Field extensions, derivations, and matroids over skew hyperfields

FIELD EXTENSIONS, DERIVATIONS, AND
MATROIDS OVER SKEW HYPERFIELDS

RUDI PENDAVINGH

ABSTRACT. We show that a field extension $K \subseteq L$ in positive characteristic p and elements $x_e \in L$ for $e \in E$ gives rise to a matroid M^σ on ground set E with coefficients in a certain skew hyperfield L^σ. This skew hyperfield L^σ is defined in terms of L and its Frobenius action $\sigma : x \mapsto x^p$. The matroid underlying M^σ describes the algebraic dependencies over K among the $x_e \in L$, and M^σ itself comprises, for each $m \in \mathbb{Z}^E$, the linear space of K-derivations of $K\left(x_e^{p^{m_e}} : e \in E\right)$.

The theory of matroid representation over hyperfields was developed by Baker and Bowler for commutative hyperfields. We partially extend their theory to skew hyperfields. To prove the duality theorems we need, we use a new axiom scheme in terms of *quasi-Plucker coordinates*.

1. INTRODUCTION

Let K be a field and let $(x_e : e \in E)$ be elements from an extension field L. A subcollection $(x_e : e \in F)$ is *algebraically dependent over K* if there is a polynomial $q \in K[X_e : e \in F]$ so that $q(x_e : e \in F) = 0$. By a theorem of Steinitz, the set $\mathcal{I} := \{F \subseteq E : (x_e : e \in F)$ algebraically independent over $K\}$ satisfies

(I0) $\emptyset \in \mathcal{I}$
(I1) if $A \in \mathcal{I}$ and $B \subseteq A$, then $B \in \mathcal{I}$
(I2) if $A, B \in \mathcal{I}$ and $|A| < |B|$, then $A \cup \{e\} \in \mathcal{I}$ for some $e \in B \setminus A$

Algebraic independence has these properties in common with linear independence. This formalizes the analogy beween algebraic closure and linear span, transcendence degree and dimension of a linear space, and in general gives a geometric perspective on field extensions.

A *matroid* is a pair $M = (E, \mathcal{I})$ where E is a finite set and $\mathcal{I}$ is any set of subsets of E satisfying (I0), (I1), and (I2). The above pair K, x thus gives an *algebraic matroid* $M(K, x)$, and a collection of vectors $(v_e)_{e \in E}$ will determine a *linear matroid* on E.

Taking poetic license, a matroid may be described as a *linear space without coefficients*. In a linear space over a field K of dimension d, any subset of d vectors is associated with a value in K, the determinant. The corresponding matroid merely distinguishes between bases and non-bases. There are less Spartan matroid variants, such as oriented matroids and valuated matroids, which can be seen as matroids with coefficients in the set of signs $\{+, -\}$ and in a linearly ordered group, respectively. This intuitive perspective was developed rigorously by Dress and Wenzel [DW91], who defined *matroids with coefficients* from a *fuzzy field*, and more recently by Baker and Bowler [BB17], who defined *matroids over hyperfields*. In both approaches, linear spaces as well as oriented-, valuated-, and ordinary matroids are matroids with coefficients in a corresponding fuzzy field or hyperfield.

Hyperfields generalize fields, and their more relaxed additive structure translates to a richer collection of homomorphisms. A hyperfield homomorphism $f : H \to H'$ induces a map f_* which takes a matroid over H and turns it into a matroid over H', simply by applying f to the coefficients. This elegantly describes how a matroid with coefficients in a field K (essentially a collection of vectors in a K-vector space) gives rise to an ordinary matroid. From any field K, there is a hyperfield homomorphism κ to the *Krasner hyperfield* $\mathbb{K} = \{0, 1\}$, which maps $0 \mapsto 0$ and all nonzero $x \mapsto 1$. The induced map κ_* is a forgetful operation which retains only the destinction between bases and nonbases, independent and dependent sets. If the ordinary matroid that arises from applying κ_* appears too coarse an abstraction, then one may consider a hyperfield homomorphism from K to a more detailed hyperfield. For example, the natural homomorphism from the reals to the hyperfield of signs induces a map from collections of vectors in Euclidean space to oriented matroids.

So in the study of linear spaces, matroids over hyperfields may serve to attain the 'right' abstraction level. In relation to field extensions, they have a different role. Unlike a linear space, a field extension is not itself a matroid over some hyperfield. The algebraic matroid $M(K, x)$ is, but the information on (K, x) it contains is quite sparse. Given a matroid N, there is no known general method to decide if $N = M(K, x)$ for some (K, x). In [BDP18], it was show that a pair (K, x) also determines a matroid valuation of $M(K, x)$, the *Lindström valuation*. That is, (K, x) determines a matroid over the hyperfield $\mathbb{Z}_{\min}$ with underlying matroid $M(K, x)$. In this paper, we show that $M(K, x)$ can even be decorated with coefficients in a certain hyperfield L^σ, which is defined in terms of L and the Frobenius action $\sigma : x \mapsto x^p$. The *left L^σ-matroid $M^\sigma(K, x)$* that arises is still a geometric object, but comprises more detailed information about the pair (K, x), such as the space of K-derivations of $K(x_e : e \in E)$.

Writing $K' := K(x_e : e \in E)$, a *K-derivation of K'* is any map $D : K' \to K'$ which is trivial on K, is additive, and satisfies the *Leibnitz rule $D(xy) = D(x)y + xD(y)$*. The collection $Der(K, K')$ of all K-derivations of K' is a linear space whose dimension in general equals the transcendence degree of K' over K. In this case, the dimension of $Der(K, K')$ equals the rank of the algebraic matroid $M(K, x)$. The linear space $Der(K, K')$ induces a matroid M on E of the same rank as $M(K, x)$, in which a set $B \subseteq E$ is a basis if and only if for each $u \in (K')^B$, there is a unique K-derivation D of K' such that $D(x_e) = u_e$ for all $e \in B$. Such a basis of M is necessarily a basis of $M(K, x)$, but the converse need not be true. In other words, the matroid of derivations M is a *weak image* of $M(K, x)$.

For any $m \in \mathbb{Z}^E$ and $x \in L^E$, let $\sigma^m(x) := (\sigma^{m_e}(x_e) : e \in E)$. Passing from x to $\sigma^m(x)$ does not affect algebraic dependence, and we have $M(K, x) = M(K, \sigma^m(x))$ for any $m \in \mathbb{Z}^E$. The matroid $M^\sigma(K, \sigma^m(x))$ arises from $M^\sigma(K, x)$ by *rescaling*, an operation which is defined generally for matroids over hyperfields. Via rescaling, $M^\sigma(K, x)$ describes the space of K-derivations of $K(\sigma^m(x))$ for each $m \in \mathbb{Z}^E$. We have the following diagram.
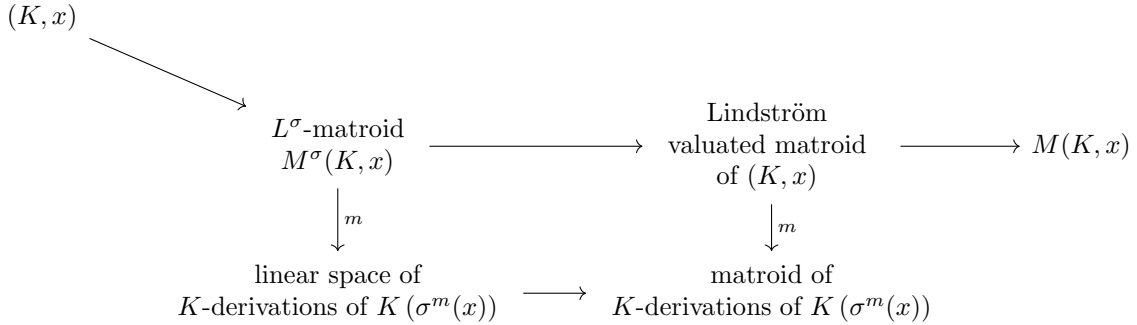
With the exception of (K, x) on the left, each node in this diagram is a matroid over a hyperfield, and each arrow represents a well-defined forgetful operation. Horizontal arrows indicate the application of a hyperfield homomorphism to the matroid coefficients, preserving the underlying matroid. Vertical arrows represent a new operation on matroids over certain hyperfields, which in general replaces the underlying matroid with a weak image of that matroid, and restricts the hyperfield to a sub-hyperfield.

As the diagram indicates, $M^\sigma(K, x)$ determines a map

$$\mathcal{V} : m \mapsto \{K\text{-derivations of } K(\sigma^m(x))\}.$$

Essentially this object was called a *Frobenius flock* in [BDP18]. It was show in that paper that the related *matroid flock $\mathcal{M} : m \mapsto M(\mathcal{V}_m)$* is a cryptomophic description of a matroid valuation of $M(K, x)$, which we named the Lindström valuation. This definition the Lindström valuation via flocks is indirect, but shortly after a preprint of [BDP18] appeared on arXiv, Dustin Cartwright presented a direct construction of the Lindstrom valuation in [Car17].

