f}\boldsymbol{x}}(\tau-t)}\|\tilde{\boldsymbol{x}}\|^2.\end{aligned} \tag{79}$$

Then we obtain, using (72) and (79), that

$$\begin{aligned}V_{\boldsymbol{x}}(t,\tilde{\boldsymbol{x}},\boldsymbol{u}) &= \int_t^{t+\delta_{\boldsymbol{x}}}\|\boldsymbol{\phi}(\tau;t,\tilde{\boldsymbol{x}},\boldsymbol{u})\|^2 d\tau \\ &\geq \int_t^{t+\delta_{\boldsymbol{x}}} e^{-2L_{\boldsymbol{f}\boldsymbol{x}}(\tau-t)}dt\|\tilde{\boldsymbol{x}}\|^2 = \frac{1}{2L_{\boldsymbol{f}\boldsymbol{x}}}\left(1-e^{-2L_{\boldsymbol{f}\boldsymbol{x}}\delta_{\boldsymbol{x}}}\right)\|\tilde{\boldsymbol{x}}\|^2.\end{aligned} \tag{80}$$

The bounds on $V_{\boldsymbol{x}}$ in (73) and (80) imply that the inequalities in (30) are satisfied with

$$\gamma_{\boldsymbol{x}1} = \frac{1}{2L_{\boldsymbol{f}\boldsymbol{x}}}\left(1-e^{-2L_{\boldsymbol{f}\boldsymbol{x}}\delta_{\boldsymbol{x}}}\right), \quad \text{and} \quad \gamma_{\boldsymbol{x}2} = \frac{\bar{\mu}_{\boldsymbol{x}}^2}{2\underline{\nu}_{\boldsymbol{x}}}\left(1-e^{-2\underline{\nu}_{\boldsymbol{x}}\delta_{\boldsymbol{x}}}\right), \tag{81}$$

and since $L_{\boldsymbol{f}\boldsymbol{x}}$, $\bar{\mu}_{\boldsymbol{x}}$, $\underline{\nu}_{\boldsymbol{x}}$, and $\delta_{\boldsymbol{x}}$ are positive constants, we have that $\gamma_{\boldsymbol{x}1} > 0$ and $\gamma_{\boldsymbol{x}2} > 0$.

Secondly, we prove that the inequality in (31) holds. By Leibniz's rule for differentiation, the derivative of $V_{\boldsymbol{x}}$ along the trajectories of the plant is given as follows:

$$\begin{aligned}\frac{\partial V_{\boldsymbol{x}}}{\partial t} + \frac{\partial V_{\boldsymbol{x}}}{\partial\tilde{\boldsymbol{x}}}\tilde{\boldsymbol{f}}(t,\tilde{\boldsymbol{x}},\boldsymbol{u}) &= \boldsymbol{\phi}^\top(t+\delta_{\boldsymbol{x}};t,\tilde{\boldsymbol{x}},\boldsymbol{u})\boldsymbol{\phi}(t+\delta_{\boldsymbol{x}};t,\tilde{\boldsymbol{x}},\boldsymbol{u}) - \boldsymbol{\phi}^\top(t;t,\tilde{\boldsymbol{x}},\boldsymbol{u})\boldsymbol{\phi}(t;t,\tilde{\boldsymbol{x}},\boldsymbol{u}) \\ &\quad + \int_t^{t+\delta_{\boldsymbol{x}}} 2\boldsymbol{\phi}^\top(\tau;t,\tilde{\boldsymbol{x}},\boldsymbol{u})\frac{\partial\boldsymbol{\phi}}{\partial t}(\tau;t,\tilde{\boldsymbol{x}},\boldsymbol{u})d\tau \\ &\quad + \int_t^{t+\delta_{\boldsymbol{x}}} 2\boldsymbol{\phi}^\top(\tau;t,\tilde{\boldsymbol{x}},\boldsymbol{u})\frac{\partial\boldsymbol{\phi}}{\partial\tilde{\boldsymbol{x}}}(\tau;t,\tilde{\boldsymbol{x}},\boldsymbol{u})d\tau\tilde{\boldsymbol{f}}(t,\tilde{\boldsymbol{x}},\boldsymbol{u}) \\ &= \boldsymbol{\phi}^\top(t+\delta_{\boldsymbol{x}};t,\tilde{\boldsymbol{x}},\boldsymbol{u})\boldsymbol{\phi}(t+\delta_{\boldsymbol{x}};t,\tilde{\boldsymbol{x}},\boldsymbol{u}) - \|\tilde{\boldsymbol{x}}\|^2 \\ &\quad + \int_t^{t+\delta_{\boldsymbol{x}}} 2\boldsymbol{\phi}^\top(\tau;t,\tilde{\boldsymbol{x}},\boldsymbol{u})\left(\frac{\partial\boldsymbol{\phi}}{\partial t}(\tau;t,\tilde{\boldsymbol{x}},\boldsymbol{u}) + \frac{\partial\boldsymbol{\phi}}{\partial\tilde{\boldsymbol{x}}}(\tau;t,\tilde{\boldsymbol{x}},\boldsymbol{u})\tilde{\boldsymbol{f}}(t,\tilde{\boldsymbol{x}},\boldsymbol{u})\right)d\tau.\end{aligned} \tag{82}$$

In order to evaluate the third term in the right-hand side of (82), we integrate both sides of (70) with respect to time over the domain $[\tau, t]$ such that we obtain

$$\boldsymbol{\phi}(\tau;t,\tilde{\boldsymbol{x}},\boldsymbol{u}) = \tilde{\boldsymbol{x}} + \int_t^\tau \tilde{\boldsymbol{f}}(s,\boldsymbol{\phi}(s;t,\tilde{\boldsymbol{x}},\boldsymbol{u}),\boldsymbol{u})ds. \tag{83}$$

Taking the partial derivative to t and $\tilde{\boldsymbol{x}}$, by Leibniz's rule for differentiation we obtain

$$\begin{aligned}
&\frac{\partial \boldsymbol{\phi}}{\partial t}(\tau;t,\tilde{\boldsymbol{x}},\boldsymbol{u}) = -\tilde{\boldsymbol{f}}(t,\tilde{\boldsymbol{x}},\boldsymbol{u}) + \int_t^\tau \frac{\partial \tilde{\boldsymbol{f}}}{\partial \tilde{\boldsymbol{x}}}(s,\boldsymbol{\phi}(s;t,\tilde{\boldsymbol{x}},\boldsymbol{u}),\boldsymbol{u})\frac{\partial \boldsymbol{\phi}}{\partial t}(s;t,\tilde{\boldsymbol{x}},\boldsymbol{u})ds,\\
&\frac{\partial \boldsymbol{\phi}}{\partial \tilde{\boldsymbol{x}}}(\tau;t,\tilde{\boldsymbol{x}},\boldsymbol{u}) = \boldsymbol{I} + \int_t^\tau \frac{\partial \tilde{\boldsymbol{f}}}{\partial \tilde{\boldsymbol{x}}}(s,\boldsymbol{\phi}(s;t,\tilde{\boldsymbol{x}},\boldsymbol{u}),\boldsymbol{u})\frac{\partial \boldsymbol{\phi}}{\partial \tilde{\boldsymbol{x}}}(s;t,\tilde{\boldsymbol{x}},\boldsymbol{u})ds.
\end{aligned} \tag{84}$$

Therefore,

$$\begin{aligned}
&\frac{\partial \boldsymbol{\phi}}{\partial t}(\tau;t,\tilde{\boldsymbol{x}},\boldsymbol{u}) + \frac{\partial \boldsymbol{\phi}}{\partial \tilde{\boldsymbol{x}}}(\tau;t,\tilde{\boldsymbol{x}},\boldsymbol{u})\tilde{\boldsymbol{f}}(t,\tilde{\boldsymbol{x}},\boldsymbol{u}) =\\
&\qquad \int_t^\tau \frac{\partial \tilde{\boldsymbol{f}}}{\partial \tilde{\boldsymbol{x}}}(s,\boldsymbol{\phi}(s;t,\tilde{\boldsymbol{x}},\boldsymbol{u}),\boldsymbol{u})\left(\frac{\partial \boldsymbol{\phi}}{\partial t}(s;t,\tilde{\boldsymbol{x}},\boldsymbol{u}) + \frac{\partial \boldsymbol{\phi}}{\partial \tilde{\boldsymbol{x}}}(s;t,\tilde{\boldsymbol{x}},\boldsymbol{u})\tilde{\boldsymbol{f}}(t,\tilde{\boldsymbol{x}},\boldsymbol{u})\right)ds.
\end{aligned} \tag{85}$$

By differentiation of (85) with respect to τ, we obtain the following differential equation

$$\begin{aligned}
&\frac{\partial}{\partial \tau}\left(\frac{\partial \boldsymbol{\phi}}{\partial t}(\tau;t,\tilde{\boldsymbol{x}},\boldsymbol{u}) + \frac{\partial \boldsymbol{\phi}}{\partial \tilde{\boldsymbol{x}}}(\tau;t,\tilde{\boldsymbol{x}},\boldsymbol{u})\tilde{\boldsymbol{f}}(t,\tilde{\boldsymbol{x}},\boldsymbol{u})\right) =\\
&\qquad \frac{\partial \tilde{\boldsymbol{f}}}{\partial \tilde{\boldsymbol{x}}}(\tau,\boldsymbol{\phi}(\tau;t,\tilde{\boldsymbol{x}},\boldsymbol{u}),\boldsymbol{u})\left(\frac{\partial \boldsymbol{\phi}}{\partial t}(\tau;t,\tilde{\boldsymbol{x}},\boldsymbol{u}) + \frac{\partial \boldsymbol{\phi}}{\partial \tilde{\boldsymbol{x}}}(\tau;t,\tilde{\boldsymbol{x}},\boldsymbol{u})\tilde{\boldsymbol{f}}(t,\tilde{\boldsymbol{x}},\boldsymbol{u})\right),
\end{aligned} \tag{86}$$

with the initial condition (which follows from (84))

$$\frac{\partial \boldsymbol{\phi}}{\partial t}(t;t,\tilde{\boldsymbol{x}},\boldsymbol{u}) + \frac{\partial \boldsymbol{\phi}}{\partial \tilde{\boldsymbol{x}}}(t;t,\tilde{\boldsymbol{x}},\boldsymbol{u})\tilde{\boldsymbol{f}}(t,\tilde{\boldsymbol{x}},\boldsymbol{u}) = -\tilde{\boldsymbol{f}}(t,\tilde{\boldsymbol{x}},\boldsymbol{u}) + \tilde{\boldsymbol{f}}(t,\tilde{\boldsymbol{x}},\boldsymbol{u}) = 0. \tag{87}$$

From the differential equation in (86) and the initial condition in (87), it follows that

$$\frac{\partial \boldsymbol{\phi}}{\partial t}(\tau;t,\tilde{\boldsymbol{x}},\boldsymbol{u}) + \frac{\partial \boldsymbol{\phi}}{\partial \tilde{\boldsymbol{x}}}(\tau;t,\tilde{\boldsymbol{x}},\boldsymbol{u})\tilde{\boldsymbol{f}}(t,\tilde{\boldsymbol{x}},\boldsymbol{u}) = 0, \quad \forall\, \tau \geq t, \tag{88}$$

which renders the third term in the right-hand side of (82) zero.

In order to evaluate the first term in the right-hand side of (82), we use (69) from which it follows that

$$\begin{aligned}
\boldsymbol{\phi}^\top(t+\delta_{\boldsymbol{x}};t,\tilde{\boldsymbol{x}},\boldsymbol{u})\boldsymbol{\phi}(t+\delta_{\boldsymbol{x}};t,\tilde{\boldsymbol{x}},\boldsymbol{u}) &= \|\boldsymbol{\phi}(t+\delta_{\boldsymbol{x}};t,\tilde{\boldsymbol{x}},\boldsymbol{u})\|^2\\
&\leq \bar{\mu}_{\boldsymbol{x}}^2 e^{-2\underline{\nu}_{\boldsymbol{x}}\delta_{\boldsymbol{x}}}\|\boldsymbol{\phi}(t;t,\tilde{\boldsymbol{x}},\boldsymbol{u})\|^2 = \bar{\mu}_{\boldsymbol{x}}^2 e^{-2\underline{\nu}_{\boldsymbol{x}}\delta_{\boldsymbol{x}}}\|\tilde{\boldsymbol{x}}\|^2.
\end{aligned} \tag{89}$$

By using the results in (88) and (89), the derivative of $V_{\boldsymbol{x}}$ along the trajectories of the plant in (82) yields

$$\frac{\partial V_{\boldsymbol{x}}}{\partial t}(t,\tilde{\boldsymbol{x}},\boldsymbol{u}) + \frac{\partial V_{\boldsymbol{x}}}{\partial \tilde{\boldsymbol{x}}}(t,\tilde{\boldsymbol{x}},\boldsymbol{u})\tilde{\boldsymbol{f}}(t,\tilde{\boldsymbol{x}},\boldsymbol{u}) \leq -\left(1 - \bar{\mu}_{\boldsymbol{x}}^2 e^{-2\underline{\nu}_{\boldsymbol{x}}\delta_{\boldsymbol{x}}}\right)\|\tilde{\boldsymbol{x}}\|^2. \tag{90}$$

The bound in (90) implies that (31) is satisfied with $\gamma_{\boldsymbol{x}3} = \left(1 - \bar{\mu}_{\boldsymbol{x}}^2 e^{-2\underline{\nu}_{\boldsymbol{x}}\delta_{\boldsymbol{x}}}\right)$. To show that $\gamma_{\boldsymbol{x}3} > 0$, we choose δ_x in (72) such that $\delta_x > \frac{\ln(\bar{\mu}_{\boldsymbol{x}})}{\underline{\nu}_{\boldsymbol{x}}} > 0$, where $\bar{\mu}_{\boldsymbol{x}}, \underline{\nu}_{\boldsymbol{x}} \in \mathbb{R}_{>0}$. Without loss of generality, this implies that

$$\begin{aligned}
\bar{\mu}_{\boldsymbol{x}}^2 e^{-2\ln(\bar{\mu}_{\boldsymbol{x}})} &> \bar{\mu}_{\boldsymbol{x}}^2 e^{-2\underline{\nu}_{\boldsymbol{x}}\delta_{\boldsymbol{x}}} \geq 0,\\
\bar{\mu}_{\boldsymbol{x}}^2 e^{\ln\left(1/\bar{\mu}_{\boldsymbol{x}}^2\right)} &> \bar{\mu}_{\boldsymbol{x}}^2 e^{-2\underline{\nu}_{\boldsymbol{x}}\delta_{\boldsymbol{x}}} \geq 0,\\
1 &> \bar{\mu}_{\boldsymbol{x}}^2 e^{-2\underline{\nu}_{\boldsymbol{x}}\delta_{\boldsymbol{x}}} \geq 0.
\end{aligned} \tag{91}$$

As such, we have that $\gamma_{\boldsymbol{x}3} > 0$.

To show the validity of the first inequality in (32), consider the derivative of $V_{\boldsymbol{x}}$ with respect to $\tilde{\boldsymbol{x}}$:

$$\frac{\partial V_{\boldsymbol{x}}}{\partial \tilde{\boldsymbol{x}}} = \int_{t}^{t+\delta_{\boldsymbol{x}}} 2\boldsymbol{\phi}^{\top}(\tau; t, \tilde{\boldsymbol{x}}, \boldsymbol{u}) \frac{\partial \boldsymbol{\phi}}{\partial \tilde{\boldsymbol{x}}}(\tau; t, \tilde{\boldsymbol{x}}, \boldsymbol{u}) d\tau. \tag{92}$$

Then we obtain

$$\begin{aligned} \left\| \frac{\partial V_{\boldsymbol{x}}}{\partial \tilde{\boldsymbol{x}}} \right\| &= \left\| \int_{t}^{t+\delta_{\boldsymbol{x}}} 2\boldsymbol{\phi}^{\top}(\tau; t, \tilde{\boldsymbol{x}}, \boldsymbol{u}) \frac{\partial \boldsymbol{\phi}}{\partial \tilde{\boldsymbol{x}}}(\tau; t, \tilde{\boldsymbol{x}}, \boldsymbol{u}) d\tau \right\| \\ &\leq \int_{t}^{t+\delta_{\boldsymbol{x}}} 2 \left\| \boldsymbol{\phi}(\tau; t, \tilde{\boldsymbol{x}}, \boldsymbol{u}) \right\| \left\| \frac{\partial \boldsymbol{\phi}}{\partial \tilde{\boldsymbol{x}}}(\tau; t, \tilde{\boldsymbol{x}}, \boldsymbol{u}) \right\| d\tau. \end{aligned} \tag{93}$$

From (71) it follows that

$$\left\| \boldsymbol{\phi}(\tau; t, \tilde{\boldsymbol{x}}, \boldsymbol{u}) \right\| \leq \bar{\mu}_{\boldsymbol{x}} e^{-\underline{\nu}_{\boldsymbol{x}}(\tau - t)} \|\tilde{\boldsymbol{x}}\|, \quad \forall \tau \geq t. \tag{94}$$

Moreover, by differentiation of the second equation in (84) with respect to τ we obtain

$$\frac{\partial}{\partial \tau}\left(\frac{\partial \boldsymbol{\phi}}{\partial \tilde{\boldsymbol{x}}}(\tau; t, \tilde{\boldsymbol{x}}, \boldsymbol{u}) \right) = \frac{\partial \tilde{\boldsymbol{f}}}{\partial \tilde{\boldsymbol{x}}}(\tau, \boldsymbol{\phi}(\tau; t, \tilde{\boldsymbol{x}}, \boldsymbol{u}), \boldsymbol{u}) \frac{\partial \boldsymbol{\phi}}{\partial \tilde{\boldsymbol{x}}}(\tau; t, \tilde{\boldsymbol{x}}, \boldsymbol{u}), \quad \frac{\partial \boldsymbol{\phi}}{\partial \tilde{\boldsymbol{x}}}(t; t, \tilde{\boldsymbol{x}}, \boldsymbol{u}) = \boldsymbol{I}. \tag{95}$$