So matroid flocks are cryptomorphic to valuated matroids, and valuated matroids 'are' matroids represented over the tropical hyperfield. Matroid flocks arise by a forgetful operation from Frobenius flocks. This suggested that perhaps, Frobenius flocks are also cryptomorphic to matroids represented over a certain hyperfield, and that the operation by which a Frobenius flock begets a matroid flock is just the pushing

forward along an appropriate hyperfield homomorphism. In this paper, we show that this is exactly the case, the cryptomorphic description of the Frobenius flock of (K, x) being the L^σ-matroid $M^\sigma(K, x)$.

The hyperfield L^σ we use to alternatively describe Frobenius flocks as left L^σ-matroids turns out to be non-commutative. This was an obstacle, since the theory of Baker and Bowler is developed on the assumption that hyperfields be commutative. In the center of their theory is the notion of a *Grassmann-Plucker function* of a matroid over a hyperfield, which generalizes the Plucker coordinates of a linear subspace. There is no proper analogue of the Grassmann-Plucker function in the context of skew hyperfields, just as there is no clean way to define the determinant of a matrix over a skew field.

However, Gelfand, Gelfand, Retakh, and Wilson [GGRW05] show that matrices over skew fields do admit *quasi-determinants*, which in the commutative setting equal ratios of certain adjacent determinants. As it turns out, this concept blends perfectly with matroids over hyperfields, and this allows us to replace the Grassmann-Plucker functions with *quasi-Plucker coordinates* in the context of skew hyperfields.

The structure of the paper is as follows. After giving some preliminaries on matroids and hyperfields in Section 2, we develop *(weak) matroids over skew hyperfields* in Section 3. Although we do not point this out on each occasion, *all* the concepts defined in this section except *quasi-Plucker coordinate*, *cross ratio*, and *weak image* were lifted more or less verbatim from the paper of Baker and Bowler, only to take a slightly more general meaning in the context of skew hyperfields. As noted above, the concept of a *quasi-Plucker coordinate* is inspired by the work of Gelfand, Gelfand, Retakh, and Wilson, and the definition of a *cross ratio* in terms of quasi-Plucker coordinates also follows their work.

In Section 4, we describe how to construct a *skew hyperfield of monomials H^σ* from any hyperfield H with automorphism σ. We describe the operation indicated by the vertical arrows in the diagram, which in general takes a matroid M with coefficients in H^σ and produces a matroid with coefficients in H, the *boundary matroid M_0*.

In Section 5, we show that each algebraic matroid representation x in a field extension L/K gives rise to a left L^σ-matroid, the *matroid of σ-derivations $M_\sigma(K, x)$*. The spaces of derivations as in the diagram arise from $M^\sigma(K, x)$ by rescaling and then taking the boundary matroid. Thus $M^\sigma(K, x)$ indeed determines the Frobenius flock. In general, a H^σ-matroid M will determine a flock of H-matroids. In Section 6, we prove that this flock in turn determines M.

In the final section of the paper, we make a few more related comments and present some conjectures.

2. PRELIMINARIES

2.1. **Hypergroups, hyperrings, and hyperfields.** A *hyperoperation* on G is a map $\boxplus : G \times G \to 2^G$. Any hyperoperation induces a map $\overline{\boxplus} : 2^G \times 2^G \to 2^G$ by setting

$$X \overline{\boxplus} Y := \bigcup \{x \boxplus y : x \in X, y \in Y\}.$$

Slightly abusing notation, one writes $x \boxplus Y := \{x\} \overline{\boxplus} Y$, $X \boxplus y := X \overline{\boxplus} \{y\}$, and $X \boxplus Y := X \overline{\boxplus} Y$. The hyperoperation $\boxplus$ then is *associative* if $x \boxplus (y \boxplus z) = (x \boxplus y) \boxplus z$ for all $x, y, z \in G$.

A *hypergroup* is a triple $(G, \boxplus, 0)$, where $0 \in G$ and $\boxplus : G \times G \to 2^G \setminus \{\emptyset\}$ is a commutative and associative hyperoperation, such that

(H0) $x \boxplus 0 = \{x\}$
(H1) for each $x \in G$ there is a unique $y \in G$ so that $0 \in x \boxplus y$. We write $-x := y$
(H2) $x \in y \boxplus z$ if and only if $z \in x \boxplus (-y)$

If G, H are hypergroups, then a map $f : G \to H$ is a *hypergroup homomorphism* if $f(x \boxplus y) \subseteq f(x) \boxplus f(y)$ for all $x, y \in G$, and $f(0) = 0$.

A *hyperring* is a tuple $(R, \cdot, \boxplus, 1, 0)$ so that

(R0) $(R, \boxplus, 0)$ is a hypergroup
(R1) $(R^\star, \cdot, 1)$ is monoid, where we denote $R^\star := R \setminus \{0\}$
(R2) $0 \cdot x = x \cdot 0 = 0$ for all $x \in R$
(R3) $\alpha(x \boxplus y) = \alpha x \boxplus \alpha y$ and $(x \boxplus y)\alpha = x\alpha \boxplus y\alpha$ for all $\alpha, x, y \in R$

If R, S are hyperrings, then $f : R \to S$ is a *hyperring homomorphism* if f is a hypergroup homomorphism, $f(1) = 1$, and $f(x \cdot y) = f(x) \cdot f(y)$ for all $x, y \in R$.

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A *skew hyperfield* is a hyperring such that $0 \neq 1$, and each nonzero element has a multiplicative inverse. A *hyperfield* is then a skew hyperfield with commutative multiplication. A (skew) hyperfield homomorphism is just a homomorphism of the underlying hyperrings.

The *Krasner hyperfield* is $\mathbb{K} = (\{0, 1\}, \cdot, \boxplus, 1, 0)$, where $1 \boxplus 1 = \{0, 1\}$. All hyperfields H admit a hyperfield homomorphism $\kappa : H \to \mathbb{K}$ so that $\kappa(x) = 1$ for all nonzero $x \in H$. Any skew field can be considered a skew hyperfield with hyperaddition $x \boxplus y = \{x + y\}$. If $(\Gamma, 0, +, <)$ is a linearly ordered abelian group, then $\Gamma_{\min} := (\Gamma \cup \{\infty\}, 0, \infty, \odot, \boxplus)$ is a hyperfield, where we denoted $i \odot j := i + j$ and

$$i \boxplus j := \begin{cases} \{\min\{i, j\}\} & \text{if } i \neq j \\ \{m \in \Gamma : m \geq i\} \cup \{\infty\} & \text{if } i = j \end{cases}$$

Replacing min with max, $\geq$ with $\leq$, and ∞ with $-\infty$, we analogously obtain $\Gamma_{\max}$. In this paper, we use the hyperfield $\mathbb{Z}_{\min}$ as obtained from this construction.

The smallest non-abelian group can be fitted with a hyperaddition to form a skew hyperfield. Consider $\mathbb{D}_3 := (D_3 \cup \{0\}, \cdot, \boxplus, 1, 0)$, where $(D_3, \cdot, 1)$ is the dihedral group presented as $D_3 = \{d_i : i \in \mathbb{Z}_6\}$ with $1 := d_0$, with multiplication and hyperaddition fixed by

$$d_i \cdot d_j = \begin{cases} d_{i+j} & \text{if } i \in \{0, 2, 4\} \\ d_{i-j} & \text{if } i \in \{1, 3, 5\} \end{cases} \quad \text{and} \quad d_i \boxplus d_j = \begin{cases} \{d_i\} & \text{if } j = i \\ \{d_i, d_j\} & \text{if } j = i + 1 \\ \{d_i, d_{i+1}, d_j\} & \text{if } j = i + 2 \\ D_3 \cup \{0\} & \text{if } j = i + 3 \end{cases}$$

Verifying that $\mathbb{D}_3$ is indeed a skew hyperfield amounts to a finite check, which we omit.

2.2. **Matroids.** A *matroid* is a pair $(E, \mathcal{C})$, where E is a finite set and $\mathcal{C}$ is a set of subsets of E, such that

(MC0) $\emptyset \notin \mathcal{C}$
(MC1) if $C, C' \in \mathcal{C}$ and $C \subseteq C'$, then $C = C'$
(MC2) for all distinct $C, C' \in \mathcal{C}$ and all $e \in C \cap C'$, there exists a $C'' \in \mathcal{C}$ such that $e \notin C'' \subseteq C \cup C'$

The elements of $\mathcal{C}$ are the *circuits* of the matroid $M = (E, \mathcal{C})$, and E is the *ground set*. A subset F of E is *dependent* if $F \supseteq C$ for some $C \in \mathcal{C}$, and is *independent* otherwise. An inclusion-wise maximal independent set is called a *basis*. In a matroid M, all bases have the same cardinality, and this common cardinality is called the *rank* of M.