Then we obtain the following bound:

$$\left\| \frac{\partial}{\partial \tau}\left(\frac{\partial \boldsymbol{\phi}}{\partial \tilde{\boldsymbol{x}}}(\tau; t, \tilde{\boldsymbol{x}}, \boldsymbol{u}) \right) \right\| \leq \left\| \frac{\partial \tilde{\boldsymbol{f}}}{\partial \tilde{\boldsymbol{x}}}(\tau, \boldsymbol{\phi}(\tau; t, \tilde{\boldsymbol{x}}, \boldsymbol{u}), \boldsymbol{u}) \right\| \left\| \frac{\partial \boldsymbol{\phi}}{\partial \tilde{\boldsymbol{x}}}(\tau; t, \tilde{\boldsymbol{x}}, \boldsymbol{u}) \right\|. \tag{96}$$

Using the fact that

$$\frac{\partial}{\partial \tau} \left\| \frac{\partial \boldsymbol{\phi}}{\partial \tilde{\boldsymbol{x}}}(\tau; t, \tilde{\boldsymbol{x}}, \boldsymbol{u}) \right\| \leq \left\| \frac{\partial}{\partial \tau}\left(\frac{\partial \boldsymbol{\phi}}{\partial \tilde{\boldsymbol{x}}}(\tau; t, \tilde{\boldsymbol{x}}, \boldsymbol{u}) \right) \right\|, \tag{97}$$

and from (74) we have that

$$\begin{aligned} \left\| \frac{\partial \tilde{\boldsymbol{f}}}{\partial \tilde{\boldsymbol{x}}}(t, \tilde{\boldsymbol{x}}, \boldsymbol{u}) \right\| &= \left\| \frac{\partial \boldsymbol{x}}{\partial \tilde{\boldsymbol{x}}} \frac{\partial}{\partial \boldsymbol{x}} \Big(\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{w}) - \boldsymbol{f}(\bar{\boldsymbol{x}}_{\boldsymbol{w}}, \boldsymbol{u}, \boldsymbol{w}) \Big) \right\| \\ &= \left\| \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{w}) \right\| \\ &\leq L_{\boldsymbol{f}\boldsymbol{x}}, \end{aligned} \tag{98}$$

it follows from (96), (97), and (98) that

$$\frac{\partial}{\partial \tau} \left\| \frac{\partial \boldsymbol{\phi}}{\partial \tilde{\boldsymbol{x}}}(\tau; t, \tilde{\boldsymbol{x}}, \boldsymbol{u}) \right\| \leq L_{\boldsymbol{f}\boldsymbol{x}} \left\| \frac{\partial \boldsymbol{\phi}}{\partial \tilde{\boldsymbol{x}}}(\tau; t, \tilde{\boldsymbol{x}}, \boldsymbol{u}) \right\|. \tag{99}$$

The inequality in (99) can be rewritten as

$$\frac{\partial}{\partial \tau} \left(\left\| \frac{\partial \boldsymbol{\phi}}{\partial \tilde{\boldsymbol{x}}}(\tau; t, \tilde{\boldsymbol{x}}, \boldsymbol{u}) \right\| e^{-L_{\boldsymbol{f}\boldsymbol{x}}\tau} \right) \leq 0. \tag{100}$$

By integrating both sides with respect to time over the domain $[t, \tau]$ and using the initial condition in (95), we obtain

$$\begin{aligned} \left\| \frac{\partial \boldsymbol{\phi}}{\partial \tilde{\boldsymbol{x}}}(\tau; t, \tilde{\boldsymbol{x}}, \boldsymbol{u}) \right\| e^{-L_{\boldsymbol{f}\boldsymbol{x}}\tau} &\leq \left\| \frac{\partial \boldsymbol{\phi}}{\partial \tilde{\boldsymbol{x}}}(t; t, \tilde{\boldsymbol{x}}, \boldsymbol{u}) \right\| e^{-L_{\boldsymbol{f}\boldsymbol{x}}t}, \\ \left\| \frac{\partial \boldsymbol{\phi}}{\partial \tilde{\boldsymbol{x}}}(\tau; t, \tilde{\boldsymbol{x}}, \boldsymbol{u}) \right\| &\leq e^{L_{\boldsymbol{f}\boldsymbol{x}}(\tau - t)}. \end{aligned} \tag{101}$$

Using (94) and (101) in (93), we obtain

$$\begin{aligned} \left\| \frac{\partial V_{\boldsymbol{x}}}{\partial \tilde{\boldsymbol{x}}} \right\| &\leq \int_{t}^{t+\delta_{\boldsymbol{x}}} 2 \left\| \boldsymbol{\phi}(\tau; t, \tilde{\boldsymbol{x}}, \boldsymbol{u}) \right\| \left\| \frac{\partial \boldsymbol{\phi}}{\partial \tilde{\boldsymbol{x}}}(\tau; t, \tilde{\boldsymbol{x}}, \boldsymbol{u}) \right\| d\tau, \\ &\leq \int_{t}^{t+\delta_{\boldsymbol{x}}} 2\bar{\mu}_{\boldsymbol{x}} e^{-(\underline{\nu}_{\boldsymbol{x}} - L_{\boldsymbol{f}\boldsymbol{x}})(\tau - t)} d\tau \|\tilde{\boldsymbol{x}}\|, \\ &\leq \frac{2\bar{\mu}_{\boldsymbol{x}}}{\underline{\nu}_{\boldsymbol{x}} - L_{\boldsymbol{f}\boldsymbol{x}}} \left(1 - e^{-(\underline{\nu}_{\boldsymbol{x}} - L_{\boldsymbol{f}\boldsymbol{x}})\delta_{\boldsymbol{x}}} \right) \|\tilde{\boldsymbol{x}}\|. \end{aligned} \tag{102}$$

The bound in (102) implies that the first inequality in (32) is satisfied with

$$\gamma_{\boldsymbol{x}4} = \frac{2\bar{\mu}_{\boldsymbol{x}}}{L_{\boldsymbol{f}\boldsymbol{x}} - \underline{\nu}_{\boldsymbol{x}}} \left(e^{(L_{\boldsymbol{f}\boldsymbol{x}} - \underline{\nu}_{\boldsymbol{x}})\delta_{\boldsymbol{x}}} - 1 \right). \tag{103}$$

Given the fact that $L_{\boldsymbol{f}\boldsymbol{x}}, \bar{\mu}_{\boldsymbol{x}}, \underline{\nu}_{\boldsymbol{x}}, \delta_{\boldsymbol{x}} \in \mathbb{R}_{>0}$, consider the following two cases:

1) If $L_{\boldsymbol{f}\boldsymbol{x}} > \underline{\nu}_{\boldsymbol{x}}$, it follows that

$$e^{(L_{\boldsymbol{f}\boldsymbol{x}} - \underline{\nu}_{\boldsymbol{x}})\delta_{\boldsymbol{x}}} - 1 > 0. \tag{104}$$

Moreover, since $\frac{2\bar{\mu}_{\boldsymbol{x}}}{L_{\boldsymbol{f}\boldsymbol{x}} - \underline{\nu}_{\boldsymbol{x}}} > 0$, it follows that $\gamma_{\boldsymbol{x}4} > 0$.

2) If $L_{\boldsymbol{f}\boldsymbol{x}} < \underline{\nu}_{\boldsymbol{x}}$, it follows that

$$-1 \leq e^{(L_{\boldsymbol{f}\boldsymbol{x}} - \underline{\nu}_{\boldsymbol{x}})\delta_{\boldsymbol{x}}} - 1 < 0. \tag{105}$$

Moreover, since $\frac{2\bar{\mu}_{\boldsymbol{x}}}{L_{\boldsymbol{f}\boldsymbol{x}} - \underline{\nu}_{\boldsymbol{x}}} < 0$, it follows that $\gamma_{\boldsymbol{x}4} > 0$.

Without loss of generality, we assume that $L_{\boldsymbol{f}\boldsymbol{x}} \neq \underline{\nu}_{\boldsymbol{x}}$, such that $\gamma_{\boldsymbol{x}4} > 0$.

To show the validity of the second inequality in (32), consider the derivative of $V_{\boldsymbol{x}}$ with respect to $\boldsymbol{u}$ which reads

$$\frac{\partial V_{\boldsymbol{x}}}{\partial \boldsymbol{u}} = \int_t^{t+\delta_{\boldsymbol{x}}} 2\boldsymbol{\phi}^\top(\tau; t, \tilde{\boldsymbol{x}}, \boldsymbol{u}) \frac{\partial \boldsymbol{\phi}}{\partial \boldsymbol{u}}(\tau; t, \tilde{\boldsymbol{x}}, \boldsymbol{u}) d\tau, \tag{106}$$

from which it follows that

$$\begin{aligned} \left\| \frac{\partial V_{\boldsymbol{x}}}{\partial \boldsymbol{u}} \right\| &= \left\| \int_t^{t+\delta_{\boldsymbol{x}}} 2\boldsymbol{\phi}^\top(\tau; t, \tilde{\boldsymbol{x}}, \boldsymbol{u}) \frac{\partial \boldsymbol{\phi}}{\partial \boldsymbol{u}}(\tau; t, \tilde{\boldsymbol{x}}, \boldsymbol{u}) d\tau \right\| \\ &\leq \int_t^{t+\delta_{\boldsymbol{x}}} 2\|\boldsymbol{\phi}(\tau; t, \tilde{\boldsymbol{x}}, \boldsymbol{u})\| \left\| \frac{\partial \boldsymbol{\phi}}{\partial \boldsymbol{u}}(\tau; t, \tilde{\boldsymbol{x}}, \boldsymbol{u}) \right\| d\tau. \end{aligned} \tag{107}$$

Differentiation of both sides of (83) with respect to $\boldsymbol{u}$ and by using (27) we have that

$$\begin{aligned} \frac{\partial \boldsymbol{\phi}}{\partial \boldsymbol{u}}(\tau; t, \tilde{\boldsymbol{x}}, \boldsymbol{u}) &= \frac{\partial}{\partial \boldsymbol{u}} \left(\int_t^\tau \tilde{\boldsymbol{f}}(s, \boldsymbol{\phi}(s; t, \tilde{\boldsymbol{x}}, \boldsymbol{u}), \boldsymbol{u}) ds \right), \\ &= \int_t^\tau \frac{\partial \tilde{\boldsymbol{f}}}{\partial \boldsymbol{u}}(s, \boldsymbol{\phi}(s; t, \tilde{\boldsymbol{x}}, \boldsymbol{u}), \boldsymbol{u}) + \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}(\boldsymbol{\phi}(s; t, \tilde{\boldsymbol{x}}, \boldsymbol{u}) + \bar{\boldsymbol{x}}_{\boldsymbol{w}}(s, \boldsymbol{u}), \boldsymbol{u}, \boldsymbol{w}) \frac{\partial \boldsymbol{\phi}}{\partial \boldsymbol{u}}(s; t, \tilde{\boldsymbol{x}}, \boldsymbol{u}) ds. \end{aligned} \tag{108}$$

Taking the norm, it follows that

$$\begin{aligned} \left\| \frac{\partial \boldsymbol{\phi}}{\partial \boldsymbol{u}}(\tau; t, \tilde{\boldsymbol{x}}, \boldsymbol{u}) \right\| \leq \int_t^\tau & \left\| \frac{\partial \tilde{\boldsymbol{f}}}{\partial \boldsymbol{u}}(s, \boldsymbol{\phi}(s; t, \tilde{\boldsymbol{x}}, \boldsymbol{u}), \boldsymbol{u}) \right\| \\ & + \left\| \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}(\boldsymbol{\phi}(s; t, \tilde{\boldsymbol{x}}, \boldsymbol{u}) + \bar{\boldsymbol{x}}_{\boldsymbol{w}}(s, \boldsymbol{u}), \boldsymbol{u}, \boldsymbol{w}) \right\| \left\| \frac{\partial \boldsymbol{\phi}}{\partial \boldsymbol{u}}(s; t, \tilde{\boldsymbol{x}}, \boldsymbol{u}) \right\| ds \end{aligned} \tag{109}$$

Using (74), Assumption 3, (98), and

$$\begin{aligned} \left\| \frac{\partial \tilde{\boldsymbol{f}}}{\partial \boldsymbol{u}}(t, \tilde{\boldsymbol{x}}, \boldsymbol{u}) \right\| &= \left\| \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{w}) - \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}}(\bar{\boldsymbol{x}}_{\boldsymbol{w}}, \boldsymbol{u}, \boldsymbol{w}) \right\|, \\ &\leq \left\| \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{w}) \right\| + \left\| \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}}(\bar{\boldsymbol{x}}_{\boldsymbol{w}}, \boldsymbol{u}, \boldsymbol{w}) \right\| \\ &\leq 2L_{\boldsymbol{f}\boldsymbol{u}}, \end{aligned} \tag{110}$$

we obtain the following inequality

$$\begin{aligned}\left\|\frac{\partial \boldsymbol{\phi}}{\partial \boldsymbol{u}}(\tau;t,\tilde{\boldsymbol{x}},\boldsymbol{u})\right\| &\leq \int_t^\tau \left(2L_{\boldsymbol{fu}} + L_{\boldsymbol{fx}}\left\|\frac{\partial \boldsymbol{\phi}}{\partial \boldsymbol{u}}(s;t,\tilde{\boldsymbol{x}},\boldsymbol{u})\right\|\right)ds,\\ &\leq 2L_{\boldsymbol{fu}}(\tau-t) + L_{\boldsymbol{fx}}\int_t^\tau \left\|\frac{\partial \boldsymbol{\phi}}{\partial \boldsymbol{u}}(s;t,\tilde{\boldsymbol{x}},\boldsymbol{u})\right\|ds.\end{aligned} \tag{111}$$

By using Grönwall's inequality, it follows that

$$\left\|\frac{\partial \boldsymbol{\phi}}{\partial \boldsymbol{u}}(\tau;t,\tilde{\boldsymbol{x}},\boldsymbol{u})\right\| \leq 2L_{\boldsymbol{fu}}(\tau-t)e^{L_{\boldsymbol{fx}}(\tau-t)}. \tag{112}$$

Substitution of the exponentially decaying bound on the trajectories in (69) and (112) in (107), it follows that

$$\begin{aligned}\left\|\frac{\partial V_{\boldsymbol{x}}}{\partial \boldsymbol{u}}\right\| &\leq \int_t^{t+\delta_{\boldsymbol{x}}} 2\|\boldsymbol{\phi}(\tau;t,\tilde{\boldsymbol{x}},\boldsymbol{u})\|\left\|\frac{\partial \boldsymbol{\phi}}{\partial \boldsymbol{u}}(\tau;t,\tilde{\boldsymbol{x}},\boldsymbol{u})\right\|d\tau,\\ &\leq \int_t^{t+\delta_{\boldsymbol{x}}} 4\bar{\mu}_{\boldsymbol{x}}L_{\boldsymbol{fu}}(\tau-t)e^{(L_{\boldsymbol{fx}}-\underline{\nu}_{\boldsymbol{x}})(\tau-t)}d\tau\|\tilde{\boldsymbol{x}}\|,\\ &\leq 4\bar{\mu}_{\boldsymbol{x}}L_{\boldsymbol{fu}}e^{-(L_{\boldsymbol{fx}}-\underline{\nu}_{\boldsymbol{x}})t}\left(\int_t^{t+\delta_{\boldsymbol{x}}} e^{(L_{\boldsymbol{fx}}-\underline{\nu}_{\boldsymbol{x}})\tau}\tau d\tau - t\int_t^{t+\delta_{\boldsymbol{x}}} e^{(L_{\boldsymbol{fx}}-\underline{\nu}_{\boldsymbol{x}})\tau}d\tau\right)\|\tilde{\boldsymbol{x}}\|.\end{aligned} \tag{113}$$

The integrals in (113) are given by

$$\begin{aligned}\int_t^{t+\delta_{\boldsymbol{x}}} e^{(L_{\boldsymbol{fx}}-\underline{\nu}_{\boldsymbol{x}})\tau}\tau d\tau &= e^{(L_{\boldsymbol{fx}}-\underline{\nu}_{\boldsymbol{x}})(t+\delta_{\boldsymbol{x}})}\left(\frac{(L_{\boldsymbol{fx}}-\underline{\nu}_{\boldsymbol{x}})(t+\delta_{\boldsymbol{x}})-1}{(L_{\boldsymbol{fx}}-\underline{\nu}_{\boldsymbol{x}})^2}\right) - e^{(L_{\boldsymbol{fx}}-\underline{\nu}_{\boldsymbol{x}})t}\left(\frac{(L_{\boldsymbol{fx}}-\underline{\nu}_{\boldsymbol{x}})t-1}{(L_{\boldsymbol{fx}}-\underline{\nu}_{\boldsymbol{x}})^2}\right)\\ &= \frac{e^{(L_{\boldsymbol{fx}}-\underline{\nu}_{\boldsymbol{x}})t}}{(L_{\boldsymbol{fx}}-\underline{\nu}_{\boldsymbol{x}})^2}\left(e^{(L_{\boldsymbol{fx}}-\underline{\nu}_{\boldsymbol{x}})\delta_{\boldsymbol{x}}}\big((L_{\boldsymbol{fx}}-\underline{\nu}_{\boldsymbol{x}})(t+\delta_{\boldsymbol{x}})-1\big) - \big((L_{\boldsymbol{fx}}-\underline{\nu}_{\boldsymbol{x}})t-1\big)\right),\end{aligned} \tag{114}$$

and

$$\begin{aligned}-t\int_t^{t+\delta_{\boldsymbol{x}}} e^{(L_{\boldsymbol{fx}}-\underline{\nu}_{\boldsymbol{x}})\tau}d\tau &= -\frac{t}{L_{\boldsymbol{fx}}-\underline{\nu}_{\boldsymbol{x}}}\left(e^{(L_{\boldsymbol{fx}}-\underline{\nu}_{\boldsymbol{x}})(t+\delta_{\boldsymbol{x}})} - e^{(L_{\boldsymbol{fx}}-\underline{\nu}_{\boldsymbol{x}})t}\right)\\ &= -\frac{e^{(L_{\boldsymbol{fx}}-\underline{\nu}_{\boldsymbol{x}})t}}{(L_{\boldsymbol{fx}}-\underline{\nu}_{\boldsymbol{x}})^2}\left(e^{(L_{\boldsymbol{fx}}-\underline{\nu}_{\boldsymbol{x}})\delta_{\boldsymbol{x}}}(L_{\boldsymbol{fx}}-\underline{\nu}_{\boldsymbol{x}})t - (L_{\boldsymbol{fx}}-\underline{\nu}_{\boldsymbol{x}})t\right),\end{aligned} \tag{115}$$

respectively. As a result, (113) can be written as

$$\left\|\frac{\partial V_{\boldsymbol{x}}}{\partial \boldsymbol{u}}\right\| \leq \frac{4\bar{\mu}_{\boldsymbol{x}}L_{\boldsymbol{fu}}}{(L_{\boldsymbol{fx}}-\underline{\nu}_{\boldsymbol{x}})^2}\left(e^{(L_{\boldsymbol{fx}}-\underline{\nu}_{\boldsymbol{x}})\delta_{\boldsymbol{x}}}\big((L_{\boldsymbol{fx}}-\underline{\nu}_{\boldsymbol{x}})\delta_{\boldsymbol{x}}-1\big)+1\right)\|\tilde{\boldsymbol{x}}\| \tag{116}$$