In the context of a matroid M with ground set E, we will write subsets of E concisely as e.g. $Fabc := F \cup \{a, b, c\}$. When we use this format, it is assumed implicitly that a, b, c are distinct elements of $E \setminus F$. So a phrase 'suppose Fab is a basis of M' hides the more elaborate setup 'suppose $F \subseteq E$, and a, b are distinct elements of $E \setminus F$ so that $F \cup \{a, b\}$ is a basis of M'.

If E is a finite set, K is a field V is a K-linear vector space, and $v_e \subseteq V$ for each $e \in E$, then for each $F \subseteq E$, the set $\{v_e : e \in F\}$ is either linearly dependent or independent over K. This distinction between dependent and independent sets is matroidal: if $\mathcal{C}$ denotes the set of inclusion-wise minimal nonempty sets F corresponding to a dependent set of vectors $\{v_e : e \in F\}$, then $\mathcal{C}$ satisfies the circuit axioms (MC0), (MC1), and (MC2), and thus $M = (E, \mathcal{C})$ is a matroid.

If $\mathcal{C} \subseteq 2^E$, then we say that a pair of distinct elements $C, C' \in \mathcal{C}$ is *modular* if $C \cup C'$ does not properly contain the union of to distinct elements of $\mathcal{C}$. Consider the *modular circuit elimination axiom*:

(MC2)' for all modular $C, C' \in \mathcal{C}$ and all $e \in C \cap C'$, there exists a $C'' \in \mathcal{C}$ such that $e \notin C'' \subseteq C \cup C'$.

Then in the presence of (MC0) and (MC1), the ordinary circuit elimination axiom (MC2) is implied by its seemingly weaker modular counterpart (MC2)', so that we could alternatively define a matroid as a pair $(E, \mathcal{C})$ for which (MC0), (MC1), and (MC2)' hold. The definition of *weak matroids over hyperfields* in [BB17] generalizes these modular circuit axioms.

Matroids over hyperfields generalize both matroids and collections of vectors from a vector space over K: the former can be regarded as 'matroids over the Krasner hyperfield $\mathbb{K}$', and the latter are 'matroids over the field K'.

We refer to Oxley's book [Oxl11] for further matroid-related notation and results and to the paper of Baker and Bowler [BB17] for the theory of matroids over (commutative) hyperfields.

3. Matroids over skew hyperfields

3.1. Circuit axioms.
Let H be a skew hyperfield. A pair $(E, \mathcal{C})$ is a *left H-matroid on E* if E is a finite set, and $\mathcal{C} \subseteq H^E$ satisfies the following *circuit axioms*.

(C0) $0 \notin \mathcal{C}$.
(C1) if $X \in \mathcal{C}$ and $\alpha \in H^\star$, then $\alpha \cdot X \in \mathcal{C}$.
(C2) if $X, Y \in \mathcal{C}$ and $\underline{X} \subseteq \underline{Y}$, then there exists an $\alpha \in H^\star$ so that $Y = \alpha \cdot X$.
(C3) if $X, Y \in \mathcal{C}$ are a modular pair in $\mathcal{C}$ and $e \in E$ is such that $X_e = -Y_e \neq 0$, then there exists a $Z \in \mathcal{C}$ so that $Z_e = 0$ and $Z \in X \boxplus Y$.

In (C3), a pair $X, Y \in \mathcal{C}$ is *modular* if $\underline{X}, \underline{Y}$ are modular in $\underline{\mathcal{C}} := \{\underline{X} : X \in \mathcal{C}\}$. A *right H-matroid M on E* is defined analogously, with $\alpha \cdot X$ replaced by $X \cdot \alpha$ in (C1) and (C2). If H is commutative, then left- and right H-matroids coincide, and we speak of *H-matroids*[1].

Suppose E is a finite set, K is a skew field, V is a left vector space over K, and $v_e \in V$ for each $e \in E$. Then the set of linear dependencies among the vectors v_e, $D := \{X \in K^E : \sum_{e \in E} X_e v_e = 0\}$ is a left linear space over K. The collection of dependencies of minimal support

$$\mathcal{C} := \{X \in D \setminus \{0\} : \text{if } Y \in D \setminus \{0\} \text{ and } \underline{Y} \subseteq \underline{X}, \text{ then } \underline{Y} = \underline{X}\},$$

satisfies the above left circuit axioms (C0)–(C3), so that $M(v) := (E, \mathcal{C})$ is a left K-matroid.

3.2. The underlying matroid, circuit signatures, and coordinates.
If $M = (E, \mathcal{C})$ is a left- or right H-matroid, then M determines an *underlying* matroid $\underline{M} := (E, \underline{\mathcal{C}})$, where

$$\underline{\mathcal{C}} := \{\underline{X} : X \in \mathcal{C}\}.$$

If H is the Krasner hyperfield, then M in turn is uniquely determined by $\underline{M}$. Thus a matroid M over the Krasner hyperfield $\mathbb{K}$ is essentially a matroid.

If N is a matroid on E and H is a skew hyperfield, then a collection $\mathcal{C} \subseteq H^E$ is a *left H-signature* of N if $\mathcal{C}$ satisfies (C0), (C1), and (C2), and $\underline{\mathcal{C}}$ is the collection of circuits of N.

If N is a matroid with bases $\mathcal{B}$, we name the set of ordered pairs of adjacent bases

$$A_N := \{(B, B') \in \mathcal{B} \times \mathcal{B} : |B \setminus B'| = 1\}.$$

Then a function $[.] : A_N \to H$ comprises *left H-coordinates* for N if

(CC0) $[Fa, Fb] \cdot [Fb, Fa] = 1$ if $Fa, Fb \in \mathcal{B}$
(CC1) $[Fac, Fbc] \cdot [Fab, Fac] \cdot [Fbc, Fab] = 1$ if $Fab, Fac, Fbc \in \mathcal{B}$
(CC2) $[Fac, Fbc] = [Fad, Fbd]$ if $Fac, Fad, Fbc, Fbd \in \mathcal{B}$, but $Fab \notin \mathcal{B}$

As we will demonstrate, a left H-signature encodes the same information as left H-coordinates. If $\mathcal{C}$ is a left H-signature of N, then we may define a map $[.] : A_N \to H$ by setting

$$[Fa, Fb]_{\mathcal{C}} := X_a^{-1} X_b$$

where $X \in \mathcal{C}$ is any circuit such that $\underline{X} \subseteq Fab$. Conversely, given left coordinates $[.]$ for N, we put

$$\mathcal{C}_{N, [.]} := \{X \in H^E : \underline{X} \text{ a circuit of } N \text{ and } X_a^{-1} X_b = [Fa, Fb] \text{ whenever } a, b \in \underline{X} \subseteq Fab\}.$$

We will usually omit the reference to N when the choice of N is unambiguous, and write $\mathcal{C}_{[.]}$.

Lemma 1. *Let N be a matroid, let $\mathcal{C} \subseteq H^E$ and let $[.] : A_N \to H$. The following are equivalent.*

(1) *$\mathcal{C}$ is a left H-signature of N, and $[.] = [.]_{\mathcal{C}}$*
(2) *$[.]$ are left H-coordinates, and $\mathcal{C} = \mathcal{C}_{[.]}$*

Proof. We show that (1) implies (2). Let $\mathcal{C}$ be a left H-signature of N, and let $[.] = [.]_{\mathcal{C}}$. It suffices to show that the three axiom (CC0), (CC1), (CC2) hold for $[.]$.

(CC0): Note that if Fa, Fb are both bases of N, and $X \in \mathcal{C}$ is any circuit so that $a, b \in \underline{X} \subseteq Fab$, then

$$[Fa, Fb][Fb, Fa] = (X_a^{-1} X_b)(X_b^{-1} X_a) = 1.$$

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[1]In [BB17], Baker and Bowler consider both *weak* and *strong* matroids over a hyperfield; our H-matroids are their *weak* H-matroids.

(CC1): Assume Fab, Fac, Fbc are bases of N. Then there exists a circuit $X \in \mathcal{C}$ so that $a, b, c \in \underline{X} \subseteq Fabc$. It follows that

$$[Fac, Fbc] \cdot [Fab, Fac] \cdot [Fbc, Fab] = (X_a^{-1} X_b)(X_b^{-1} X_c)(X_c^{-1} X_a) = 1.$$

(CC2): Assume that Fac, Fad, Fbc, Fbd are bases of N. Then there are circuits $X, Y \in \mathcal{C}$, so that $a, b \in \underline{X} \subseteq Fabc$, and $a, b \in \underline{Y} \subseteq Fabd$. If Fab is not a basis of N, then Fab contains a circuit, so that $\underline{X} = \underline{Y}$. By (C2), $Y = \alpha X$. Then

$$[Fac, Fbc] = X_a^{-1} X_b = (\alpha X_a)^{-1}(\alpha X_b) = Y_a^{-1} Y_b = [Fad, Fbd].$$

We now argue that (2) implies (1). So suppose [.] are left H-coordinates, and that $\mathcal{C} = \mathcal{C}_{[.]}$. We will first argue that for each circuit C of N, there is an $X \in \mathcal{C}_{[.]}$ so that $\underline{X} = C$. So let C be a circuit of N.