The bound in (116) implies that the second inequality in (32) is satisfied with

$$\gamma_{\boldsymbol{x}5} = \frac{4\bar{\mu}_{\boldsymbol{x}}L_{\boldsymbol{fu}}}{(L_{\boldsymbol{fx}}-\underline{\nu}_{\boldsymbol{x}})^2}\left(e^{(L_{\boldsymbol{fx}}-\underline{\nu}_{\boldsymbol{x}})\delta_{\boldsymbol{x}}}\big((L_{\boldsymbol{fx}}-\underline{\nu}_{\boldsymbol{x}})\delta_{\boldsymbol{x}}-1\big)+1\right). \tag{117}$$

Without loss of generality, if $L_{\boldsymbol{fx}} \neq \underline{\nu}_{\boldsymbol{x}}$ and with $\bar{\mu}_{\boldsymbol{x}}$, $\underline{\nu}_{\boldsymbol{x}}$, $L_{\boldsymbol{x}}$, $L_{\boldsymbol{fu}}$, $L_{\boldsymbol{fx}}$, $\delta_{\boldsymbol{x}} \in \mathbb{R}_{>0}$, it can be shown that $\gamma_{\boldsymbol{x}5} > 0$; the fraction in the expression for $\gamma_{\boldsymbol{x}5}$ is defined and positive whenever $L_{\boldsymbol{fx}} \neq \underline{\nu}_{\boldsymbol{x}}$. The expression between brackets is a function of the form $q(x) = 1 + e^x(x-1)$. The derivative of $q(x)$ with respect to x is given by ${dq}/{dx} = e^x x$. From this follows that ${dq}/{dx} = 0$ if $x = 0$ (the limit $x \to -\infty$ is not considered here). For $x = 0$ it follows that $q(0) = 0$. Furthermore, ${dq}/{dx} < 0$ for $x < 0$, and ${dq}/{dx} > 0$ for $x > 0$. As such, $q(x)$ is positive for all $x \neq 0$. As a result, if $L_{\boldsymbol{fx}} \neq \underline{\nu}_{\boldsymbol{x}}$, then $\gamma_{\boldsymbol{x}5} > 0$. This completes the proof of Lemma 1. □

6.2 Proof of Lemma 2.

By using the function $V_{\boldsymbol{x}}$ in Lemma 1 as a Lyapunov function candidate for the $\tilde{\boldsymbol{x}}$-dynamics with time-varying inputs $\boldsymbol{u}(t)$ in (33) we obtain the following expression for $\dot{V}_{\boldsymbol{x}}$:

$$\begin{aligned}\dot{V}_{\boldsymbol{x}}(t,\tilde{\boldsymbol{x}},\boldsymbol{u}) &= \frac{\partial V_{\boldsymbol{x}}}{\partial t}(t,\tilde{\boldsymbol{x}},\boldsymbol{u}) + \frac{\partial V_{\boldsymbol{x}}}{\partial \tilde{\boldsymbol{x}}}(t,\tilde{\boldsymbol{x}},\boldsymbol{u})\dot{\tilde{\boldsymbol{x}}} + \frac{\partial V_{\boldsymbol{x}}}{\partial \boldsymbol{u}}(t,\tilde{\boldsymbol{x}},\boldsymbol{u})\dot{\boldsymbol{u}} \\ &= \frac{\partial V_{\boldsymbol{x}}}{\partial t}(t,\tilde{\boldsymbol{x}},\boldsymbol{u}) + \frac{\partial V_{\boldsymbol{x}}}{\partial \tilde{\boldsymbol{x}}}(t,\tilde{\boldsymbol{x}},\boldsymbol{u})\left(\tilde{\boldsymbol{f}}(t,\tilde{\boldsymbol{x}},\boldsymbol{u}) - \frac{\partial \bar{\boldsymbol{x}}_{\boldsymbol{w}}}{\partial \boldsymbol{u}}(t,\boldsymbol{u})\dot{\boldsymbol{u}}\right) + \frac{\partial V_{\boldsymbol{x}}}{\partial \boldsymbol{u}}(t,\tilde{\boldsymbol{x}},\boldsymbol{u})\dot{\boldsymbol{u}} \\ &= \frac{\partial V_{\boldsymbol{x}}}{\partial t}(t,\tilde{\boldsymbol{x}},\boldsymbol{u}) + \frac{\partial V_{\boldsymbol{x}}}{\partial \tilde{\boldsymbol{x}}}(t,\tilde{\boldsymbol{x}},\boldsymbol{u})\tilde{\boldsymbol{f}}(t,\tilde{\boldsymbol{x}},\boldsymbol{u}) + \left(\frac{\partial V_{\boldsymbol{x}}}{\partial \boldsymbol{u}}(t,\tilde{\boldsymbol{x}},\boldsymbol{u}) - \frac{\partial V_{\boldsymbol{x}}}{\partial \tilde{\boldsymbol{x}}}(t,\tilde{\boldsymbol{x}},\boldsymbol{u})\frac{\partial \bar{\boldsymbol{x}}_{\boldsymbol{w}}}{\partial \boldsymbol{u}}(t,\boldsymbol{u})\right)\dot{\boldsymbol{u}},\end{aligned} \tag{118}$$

for all $\boldsymbol{x} \in \mathcal{X}$, all (time-varying) inputs $\boldsymbol{u}(t) \in \mathcal{U}$, for all t, and all (time-varying) disturbances $\boldsymbol{w}(t) \in \mathcal{W}$, for all t. Note that in (118) we have omitted the implicit time-dependency for notational clarity. Using Lemma 1 and Assumption 3, it follows that

$$\begin{aligned}\dot{V}_{\boldsymbol{x}}(t,\tilde{\boldsymbol{x}},\boldsymbol{u}) &\le -\gamma_{\boldsymbol{x}3}\|\tilde{\boldsymbol{x}}\|^2 + \left(\left\|\frac{\partial V_{\boldsymbol{x}}}{\partial \boldsymbol{u}}(t,\tilde{\boldsymbol{x}},\boldsymbol{u})\right\| + \left\|\frac{\partial V_{\boldsymbol{x}}}{\partial \tilde{\boldsymbol{x}}}(t,\tilde{\boldsymbol{x}},\boldsymbol{u})\right\|\left\|\frac{\partial \bar{\boldsymbol{x}}_{\boldsymbol{w}}}{\partial \boldsymbol{u}}(t,\boldsymbol{u})\right\|\right)\|\dot{\boldsymbol{u}}\| \\ &\le -\gamma_{\boldsymbol{x}3}\|\tilde{\boldsymbol{x}}\|^2 + (\gamma_{\boldsymbol{x}4}L_{\boldsymbol{xu}} + \gamma_{\boldsymbol{x}5})\|\tilde{\boldsymbol{x}}\|\|\dot{\boldsymbol{u}}\|, \\ &\le -\gamma_{\boldsymbol{x}3}\|\tilde{\boldsymbol{x}}\|^2 + \sqrt{\gamma_{\boldsymbol{x}3}}\|\tilde{\boldsymbol{x}}\|\frac{(\gamma_{\boldsymbol{x}4}L_{\boldsymbol{xu}} + \gamma_{\boldsymbol{x}5})}{\sqrt{\gamma_{\boldsymbol{x}3}}}\|\dot{\boldsymbol{u}}\|.\end{aligned} \tag{119}$$

Using Young's inequality and from (30) in Lemma 1 we obtain

$$\begin{aligned}\dot{V}_{\boldsymbol{x}}(t,\tilde{\boldsymbol{x}},\boldsymbol{u}) &\le -\gamma_{\boldsymbol{x}3}\|\tilde{\boldsymbol{x}}\|^2 + \frac{1}{2}\gamma_{\boldsymbol{x}3}\|\tilde{\boldsymbol{x}}\|^2 + \frac{(\gamma_{\boldsymbol{x}4}L_{\boldsymbol{xu}} + \gamma_{\boldsymbol{x}5})^2}{2\gamma_{\boldsymbol{x}3}}\|\dot{\boldsymbol{u}}\|^2, \\ &\le -\frac{\gamma_{\boldsymbol{x}3}}{2\gamma_{\boldsymbol{x}2}}V_{\boldsymbol{x}}(t,\tilde{\boldsymbol{x}},\boldsymbol{u}) + \frac{(\gamma_{\boldsymbol{x}4}L_{\boldsymbol{xu}} + \gamma_{\boldsymbol{x}5})^2}{2\gamma_{\boldsymbol{x}3}}\|\dot{\boldsymbol{u}}\|^2.\end{aligned} \tag{120}$$

To find an upperbound for $\|\dot{\boldsymbol{u}}\|$, it follows from (14) that $\dot{\boldsymbol{u}} = \dot{\hat{\boldsymbol{u}}} + \alpha_{\boldsymbol{\omega}}\dot{\boldsymbol{\omega}}$. From (15) we have that there exists a constant $L_{\boldsymbol{\omega}1} \in \mathbb{R}_{>0}$ such that

$$\|\dot{\boldsymbol{\omega}}\| \le \eta_{\boldsymbol{\omega}}L_{\boldsymbol{\omega}1}. \tag{121}$$

Furthermore, from (22) we have that $\|\dot{\hat{\boldsymbol{u}}}\| \le \eta_{\boldsymbol{u}}$. Therefore, we obtain an upperbound on $\|\dot{\boldsymbol{u}}\|$ which is given by

$$\|\dot{\boldsymbol{u}}\| \le \eta_{\boldsymbol{u}} + \alpha_{\boldsymbol{\omega}}\eta_{\boldsymbol{\omega}}L_{\boldsymbol{\omega}1}, \tag{122}$$

which implies that

$$\|\dot{\boldsymbol{u}}\| \le \alpha_{\boldsymbol{\omega}}\eta_{\boldsymbol{\omega}}\left(\epsilon_3\epsilon_5 + L_{\boldsymbol{\omega}1}\right), \tag{123}$$

for all $\epsilon_3, \epsilon_5 \in \mathbb{R}_{>0}$, all $\eta_{\boldsymbol{m}} \le \eta_{\boldsymbol{\omega}}\epsilon_3$, and all $\eta_{\boldsymbol{u}} \le \alpha_{\boldsymbol{\omega}}\eta_{\boldsymbol{m}}\epsilon_5$. Substitution of (123) in (120) gives

$$\dot{V}_{\boldsymbol{x}}(t,\tilde{\boldsymbol{x}},\boldsymbol{u}) \le -\frac{\gamma_{\boldsymbol{x}3}}{2\gamma_{\boldsymbol{x}2}}V_{\boldsymbol{x}}(t,\tilde{\boldsymbol{x}},\boldsymbol{u}) + \alpha_{\boldsymbol{\omega}}^2\eta_{\boldsymbol{\omega}}^2\frac{(\gamma_{\boldsymbol{x}4}L_{\boldsymbol{xu}} + \gamma_{\boldsymbol{x}5})^2}{2\gamma_{\boldsymbol{x}3}}\left(\epsilon_3\epsilon_5 + L_{\boldsymbol{\omega}1}\right)^2. \tag{124}$$

From the comparison lemma in Lemma 3.4 in Khalil (2002) follows that

$$\begin{aligned}V_{\boldsymbol{x}}(t,\tilde{\boldsymbol{x}}(t),\boldsymbol{u}(t)) &\le V_{\boldsymbol{x}}(0,\tilde{\boldsymbol{x}}(0),\boldsymbol{u}(0))e^{-\frac{\gamma_{\boldsymbol{x}3}}{2\gamma_{\boldsymbol{x}2}}t} + \alpha_{\boldsymbol{\omega}}^2\eta_{\boldsymbol{\omega}}^2\gamma_{\boldsymbol{x}2}\frac{(\gamma_{\boldsymbol{x}4}L_{\boldsymbol{xu}} + \gamma_{\boldsymbol{x}5})^2}{\gamma_{\boldsymbol{x}3}^2}\left(\epsilon_3\epsilon_5 + L_{\boldsymbol{\omega}1}\right)^2\left(1 - e^{-\frac{\gamma_{\boldsymbol{x}3}}{2\gamma_{\boldsymbol{x}2}}t}\right) \\ &\le V_{\boldsymbol{x}}(0,\tilde{\boldsymbol{x}}(0),\boldsymbol{u}(0))e^{-\frac{\gamma_{\boldsymbol{x}3}}{2\gamma_{\boldsymbol{x}2}}t} + \alpha_{\boldsymbol{\omega}}^2\eta_{\boldsymbol{\omega}}^2\gamma_{\boldsymbol{x}2}\frac{(\gamma_{\boldsymbol{x}4}L_{\boldsymbol{xu}} + \gamma_{\boldsymbol{x}5})^2}{\gamma_{\boldsymbol{x}3}^2}\left(\epsilon_3\epsilon_5 + L_{\boldsymbol{\omega}1}\right)^2,\end{aligned} \tag{125}$$

for all $t \ge 0$, all $\boldsymbol{x}(0) \in \mathcal{X}_0$, and all time-varying $\boldsymbol{u}(t) \in \mathcal{U}$. From (30) in Lemma 1 we obtain

$$\|\tilde{\boldsymbol{x}}(t)\|^2 \le \frac{\gamma_{\boldsymbol{x}2}}{\gamma_{\boldsymbol{x}1}}\|\tilde{\boldsymbol{x}}(0)\|^2 e^{-\frac{\gamma_{\boldsymbol{x}3}}{2\gamma_{\boldsymbol{x}2}}t} + \alpha_{\boldsymbol{\omega}}^2\eta_{\boldsymbol{\omega}}^2\gamma_{\boldsymbol{x}2}\frac{(\gamma_{\boldsymbol{x}4}L_{\boldsymbol{xu}} + \gamma_{\boldsymbol{x}5})^2}{\gamma_{\boldsymbol{x}1}\gamma_{\boldsymbol{x}3}^2}\left(\epsilon_3\epsilon_5 + L_{\boldsymbol{\omega}1}\right)^2, \tag{126}$$

such that the bound on $\|\tilde{\boldsymbol{x}}(t)\|$ reads

$$\|\tilde{\boldsymbol{x}}(t)\| \le \sqrt{\frac{\gamma_{\boldsymbol{x}2}}{\gamma_{\boldsymbol{x}1}}\|\tilde{\boldsymbol{x}}(0)\|^2 e^{-\frac{\gamma_{\boldsymbol{x}3}}{2\gamma_{\boldsymbol{x}2}}t} + \alpha_{\boldsymbol{\omega}}^2\eta_{\boldsymbol{\omega}}^2\gamma_{\boldsymbol{x}2}\frac{(\gamma_{\boldsymbol{x}4}L_{\boldsymbol{xu}} + \gamma_{\boldsymbol{x}5})^2}{\gamma_{\boldsymbol{x}1}\gamma_{\boldsymbol{x}3}^2}\left(\epsilon_3\epsilon_5 + L_{\boldsymbol{\omega}1}\right)^2}. \tag{127}$$

The last step is obtain as follows. The inequality in (127) is of the form $\sqrt{C_0^2} \leq \sqrt{C_1^2 + C_2^2}$, with $C_1, C_2 \geq 0$. If $C_1 \geq C_2$, then $\sqrt{C_0^2} \leq \sqrt{2C_1^2} \leq \sqrt{2}C_1$. If $C_2 \geq C_1$, then $\sqrt{C_0^2} \leq \sqrt{2C_2^2} \leq \sqrt{2}C_2$. As a result, $\sqrt{C_0^2} \leq \max\{\sqrt{2}C_1, \sqrt{2}C_2\}$, and thus the bound on $\|\tilde{\boldsymbol{x}}(t)\|$ reads

$$\|\tilde{\boldsymbol{x}}(t)\| \leq \max\left\{c_{\boldsymbol{x}1}\|\tilde{\boldsymbol{x}}(0)\|e^{-\beta_{\boldsymbol{x}}t}, \alpha_{\boldsymbol{\omega}}\eta_{\boldsymbol{\omega}}c_{\boldsymbol{x}2}\right\}, \tag{128}$$

with

$$c_{\boldsymbol{x}1} = \sqrt{\frac{2\gamma_{\boldsymbol{x}2}}{\gamma_{\boldsymbol{x}1}}}, c_{\boldsymbol{x}2} = \sqrt{\frac{2\gamma_{\boldsymbol{x}2}}{\gamma_{\boldsymbol{x}1}}}\frac{\gamma_{\boldsymbol{x}4}L_{\boldsymbol{xu}} + \gamma_{\boldsymbol{x}5}}{\gamma_{\boldsymbol{x}3}}\left(\epsilon_3\epsilon_5 + L_{\boldsymbol{\omega}1}\right), \tag{129}$$

and $\beta_{\boldsymbol{x}} = \dfrac{\gamma_{\boldsymbol{x}3}}{4\gamma_{\boldsymbol{x}2}}$, which completes the proof of Lemma 2. □