Consider two elements $a, b \in C$. We claim that if $Fa, Fb, F'a, F'b$ are bases of N so that if $C \subseteq Fab, F'ab$, then $[Fa, Fb] = [F'a, F'b]$. To show this, we use induction on $|F \setminus F'|$. Assume first that $|F \setminus F'| = 1$. Then $F = F''c$ and $F' = F''d$ for some F'', c, d. Since $C \subseteq (Fab) \cap (F'ab) = F''ab$, $F''ab$ is not a basis of N. By (CC2), it folllows that

$$[Fa, Fb] = [F''ac, F''bc] = [F''ad, F''bd] = [F'a, F'b].$$

If $|F \setminus F'| > 1$, then pick any $c \in (F \setminus F') \setminus C$. By virtue of the base exchange axiom in N/C, there exists a $d \in (F' \setminus F) \setminus C$ so that $F''a, F''b$ are bases of N, where $F'' = F - c + d$. By the induction hypothesis, we obtain

$$[Fa, Fb] = [F''a, F''b] = [F'a, F'b].$$

This proves the claim.

Fix any $c \in C$, let B be a basis of N containing $C - c$, and let $X \in H^E$ be such that $\underline{X} = C$, $X_c = 1$, and $X_a := [B - a + c, B]$ for all $a \in C - c$. By the claim, X does not depend on the choice of B. By (CC0) and (CC1), we have

$$X_a^{-1} X_b = (X_a^{-1} X_c)(X_c^{-1} X_b) = [Fab, Fbc][Fac, Fab] = [Fac, Fbc]$$

whenever $a, b \in \underline{X} \subseteq Fabc$, so that $X \in \mathcal{C}$. Thus $\underline{\mathcal{C}}$ is the set of circuits of N. It remains to verify that $\mathcal{C}$ satisfies (C0), (C1), (C2), but these are straightforward. $\qquad\square$

The definition of right H-signatures $\mathcal{C}$, right coordinates [.], and of the constructions $\mathcal{C}_{[.]}$ and $[.]_{\mathcal{C}}$ are obtained by reversing the order of multiplication throughout.

3.3. The push-forward.
Let $f : H \to H'$ be a hyperfield homomorphism. Denote $f_* X := (f(X_e) : e \in E)$ for any $X \in H^E$, and for a set $\mathcal{C} \subseteq H^E$ denote

$$f_* \mathcal{C} := \{\alpha' \cdot f_* X : \alpha' \in H', X \in \mathcal{C}\},$$

If $M = (E, \mathcal{C})$ is a left H-matroid, the *push-forward* is $f_* M := (E, f_* \mathcal{C})$. A straightforward verification yields that then (C0), (C1), (C2), (C3) hold for $f_* \mathcal{C}$, so that $f_* M$ is a left H'-matroid. From the definition of coordinates, we see that $[B, B']_{f_* M} = f([B, B'])_M$.

Clearly $\underline{f_* M} = \underline{M}$ for any hyperfield homomorphism f from H. In particular, if $\kappa : H \to \mathbb{K}$ then $\underline{M} = \kappa_* M$, so that the underlying matroid can be considered as the ultimate push-forward.

3.4. Quasi-Plucker coordinates.
Let H be a skew hyperfield and let N be a matroid on E with bases $\mathcal{B}$. Then $[.] : A_N \to H$ are *left quasi-Plucker coordinates* if

(P0) $[Fa, Fb] \cdot [Fb, Fa] = 1$		if $Fa, Fb \in \mathcal{B}$.
(P1) $[Fac, Fbc] \cdot [Fab, Fac] \cdot [Fbc, Fab] = 1$		if $Fab, Fac, Fbc \in \mathcal{B}$.
(P2) $[Fa, Fb] \cdot [Fb, Fc] \cdot [Fc, Fa] = -1$		if $Fa, Fb, Fc \in \mathcal{B}$.
(P3) $[Fac, Fbc] = [Fad, Fbd]$	if $Fac, Fad, Fbc, Fbd \in \mathcal{B}$, and $Fab \notin \mathcal{B}$ or $Fcd \notin \mathcal{B}$.
(P4) $1 \in [Fbd, Fab] \cdot [Fac, Fcd] \boxplus [Fad, Fab] \cdot [Fbc, Fcd]$	if $Fac, Fad, Fbc, Fbd, Fab, Fcd \in \mathcal{B}$.

We will show that in the presence of an underlying matroid N, these axioms are cryptomorphic to the left circuit axioms (C0)-(C3).

Theorem 1. *Let N be a matroid on E, let H be a skew hyperfield, let $[.] : A_N \to H$ map, and let $\mathcal{C} \subseteq H^E$. The following are equivalent:*

(1) *$M = (E, \mathcal{C})$ is a left H-matroid such that $\underline{M} = N$, and $[.] = [.]_{\mathcal{C}}$.*
(2) *[.] are left quasi-Plucker coordinates for N, and $\mathcal{C} = \mathcal{C}_{[.]}$.*

Proof. We show that (1) implies (2). Let $M = (E, \mathcal{C})$ be a left H-matroid such that $N = \underline{M}$, and let $[.] = [.]_\mathcal{C}$. By Lemma 1, $[.]$ are coordinates for N. We must show that the five axioms (P0)-(P4) hold. But (P0) is (CC0), (P1) is (CC1), and (P3) partially follows from (CC2). We verify what remains.

(P2): Suppose that Fa, Fb, Fc are bases of N, then there are circuits $X, Y, Z \in \mathcal{C}$ so that $a, b \in \underline{X} \subseteq Fab$, and $b, c \in \underline{Y} \subseteq Fbc$, and $a, c \in \underline{Z} \subseteq Fac$ which determine the quasi-Plücker coordinates

$$[Fa, Fb]_M := X_a^{-1} X_b, \ [Fb, Fc]_M := Y_b^{-1} Y_c, \ [Fc, Fa]_M := Z_c^{-1} Z_a.$$

The circuits X, Y are modular, and by (C1) we may assume without loss of generality that $X_b = -Y_b$. By (C3) there exists a circuit $Z' \in \mathcal{C}$ with $Z'_b = 0$ and $Z' \in X \boxplus Y$. Then $\underline{Z'} \subseteq Fac$, so that $\underline{Z'} = \underline{Z}$. By (C2), we may assume that $Z = Z'$. Then $Z_a = Z'_a \in X_a \boxplus Y_a = X_a \boxplus 0 = \{X_a\}$ and $Z_c = Z'_c \in X_c \boxplus Y_c = 0 \boxplus Y_c = \{Y_c\}$, so that $Z_a = X_a$ and $Z_c = Y_c$. It follows that

$$[Fa, Fb] \cdot [Fb, Fc] \cdot [Fc, Fa] = (X_a^{-1} X_b)(Y_b^{-1} Y_c)(Z_c^{-1} Z_a) = -1.$$

(P3): Assume that Fac, Fad, Fbc, Fbd are bases of N. The case that $Fab \notin \mathcal{B}$ is settled by (CC2), and we assume $Fcd \notin \mathcal{B}$. Then there are circuits $X, Y \in \mathcal{C}$, so that $a, b \in \underline{X} \subseteq Fabc$, and $a, b \in \underline{Y} \subseteq Fabd$, and we may assume that $X_a = -Y_a$ by (C2). By (C3), there is a circuit $Z \in \mathcal{C}$ so that $\underline{Z} \subseteq Fbcd$, and $Z \in X \boxplus Y$. As Fcd is dependent, we have $\underline{Z} \subseteq Fcd$, so that $0 = Z_b \in X_b \boxplus Y_b$, i.e. $X_b = -Y_b$. Then also

$$[Fac, Fbc] = X_a^{-1} X_b = Y_a^{-1} Y_b = [Fad, Fbd].$$

(P4): Assume that $Fac, Fad, Fbc, Fbd, Fab, Fcd$ are all bases of N. Then there are circuits $X, Y \in \mathcal{C}$, so that $a, c, d \in \underline{X} \subseteq Facd$, and $b, c, d \in \underline{Y} \subseteq Fbcd$. Then X, Y are modular, and by (C2) we may assume that $X_c = -Y_c$. By (C3), there is a circuit $Z \in \mathcal{C}$ so that $Z_c = 0$ and $Z \in X \boxplus Y$. Thus $Z_a \in X_a \boxplus Y_a = X_a \boxplus 0 = \{X_a\}$, $Z_b \in X_b \boxplus Y_b = 0 \boxplus Y_b = \{Y_b\}$, and $Z_d \in X_d \boxplus Y_d$. It follows that $Z_a = X_a \neq 0$, $Z_b = Y_b \neq 0$, and thus $a, b \in \underline{Z} \subseteq Fabd$. Since Fab is a basis of $\underline{M}$, we have $\underline{Z} \not\subseteq Fab$, and hence $Z_d \neq 0$. Then

$$Z_d \in X_d \boxplus Y_d = Z_a X_a^{-1} X_d \boxplus Z_b Y_b^{-1} Y_d$$

Multiplying on the left by Z_d^{-1} and using the left distributivity of the hyperring H, it follows that

$$1 = Z_d^{-1} Z_d \in (Z_d^{-1} Z_a) \cdot (X_a^{-1} X_d) \boxplus (Z_d^{-1} Z_b) \cdot (Y_b^{-1} Y_d) = [Fbd, Fab] \cdot [Fac, Fcd] \boxplus [Fad, Fab] \cdot [Fbc, Fcd].$$

This completes the proof of (1)$\Rightarrow$(2).

We next show that (2) implies (1). Let $[.] : A_N \to H$ be left quasi-Plücker coordinates for N, and suppose that $\mathcal{C} = \mathcal{C}_{[.]}$. By Lemma 1, $\mathcal{C}$ satisfies (C0), (C1), (C2). It remains to show (C3).

So let $X, Y \in \mathcal{C}_{[.]}$ be modular, and consider a $c \in \underline{X} \cap \underline{Y}$. Assume that $X_c = -Y_c$. There exists a circuit $Z \in \mathcal{C}_{[.]}$ with $\underline{Z} \subseteq \underline{X} \cup \underline{Y} - c$, and we may assume that $Z_a = X_a$ for some $a \in \underline{X} \setminus \underline{Y}$. It remains to show that $Z \in X \boxplus Y$.

Pick $b \in \underline{Y} \setminus \underline{X}$. There is an F such that $a, c \in \underline{X} \subseteq Fac$, $b, c \in \underline{Y} \subseteq Fbc$, and $a, b \in \underline{Z} \subseteq Fab$. Then by (P3)

$$Z_a^{-1} Z_b = [Fa, Fb] = -[Fa, Fc][Fc, Fb] = (X_a^{-1} X_c)(Y_c^{-1} Y_b) = X_a^{-1} Y_b,$$

so that $Z_b = Y_b \in 0 \boxplus Y_b = X_b \boxplus Y_b$, as required.

Next, consider a $d \in \underline{X} \cap \underline{Y}$, other than c. We may assume that $Z_d = 1$, again by rescaling as in (C1). By rescaling X and Y accordingly, we may assume that $X_a = Z_a$, $Y_b = Z_b$, and $X_c = -Y_c$. Then

$$X_d = X_a[Fac, Fad] = Z_a[Fac, Fcd] = Z_d[Fbd, Fab][Fac, Fad] = [Fbd, Fab][Fac, Fad]$$

and

$$Y_d = Y_b[Fbc, Fcd] = Z_b[Fbc, Fcd] = Z_d[Fad, Fab][Fbc, Fcd] = [Fad, Fab][Fbc, Fcd].$$

Hence by (P4), $Z_d = 1 \in [Fbd, Fab][Fac, Fad] \boxplus [Fad, Fab][Fbc, Fcd] = X_d \boxplus Y_d$. $\qquad\square$

3.5. **Duality.** Let H be a skew hyperfield, and let E be a finite set. We say that $X, Y \in H^E$ are *orthogonal*, denoted $X \perp Y$, if

$$0 \in \boxplus_{e \in E} X_e \cdot Y_e.$$

For sets $\mathcal{C}, \mathcal{D} \subseteq H^E$, we write $\mathcal{C} \perp_k \mathcal{D}$ if $X \perp Y$ for all $X \in \mathcal{C}$ and $Y \in \mathcal{D}$ such that $|\underline{X} \cap \underline{Y}| \leq k$.

Let N be a matroid on E and let H be a skew hyperfield. To any $[.] : A_N \to H$ we associate a *dual map* $[.]^* : A_{N^*} \to H$ by setting

$$[B, B']^* := -[E \setminus B, E \setminus B']$$

for all $(B, B') \in A_{N^*}$. It is evident from this definition that $[.]^{**} = [.]$.

Lemma 2. *Let N be a matroid on E and let H be a skew hyperfield, let $\mathcal{C}$ be a left H-signature of N, and let $\mathcal{D} \subseteq H^E$. The following are equivalent.*

    (1) $\mathcal{D}$ *is a right H-signature of N^*, and $\mathcal{C} \perp_2 \mathcal{D}$*
    (2) $[.] := [.]_{\mathcal{C}}$ *satisfies (P0), (P1), (P2), (P3), and $\mathcal{D} = \mathcal{C}_{[.]^*}$*

Proof. We show that (1) implies (2). If $\mathcal{D}$ is a right H-signature of N^*, and $\mathcal{C} \perp_2 \mathcal{D}$, then $[.]_{\mathcal{D}} = [.]_{\mathcal{C}}^*$. By Lemma 1, it follows that $\mathcal{D} = \mathcal{C}_{[.]^*}$. Being right H-coordinates, $[.]_{\mathcal{C}}^*$ satisfies (CC0), (CC1), (CC2), which in terms of $[.] := [.]_{\mathcal{C}}$ translates to

(CC0)* $[Fa, Fb] \cdot [Fb, Fa] = 1$ if $Fa, Fb \in \mathcal{B}$.
(CC1)* $[Fa, Fb] \cdot [Fb, Fc] \cdot [Fc, Fa] = -1$ if $Fa, Fb, Fc \in \mathcal{B}$.
(CC2)* $[Fac, Fbc] = [Fad, Fbd]$ if $Fac, Fad, Fbc, Fbd \in \mathcal{B}$, and $Fcd \notin \mathcal{B}$.

Together with (CC0), (CC1), (CC2) for $[.]$, we have (P0), (P1), (P2), (P3) for $[.]$.

    The proof that (2) implies (1) is a reversal of these steps. $\qquad\square$

Lemma 3. *Let N be a matroid on E and let H be a skew hyperfield, let $\mathcal{C}$ be a left H-signature of N, and let $\mathcal{D} \subseteq H^E$. The following are equivalent.*

    (1) $\mathcal{D}$ *is a right H-signature of N^*, and $\mathcal{C} \perp_3 \mathcal{D}$*
    (2) $[.] := [.]_{\mathcal{C}}$ *are left quasi-Plücker coordinates, and $\mathcal{D} = \mathcal{C}_{[.]^*}$*

Proof. In view of Lemma 2, we need to argue that if $\mathcal{C}$ is a left H-signature of N and $\mathcal{D}$ is a right H-signature of N^* so that $\mathcal{C} \perp_2 \mathcal{D}$, then

$$\mathcal{C} \perp_3 \mathcal{D} \text{ if and only if (P5) holds for } [.] := [.]_{\mathcal{C}}.$$

We first show sufficiency. So assume that $\mathcal{C} \perp_3 \mathcal{D}$, and let $Fac, Fad, Fbc, Fbd, Fab, Fcd$ be bases of N. Let $X \in \mathcal{C}$ be such that $a, b, d \in \underline{X} \subseteq Fabd$, and let $Y \in \mathcal{D}$ be such that $a, b, d \in \underline{Y} \subseteq E \setminus Fd$. Without loss of generality, we may assume that $X_d = 1$ and $Y_d = -1$. Using that $X \perp Y$,

$$0 \in X_a \cdot Y_a \boxplus X_b \cdot Y_b \boxplus X_d \cdot Y_d = [Fbd, Fab] \cdot [Fac, Fcd] \boxplus [Fad, Fab] \cdot [Fbc, Fcd] \boxplus -1,$$

and it follows that $1 \in [Fbd, Fab] \cdot [Fac, Fcd] \boxplus [Fad, Fab] \cdot [Fbc, Fcd]$.

    To see necessity, let $X \in \mathcal{C}$ and $Y \in \mathcal{D}$ be such that $\underline{X} \cap \underline{Y} = \{a, b, d\}$ for distinct $a, b, d \in E$. Since $\underline{Y} - ab$ is independent in N^*, we have $r(N \setminus (\underline{Y} - ab)) = r(N)$. Hence, there exist a basis Fab of N extending the independent set $\underline{X} - d$ of N, such that $F \cap \underline{Y} = \emptyset$. By a dual argument, there exists a basis Gab of $N^* \setminus F$ extending $\underline{Y} - d$. Since $|Fab| + |Gab| = r(N) + r(N^*) = |E|$ and $|F \cap G| = \emptyset$, $E \setminus (Fab \cup Gab)$ contains an element c besides d. Scaling, we may assume that $X_d = 1$ and $Y_d = -1$. Using (P4), we have

$$X_a \cdot Y_a \boxplus X_b \cdot Y_b \boxplus X_d \cdot Y_d = [Fbd, Fab] \cdot [Fac, Fcd] \boxplus [Fad, Fab] \cdot [Fbc, Fcd] \boxplus -1 \ni 0,$$

so that $X \perp Y$. $\qquad\square$

    We say that a left H-matroid $M = (E, \mathcal{C})$ and a right H-matroid $M' = (E, \mathcal{D})$ are *dual* if $\underline{M} = \underline{M'}^*$ and $\mathcal{C} \perp_3 \mathcal{D}$. By Lemma 3, each left or right H-matroid M has a dual, which we denote by M^*. We highlight the following direct consequence of Lemma 3.