6.3 Proof of Lemma 5

By using the function $V_{\boldsymbol{z}}$ in Lemma 4 as a Lyapunov function candidate for the $\tilde{\boldsymbol{z}}$-dynamics with time-varying inputs $\boldsymbol{u}(t)$ in (41) we obtain the following expression for $\dot{V}_{\boldsymbol{z}}$:

$$\begin{aligned}
\dot{V}_{\boldsymbol{z}}(t,\tilde{\boldsymbol{z}},\boldsymbol{u},\alpha_{\boldsymbol{z}}) &= \frac{\partial V_{\boldsymbol{z}}}{\partial t}(t,\tilde{\boldsymbol{z}},\boldsymbol{u},\alpha_{\boldsymbol{z}}) + \frac{\partial V_{\boldsymbol{z}}}{\partial \tilde{\boldsymbol{z}}}(t,\tilde{\boldsymbol{z}},\boldsymbol{u},\alpha_{\boldsymbol{z}})\dot{\tilde{\boldsymbol{z}}} + \frac{\partial V_{\boldsymbol{z}}}{\partial \boldsymbol{u}}(t,\tilde{\boldsymbol{z}},\boldsymbol{u},\alpha_{\boldsymbol{z}})\dot{\boldsymbol{u}} \\
&= \frac{\partial V_{\boldsymbol{z}}}{\partial t}(t,\tilde{\boldsymbol{z}},\boldsymbol{u},\alpha_{\boldsymbol{z}}) + \alpha_{\boldsymbol{z}}\frac{\partial V_{\boldsymbol{z}}}{\partial \tilde{\boldsymbol{z}}}(t,\tilde{\boldsymbol{z}},\boldsymbol{u},\alpha_{\boldsymbol{z}})\tilde{\boldsymbol{h}}(t,\tilde{\boldsymbol{z}},\boldsymbol{u},\alpha_{\boldsymbol{z}}) \\
&\quad + \alpha_{\boldsymbol{z}}\frac{\partial V_{\boldsymbol{z}}}{\partial \tilde{\boldsymbol{z}}}(t,\tilde{\boldsymbol{z}},\boldsymbol{u},\alpha_{\boldsymbol{z}})\Big(\boldsymbol{h}(\boldsymbol{z},\boldsymbol{y}) - \boldsymbol{h}(\boldsymbol{z},\bar{y}_{\boldsymbol{w}}(t,\boldsymbol{u}))\Big) \\
&\quad + \left(\frac{\partial V_{\boldsymbol{z}}}{\partial \boldsymbol{u}}(t,\tilde{\boldsymbol{z}},\boldsymbol{u},\alpha_{\boldsymbol{z}}) - \frac{\partial V_{\boldsymbol{z}}}{\partial \tilde{\boldsymbol{z}}}(t,\tilde{\boldsymbol{z}},\boldsymbol{u},\alpha_{\boldsymbol{z}})\frac{\partial \bar{\boldsymbol{z}}_{\boldsymbol{w}}}{\partial \boldsymbol{u}}(t,\boldsymbol{u},\alpha_{\boldsymbol{z}})\right)\dot{\boldsymbol{u}} \\
&= \frac{\partial V_{\boldsymbol{z}}}{\partial t}(t,\tilde{\boldsymbol{z}},\boldsymbol{u},\alpha_{\boldsymbol{z}}) + \alpha_{\boldsymbol{z}}\frac{\partial V_{\boldsymbol{z}}}{\partial \tilde{\boldsymbol{z}}}(t,\tilde{\boldsymbol{z}},\boldsymbol{u},\alpha_{\boldsymbol{z}})\tilde{\boldsymbol{h}}(t,\tilde{\boldsymbol{z}},\boldsymbol{u},\alpha_{\boldsymbol{z}}) \\
&\quad + \alpha_{\boldsymbol{z}}\frac{\partial V_{\boldsymbol{z}}}{\partial \tilde{\boldsymbol{z}}}(t,\tilde{\boldsymbol{z}},\boldsymbol{u},\alpha_{\boldsymbol{z}})\Big(\boldsymbol{h}(\boldsymbol{z},\boldsymbol{y}) - \boldsymbol{h}(\boldsymbol{z},\bar{y}_{\boldsymbol{w}}(t,\boldsymbol{u}))\Big) \\
&\quad + \left(\frac{\partial V_{\boldsymbol{z}}}{\partial \boldsymbol{u}}(t,\tilde{\boldsymbol{z}},\boldsymbol{u},\alpha_{\boldsymbol{z}}) - \frac{\partial V_{\boldsymbol{z}}}{\partial \tilde{\boldsymbol{z}}}(t,\tilde{\boldsymbol{z}},\boldsymbol{u},\alpha_{\boldsymbol{z}})\left(\frac{\partial \bar{\boldsymbol{z}}_{\boldsymbol{w}}}{\partial \boldsymbol{u}}(t,\boldsymbol{u},\alpha_{\boldsymbol{z}}) - \frac{d\boldsymbol{q}_{\boldsymbol{w}}}{d\boldsymbol{u}}(\boldsymbol{u}) + \frac{d\boldsymbol{q}_{\boldsymbol{w}}}{d\boldsymbol{u}}(\boldsymbol{u})\right)\right)\dot{\boldsymbol{u}}
\end{aligned} \tag{130}$$

for all $\boldsymbol{z} \in \mathcal{Z}$, all (time-varying) inputs $\boldsymbol{u}(t) \in \mathcal{U}$, for all t, all $y, \bar{y}_{\boldsymbol{w}} \in \mathbb{R}$ satisfying (4), respectively, and all (time-varying) disturbances $\boldsymbol{w}(t) \in \mathcal{W}$. Note that in (130) we have omitted the implicit time-dependency for notational clarity. Using the inequalities from Lemma 4, Assumption 6, and the fact that there exists a constant $L_{\boldsymbol{q}} \in \mathbb{R}_{>0}$ such that

$$\left\|\frac{d\boldsymbol{q}_{\boldsymbol{w}}(\boldsymbol{u})}{d\boldsymbol{u}}\right\| \le L_{\boldsymbol{q}}, \qquad \forall\, \boldsymbol{u} \in \mathcal{U} \tag{131}$$

it follows that

$$\begin{aligned}
\dot{V}_{\boldsymbol{z}}(t,\tilde{\boldsymbol{z}},\boldsymbol{u},\alpha_{\boldsymbol{z}}) &\le -\alpha_{\boldsymbol{z}}\gamma_{\boldsymbol{z}3}\|\tilde{\boldsymbol{z}}\|^2 + \alpha_{\boldsymbol{z}}\left\|\frac{\partial V_{\boldsymbol{z}}}{\partial \tilde{\boldsymbol{z}}}(t,\tilde{\boldsymbol{z}},\boldsymbol{u},\alpha_{\boldsymbol{z}})\right\|\|\boldsymbol{h}(\boldsymbol{z},y) - \boldsymbol{h}(\boldsymbol{z},\bar{y}_{\boldsymbol{w}}(t,\boldsymbol{u}))\| \\
&\quad + \left\|\frac{\partial V_{\boldsymbol{z}}}{\partial \boldsymbol{u}}(t,\tilde{\boldsymbol{z}},\boldsymbol{u},\alpha_{\boldsymbol{z}})\right\|\|\dot{\boldsymbol{u}}\| \\
&\quad + \left\|\frac{\partial V_{\boldsymbol{z}}}{\partial \tilde{\boldsymbol{z}}}(t,\tilde{\boldsymbol{z}},\boldsymbol{u},\alpha_{\boldsymbol{z}})\right\|\left(\left\|\frac{\partial \bar{\boldsymbol{z}}_{\boldsymbol{w}}}{\partial \boldsymbol{u}}(t,\boldsymbol{u},\alpha_{\boldsymbol{z}}) - \frac{d\boldsymbol{q}_{\boldsymbol{w}}}{d\boldsymbol{u}}(\boldsymbol{u})\right\| + \left\|\frac{d\boldsymbol{q}_{\boldsymbol{w}}}{d\boldsymbol{u}}(\boldsymbol{u})\right\|\right)\|\dot{\boldsymbol{u}}\|, \\
&\le -\alpha_{\boldsymbol{z}}\gamma_{\boldsymbol{z}3}\|\tilde{\boldsymbol{z}}\|^2 + \alpha_{\boldsymbol{z}}\gamma_{\boldsymbol{z}4}\|\tilde{\boldsymbol{z}}\|\|\boldsymbol{h}(\boldsymbol{z},y) - \boldsymbol{h}(\boldsymbol{z},\bar{y}_{\boldsymbol{w}}(t,\boldsymbol{u}))\| \\
&\quad + \Big(\gamma_{\boldsymbol{z}5} + \alpha_{\boldsymbol{z}}\gamma_{\boldsymbol{z}4}L_{\boldsymbol{z}1} + \gamma_{\boldsymbol{z}4}L_{\boldsymbol{q}}\Big)\|\tilde{\boldsymbol{z}}\|\|\dot{\boldsymbol{u}}\|.
\end{aligned} \tag{132}$$

On compact sets, Assumption 5 implies that there exist constants $L_{\boldsymbol{hz}}$, $L_{\boldsymbol{h}y}$, L_k, $L_{Z\boldsymbol{e}}$, $L_{Z\boldsymbol{u}} \in \mathbb{R}_{>0}$ such that

$$\left\|\frac{\partial^2 Z}{\partial \boldsymbol{e}\partial \boldsymbol{e}^\top}(\boldsymbol{e},\boldsymbol{u})\right\| \le L_{Z\boldsymbol{e}}, \qquad \left\|\frac{\partial^2 Z}{\partial \boldsymbol{e}\partial \boldsymbol{u}^\top}(\boldsymbol{e},\boldsymbol{u})\right\| \le L_{Z\boldsymbol{u}}, \tag{133}$$

and

$$\left\|\frac{\partial \boldsymbol{h}}{\partial \boldsymbol{z}}(\boldsymbol{z},y)\right\| \le L_{\boldsymbol{hz}}, \ \left\|\frac{\partial \boldsymbol{h}}{\partial y}(\boldsymbol{z},y)\right\| \le L_{\boldsymbol{h}y}, \ \left\|\frac{\partial k}{\partial \boldsymbol{z}}(\boldsymbol{z})\right\| \le L_k, \tag{134}$$

for all $\boldsymbol{x} \in \mathcal{X}$, all $\boldsymbol{u} \in \mathcal{U}$, all $\boldsymbol{z} \in \mathcal{Z}$, and all $y \in \mathbb{R}$. By defining $\tilde{y} := y - \bar{y}_{\boldsymbol{w}}(t,\boldsymbol{u})$ and using the bounds in (134), it follows that

$$\begin{aligned}
\|\boldsymbol{h}(\boldsymbol{z},y) - \boldsymbol{h}(\boldsymbol{z},\bar{y}_{\boldsymbol{w}}(t,\boldsymbol{u}))\| &= \|\boldsymbol{h}(\boldsymbol{z},\tilde{y} + \bar{y}_{\boldsymbol{w}}(t,\boldsymbol{u})) - \boldsymbol{h}(\boldsymbol{z},\bar{y}_{\boldsymbol{w}}(t,\boldsymbol{u}))\| \\
&= \left\|\int_0^1 \frac{\partial \boldsymbol{h}}{\partial y}(\boldsymbol{z},\sigma\tilde{y} + \bar{y}_{\boldsymbol{w}}(t,\boldsymbol{u}))d\sigma\tilde{y}\right\| \\
&\le \int_0^1 \left\|\frac{\partial \boldsymbol{h}}{\partial y}(\boldsymbol{z},\sigma\tilde{y} + \bar{y}_{\boldsymbol{w}}(t,\boldsymbol{u}))\right\| d\sigma\,\|\tilde{y}\| = L_{\boldsymbol{h}y}\,\|\tilde{y}\|.
\end{aligned} \tag{135}$$

By defining $\tilde{\boldsymbol{e}} = \boldsymbol{e} - \bar{\boldsymbol{e}}_{\boldsymbol{w}}(t, \boldsymbol{u})$, using (4) and the bounds in (133), it follows that

$$
\begin{aligned}
\tilde{y}(t) &= \boldsymbol{Z}(\tilde{\boldsymbol{e}} + \bar{\boldsymbol{e}}_{\boldsymbol{w}}(t, \boldsymbol{u}), \boldsymbol{u}) - \boldsymbol{Z}(\bar{\boldsymbol{e}}_{\boldsymbol{w}}(t, \boldsymbol{u}), \boldsymbol{u}) = \int_0^1 \frac{\partial \boldsymbol{Z}}{\partial \boldsymbol{e}}(\sigma\tilde{\boldsymbol{e}} + \bar{\boldsymbol{e}}_{\boldsymbol{w}}(t, \boldsymbol{u}), \boldsymbol{u}) d\sigma \tilde{\boldsymbol{e}} \\
&= \int_0^1 \left(\frac{\partial \boldsymbol{Z}}{\partial \boldsymbol{e}}(\sigma\tilde{\boldsymbol{e}} + \bar{\boldsymbol{e}}_{\boldsymbol{w}}(t, \boldsymbol{u}), \boldsymbol{u}) - \frac{\partial \boldsymbol{Z}}{\partial \boldsymbol{e}}(\bar{\boldsymbol{e}}_{\boldsymbol{w}}(t, \boldsymbol{u}), \boldsymbol{u}) \right) d\sigma \tilde{\boldsymbol{e}} \\
&\quad + \left(\frac{\partial \boldsymbol{Z}}{\partial \boldsymbol{e}}(\bar{\boldsymbol{e}}_{\boldsymbol{w}}(t, \boldsymbol{u}), \boldsymbol{u}) - \frac{\partial \boldsymbol{Z}}{\partial \boldsymbol{e}}(\bar{\boldsymbol{e}}_{\boldsymbol{w}}(t, \boldsymbol{u}^*), \boldsymbol{u}^*) \right) \tilde{\boldsymbol{e}} + \frac{\partial \boldsymbol{Z}}{\partial \boldsymbol{e}}(\bar{\boldsymbol{e}}_{\boldsymbol{w}}(t, \boldsymbol{u}^*), \boldsymbol{u}^*)\tilde{\boldsymbol{e}} \\
&= \int_0^1 \int_0^1 \frac{\partial^2 Z}{\partial \boldsymbol{e} \partial \boldsymbol{e}^\top}(\tau\sigma\tilde{\boldsymbol{e}} + \bar{\boldsymbol{e}}_{\boldsymbol{w}}(t, \boldsymbol{u}))\sigma\tilde{\boldsymbol{e}} d\tau d\sigma \tilde{\boldsymbol{e}} + \left(\frac{\partial \boldsymbol{Z}}{\partial \boldsymbol{e}}(\bar{\boldsymbol{e}}_{\boldsymbol{w}}(t, \boldsymbol{u}), \boldsymbol{u}) - \frac{\partial \boldsymbol{Z}}{\partial \boldsymbol{e}}(\bar{\boldsymbol{e}}_{\boldsymbol{w}}(t, \boldsymbol{u}^*), \boldsymbol{u}) \right) \tilde{\boldsymbol{e}} \\
&\quad + \left(\frac{\partial \boldsymbol{Z}}{\partial \boldsymbol{e}}(\bar{\boldsymbol{e}}_{\boldsymbol{w}}(t, \boldsymbol{u}^*), \boldsymbol{u}) - \frac{\partial \boldsymbol{Z}}{\partial \boldsymbol{e}}(\bar{\boldsymbol{e}}_{\boldsymbol{w}}(t, \boldsymbol{u}^*), \boldsymbol{u}^*) \right) \tilde{\boldsymbol{e}} + \frac{\partial \boldsymbol{Z}}{\partial \boldsymbol{e}}(\bar{\boldsymbol{e}}_{\boldsymbol{w}}(t, \boldsymbol{u}^*), \boldsymbol{u}^*)\tilde{\boldsymbol{e}} \\
&= \int_0^1 \int_0^1 \frac{\partial^2 Z}{\partial \boldsymbol{e} \partial \boldsymbol{e}^\top}(\tau\sigma\tilde{\boldsymbol{e}} + \bar{\boldsymbol{e}}_{\boldsymbol{w}}(t, \boldsymbol{u}))\sigma\tilde{\boldsymbol{e}} d\tau d\sigma \tilde{\boldsymbol{e}} \\
&\quad + \int_0^1 \frac{\partial^2 \boldsymbol{Z}}{\partial \boldsymbol{e} \partial \boldsymbol{e}^\top}(\bar{\boldsymbol{e}}_{\boldsymbol{w}}(t, \boldsymbol{u}^*) + \sigma\left(\bar{\boldsymbol{e}}_{\boldsymbol{w}}(t, \boldsymbol{u}) - \bar{\boldsymbol{e}}_{\boldsymbol{w}}(t, \boldsymbol{u}^*)\right), \boldsymbol{u}) d\sigma \left(\bar{\boldsymbol{e}}_{\boldsymbol{w}}(t, \boldsymbol{u}) - \bar{\boldsymbol{e}}_{\boldsymbol{w}}(t, \boldsymbol{u}^*)\right) \tilde{\boldsymbol{e}} \\
&\quad + \int_0^1 \frac{\partial^2 \boldsymbol{Z}}{\partial \boldsymbol{e} \partial \boldsymbol{u}^\top}(\bar{\boldsymbol{e}}_{\boldsymbol{w}}(t, \boldsymbol{u}^*), \boldsymbol{u}^* + \sigma(\boldsymbol{u} - \boldsymbol{u}^*)) d\sigma (\boldsymbol{u} - \boldsymbol{u}^*)\tilde{\boldsymbol{e}} + \frac{\partial \boldsymbol{Z}}{\partial \boldsymbol{e}}(\bar{\boldsymbol{e}}_{\boldsymbol{w}}(t, \boldsymbol{u}^*), \boldsymbol{u}^*)\tilde{\boldsymbol{e}}
\end{aligned}
\tag{136}
$$

Then it follows that

$$
\begin{aligned}
\|\tilde{y}\| &\leq \int_0^1 \int_0^1 \left\| \frac{\partial^2 Z}{\partial \boldsymbol{e} \partial \boldsymbol{e}^\top}(\tau\sigma\tilde{\boldsymbol{e}} + \bar{\boldsymbol{e}}_{\boldsymbol{w}}(t, \boldsymbol{u})) \right\| \sigma d\tau d\sigma \|\tilde{\boldsymbol{e}}\|^2 \\
&\quad + \int_0^1 \left\| \frac{\partial^2 \boldsymbol{Z}}{\partial \boldsymbol{e} \partial \boldsymbol{e}^\top}(\bar{\boldsymbol{e}}_{\boldsymbol{w}}(t, \boldsymbol{u}^*) + \sigma\left(\bar{\boldsymbol{e}}_{\boldsymbol{w}}(t, \boldsymbol{u}) - \bar{\boldsymbol{e}}_{\boldsymbol{w}}(t, \boldsymbol{u}^*)\right), \boldsymbol{u}) \right\| d\sigma \, \|\bar{\boldsymbol{e}}_{\boldsymbol{w}}(t, \boldsymbol{u}) - \bar{\boldsymbol{e}}_{\boldsymbol{w}}(t, \boldsymbol{u}^*)\| \, \|\tilde{\boldsymbol{e}}\| \\
&\quad + \int_0^1 \left\| \frac{\partial^2 \boldsymbol{Z}}{\partial \boldsymbol{e} \partial \boldsymbol{u}^\top}(\bar{\boldsymbol{e}}_{\boldsymbol{w}}(t, \boldsymbol{u}^*), \boldsymbol{u}^* + \sigma(\boldsymbol{u} - \boldsymbol{u}^*)) \right\| d\sigma \, \|\boldsymbol{u} - \boldsymbol{u}^*\| \, \|\tilde{\boldsymbol{e}}\| + \left\| \frac{\partial \boldsymbol{Z}}{\partial \boldsymbol{e}}(\bar{\boldsymbol{e}}_{\boldsymbol{w}}(t, \boldsymbol{u}^*), \boldsymbol{u}^*) \right\| \|\tilde{\boldsymbol{e}}\|, \\
&\leq \frac{L_{\boldsymbol{Ze}}}{2}\|\tilde{\boldsymbol{e}}\|^2 + L_{\boldsymbol{Ze}}\|\bar{\boldsymbol{e}}_{\boldsymbol{w}}(t, \boldsymbol{u}) - \bar{\boldsymbol{e}}_{\boldsymbol{w}}(t, \boldsymbol{u}^*)\|\|\tilde{\boldsymbol{e}}\| + (L_{\boldsymbol{Zu}}L_{\boldsymbol{u}} + L_{\boldsymbol{Z}*})\|\tilde{\boldsymbol{e}}\|,
\end{aligned}
\tag{137}
$$

where we have used that $\boldsymbol{u} \in \mathcal{U}$, i.e., $\|\boldsymbol{u} - \boldsymbol{u}^*\| \leq L_{\boldsymbol{u}}$ for all $\boldsymbol{u} \in \mathcal{U}$, and with

$$
L_{\boldsymbol{Z}*} = \left\| \frac{\partial \boldsymbol{Z}}{\partial \boldsymbol{e}}(\bar{\boldsymbol{e}}_{\boldsymbol{w}}(t, \boldsymbol{u}^*), \boldsymbol{u}^*) \right\|. \tag{138}
$$

From Assumption 5 and consequently the bounds in (74) it follows that

$$
\begin{aligned}
\|\tilde{\boldsymbol{e}}\| &= \|\boldsymbol{e} - \bar{\boldsymbol{e}}_{\boldsymbol{w}}(t, \boldsymbol{u})\| = \|\boldsymbol{g}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{w}) - \boldsymbol{g}(\bar{\boldsymbol{x}}_{\boldsymbol{w}}(t, \boldsymbol{u}), \boldsymbol{u}, \boldsymbol{w})\| \\
&\leq \int_0^1 \left\| \frac{\partial \boldsymbol{g}}{\partial \boldsymbol{x}}(\sigma\tilde{\boldsymbol{x}} + \bar{\boldsymbol{x}}_{\boldsymbol{w}}(t, \boldsymbol{u}), \boldsymbol{u}, \boldsymbol{w}) \right\| d\sigma \|\tilde{\boldsymbol{x}}\| = L_{\boldsymbol{gx}}\|\tilde{\boldsymbol{x}}\|,
\end{aligned}
\tag{139}
$$

and

$$
\begin{aligned}
\|\bar{\boldsymbol{e}}_{\boldsymbol{w}}(t, \boldsymbol{u}) - \bar{\boldsymbol{e}}_{\boldsymbol{w}}(t, \boldsymbol{u}^*)\| &= \|\boldsymbol{g}(\bar{\boldsymbol{x}}_{\boldsymbol{w}}(t, \boldsymbol{u}), \boldsymbol{u}, \boldsymbol{w}) - \boldsymbol{g}(\bar{\boldsymbol{x}}_{\boldsymbol{w}}(t, \boldsymbol{u}^*), \boldsymbol{u}^*, \boldsymbol{w})\| \\
&\leq \|\boldsymbol{g}(\bar{\boldsymbol{x}}_{\boldsymbol{w}}(t, \boldsymbol{u}), \boldsymbol{u}, \boldsymbol{w}) - \boldsymbol{g}(\bar{\boldsymbol{x}}_{\boldsymbol{w}}(t, \boldsymbol{u}^*), \boldsymbol{u}, \boldsymbol{w})\| \\
&\quad + \|\boldsymbol{g}(\bar{\boldsymbol{x}}_{\boldsymbol{w}}(t, \boldsymbol{u}^*), \boldsymbol{u}, \boldsymbol{w}) - \boldsymbol{g}(\bar{\boldsymbol{x}}_{\boldsymbol{w}}(t, \boldsymbol{u}^*), \boldsymbol{u}^*, \boldsymbol{w})\| \\
&\leq L_{\boldsymbol{gx}}\|\bar{\boldsymbol{x}}_{\boldsymbol{w}}(t, \boldsymbol{u}) - \bar{\boldsymbol{x}}_{\boldsymbol{w}}(t, \boldsymbol{u}^*)\| + L_{\boldsymbol{gu}}L_{\boldsymbol{u}},
\end{aligned}
\tag{140}
$$

where we again used that $\boldsymbol{u} \in \mathcal{U}$, i.e., $\|\boldsymbol{u} - \boldsymbol{u}^*\| \leq L_{\boldsymbol{u}}$ for all $\boldsymbol{u} \in \mathcal{U}$. From Assumption 3 and $\boldsymbol{u} \in \mathcal{U}$, it follows that

$$\|\bar{\boldsymbol{x}}_{\boldsymbol{w}}(t, \boldsymbol{u}) - \bar{\boldsymbol{x}}_{\boldsymbol{w}}(t, \boldsymbol{u}^*)\| \leq L_{\boldsymbol{xu}} L_{\boldsymbol{u}}. \tag{141}$$

Substitution of (139), (140), and (141) in (137) yields,

$$\|\tilde{y}\| \leq \frac{L_{\boldsymbol{Ze}}}{2} L_{\boldsymbol{gx}}^2 \|\tilde{\boldsymbol{x}}\|^2 + \left(L_{\boldsymbol{Ze}} L_{\boldsymbol{gx}}^2 L_{\boldsymbol{xu}} L_{\boldsymbol{u}} + L_{\boldsymbol{Ze}} L_{\boldsymbol{gu}} L_{\boldsymbol{gx}} L_{\boldsymbol{u}} + L_{\boldsymbol{Zu}} L_{\boldsymbol{u}} L_{\boldsymbol{gx}} + L_{\boldsymbol{Z}*} L_{\boldsymbol{gx}}\right) \|\tilde{\boldsymbol{x}}\|, \tag{142}$$

Substitution of (135) and (142) in (132) yields

$$\begin{aligned} \dot{V}_{\boldsymbol{z}}(t, \tilde{\boldsymbol{z}}, \boldsymbol{u}, \alpha_{\boldsymbol{z}}) \leq& -\alpha_{\boldsymbol{z}} \gamma_{\boldsymbol{z}3} \|\tilde{\boldsymbol{z}}\|^2 + \alpha_{\boldsymbol{z}} \gamma_{\boldsymbol{z}4} \frac{L_{\boldsymbol{Ze}}}{2} L_{\boldsymbol{gx}}^2 L_{\boldsymbol{hy}} \|\tilde{\boldsymbol{z}}\| \|\tilde{\boldsymbol{x}}\|^2 + \\ & \alpha_{\boldsymbol{z}} \gamma_{\boldsymbol{z}4} L_{\boldsymbol{hy}} \left(L_{\boldsymbol{Ze}} L_{\boldsymbol{gx}}^2 L_{\boldsymbol{xu}} L_{\boldsymbol{u}} + L_{\boldsymbol{Ze}} L_{\boldsymbol{gu}} L_{\boldsymbol{gx}} L_{\boldsymbol{u}} + L_{\boldsymbol{Zu}} L_{\boldsymbol{u}} L_{\boldsymbol{gx}} + L_{\boldsymbol{Z}*} L_{\boldsymbol{gx}}\right) \|\tilde{\boldsymbol{z}}\| \|\tilde{\boldsymbol{x}}\| \\ & + \Big(\gamma_{\boldsymbol{z}5} + \epsilon_{\boldsymbol{z}} \gamma_{\boldsymbol{z}4} L_{\boldsymbol{z}1} + \gamma_{\boldsymbol{z}4} L_{\boldsymbol{q}}\Big) \|\tilde{\boldsymbol{z}}\| \|\dot{\boldsymbol{u}}\|. \end{aligned} \tag{143}$$

for all $0 < \alpha_{\boldsymbol{z}} \leq \epsilon_{\boldsymbol{z}}$. The expression in (143) is of the form

$$\dot{V}_{\boldsymbol{z}}(t, \tilde{\boldsymbol{z}}, \boldsymbol{u}, \alpha_{\boldsymbol{z}}) \leq -\alpha_{\boldsymbol{z}} \gamma_{\boldsymbol{z}3} \|\tilde{\boldsymbol{z}}\|^2 + \alpha_{\boldsymbol{z}} z_1 \|\tilde{\boldsymbol{z}}\| \|\tilde{\boldsymbol{x}}\|^2 + \alpha_{\boldsymbol{z}} z_2 \|\tilde{\boldsymbol{z}}\| \|\tilde{\boldsymbol{x}}\| + z_3 \|\tilde{\boldsymbol{z}}\| \|\dot{\boldsymbol{u}}\|. \tag{144}$$

with

$$\begin{aligned} z_1 &= \gamma_{\boldsymbol{z}4} \frac{L_{\boldsymbol{Ze}}}{2} L_{\boldsymbol{gx}}^2 L_{\boldsymbol{hy}}, \\ z_2 &= \gamma_{\boldsymbol{z}4} L_{\boldsymbol{hy}} \left(L_{\boldsymbol{Ze}} L_{\boldsymbol{gx}}^2 L_{\boldsymbol{xu}} L_{\boldsymbol{u}} + L_{\boldsymbol{Ze}} L_{\boldsymbol{gu}} L_{\boldsymbol{gx}} L_{\boldsymbol{u}} + L_{\boldsymbol{Zu}} L_{\boldsymbol{u}} L_{\boldsymbol{gx}} + L_{\boldsymbol{Z}*} L_{\boldsymbol{gx}}\right), \\ z_3 &= \Big(\gamma_{\boldsymbol{z}5} + \epsilon_{\boldsymbol{z}} \gamma_{\boldsymbol{z}4} L_{\boldsymbol{z}1} + \gamma_{\boldsymbol{z}4} L_{\boldsymbol{q}}\Big), \end{aligned} \tag{145}$$

where we note that $z_1, z_2, z_3 \in \mathbb{R}_{>0}$. Applying Young's inequality and defining to-be-chosen positive constants $\gamma_i > 0$, with $i = 1, 2, 3$, gives

$$\begin{aligned} \dot{V}_{\boldsymbol{z}}(t, \tilde{\boldsymbol{z}}, \boldsymbol{u}, \alpha_{\boldsymbol{z}}) \leq& -\alpha_{\boldsymbol{z}} \gamma_{\boldsymbol{z}3} \left(1 - \frac{1}{2\gamma_1} - \frac{1}{2\gamma_2} - \frac{1}{2\gamma_3}\right) \|\tilde{\boldsymbol{z}}\|^2 \\ & + \frac{\alpha_{\boldsymbol{z}} \gamma_1}{2\gamma_{\boldsymbol{z}3}} z_1^2 \|\tilde{\boldsymbol{x}}\|^4 + \frac{\alpha_{\boldsymbol{z}} \gamma_2}{2\gamma_{\boldsymbol{z}3}} z_2^2 \|\tilde{\boldsymbol{x}}\|^2 + \frac{\gamma_3}{2\alpha_{\boldsymbol{z}} \gamma_{\boldsymbol{z}3}} z_3^2 \|\dot{\boldsymbol{u}}\|^2. \end{aligned} \tag{146}$$

Choosing $\gamma_1, \gamma_2, \gamma_3$ equal to 3 yields

$$\dot{V}_{\boldsymbol{z}}(t, \tilde{\boldsymbol{z}}, \boldsymbol{u}, \alpha_{\boldsymbol{z}}) \leq -\frac{\alpha_{\boldsymbol{z}} \gamma_{\boldsymbol{z}3}}{2} \|\tilde{\boldsymbol{z}}\|^2 + \frac{3\alpha_{\boldsymbol{z}} z_1^2}{2\gamma_{\boldsymbol{z}3}} \|\tilde{\boldsymbol{x}}\|^4 + \frac{3\alpha_{\boldsymbol{z}} z_2^2}{2\gamma_{\boldsymbol{z}3}} \|\tilde{\boldsymbol{x}}\|^2 + \frac{3z_3^2}{2\alpha_{\boldsymbol{z}} \gamma_{\boldsymbol{z}3}} \|\dot{\boldsymbol{u}}\|^2. \tag{147}$$

To find an upperbound for $\|\dot{\boldsymbol{u}}\|$, it follows from (14) that $\dot{\boldsymbol{u}} = \dot{\hat{\boldsymbol{u}}} + \alpha_{\boldsymbol{\omega}} \dot{\boldsymbol{\omega}}$. From (15) we have that there exists a constant $L_{\boldsymbol{\omega}1} \in \mathbb{R}_{>0}$ such that

$$\|\dot{\boldsymbol{\omega}}\| \leq \eta_{\boldsymbol{\omega}} L_{\boldsymbol{\omega}1}. \tag{148}$$

Furthermore, from (22) we have that $\|\dot{\hat{\boldsymbol{u}}}\| \leq \eta_{\boldsymbol{u}}$. Therefore, we obtain an upperbound on $\|\dot{\boldsymbol{u}}\|$ which is given by $\|\dot{\boldsymbol{u}}\| \leq \eta_{\boldsymbol{u}} + \alpha_{\boldsymbol{\omega}} \eta_{\boldsymbol{\omega}} L_{\boldsymbol{\omega}1}$ which implies that

$$\|\dot{\boldsymbol{u}}\| \leq \alpha_{\boldsymbol{\omega}} \eta_{\boldsymbol{\omega}} \left(\epsilon_3 \epsilon_5 + L_{\boldsymbol{\omega}1}\right), \tag{149}$$

for all $\epsilon_3, \epsilon_5 \in \mathbb{R}_{>0}$, all $\eta_{\boldsymbol{m}} \leq \eta_{\boldsymbol{\omega}} \epsilon_3$, and all $\eta_{\boldsymbol{u}} \leq \alpha_{\boldsymbol{\omega}} \eta_{\boldsymbol{m}} \epsilon_5$. Using the bound on $\|\dot{\boldsymbol{u}}\|$ in (149) and (38) in Lemma 4 we obtain

$$\begin{aligned} \dot{V}_{\boldsymbol{z}}(t, \tilde{\boldsymbol{z}}, \boldsymbol{u}, \alpha_{\boldsymbol{z}}) \leq& -\frac{\alpha_{\boldsymbol{z}} \gamma_{\boldsymbol{z}3}}{2\gamma_{\boldsymbol{z}2}} V_{\boldsymbol{z}}(t, \tilde{\boldsymbol{z}}, \boldsymbol{u}, \alpha_{\boldsymbol{z}}) + \frac{3\alpha_{\boldsymbol{z}} z_1^2}{2\gamma_{\boldsymbol{z}3}} \|\tilde{\boldsymbol{x}}\|^4 + \frac{3\alpha_{\boldsymbol{z}} z_2^2}{2\gamma_{\boldsymbol{z}3}} \|\tilde{\boldsymbol{x}}\|^2 \\ & + \frac{3z_3^2}{2\alpha_{\boldsymbol{z}} \gamma_{\boldsymbol{z}3}} \alpha_{\boldsymbol{\omega}}^2 \eta_{\boldsymbol{\omega}}^2 \left(\epsilon_3 \epsilon_5 + L_{\boldsymbol{\omega}1}\right)^2 \end{aligned} \tag{150}$$

for all $0 < \alpha_{\boldsymbol{z}} \leq \epsilon_{\boldsymbol{z}}$, all $\eta_{\boldsymbol{m}} \leq \eta_{\boldsymbol{\omega}}\epsilon_3$, and all $\eta_{\boldsymbol{u}} \leq \alpha_{\boldsymbol{\omega}}\eta_{\boldsymbol{m}}\epsilon_5$. From the comparison lemma and (150) we obtain