Theorem 2. *Let N be a matroid on E and let H be a skew hyperfield. If $\mathcal{C}$ is a left H-signature of N and $\mathcal{D}$ is a right H-signature of N^* so that $\mathcal{C} \perp_3 \mathcal{D}$, then $M = (E, \mathcal{C})$ is a left H-matroid and $M^* = (E, \mathcal{D})$.*

3.6. **Minors.** Let N be a matroid on E, and let $\mathcal{C}$ be a left H-signature of N. For any disjoint sets $S, T \subseteq E$, put

$$\mathcal{C}/S \setminus T := \{X \setminus (S \cup T) : X \in \mathcal{C}, \underline{X} \subseteq E \setminus T, \text{ and } \underline{X} \setminus S \text{ a circuit of } N/S \setminus T\}.$$

If $M = (E, \mathcal{C})$, the *minor* of M obtained by *contracting S* and *deleting T* is $M/S \setminus T := (E \setminus (S \cup T), \mathcal{C}/S \setminus T)$. By construction, this minor $M/S \setminus T$ is a left H-signature of $N/S \setminus T$.

    Now consider coordinates $[.] : A_N \to H$. For any pair of disjoint sets $S, T \subseteq E$ so that S is independent in N and T is independent in N^*, we define $[.]/S \setminus T : A_{N/S \setminus T} \to H$ by setting

$$[B, B']/S \setminus T := [S \cup B, S \cup B']$$

for any pair of adjacent bases B, B' of the minor $N/S \setminus T$ of N. It is straightforward that $[.]_{\mathcal{C}}/S \setminus T = [.]_{\mathcal{C}/S \setminus T}$.

Lemma 4. *Let N be a matroid on E, let H be a skew hyperfield, and let $[.] : A_N \to H$ be H coordinates for N. The following are equivalent:*

    (1) *$[.]$ are left quasi-Plücker coordinates*

    (2) *$[.]/S \setminus T$ are left quasi-Plücker coordinates for all disjoint $S, T \subseteq E$ so that*

        (a) *S is independent in N and T is independent in N^*; and*

        (b) *$N/S \setminus T$ has rank ≤ 2 and corank ≤ 2.*

Proof. That (1) implies (2) is straightforward. We prove that (2) implies (1). Assume (1). To see that (P0) holds for $[.]$, let Fa, Fb be a bases. Then

$$[Fa, Fb] \cdot [Fb, Fa] = [a, b]/S \setminus T \cdot [b, a]/S \setminus T = 1$$

by (P0) for $[.]/S \setminus T$, where $S = F$ and $T = E \setminus Fab$. Then N has ground set ab and rank and corank both equal to 1.

An analogous argument applies to each of the other axioms: contract $S = F$ and delete $T = E \setminus Fabc$ (for (P1), (P2)) or $T = E \setminus Fabcd$ (for (P3), (P4)). It is easy to see that each time, the minors have both rank ≤ 2 and corank ≤ 2. $\qquad\square$

Translating back to circuit signatures, we obtain:

Theorem 3. *Let N be a matroid on E, let H be a skew hyperfield, and let $\mathcal{C}$ be a left H-signature of N. Then $M = (E, \mathcal{C})$ is a left H-matroid if and only if $M/S \setminus T$ is a left H-matroid, for all $S, T \subseteq E$ so that $N/S \setminus T$ has rank and corank ≤ 2.*

3.7. The weak order. Let $M = (E, \mathcal{C}), M' = (E, \mathcal{C})$ be left H-matroids. We say that M' is a *weak image* of M, notation $M' \preceq M$, if for all $X \in \mathcal{C}$ there exists an $X' \in \mathcal{C}'$ so that $X_e = X'_e$ for all $e \in \underline{X'}$. So in particular, each circuit of $\underline{M}$ contains a circuit of $\underline{M'}$, and hence $\underline{M'}$ is a weak image of $\underline{M}$ in the usual sense for matroids.

If $\underline{M'} \preceq \underline{M}$ and $r(M') = r(M)$, then each basis of M' is necessarily a basis of M. In this case it, we have $M' \preceq M$ if and only if $[B, B']_{M'} = [B, B']_M$ for all adjacent bases B, B' of M', i.e. if $[.]_{M'}$ is the restriction of $[.]_M$ to $A_{\underline{M'}}$.

Lemma 5. *Let M be a left H-matroid and let N be a matroid, so that N is a rank-preserving weak image of $\underline{M}$. Let $[.] : A_N \to H$ be the restriction of $[.]_M$ to A_N. Then $[.]$ are quasi-Plücker coordinates for N if and only if $[.]$ satisfies (P3).*

Proof. As M is a left H-matroid, (P0), (P1), (P2), and (P4) hold for $[.]_M$. The premise of each of these axioms is purely that certain bases exist. Since each basis of N is necessarily a basis of $\underline{M}$, the same axioms will hold true for the restriction $[.]$ of $[.]_M$. Hence if (P3) also holds for $[.]$, then $[.]$ are quasi-Plücker coordinates. $\qquad\square$

3.8. Rescaling. If N is a matroid on E, $\mathcal{C}$ is a left H-signature of N, and $\rho : E \to H^\star$, then *left rescaling* $\mathcal{C}$ *by* ρ yields

$$\mathcal{C}^\rho := \{(X_e \rho_e : e \in E) : X \in \mathcal{C}\}.$$

If $\mathcal{D}$ is a right H-signature of N^*, and $\rho : E \to H^\star$, then *right rescaling* $\mathcal{D}$ *by* ρ yields

$$\mathcal{D}^\rho := \{(\rho_e Y_e : e \in E) : Y \in \mathcal{D}\}.$$

Lemma 6. *Let N be a matroid on E, let $\mathcal{C}$ be a left H-signature of N and let $\mathcal{D}$ is a right H-signature of N^*. Then for any $\rho : E \to H$ we have $\mathcal{C} \perp_k \mathcal{D}$ if and only if $\mathcal{C}^\rho \perp_k \mathcal{D}^{\rho^{-1}}$, where $\rho^{-1} : e \mapsto \rho_e^{-1}$.*

Rescaled signatures $\mathcal{C}^\rho$ and $\mathcal{D}^\rho$ have coordinates

$$[Fa, Fb]_{\mathcal{C}^\rho} = (X_a \rho_a)^{-1}(X_b \rho_b) = \rho_a^{-1} X_a^{-1} X_b \rho_b = \rho_a^{-1} [Fa, Fb]_{\mathcal{C}} \rho_b$$

for any $X \in \mathcal{C}$ so that $a, b \in \underline{X} \subseteq Fab$, and

$$[Fa, Fb]_{\mathcal{D}^\rho} = (\rho_b Y_b)(\rho_a Y_a)^{-1} = \rho_b Y_b Y_a^{-1} \rho_a^{-1} = \rho_b [Fa, Fb]_{\mathcal{D}} \rho_a^{-1}$$

for any $Y \in \mathcal{D}$ so that $a, b \in \underline{Y} \subseteq Fab$. We define *rescaling* of left and right coordinates by ρ accordingly, setting $[Fa, Fb]^\rho := \rho_a^{-1} [Fa, Fb] \rho_b$ for left coordinates and $[Fa, Fb]^\rho := \rho_b [Fa, Fb] \rho_a^{-1}$ for right coordinates.

Write $\mathcal{C} \sim \mathcal{C}'$ if $\mathcal{C}^\rho = \mathcal{C}'$ for some $\rho : E \mapsto H^\star$. We investigate the rescaling classes of $U_{2,4}$. For any $x, y \in H^\star$, let $\mathcal{U}_H(x, y)$ denote the unique H-signature of $U_{2,4}$ containing

$$(0, 1, 1, 1), (1, 0, -1, -x), (1, 1, 0, y), (1, -x, y, 0).$$

Lemma 7. *Let H be a skew hyperfield, and let $M = (E, \mathcal{C})$ be a left H-matroid so that $\underline{M} = U_{2,4}$. Then there are $x, y \in H^\star$ with $1 \in x \boxplus y$ so that $\mathcal{C} \sim \mathcal{U}_H(x, y)$. Moreover,*

$$\{(x', y') : \mathcal{C} \sim \mathcal{U}_H(x', y')\} = \{\beta^{-1}(x, y)\beta : \beta \in H^\star\}.$$

Proof. Write $E = \{a, b, c, d\}$, and pick $W, X, Y, Z \in \mathcal{C}$ such that $\underline{W} = bcd, \underline{X} = acd, \underline{Y} = abd, \underline{Z} = abc$. Using (C2), we may assume that $X_a = Y_a = Z_a = 1$, and $W_b = Y_b$. Define $\rho \in E \to H^\star$ by

$$\rho_a = 1, \rho_b = W_b^{-1}, \rho_c = W_c^{-1}, \rho_d = W_d^{-1}.$$

Replacing $\mathcal{C}$ with $\mathcal{C}^\rho \sim \mathcal{C}$, we have

$$W = (0, 1, 1, 1), X = (1, 0, s, -x), Y = (1, 1, 0, y), Z = (1, -x', y', 0)$$

for some $s, x, y, x', y' \in H^\star$. Note that each pair of these circuits is modular in $\mathcal{C}$. Applying (C3), we have

(1) $X \in (-W) \boxplus Y$, so that $s = X_c \in (-W_c) \boxplus Y_c = \{-1\}$, so $s = -1$;
(2) $Z \in (-xW) \boxplus X$, so that $-x' = Z_b \in (-xW_b) \boxplus -X_b = -x \boxplus 0$, so $x' = x$;
(3) $Z \in yW \boxplus Y$, so that $y' = Z_c \in yW_c \boxplus Y_c = y \boxplus 0$, so $y' = y$; and
(4) $W \in (-X) \boxplus Y$, so that $1 = W_d \in (-X_d) \boxplus Y_d = x \boxplus y$.

Then $\mathcal{C} = \mathcal{U}_H(x, y)$ and $1 \in x \boxplus y$, as required. Finally, if $\mathcal{U}_H(x', y') \sim \mathcal{U}_H(x, y)$, then we must have $\mathcal{U}_H(x', y') = \mathcal{U}_H(x, y)^\rho$ with $\rho = \beta \mathbf{1}_E$ for some β. It then follows that $(x', y') = (\beta^{-1}x\beta, \beta^{-1}y\beta)$. $\square$

Thus the conjugacy class of the pair (x, y) as in the lemma is a scaling invariant of any H-orientation of $U_{2,4}$, and more generally, gives an invariant for each $U_{2,4}$-minor of each left H-matroid M.

3.9. **Cross ratios.** Let M be a left H-matroid on E. The *cross ratio* is defined as

$$cr_M(F, a, b, c, d) := [Fac, Fad]_M[Fbd, Fbc]_M.$$

Formally $cr_M : CR_{\underline{M}} \to H$, where $CR_N := \{(F, a, b, c, d) : Fac, Fad, Fbd, Fbc \text{ are bases of } N\}$. The following properties are straightforward by substituting the definition of cross ratio and applying the quasi-Plücker axioms.

(CR0) $cr(F, a, b, c, d)cr(F, b, a, c, d) = 1$
(CR1) $cr(F, a, b, d, e)cr(F, b, c, d, e)cr(F, c, a, d, e) = 1$
(CR2) $cr(Fa, b, c, d, e)cr(Fc, a, b, d, e)cr(Fb, c, a, c, d) = 1$
(CR3) $cr(F, a, b, c, d) = 1$ whenever Fab is not a basis of $\underline{M}$
(CR4) $1 \in cr(F, b, c, d, a) \boxplus cr(F, a, c, d, b)$
(CRP) $[Fad, Fcd]cr(F, a, b, c, d) = cr(F, c, b, a, d)[Fbd, Fcd]$

In the context of quasi-determinants, the cross ratio was similarly defined by Gelfand, Gelfand, Retakh, and Wilson, who also note such properties [GGRW05, Ret14].

3.10. **Matroids over commutative hyperfields.** The following paraphrases Baker and Bowler [BB17].

Theorem 4. *Let H be a hyperfield and let $M = (E, \mathcal{C})$ an H-matroid. There exists an alternating function $\phi : E^r \to H$ such that $\phi(B) \neq 0$ if and only if B is an ordered basis of $\underline{M}$, and*

$$\frac{\phi(f_1, \ldots, f_{r-1}, a)}{\phi(f_1, \ldots, f_{r-1}, b)} = \frac{X_b}{X_a}$$

for all distinct $f_1, \ldots, f_{r-1}, a, b \in E$ so that $\{f_1, \ldots, f_{r-1}, b\}$ is a basis of $\underline{M}$, and all circuits $X \in \mathcal{C}$ so that $\underline{X} \subseteq \{f_1, \ldots, f_{r-1}, a, b\}$. Moreover, we have

$$0 \in \phi(f_1, \ldots, f_{r-2}, a, b) \cdot \phi(f_1, \ldots, f_{r-2}, c, d) \boxplus \phi(f_1, \ldots, f_{r-2}, a, c) \cdot \phi(f_1, \ldots, f_{r-2}, d, b) \boxplus$$
$$\boxplus \phi(f_1, \ldots, f_{r-2}, a, d) \cdot \phi(f_1, \ldots, f_{r-2}, b, c)$$

for any distinct $f_1, \ldots, f_{r-1}, a, b, c, d \in E$.

The function ϕ is called the *Grassmann-Plucker* function of M, denoted ϕ_M. From their definitions, the relation between (quasi-Plucker) coordinates and the Grassmann-Plucker function is immediately clear.

Lemma 8. *Let H be a commutative hyperfield, let M be an H-matroid on E of rank r. Then*

$$[Fa, Fb]_M = \frac{\phi_M(f_1, \ldots, f_{r-1}, a)}{\phi_M(f_1, \ldots, f_{r-1}, b)}$$

for all distinct $f_1, \ldots, f_{r-1}, a, b \in E$ so that Fa, Fb are basis of $\underline{M}$, where $F = \{f_1, \ldots, f_{r-1}\}$.

Over skew hyperfields, there seems to be no proper analogue of the Grassmann-Plucker function. However, with each skew hyperfield H we may associate a commutative hyperfield H^{ab}, which arises by dividing out the commutator subgroup of $H^\star$, and there is a canonical homomorphism $\delta : H \to H^{ab}$. If M is a left- or right H-matroid, then it may not be possible to define a Grassmann-Plucker function for M, but the push-forward $\delta_* M$ is a matroid over a commutative hyperfield, which does admit a Grassmann-Plucker function.

If K is a skew field and $\delta : K \to K^{ab}$ is the canonical hyperfield homomorphism, the push-forward construction illustrates the definition of the *Dieudonne determinant* of a matrix over a skew field K [Die43].

4. A SKEW HYPERFIELD

4.1. **The skew hyperfield of monomials.** Let H be any hyperring, and let $\sigma : H \to H$ be an automorphism. We define a new hyperring

$$H(T, \sigma, \min) = (\{T^\infty\} \cup \{aT^i : a \in H^\star, i \in \mathbb{Z}\}, 1, 0, \cdot, \boxplus),$$

as follows. As the notation suggests, we identify $a \in H$ with aT^0 and write T^i for $1T^i$. We put $1 := T^0$ and $0 := T^\infty$. Multiplication follows the rules $0 \cdot aT^i = aT^i \cdot 0 = 0$ and

$$aT^i \cdot bT^j := a\sigma^i(b)T^{i+j}$$

for all $a, b \in H^\star$ and $i, j \in \mathbb{Z}$. In particular, $a \cdot T^j = aT^j$. The hypersum is given by $0 \boxplus x = x \boxplus 0 = \{x\}$ and

$$aT^i \boxplus bT^j := \begin{cases} \{aT^i\} & \text{if } i < j \\ \{bT^j\} & \text{if } i > j \\ (a+b) \cdot T^i & \text{if } i = j \text{ and } a \neq -b \\ (a+b) \cdot T^i \cup H^\star \cdot \{T^k : k \in \mathbb{Z}, k > i\} & \text{if } i = j \text{ and } a = -b \end{cases}$$

for $a, b \in H^\star$ and $i, j \in \mathbb{Z}$, where $+$ is the hyperaddition of H. Note that in the last line of this definition, we have $0 = 0 \cdot T^i \in (a+b) \cdot T^i$ as $a = -b$.

There is a variant $H(T, \sigma, \max)$ which arises by reversing $<$ and $>$ in the above definition. In the present paper, we will hardly use this variant, and we will not substitute the symbol T. For brevity, we write $H^\sigma := H(T, \sigma, \min)$ in what follows.

Lemma 9. *Let H be a hyperring, and let σ be an automorphism of H. Then H^σ is a hyperring. Moreover, if H is a skew hyperfield, then H^σ is a skew hyperfield.*

Proof. We must first verify that $\boxplus$ is commutative and associative. Commutativity is clear from the symmetry in the definition. To see associativity, consider aT^i, bT^j, cT^k. If $i < j$, then

$$(aT^i \boxplus bT^j) \boxplus cT^k = aT^i \boxplus cT^k = aT^i \boxplus (bT^j \boxplus cT^k)$$

If $i > j$ then

$$(aT^i \boxplus bT^j) \boxplus cT^k = bT^j \boxplus cT^k = aT^i \boxplus (bT^j \boxplus cT^k)$$

So $i = j$, and by symmetry $j = m$. Then

$$(aT^i \boxplus bT^i) \boxplus cT^i = (a+b)T^i \boxplus cT^i = aT^i \boxplus (c+b)T^i = aT^i \boxplus (bT^i \boxplus cT^i).$$

Next, we show that $(H^\sigma, T^\infty, \boxplus)$ satisfies the hypergroup axioms (H0), (H1).

(H0): $aT^i \boxplus T^\infty = aT^i$ by definition.