$$\begin{aligned} V_{\boldsymbol{z}}(t,\tilde{\boldsymbol{z}}(t),\boldsymbol{u}(t),\alpha_{\boldsymbol{z}}) &\leq V_{\boldsymbol{z}}(0,\tilde{\boldsymbol{z}}(0),\boldsymbol{u}(0),\alpha_{\boldsymbol{z}})e^{-\frac{\alpha_{\boldsymbol{z}}\gamma_{\boldsymbol{z}3}}{2\gamma_{\boldsymbol{z}2}}t} + \Big(\frac{3\gamma_{\boldsymbol{z}2}z_1^2}{\gamma_{\boldsymbol{z}3}^2}\sup_{t\geq 0}\|\tilde{\boldsymbol{x}}(t)\|^4 + \frac{3\gamma_{\boldsymbol{z}2}z_2^2}{\gamma_{\boldsymbol{z}3}^2}\sup_{t\geq 0}\|\tilde{\boldsymbol{x}}(t)\|^2 \\ &\quad + \frac{3\gamma_{\boldsymbol{z}2}z_3^2}{\alpha_{\boldsymbol{z}}^2\gamma_{\boldsymbol{z}3}^2}\alpha_{\boldsymbol{\omega}}^2\eta_{\boldsymbol{\omega}}^2\left(\epsilon_3\epsilon_5 + L_{\boldsymbol{\omega}1}\right)^2\Big)\Big(1 - e^{-\frac{\alpha_{\boldsymbol{z}}\gamma_{\boldsymbol{z}3}}{2\gamma_{\boldsymbol{z}2}}t}\Big) \\ &\leq V_{\boldsymbol{z}}(0,\tilde{\boldsymbol{z}}(0),\boldsymbol{u}(0),\alpha_{\boldsymbol{z}})e^{-\frac{\alpha_{\boldsymbol{z}}\gamma_{\boldsymbol{z}3}}{2\gamma_{\boldsymbol{z}2}}t} + \frac{3\gamma_{\boldsymbol{z}2}z_1^2}{\gamma_{\boldsymbol{z}3}^2}\sup_{t\geq 0}\|\tilde{\boldsymbol{x}}(t)\|^4 + \frac{3\gamma_{\boldsymbol{z}2}z_2^2}{\gamma_{\boldsymbol{z}3}^2}\sup_{t\geq 0}\|\tilde{\boldsymbol{x}}(t)\|^2 \\ &\quad + \frac{3\gamma_{\boldsymbol{z}2}z_3^2}{\alpha_{\boldsymbol{z}}^2\gamma_{\boldsymbol{z}3}^2}\alpha_{\boldsymbol{\omega}}^2\eta_{\boldsymbol{\omega}}^2\left(\epsilon_3\epsilon_5 + L_{\boldsymbol{\omega}1}\right)^2 \\ &\leq V_{\boldsymbol{z}}(0,\tilde{\boldsymbol{z}}(0),\boldsymbol{u}(0),\alpha_{\boldsymbol{z}})e^{-\frac{\alpha_{\boldsymbol{z}}\gamma_{\boldsymbol{z}3}}{2\gamma_{\boldsymbol{z}2}}t} + \frac{3\gamma_{\boldsymbol{z}2}z_1^2}{\gamma_{\boldsymbol{z}3}^2}\sup_{t\geq 0}\|\tilde{\boldsymbol{x}}(t)\|^4 + \frac{3\gamma_{\boldsymbol{z}2}z_2^2}{\alpha_{\boldsymbol{z}}^2\gamma_{\boldsymbol{z}3}^2}\epsilon_1^2\sup_{t\geq 0}\|\tilde{\boldsymbol{x}}(t)\|^2 \\ &\quad + \frac{3\gamma_{\boldsymbol{z}2}z_3^2}{\alpha_{\boldsymbol{z}}^2\gamma_{\boldsymbol{z}3}^2}\alpha_{\boldsymbol{\omega}}^2\eta_{\boldsymbol{\omega}}^2\left(\epsilon_3\epsilon_5 + L_{\boldsymbol{\omega}1}\right)^2 \end{aligned} \tag{151}$$

for all $0 < \alpha_{\boldsymbol{z}} \leq \epsilon_{\boldsymbol{z}}$, all $\alpha_{\boldsymbol{z}} \leq \epsilon_1$, all $t \geq 0$, all $\boldsymbol{z}(0) \in \mathcal{Z}_0$, all $\eta_{\boldsymbol{m}} \leq \eta_{\boldsymbol{\omega}}\epsilon_3$, and all $\eta_{\boldsymbol{u}} \leq \alpha_{\boldsymbol{\omega}}\eta_{\boldsymbol{m}}\epsilon_5$. From (38) in Lemma 4 and (151) we obtain

$$\begin{aligned} \|\tilde{\boldsymbol{z}}(t)\|^2 &\leq \frac{\gamma_{\boldsymbol{z}2}}{\gamma_{\boldsymbol{z}1}}\|\tilde{\boldsymbol{z}}(0)\|^2 e^{-\frac{\alpha_{\boldsymbol{z}}\gamma_{\boldsymbol{z}3}}{2\gamma_{\boldsymbol{z}2}}t} + \frac{3\gamma_{\boldsymbol{z}2}z_1^2}{\gamma_{\boldsymbol{z}1}\gamma_{\boldsymbol{z}3}^2}\sup_{t\geq 0}\|\tilde{\boldsymbol{x}}(t)\|^4 + \frac{3\gamma_{\boldsymbol{z}2}z_2^2}{\gamma_{\boldsymbol{z}1}\alpha_{\boldsymbol{z}}^2\gamma_{\boldsymbol{z}3}^2}\epsilon_1^2\sup_{t\geq 0}\|\tilde{\boldsymbol{x}}(t)\|^2 \\ &\quad + \frac{3\gamma_{\boldsymbol{z}2}z_3^2}{\gamma_{\boldsymbol{z}1}\alpha_{\boldsymbol{z}}^2\gamma_{\boldsymbol{z}3}^2}\alpha_{\boldsymbol{\omega}}^2\eta_{\boldsymbol{\omega}}^2\left(\epsilon_3\epsilon_5 + L_{\boldsymbol{\omega}1}\right)^2, \end{aligned} \tag{152}$$

for all $t \geq 0$. From Lemma 2 it follows that, for any finite time $t_1 \geq 0$, the solutions of $\tilde{\boldsymbol{x}}(t)$ are bounded for all $0 \leq t \leq t_1$. As such, we obtain that the last three terms in the right-hand side of (152) are bounded for all $0 \leq t \leq t_1$, and thus the solutions of $\tilde{\boldsymbol{z}}(t)$ are bounded for all $0 \leq t \leq t_1$.

From Lemma 2 it follows that here exists a time instance $t_1 \geq 0$, such that

$$\|\tilde{\boldsymbol{x}}(t)\| \leq \alpha_{\boldsymbol{\omega}}\eta_{\boldsymbol{\omega}}c_{\boldsymbol{x}2}, \quad \forall\, t \geq t_1, \tag{153}$$

From (153) and (152) we obtain

$$\begin{aligned} \|\tilde{\boldsymbol{z}}(t)\|^2 &\leq \frac{\gamma_{\boldsymbol{z}2}}{\gamma_{\boldsymbol{z}1}}\|\tilde{\boldsymbol{z}}(t_1)\|^2 e^{-\frac{\alpha_{\boldsymbol{z}}\gamma_{\boldsymbol{z}3}}{2\gamma_{\boldsymbol{z}2}}(t-t_1)} + \frac{3\gamma_{\boldsymbol{z}2}z_1^2}{\gamma_{\boldsymbol{z}1}\gamma_{\boldsymbol{z}3}^2}\alpha_{\boldsymbol{\omega}}^4\eta_{\boldsymbol{\omega}}^4c_{\boldsymbol{x}2}^4 + \frac{3\gamma_{\boldsymbol{z}2}z_2^2}{\gamma_{\boldsymbol{z}1}\alpha_{\boldsymbol{z}}^2\gamma_{\boldsymbol{z}3}^2}\alpha_{\boldsymbol{\omega}}^2\eta_{\boldsymbol{\omega}}^2\epsilon_1^2c_{\boldsymbol{x}2}^2 \\ &\quad + \frac{3\gamma_{\boldsymbol{z}2}z_3^2}{\gamma_{\boldsymbol{z}1}\alpha_{\boldsymbol{z}}^2\gamma_{\boldsymbol{z}3}^2}\alpha_{\boldsymbol{\omega}}^2\eta_{\boldsymbol{\omega}}^2\left(\epsilon_3\epsilon_5 + L_{\boldsymbol{\omega}1}\right)^2. \\ &\leq \frac{\gamma_{\boldsymbol{z}2}}{\gamma_{\boldsymbol{z}1}}\|\tilde{\boldsymbol{z}}(t_1)\|^2 e^{-\frac{\alpha_{\boldsymbol{z}}\gamma_{\boldsymbol{z}3}}{2\gamma_{\boldsymbol{z}2}}(t-t_1)} + \frac{3\gamma_{\boldsymbol{z}2}z_1^2}{\gamma_{\boldsymbol{z}1}\gamma_{\boldsymbol{z}3}^2}\alpha_{\boldsymbol{\omega}}^4\eta_{\boldsymbol{\omega}}^4c_{\boldsymbol{x}2}^4 \\ &\quad + \frac{3\gamma_{\boldsymbol{z}2}}{\gamma_{\boldsymbol{z}1}\alpha_{\boldsymbol{z}}^2\gamma_{\boldsymbol{z}3}^2}\alpha_{\boldsymbol{\omega}}^2\eta_{\boldsymbol{\omega}}^2\left(z_3^2\left(\epsilon_3\epsilon_5 + L_{\boldsymbol{\omega}1}\right)^2 + z_2^2\epsilon_1^2c_{\boldsymbol{x}2}^2\right), \end{aligned} \tag{154}$$

for $t \geq t_1$. Similar as in the proof of Lemma 2, we obtain the bound on $\|\tilde{\boldsymbol{z}}(t)\|$ as

$$\|\tilde{\boldsymbol{z}}(t)\| \leq \max\left\{c_{\boldsymbol{z}1}\|\tilde{\boldsymbol{z}}(t_1)\|e^{-\alpha_{\boldsymbol{z}}\beta_{\boldsymbol{z}}(t-t_1)}, \alpha_{\boldsymbol{\omega}}^2\eta_{\boldsymbol{\omega}}\alpha_{\boldsymbol{z}}c_{\boldsymbol{z}2}, \frac{\alpha_{\boldsymbol{\omega}}\eta_{\boldsymbol{\omega}}}{\alpha_{\boldsymbol{z}}}c_{\boldsymbol{z}3}\right\} \tag{155}$$

for all $0 < \alpha_{\boldsymbol{z}} \leq \epsilon_{\boldsymbol{z}}$, all $t \geq t_1$, all $\alpha_{\boldsymbol{z}} \leq \epsilon_1$, all $\eta_{\boldsymbol{\omega}} \leq \alpha_{\boldsymbol{z}}\epsilon_2$, all $\eta_{\boldsymbol{m}} \leq \eta_{\boldsymbol{\omega}}\epsilon_3$, and all $\eta_{\boldsymbol{u}} \leq \alpha_{\boldsymbol{\omega}}\eta_{\boldsymbol{m}}\epsilon_5$, with

$$\begin{aligned} c_{\boldsymbol{z}1} &= \sqrt{3\frac{\gamma_{\boldsymbol{z}2}}{\gamma_{\boldsymbol{z}1}}}, \quad c_{\boldsymbol{z}2} = 3\epsilon_2\sqrt{\frac{\gamma_{\boldsymbol{z}2}}{\gamma_{\boldsymbol{z}1}}}\frac{z_1}{\gamma_{\boldsymbol{z}3}}c_{\boldsymbol{x}2}^2, \\ c_{\boldsymbol{z}3} &= 3\sqrt{\frac{\gamma_{\boldsymbol{z}2}}{\gamma_{\boldsymbol{z}1}\gamma_{\boldsymbol{z}3}^2}\left(z_3^2\left(\epsilon_3\epsilon_5 + L_{\boldsymbol{\omega}1}\right)^2 + z_2^2\epsilon_1^2c_{\boldsymbol{x}2}^2\right)} \end{aligned} \tag{156}$$

and

$$\beta_{\boldsymbol{z}} = \frac{\gamma_{\boldsymbol{z}3}}{4\gamma_{\boldsymbol{z}2}}. \tag{157}$$

The boundedness of the solutions of $\tilde{\boldsymbol{z}}(t)$ for $0 \leq t \leq t_1$ follows from (152) for any finite time t_1. The bound in (42) of Lemma 5 follows from (155), which completes the proof of Lemma 5. □

6.4 Proof of Lemma 7

The proof of Lemma 7 is inspired by, and partially adopted from the one in [12, Ch. 2]. We define the following Lyapunov function candidate for the $\tilde{\boldsymbol{m}}$-dynamics in (47):

$$V_{\boldsymbol{m}}(\tilde{\boldsymbol{m}}, \boldsymbol{Q}) = \tilde{\boldsymbol{m}}^\top \boldsymbol{Q}^{-1} \tilde{\boldsymbol{m}}. \tag{158}$$

For notational clarity, from this point on we omit the time argument. We note that

$$\lambda_{min}(\boldsymbol{Q}^{-1}) \|\tilde{\boldsymbol{m}}\|^2 \leq V_{\boldsymbol{m}}(\tilde{\boldsymbol{m}}, \boldsymbol{Q}) \leq \lambda_{max}(\boldsymbol{Q}^{-1}) \|\tilde{\boldsymbol{m}}\|^2, \tag{159}$$

where $\lambda_{min}(\boldsymbol{Q}^{-1})$ and $\lambda_{max}(\boldsymbol{Q}^{-1})$ are the smallest and largest eigenvalue of $\boldsymbol{Q}^{-1}$, respectively. For further details on $\boldsymbol{Q}^{-1}$, the reader is referred to [12, Ch. 2]. From the observer in (21) and (47) we obtain the time derivative of $V_{\boldsymbol{m}}$ as

$$\begin{aligned}
\dot{V}_{\boldsymbol{m}}(\tilde{\boldsymbol{m}}, \boldsymbol{Q}) &= \tilde{\boldsymbol{m}}^\top \left(\boldsymbol{Q}^{-\top} + \boldsymbol{Q}^{-1}\right) \dot{\tilde{\boldsymbol{m}}} - \tilde{\boldsymbol{m}}^\top \boldsymbol{Q}^{-1} \dot{\boldsymbol{Q}} \boldsymbol{Q}^{-1} \tilde{\boldsymbol{m}} \\
&= 2\tilde{\boldsymbol{m}}^\top \boldsymbol{Q}^{-1} \dot{\tilde{\boldsymbol{m}}} - \tilde{\boldsymbol{m}}^\top \boldsymbol{Q}^{-1} \dot{\boldsymbol{Q}} \boldsymbol{Q}^{-1} \tilde{\boldsymbol{m}} \\
&= -\eta_{\boldsymbol{m}} \tilde{\boldsymbol{m}}^\top \boldsymbol{Q}^{-1} \tilde{\boldsymbol{m}} - \eta_{\boldsymbol{m}} \tilde{\boldsymbol{m}}^\top \left(\boldsymbol{C}^\top \boldsymbol{C} + \sigma_r \boldsymbol{D}^\top \boldsymbol{D}\right) \tilde{\boldsymbol{m}} - 2\alpha_{\boldsymbol{\omega}}^2 \tilde{\boldsymbol{m}}^\top \boldsymbol{Q}^{-1} \boldsymbol{B} \boldsymbol{s} \\
&\quad - 2\eta_{\boldsymbol{m}} \tilde{\boldsymbol{m}}^\top \boldsymbol{C}^\top (-\alpha_{\boldsymbol{\omega}}^2 v - r - d) - 2\eta_{\boldsymbol{m}} \sigma_r \alpha_{\boldsymbol{\omega}} \tilde{\boldsymbol{m}}^\top \boldsymbol{D}^\top \frac{dF_{\boldsymbol{w}}}{d\boldsymbol{u}^\top}(\hat{\boldsymbol{u}}),
\end{aligned} \tag{160}$$

where we have used the fact that $\boldsymbol{Q}^{-1}$ is real and symmetric, i.e., $\boldsymbol{Q}^{-1} = \boldsymbol{Q}^{-\top}$, and, given $\boldsymbol{A}$ in (19), that $\tilde{\boldsymbol{m}}^\top \left(\boldsymbol{Q}^{-1}\boldsymbol{A} - \boldsymbol{A}^\top \boldsymbol{Q}^{-1}\right) \tilde{\boldsymbol{m}} = 0$. Furthermore, given $\boldsymbol{C}$ in (19) and $\boldsymbol{D} = [\boldsymbol{0} \quad \boldsymbol{I}]$ we have the following inequality

$$\begin{aligned}
\dot{V}_{\boldsymbol{m}}(\tilde{\boldsymbol{m}}, \boldsymbol{Q}) \leq{}& -\eta_{\boldsymbol{m}} \tilde{\boldsymbol{m}}^\top \boldsymbol{Q}^{-1} \tilde{\boldsymbol{m}} - \eta_{\boldsymbol{m}} \|\boldsymbol{C}\tilde{\boldsymbol{m}}\|^2 - \eta_{\boldsymbol{m}} \sigma_r \|\boldsymbol{D}\tilde{\boldsymbol{m}}\|^2 \\
& - \sqrt{\eta_{\boldsymbol{m}}} \tilde{\boldsymbol{m}}^\top \boldsymbol{Q}^{-1} \frac{2\alpha_{\boldsymbol{\omega}}^2}{\sqrt{\eta_{\boldsymbol{m}}}} \boldsymbol{B} \boldsymbol{s} + \sqrt{\frac{2}{3}\eta_{\boldsymbol{m}}} \|\boldsymbol{C}\tilde{\boldsymbol{m}}\| \sqrt{6\eta_{\boldsymbol{m}}} \alpha_{\boldsymbol{\omega}}^2 |v| \\
& + \sqrt{\frac{2}{3}\eta_{\boldsymbol{m}}} \|\boldsymbol{C}\tilde{\boldsymbol{m}}\| \sqrt{6\eta_{\boldsymbol{m}}} |r| + \sqrt{\frac{2}{3}\eta_{\boldsymbol{m}}} \|\boldsymbol{C}\tilde{\boldsymbol{m}}\| \sqrt{6\eta_{\boldsymbol{m}}} |d| \\
& + \sqrt{2\eta_{\boldsymbol{m}}\sigma_r} \|\boldsymbol{D}\tilde{\boldsymbol{m}}\| \sqrt{2\eta_{\boldsymbol{m}}\sigma_r} \alpha_{\boldsymbol{\omega}} \left\| \frac{dF_{\boldsymbol{w}}}{d\boldsymbol{u}^\top}(\hat{\boldsymbol{u}}) \right\|,
\end{aligned} \tag{161}$$

where we have used that $-\tilde{\boldsymbol{m}}^\top \boldsymbol{C}^\top \boldsymbol{C} \tilde{\boldsymbol{m}} = -\left\|\tilde{\boldsymbol{m}}^\top \boldsymbol{C}^\top \boldsymbol{C} \tilde{\boldsymbol{m}}\right\| = -\|\boldsymbol{C}\tilde{\boldsymbol{m}}\|^2$ and $\left\|\tilde{\boldsymbol{m}}^\top \boldsymbol{C}^\top\right\| = \|\boldsymbol{C}\tilde{\boldsymbol{m}}\|$. Using Young's inequality we have

$$\begin{aligned}
\dot{V}_{\boldsymbol{m}}(\tilde{\boldsymbol{m}}, \boldsymbol{Q}) \leq{}& -\eta_{\boldsymbol{m}} \tilde{\boldsymbol{m}}^\top \boldsymbol{Q}^{-1} \tilde{\boldsymbol{m}} - \sqrt{\eta_{\boldsymbol{m}}} \tilde{\boldsymbol{m}}^\top \boldsymbol{Q}^{-1} \frac{2\alpha_{\boldsymbol{\omega}}^2}{\sqrt{\eta_{\boldsymbol{m}}}} \boldsymbol{B}(\boldsymbol{s}) \\
& + 3\eta_{\boldsymbol{m}} \alpha_{\boldsymbol{\omega}}^4 |v|^2 + 3\eta_{\boldsymbol{m}} |r|^2 + 3\eta_{\boldsymbol{m}} |d|^2 + \eta_{\boldsymbol{m}} \sigma_r \alpha_{\boldsymbol{\omega}}^2 \left\| \frac{dF_{\boldsymbol{w}}}{d\boldsymbol{u}^\top}(\hat{\boldsymbol{u}}) \right\|^2.
\end{aligned} \tag{162}$$

Since $\boldsymbol{Q}^{-1}$ is real, symmetric and positive definite, we can write $\boldsymbol{Q}^{-1} = \boldsymbol{L}\boldsymbol{L}^\top$ for some real, positive definite matrix L. Then, $\|\boldsymbol{Q}^{-1}\| = \|\boldsymbol{L}\|\|\boldsymbol{L}^\top\| = \|\boldsymbol{L}\|\|\boldsymbol{L}\| = \|\boldsymbol{L}\|^2$. From this, Young's inequality and by using (158), we obtain

$$\begin{aligned}
\dot{V}_{\boldsymbol{m}}(\tilde{\boldsymbol{m}}, \boldsymbol{Q}) \leq{}& -\frac{\eta_{\boldsymbol{m}}}{2} V_{\boldsymbol{m}}(\tilde{\boldsymbol{m}}, \boldsymbol{Q}) - \frac{\eta_{\boldsymbol{m}}}{2} \tilde{\boldsymbol{m}}^\top \boldsymbol{L}\boldsymbol{L}^\top \tilde{\boldsymbol{m}} + \sqrt{\eta_{\boldsymbol{m}}} \tilde{\boldsymbol{m}}^\top \boldsymbol{L}\boldsymbol{L}^\top \frac{2\alpha_{\boldsymbol{\omega}}^2}{\sqrt{\eta_{\boldsymbol{m}}}} \boldsymbol{B}\boldsymbol{s} \\
& + 3\eta_{\boldsymbol{m}} \alpha_{\boldsymbol{\omega}}^4 |v|^2 + 3\eta_{\boldsymbol{m}} |r|^2 + 3\eta_{\boldsymbol{m}} |d|^2 + \eta_{\boldsymbol{m}} \sigma_r \alpha_{\boldsymbol{\omega}}^2 \left\| \frac{dF_{\boldsymbol{w}}}{d\boldsymbol{u}^\top}(\hat{\boldsymbol{u}}) \right\|^2 \\
\leq{}& -\frac{\eta_{\boldsymbol{m}}}{2} V_{\boldsymbol{m}}(\tilde{\boldsymbol{m}}, \boldsymbol{Q}) - \frac{\eta_{\boldsymbol{m}}}{2} \left\|\tilde{\boldsymbol{m}}^\top \boldsymbol{L}\right\|^2 + \frac{\eta_{\boldsymbol{m}}}{2} \left\|\tilde{\boldsymbol{m}}^\top \boldsymbol{L}\right\|^2 + \frac{2\alpha_{\boldsymbol{\omega}}^4}{\eta_{\boldsymbol{m}}} \|\boldsymbol{L}\|^2 \|\boldsymbol{B}\|^2 \|\boldsymbol{s}\|^2 \\
& + 3\eta_{\boldsymbol{m}} \alpha_{\boldsymbol{\omega}}^4 |v|^2 + 3\eta_{\boldsymbol{m}} |r|^2 + 3\eta_{\boldsymbol{m}} |d|^2 + \eta_{\boldsymbol{m}} \sigma_r \alpha_{\boldsymbol{\omega}}^2 \left\| \frac{dF_{\boldsymbol{w}}}{d\boldsymbol{u}^\top}(\hat{\boldsymbol{u}}) \right\|^2,
\end{aligned} \tag{163}$$

which can be written as

$$\dot{V}_{\boldsymbol{m}}(\tilde{\boldsymbol{m}}, \boldsymbol{Q}) \leq -\frac{\eta_{\boldsymbol{m}}}{2} V_{\boldsymbol{m}}(\tilde{\boldsymbol{m}}, \boldsymbol{Q}) + \frac{2\alpha_{\boldsymbol{\omega}}^4}{\eta_{\boldsymbol{m}}} \|\boldsymbol{Q}^{-1}\| \|\boldsymbol{B}\|^2 \|\boldsymbol{s}\|^2 + 3\eta_{\boldsymbol{m}} \alpha_{\boldsymbol{\omega}}^4 |v|^2 + 3\eta_{\boldsymbol{m}} |r|^2 + 3\eta_{\boldsymbol{m}} |d|^2 + \eta_{\boldsymbol{m}} \sigma_r \alpha_{\boldsymbol{\omega}}^2 \left\| \frac{dF_{\boldsymbol{w}}}{d\boldsymbol{u}^\top}(\hat{\boldsymbol{u}}) \right\|^2 \tag{164}$$

Assumption 7 implies that there exists a constant $L_{F2} \in \mathbb{R}_{>0}$ such that

$$\left\| \frac{d^2 F_{\boldsymbol{w}}}{d\boldsymbol{u} d\boldsymbol{u}^\top}(\boldsymbol{u}) \right\| \leq L_{F2}, \quad \forall\, \boldsymbol{u} \in \mathcal{U} \tag{165}$$

From (20) and the bound in (165) we obtain

$$\begin{aligned} \|\boldsymbol{s}\| &= \frac{1}{\boldsymbol{\alpha_\omega}} \left\| \frac{d^2 F_{\boldsymbol{w}}}{d\boldsymbol{u} d\boldsymbol{u}^\top}(\hat{\boldsymbol{u}}) \dot{\hat{\boldsymbol{u}}} \right\| \\ &\leq \frac{1}{\boldsymbol{\alpha_\omega}} L_{F2} \|\dot{\hat{\boldsymbol{u}}}\|. \end{aligned} \tag{166}$$

From (20), the definition of $\boldsymbol{\omega}$ in (15), which implies that there exists a constant $L_{\boldsymbol{\omega}2} \in \mathbb{R}_{>0}$ such that $\|\boldsymbol{\omega}\| \leq L_{\boldsymbol{\omega}2}$, and the bound in (165) we obtain

$$\begin{aligned} |v| &\leq \left\| \int_0^1 (1-\sigma) \frac{d^2 F_{\boldsymbol{w}}}{d\boldsymbol{u} d\boldsymbol{u}^\top}(\hat{\boldsymbol{u}} + \sigma \boldsymbol{\alpha_\omega \omega}) d\sigma \right\| \|\boldsymbol{\omega}\|^2 \\ &\leq \int_0^1 (1-\sigma) \left\| \frac{d^2 F_{\boldsymbol{w}}}{d\boldsymbol{u} d\boldsymbol{u}^\top}(\hat{\boldsymbol{u}} + \sigma \boldsymbol{\alpha_\omega \omega}) \right\| d\sigma\, \|\boldsymbol{\omega}\|^2 \\ &\leq \frac{1}{2} L_{F2} L_{\boldsymbol{\omega}2}^2. \end{aligned} \tag{167}$$

From (20) and Assumption 5 it follows that

$$\begin{aligned} |r| &= \|k(\boldsymbol{z}) - k(\bar{\boldsymbol{z}}_{\boldsymbol{w}})\| = \|k(\tilde{\boldsymbol{z}} + \bar{\boldsymbol{z}}_{\boldsymbol{w}}) - k(\bar{\boldsymbol{z}}_{\boldsymbol{w}})\|, \\ &= \left\| \int_0^1 \frac{\partial k}{\partial \boldsymbol{z}}(\sigma \tilde{\boldsymbol{z}} + \bar{\boldsymbol{z}}_{\boldsymbol{w}}) d\sigma \tilde{\boldsymbol{z}} \right\| \leq L_k \|\tilde{\boldsymbol{z}}\|. \end{aligned} \tag{168}$$

From (20) and Assumptions 5 and 6 we obtain

$$\begin{aligned} |d| &= \|k(\bar{\boldsymbol{z}}_{\boldsymbol{w}}(t, \boldsymbol{u}, \alpha_{\boldsymbol{z}})) - k(\boldsymbol{q}_{\boldsymbol{w}}(\boldsymbol{u}))\|, \\ &= \|k((\bar{\boldsymbol{z}}_{\boldsymbol{w}}(t, \boldsymbol{u}, \alpha_{\boldsymbol{z}}) - \boldsymbol{q}_{\boldsymbol{w}}(\boldsymbol{u})) + \boldsymbol{q}_{\boldsymbol{w}}(\boldsymbol{u})) - k(\boldsymbol{q}_{\boldsymbol{w}}(\boldsymbol{u}))\|, \\ &= \left\| \int_0^1 \frac{\partial k}{\partial \boldsymbol{z}}(\sigma(\bar{\boldsymbol{z}}_{\boldsymbol{w}}(t, \boldsymbol{u}, \alpha_{\boldsymbol{z}}) - \boldsymbol{q}_{\boldsymbol{w}}(\boldsymbol{u})) + \boldsymbol{q}_{\boldsymbol{w}}(\boldsymbol{u})) d\sigma (\bar{\boldsymbol{z}}_{\boldsymbol{w}}(t, \boldsymbol{u}, \alpha_{\boldsymbol{z}}) - \boldsymbol{q}_{\boldsymbol{w}}(\boldsymbol{u})) \right\| \\ &\leq \int_0^1 \left\| \frac{\partial k}{\partial \boldsymbol{z}}(\sigma(\bar{\boldsymbol{z}}_{\boldsymbol{w}}(t, \boldsymbol{u}, \alpha_{\boldsymbol{z}}) - \boldsymbol{q}_{\boldsymbol{w}}(\boldsymbol{u})) + \boldsymbol{q}_{\boldsymbol{w}}(\boldsymbol{u})) \right\| d\sigma\, \|\bar{\boldsymbol{z}}_{\boldsymbol{w}}(t, \boldsymbol{u}, \alpha_{\boldsymbol{z}}) - \boldsymbol{q}_{\boldsymbol{w}}(\boldsymbol{u})\|, \\ &\leq L_k \|\bar{\boldsymbol{z}}_{\boldsymbol{w}}(t, \boldsymbol{u}, \alpha_{\boldsymbol{z}}) - \boldsymbol{q}_{\boldsymbol{w}}(\boldsymbol{u})\| = L_k \alpha_{\boldsymbol{z}} \delta_{\boldsymbol{w}}. \end{aligned} \tag{169}$$

From the coordinate transformation in (24) and the bound in (165) we obtain

$$\begin{aligned} \left\| \frac{dF_{\boldsymbol{w}}}{d\boldsymbol{u}}(\hat{\boldsymbol{u}}) \right\| &= \left\| \frac{dF_{\boldsymbol{w}}}{d\boldsymbol{u}}(\tilde{\boldsymbol{u}} + \boldsymbol{u}^*) - \frac{dF_{\boldsymbol{w}}}{d\boldsymbol{u}}(\boldsymbol{u}^*) \right\| \\ &= \left\| \int_0^1 \frac{d^2 F_{\boldsymbol{w}}}{d\boldsymbol{u} d\boldsymbol{u}^\top}(\sigma \tilde{\boldsymbol{u}} + \boldsymbol{u}^*) d\sigma \tilde{\boldsymbol{u}} \right\| \\ &\leq \int_0^1 \left\| \frac{d^2 F_{\boldsymbol{w}}}{d\boldsymbol{u} d\boldsymbol{u}^\top}(\sigma \tilde{\boldsymbol{u}} + \boldsymbol{u}^*) \right\| d\sigma\, \|\tilde{\boldsymbol{u}}\| = L_{F2} \|\tilde{\boldsymbol{u}}\|. \end{aligned} \tag{170}$$

By combining (164)-(170) and since we have from (19) that $\|\boldsymbol{B}\| = 1$, we obtain

$$\begin{aligned}\dot{V}_{\boldsymbol{m}}(\tilde{\boldsymbol{m}}, \boldsymbol{Q}) \leq & -\frac{\eta_{\boldsymbol{m}}}{2} V_{\boldsymbol{m}}(\tilde{\boldsymbol{m}}, \boldsymbol{Q}) + \frac{2\alpha_{\boldsymbol{\omega}}^2}{\eta_{\boldsymbol{m}}} L_{F2}^2 \|\boldsymbol{Q}^{-1}\| \|\dot{\hat{\boldsymbol{u}}}\|^2 \\ & + \frac{3}{4}\eta_{\boldsymbol{m}} \alpha_{\boldsymbol{\omega}}^4 L_{F2}^2 L_{\boldsymbol{\omega}2}^4 + 3\eta_{\boldsymbol{m}} L_k^2 \|\tilde{\boldsymbol{z}}\|^2 + 3\eta_{\boldsymbol{m}} L_k^2 \alpha_{\boldsymbol{z}}^2 \delta_{\boldsymbol{w}}^2 + \eta_{\boldsymbol{m}} \sigma_r \alpha_{\boldsymbol{\omega}}^2 L_{F2}^2 \|\tilde{\boldsymbol{u}}\|^2 .\end{aligned} \tag{171}$$

From Lemmas 4 and 6 we have that, for any finite time $t_2 \geq 0$, the solutions of $\tilde{\boldsymbol{z}}$ and $\tilde{\boldsymbol{u}}$ are bounded for all $0 \leq t \leq t_2$. Moreover, from the proof of Lemma 2.11 in [12, Ch. 2] we have that $\boldsymbol{Q}^{-1}$ is positive definite and bounded for all $0 \leq t \leq t_2$. From these facts and $\|\dot{\hat{\boldsymbol{u}}}\| \leq \eta_{\boldsymbol{u}}$, which follows from (22), we obtain that the right-hand side of (171) is bounded for all $0 \leq t \leq t_2$. Therefore, by applying the comparison lemma we obtain that $V_{\boldsymbol{m}}(\tilde{\boldsymbol{m}}(t), \boldsymbol{Q}(t))$ will be bounded for all $0 \leq t \leq t_2$. Because $V_{\boldsymbol{m}}(\tilde{\boldsymbol{m}}(t), \boldsymbol{Q}(t))$ is bounded for all $0 \leq t \leq t_2$ and $\boldsymbol{Q}^{-1}$ is positive definite and bounded for all $0 \leq t \leq t_2$, it follows from (159) that the solutions of $\tilde{\boldsymbol{m}}$ are bounded for all $0 \leq t \leq t_2$.

Let us define $t_2 \geq 0$. From Lemma 2.11 in [12, Ch. 2] we have that

$$\frac{1}{4}\boldsymbol{I} \preceq \boldsymbol{Q}^{-1} \preceq \frac{5}{4}\boldsymbol{I} \tag{172}$$

for all $t \geq t_2$, all $\eta_{\boldsymbol{m}} \leq \eta_{\boldsymbol{\omega}} \epsilon_3$ and all $\sigma_r \leq \epsilon_6$, and ϵ_3 and ϵ_6 sufficiently small. Moreover, it follows that

$$\frac{1}{4}\|\tilde{\boldsymbol{m}}\|^2 \leq V_{\boldsymbol{m}}(\tilde{\boldsymbol{m}}, \boldsymbol{Q}) \leq \frac{5}{4}\|\tilde{\boldsymbol{m}}\|^2, \tag{173}$$

for all $t \geq t_2$, and $\|\boldsymbol{Q}^{-1}\| \leq \dfrac{5}{4}$ for all $t \geq t_2$. From (16), (22), (24) and $\|\boldsymbol{D}\| = 1$, it follows that

$$\|\dot{\hat{\boldsymbol{u}}}\| \leq \lambda_{\boldsymbol{u}} \|\boldsymbol{D}\hat{\boldsymbol{m}}\| \leq \lambda_{\boldsymbol{u}} \left(\alpha_{\boldsymbol{\omega}} \left\| \frac{dF_{\boldsymbol{w}}}{d\boldsymbol{u}}(\hat{\boldsymbol{u}}) \right\| + \|\tilde{\boldsymbol{m}}\| \right). \tag{174}$$

Subsequently, from (174) and the bound in (165) we obtain

$$\|\dot{\hat{\boldsymbol{u}}}\| \leq \lambda_{\boldsymbol{u}} \left(\alpha_{\boldsymbol{\omega}} L_{F2} \|\tilde{\boldsymbol{u}}\| + \|\tilde{\boldsymbol{m}}\| \right). \tag{175}$$

From (173) and (175), it follows that

$$\|\dot{\hat{\boldsymbol{u}}}\|^2 \leq 8\lambda_{\boldsymbol{u}}^2 V_{\boldsymbol{m}}(\tilde{\boldsymbol{m}}, \boldsymbol{Q}) + 2\alpha_{\boldsymbol{\omega}}^2 \lambda_{\boldsymbol{u}}^2 L_{F2}^2 \|\tilde{\boldsymbol{u}}\|^2, \tag{176}$$

for all $t_2 \geq 0$. We assume that ϵ_4 in Theorem 1 is sufficiently small such that we obtain from (171) and (176) that

$$\begin{aligned}\dot{V}_{\boldsymbol{m}}(\tilde{\boldsymbol{m}}, \boldsymbol{Q}) \leq & -\frac{\eta_{\boldsymbol{m}}}{2} V_{\boldsymbol{m}}(\tilde{\boldsymbol{m}}, \boldsymbol{Q}) + \frac{20\alpha_{\boldsymbol{\omega}}^2 \lambda_{\boldsymbol{u}}^2}{\eta_{\boldsymbol{m}}} L_{F2}^2 V_{\boldsymbol{m}}(\tilde{\boldsymbol{m}}, \boldsymbol{Q}) + \frac{5\alpha_{\boldsymbol{\omega}}^4 \lambda_{\boldsymbol{u}}^2}{\eta_{\boldsymbol{m}}} L_{F2}^4 \|\tilde{\boldsymbol{u}}\|^2 \\ & + \frac{3}{4}\eta_{\boldsymbol{m}} \alpha_{\boldsymbol{\omega}}^4 L_{F2}^2 L_{\boldsymbol{\omega}2}^4 + 3\eta_{\boldsymbol{m}} L_k^2 \|\tilde{\boldsymbol{z}}\|^2 + 3\eta_{\boldsymbol{m}} L_k^2 \alpha_{\boldsymbol{z}}^2 \delta_{\boldsymbol{w}}^2 + \eta_{\boldsymbol{m}} \sigma_r \alpha_{\boldsymbol{\omega}}^2 L_{F2}^2 \|\tilde{\boldsymbol{u}}\|^2, \\ \leq & -\frac{\eta_{\boldsymbol{m}}}{4} V_{\boldsymbol{m}}(\tilde{\boldsymbol{m}}, \boldsymbol{Q}) + \frac{5\alpha_{\boldsymbol{\omega}}^4 \lambda_{\boldsymbol{u}}^2}{\eta_{\boldsymbol{m}}} L_{F2}^4 \|\tilde{\boldsymbol{u}}\|^2 \\ & + \frac{3}{4}\eta_{\boldsymbol{m}} \alpha_{\boldsymbol{\omega}}^4 L_{F2}^2 L_{\boldsymbol{\omega}2}^4 + 3\eta_{\boldsymbol{m}} L_k^2 \|\tilde{\boldsymbol{z}}\|^2 + 3\eta_{\boldsymbol{m}} L_k^2 \alpha_{\boldsymbol{z}}^2 \delta_{\boldsymbol{w}}^2 + \eta_{\boldsymbol{m}} \sigma_r \alpha_{\boldsymbol{\omega}}^2 L_{F2}^2 \|\tilde{\boldsymbol{u}}\|^2,\end{aligned} \tag{177}$$

for all $t \geq t_2$, and all $\alpha_{\boldsymbol{\omega}} \lambda_{\boldsymbol{u}} \leq \eta_{\boldsymbol{m}} \epsilon_4$. From the comparison lemma and (177) we obtain

$$\begin{aligned}V_{\boldsymbol{m}}(\tilde{\boldsymbol{m}}(t), \boldsymbol{Q}(t)) \leq & \, e^{-\frac{\eta_{\boldsymbol{m}}}{4}(t-t_2)} V_{\boldsymbol{m}}(\tilde{\boldsymbol{m}}(t_2), \boldsymbol{Q}(t_2)) \\ & + \left(1 - e^{-\frac{\eta_{\boldsymbol{m}}}{4}(t-t_2)}\right) \frac{4}{\eta_{\boldsymbol{m}}} \left(\frac{5\alpha_{\boldsymbol{\omega}}^4 \lambda_{\boldsymbol{u}}^2}{\eta_{\boldsymbol{m}}} L_{F2}^4 \sup_{t \geq t_2} \|\tilde{\boldsymbol{u}}(t)\|^2 \right. \\ & + \frac{3}{4}\eta_{\boldsymbol{m}} \alpha_{\boldsymbol{\omega}}^4 L_{F2}^2 L_{\boldsymbol{\omega}2}^4 + 3\eta_{\boldsymbol{m}} L_k^2 \sup_{t \geq t_2} \|\tilde{\boldsymbol{z}}(t)\|^2 \\ & \left. + 3\eta_{\boldsymbol{m}} L_k^2 \alpha_{\boldsymbol{z}}^2 \delta_{\boldsymbol{w}}^2 + \eta_{\boldsymbol{m}} \sigma_r \alpha_{\boldsymbol{\omega}}^2 L_{F2}^2 \sup_{t \geq t_2} \|\tilde{\boldsymbol{u}}(t)\|^2 \right).\end{aligned} \tag{178}$$

Finally, from Lemma 5 it follows that here exists a time instance $t_2 \geq 0$, such that

$$\|\tilde{\boldsymbol{z}}(t)\| \leq \max\{\alpha_{\boldsymbol{\omega}}^2 \eta_{\boldsymbol{\omega}} \alpha_{\boldsymbol{z}} c_{\boldsymbol{z}2}, \frac{\alpha_{\boldsymbol{\omega}} \eta_{\boldsymbol{\omega}}}{\alpha_{\boldsymbol{z}}} c_{\boldsymbol{z}3}\}, \tag{179}$$

By utilizing the bound in (179) and applying a similar approach as in Lemma 5 it follows from (178) that

$$\begin{aligned}\sup_{t\geq t_2} V_{\boldsymbol{m}}(\tilde{\boldsymbol{m}}(t), \boldsymbol{Q}(t)) \leq V_{\boldsymbol{m}}(\tilde{\boldsymbol{m}}(t_2), \boldsymbol{Q}(t_2)) + \frac{20\alpha_{\boldsymbol{\omega}}^4 \lambda_{\boldsymbol{u}}^2}{\eta_{\boldsymbol{m}}^2} L_{F2}^4 \sup_{t\geq t_2} \|\tilde{\boldsymbol{u}}(t)\|^2 \\ +3\alpha_{\boldsymbol{\omega}}^4 L_{F2}^2 L_{\boldsymbol{\omega}2}^4 + 12 L_k^2 \sup_{t\geq t_2} \|\tilde{\boldsymbol{z}}(t)\|^2 \\ + 12 L_k^2 \alpha_{\boldsymbol{z}}^2 \delta_{\boldsymbol{w}}^2 + 4\sigma_r \alpha_{\boldsymbol{\omega}}^2 L_{F2}^2 \sup_{t\geq t_2} \|\tilde{\boldsymbol{u}}(t)\|^2 ,\end{aligned} \tag{180}$$

and as a result it follows that

$$\begin{aligned}\sup_{t\geq t_2} V_{\boldsymbol{m}}(\tilde{\boldsymbol{m}}(t), \boldsymbol{Q}(t)) \leq 6 \sup_{t\geq t_2} \max \Bigg\{ V_{\boldsymbol{m}}(\tilde{\boldsymbol{m}}(t_2), \boldsymbol{Q}(t_2)), \frac{20\alpha_{\boldsymbol{\omega}}^4 \lambda_{\boldsymbol{u}}^2}{\eta_{\boldsymbol{m}}^2}^2 L_{F2}^4 \|\tilde{\boldsymbol{u}}(t)\|^2, \\ 3\alpha_{\boldsymbol{\omega}}^4 L_{F2}^2 L_{\boldsymbol{\omega}2}^4, 12 L_k^2 \alpha_{\boldsymbol{\omega}}^4 \eta_{\boldsymbol{\omega}}^2 \alpha_{\boldsymbol{z}}^2 c_{\boldsymbol{z}2}^2, 12 L_k^2 \frac{\alpha_{\boldsymbol{\omega}}^2 \eta_{\boldsymbol{\omega}}^2}{\alpha_{\boldsymbol{z}}^2} c_{\boldsymbol{z}3}^2, \\ 12 L_k^2 \alpha_{\boldsymbol{z}}^2 \delta_{\boldsymbol{w}}^2, 4\sigma_r \alpha_{\boldsymbol{\omega}}^2 L_{F2}^2 \|\tilde{\boldsymbol{u}}(t)\|^2 \Big\} ,\end{aligned} \tag{181}$$

and

$$\begin{aligned}\limsup_{t\to\infty} V_{\boldsymbol{m}}(\tilde{\boldsymbol{m}}(t), \boldsymbol{Q}(t)) \leq 6 \limsup_{t\to\infty} \max \Bigg\{ \frac{20\alpha_{\boldsymbol{\omega}}^4 \lambda_{\boldsymbol{u}}^2}{\eta_{\boldsymbol{m}}^2} L_{F2}^4 \|\tilde{\boldsymbol{u}}(t)\|^2, \\ 3\alpha_{\boldsymbol{\omega}}^4 L_{F2}^2 L_{\boldsymbol{\omega}2}^4, 12 L_k^2 \alpha_{\boldsymbol{\omega}}^4 \eta_{\boldsymbol{\omega}}^2 \alpha_{\boldsymbol{z}}^2 c_{\boldsymbol{z}2}^2, 12 L_k^2 \frac{\alpha_{\boldsymbol{\omega}}^2 \eta_{\boldsymbol{\omega}}^2}{\alpha_{\boldsymbol{z}}^2} c_{\boldsymbol{z}3}^2, \\ 12 L_k^2 \alpha_{\boldsymbol{z}}^2 \delta_{\boldsymbol{w}}^2, 4\sigma_r \alpha_{\boldsymbol{\omega}}^2 L_{F2}^2 \|\tilde{\boldsymbol{u}}(t)\|^2 \Big\} ,\end{aligned} \tag{182}$$

From (173) and (181) we have that

$$\begin{aligned}\sup_{t\geq t_2} \|\tilde{\boldsymbol{m}}(t)\| \leq 2\sqrt{6} \sup_{t\geq t_2} \max \Bigg\{ \sqrt{\frac{5}{4}} \|\tilde{\boldsymbol{m}}(t_2)\|, \frac{2\sqrt{5}\alpha_{\boldsymbol{\omega}}^2 \lambda_{\boldsymbol{u}}}{\eta_{\boldsymbol{m}}} L_{F2}^2 \|\tilde{\boldsymbol{u}}(t)\|, \\ \sqrt{3}\alpha_{\boldsymbol{\omega}}^2 L_{F2} L_{\boldsymbol{\omega}2}^2, 2\sqrt{3} L_k \alpha_{\boldsymbol{\omega}}^2 \eta_{\boldsymbol{\omega}} \alpha_{\boldsymbol{z}} c_{\boldsymbol{z}2}, 2\sqrt{3} L_k \frac{\alpha_{\boldsymbol{\omega}} \eta_{\boldsymbol{\omega}}}{\alpha_{\boldsymbol{z}}} c_{\boldsymbol{z}3}, \\ 2\sqrt{3} L_k \alpha_{\boldsymbol{z}} \delta_{\boldsymbol{w}}, 2\sqrt{\sigma_r} \alpha_{\boldsymbol{\omega}} L_{F2} \|\tilde{\boldsymbol{u}}(t)\| \Bigg\} ,\end{aligned} \tag{183}$$

Similarly, from (173) and (182) we have that

$$\begin{aligned}\limsup_{t\to\infty} \|\tilde{\boldsymbol{m}}(t)\| \leq 2\sqrt{6} \limsup_{t\to\infty} \Bigg\{ \frac{2\sqrt{5}\alpha_{\boldsymbol{\omega}}^2 \lambda_{\boldsymbol{u}}}{\eta_{\boldsymbol{m}}} L_{F2}^2 \|\tilde{\boldsymbol{u}}(t)\|, \\ \sqrt{3}\alpha_{\boldsymbol{\omega}}^2 L_{F2} L_{\boldsymbol{\omega}2}^2, 2\sqrt{3} L_k \alpha_{\boldsymbol{\omega}}^2 \eta_{\boldsymbol{\omega}} \alpha_{\boldsymbol{z}} c_{\boldsymbol{z}2}, 2\sqrt{3} L_k \frac{\alpha_{\boldsymbol{\omega}} \eta_{\boldsymbol{\omega}}}{\alpha_{\boldsymbol{z}}} c_{\boldsymbol{z}3}, \\ 2\sqrt{3} L_k \alpha_{\boldsymbol{z}} \delta_{\boldsymbol{w}}, 2\sqrt{\sigma_r} \alpha_{\boldsymbol{\omega}} L_{F2} \|\tilde{\boldsymbol{u}}(t)\| \Bigg\} ,\end{aligned} \tag{184}$$

The bounds in (48) and (49) of Lemma 7 follow from (183) and (184), respectively, which completes the proof of the lemma. □

Bibliography

[1] Ariyur, K.B. and Krstić, M., Real-time optimization by extremum-seeking control. Wiley-Interscience, John Wiley & Sons, Inc, 2003.

[2] Krstić, M. and Wang, H-H., Stability of extremum- seeking feedback for general nonlinear dynamic systems, Automatica, vol. 36, no. 4, pp. 889-903, 2006

[3] Teel, A.R. and Popović, D., Solving smooth and nonsmooth multivariable extremum seeking problems by the methods of nonlinear programming. In Proceedings of the American Control Conference, ACC, 2001, pp 25–27, Arlington, VA, 2001.

[4] Popovic, D. and Jankovic, M. and Magner, S. and Teel, A.R., Extremum seeking methods for optimization of variable cam timing engine operation, IEEE Transactions on Control Systems Technology, Vol. 14, no. 3, pp. 398–407, 2006.

[5] Zhang, C. and Ordonez, R., Numerical optimization-based extremum-seeking control with application to ABS design, IEEE Transactions on automatic control, Vol. 52, no. 3, pp. 454–467, 2007.

[6] Bolder, J.J. and Witvoet, G. and de Baar, M.R. and van de Wouw, N. and Haring, M.A.M. and Westerhof, E. and Doelman, N.J. and Steinbuch, M., Robust sawtooth period control based on adaptive online optimization, Nuclear Fusion, Vol. 52, no. 7, 2012.

[7] Tan, Y. and Nešić, D. and Mareels, I., On non-local stability properties of extremum seeking control, Automatica, vol. 42, no. 6, pp. 595-601, 2000.

[8] Höffner, K. and Hudon, N. and Guay, M., On-line feedback control for optimal periodic control systems, The Canadian Journal of Chemical Engineering, Vol. 85, no. 4, pp. 479–489, 2007.

[9] Wang, H-H. and Krstić, M., Extremum seeking for limit cycle minimization, IEEE Transactions on Automatic control, vol 45, no. 12, 2000.

[10] Guay, M. and Dochain, D. and Perrier, M. and Hudon, N., Flatness-based extremum-seeking control over periodic orbits, IEEE Transactions on Automatic Control, Vol. 52, no. 10, pp. 2005–2012, 2007.

[11] Haring, M.A.M. and van de Wouw, N. and Nešić, D., Extremum-seeking control for nonlinear systems with periodic steady-state outputs, Automatica, Vol. 49, pp. 1883–1891, 2013.

[12] Haring, M.A.M., Extremum-seeking control: convergence improvements and asymptotic stability, Ph.D. dissertation, Dept. of Engineering Cybernetics, Norwegian University of Science and Technology, Trondheim, 2016.

[13] Heertjes, M.F. and Nijmeijer, H., Self-tuning of a switching controller for scanning motion systems, Mechatronics, Vol. 22, no. 3, pp. 310–319, 2012.

[14] Mishra, S. and Coaplen, J. and Tomizuka, M., Precision positioning of wafer scanners segmented iterative learning control for nonrepetitive disturbances, IEEE Control Systems, Vol. 27, no. 4, pp. 20–25, 2007.

[15] Bolder, J. and Oomen, T. and Koekebakker, S. and Steinbuch, M., Using iterative learning control with basis functions to compensate medium deformation in a wide-form inkjet printer, Mechatronics, Vol. 24, no. 8, pp. 944-953, 2014.

[16] Kim, K. and Kasnakoglu, C. and Serrani, A. and Samimy, M., Extremum-seeking control of subsonic cavity flow, AIAA Journal 2009 47:1, pp. 195–205, 2009.

[17] Antonello, R. and Oboe, R. and Prandi, L. and Biganzoli, F., Automatic mode matching in MEMS vibrating gyroscopes using extremum-seeking control, IEEE Transactions on Industrial Electronics, Vol. 56(10), pp. 3880–3891, 2009.

[18] Hunnekens, B. G. B. and Di Dino, A. and van de Wouw, N. and van Dijk, N. and and Nijmeijer, H., Extremum-seeking control for the adaptive design of variable gain controllers, IEEE Transactions on control systems technology, vol 23, no. 3, pp 1041–1051, 2015.

[19] Pavlov, A.V. and van de Wouw, N. and Nijmeijer, H., Uniform output regulation of nonlinear systems: a convergence dynamics approach, Birkhäuser, Boston, 2005.

[20] Pavlov, A.V. and Hunnekens, B. G. B. and van de Wouw, N. and Nijmeijer, H., Steady-state performance optimization for nonlinear control systems of Lur'e type, Automatica, Vol 49, No. 7, pp. 2087–2097, 2013.

[21] Freudenberg, J. and Middleton, R. and Stefanpoulou, A., A survey of inherent design limitations, Proc. of the 2000 American Control Conf., Vol. 5, pp. 2987–3001, 2000.

[22] Liu, T. and Hill, D.J. and Jiang, Z., Lyapunov formulation of ISS cyclic-small-gain in continuous-time dynamical networks, Automatica, Vol. 47, no. 9, pp. 2088–2093, 2011.