(H1): For $aT^i \in H^\sigma$, we have $T^\infty \in aT^i \boxplus bT^j$ if and only if $i = j$ and $a = -b$. Thus $-(a \cdot T^i) = (-a) \cdot T^i$.

(H2): Suppose $aT^i \in bT^j \boxplus cT^k$. We must show $cT^k \in aT^i \boxplus -bT^j$. If $j < k$, then $aT^i = bT^j$ and hence $cT^k \in aT^i \boxplus -bT^j$, and similar if $j > k$. If $i > j = k$, then $b = -c$ and hence $cT^k \in aT^i \boxplus -bT^j$. So $i = j = k$ and $a \in b + c$, so that $c \in a + (-b)$ and hence $cT^k \in aT^i \boxplus -bT^j$.

It is evident that $(H^\sigma \setminus \{T^\infty\}, T^0, \cdot)$ is a multiplicative monoid. We have $aT^i \cdot T^\infty = T^\infty$, so the zero T^∞ is absorbing. Distributivity is straightforward.

Finally, if H is a skew hyperfield, then $1 \in H^\sigma$ is distinct from $0 \in H^\sigma$, and each aT^i has multiplicative inverse $\sigma^{-i}(a)T^{-i}$, so that $(H^\sigma \setminus \{T^\infty\}, T^0, \cdot)$ is a multiplicative group. Then H^σ is a skew hyperfield. $\square$

For any hyperring H, there is a natural homomorphism $\zeta : H^\sigma \to \mathbb{Z}_{\min}$ given by $\zeta : aT^i \mapsto i$. In the reverse direction, there is the injective homomorphism $\tau : \mathbb{Z}_{\min} \to H^\sigma$ given by $\tau : i \mapsto T^i$. If $H = \mathbb{K}$ and σ is the identity, then $\zeta \circ \tau = id = \tau \circ \zeta$, so that ζ is an isomorphism. In this sense, $\mathbb{Z}_{\min} \cong \mathbb{K}^{id}$ is a special case of this construction.

4.2. Ore extensions of fields.
The definition of the above skew hyperfield of monomials was inspired by a construction of skew fields due to Ore [Ore33].

Let K be a skew field, and let $\sigma : K \to K$ be an automorphism. The *Ore extension* $K[T, \sigma]$ is the ring of formal polynomials $\sum_{i=0}^{n} a_i T^i$ in which T commutes with elements $a \in K$ according to the rule $Ta = \sigma(a)T$.

The ring $R = K[T, \sigma]$ satisfies left and right *Ore conditions*: for each $s, t \in R$, we have $sR \cap tR \neq \emptyset$ and $Rs \cap Rt \neq \emptyset$, which allows to define the left field of fractions

$$K(T, \sigma) := \{a^{-1}b : a, b \in K[T, \sigma]\}.$$

There is a hyperring homomorphism $\nu : K[T, \sigma] \to \mathbb{Z}_\infty$ determined by

$$\nu\left(\sum_{i=0}^{n} a_i T^i\right) = \min\{i : a_i \neq 0\}$$

and $\nu(0) := \infty$. This ν extends to $\nu : K(T, \sigma) \to \mathbb{Z}_{\min}$ by setting $\nu(b^{-1}a) := -\nu(b) + \nu(a)$.

If K is a skew field, then there is a hyperring homomorphism $\mu : K[T, \sigma] \to K(T, \sigma, \min)$ determined by

$$\mu\left(\sum_{i=0}^{n} a_i T^i\right) = a_m T^m, \text{where } m = \min\{i : a_i \neq 0\}$$

This hyperring homomorphism extends to $\mu : K(T, \sigma) \to K(T, \sigma, \min)$ by setting $\mu(a^{-1}b) = \mu(a)^{-1}\mu(b)$.

Lemma 10. *μ and ζ are hyperfield homomorphisms, and $\nu = \zeta \circ \mu$.*

There is a similar homomorphism $K(T, \sigma) \to K(T, \sigma, \max)$ which picks up the leading term.

4.3. The boundary matroid of an H^σ-matroid.
Consider a $\mathbb{Z}_{\min}$-matroid M on E with Grassmann-Plucker function $\phi : E^r \to \mathbb{Z}_{\min}$. As the hyperaddition of $\mathbb{Z}_{\min}$ is idempotent, we have $x = -x$ in $\mathbb{Z}_{\min}$ and hence the otherwise alternating Grassmann-Plucker function becomes oblivious to the ordering of its argument: $\phi(b_1, \ldots, b_r) = \phi(b'_1, \ldots, b'_r)$ whenever $\{b_1, \ldots, b_r\} = \{b'_1, \ldots, b'_r\}$. Let $\nu : \binom{E}{r} \to \mathbb{Z}_{\min}$ be determined by

$$\nu(B) := \phi(b_1, \ldots, b_r)$$

whenever $B = \{b_1, \ldots, b_r\}$. Then ν is a *matroid valuation*, and it was shown by Dress and Wenzel [DW92] that

$$\mathcal{B}_0 := \left\{ B \in \binom{E}{r} : \nu(B) = \min\{\nu(B') : B' \in \binom{E}{r}\} \right\}$$

is a nonempty set satisfying the base exchange axiom. We will call the matroid M_0 with ground set E and set of bases $\mathcal{B}_0$, the *boundary matroid of M*.[2]

We will define boundary matroids more generally for H^σ- matroids. Consider the natural hyperfield homomorphism $\zeta : H^\sigma \to \mathbb{Z}_{\min}$ given by $\zeta : aT^i \mapsto i$.

Lemma 11. *Let H be a skew hyperfield and let M be a left H^σ-matroid, and let $N := (\zeta_* M)_0$. Let $[.]_0$ be the restriction of $[.]_M$ to A_N. Then $[.]_0$ are quasi-Plucker coordinates for N, taking values in H.*

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[2]Dress and Wenzel speak of a *residue class geometry* in [DW92].

Proof. Recall that by definition of the boundary matroid of a $\mathbb{Z}_{\min}$-matroid, the matroid N has bases

$$\mathcal{B}_0 := \{B \in \mathcal{B} : \nu(B) = \min\{\nu(B') : B' \in \mathcal{B}\}\},$$

where $\mathcal{B}$ is the set of bases of $\zeta_* M$ and $\nu(B) - \nu(B') = \zeta[B, B']$ for any $(B, B') \in A_M$. Assuming without loss of generality that $\min\{\nu(B') : B' \in \mathcal{B}\} = 0$, we have $\mathcal{B}_0 := \{B \in \mathcal{B} : \nu(B) = 0\}$, and $\nu(B) > 0$ if $B \in \mathcal{B} \setminus \mathcal{B}_0$. In particular $[B, B']_0 = [B, B']_M \in H$ for all $(B, B') \in A_N$, since for such (B, B') we have $\zeta([B, B']_M) = \nu(B) - \nu(B) = 0$.

To prove that $[.]_0$ are quasi-Plücker coordinates for N, we need only show that $[.]_0$ satisfies (P3) by Lemma 5. Consider F, a, b, c, d so that $Fac, Fad, Fbc, Fbd \in \mathcal{B}_0$, and $Fab \notin \mathcal{B}_0$. If Fab is not a base of M, then

$$[Fac, Fbc]_0 = [Fac, Fbc]_M = [Fad, Fbd]_M = [Fad, Fbd]_0,$$

and likewise if Fcd is not a basis of M. If on the other hand both $Fab, Fcd \in \mathcal{B}$, then

$$1 \in [Fac, Fad]_M \cdot [Fbd, Fbc]_M \boxplus [Fcd, Fad]_M \cdot [Fab, Fbc]_M$$

by the fact that the quasi-Plücker coordinates of M satisfy (P4). As $Fac, Fad, Fbc, Fbd \in \mathcal{B}_0$, and $Fab, Fcd \notin \mathcal{B}_0$, we have

$$\zeta([Fac, Fad]_M \cdot [Fbd, Fbc]_M) = \nu(Fac) - \nu(Fad) + \nu(Fbd) - \nu(Fbc) = 0$$

and

$$\zeta([Fcd, Fad]_M \cdot [Fab, Fbc]_M) = \nu(Fcd) - \nu(Fad) + \nu(Fab) - \nu(Fbc) > 0.$$

Then $1 \in [Fac, Fad]_M \cdot [Fbd, Fbc]_M \boxplus [Fcd, Fad]_M \cdot [Fab, Fbc]_M = \{[Fac, Fad]_M \cdot [Fbd, Fbc]_M\}$, and hence $[Fac, Fad]_0 \cdot [Fbd, Fbc]_0 = [Fac, Fad]_M \cdot [Fbd, Fbc]_M = 1$. $\qquad\square$

If M is a left H^σ-matroid, then the *boundary matroid of M* if is the unique matroid M_0 such that $\underline{M_0} = (\zeta_* M)_0$ and M_0 is a weak image of M. By the Lemma, such M_0 exists and is a left H-matroid.

5. Matroids over hyperfields from algebraic matroids

5.1. Preliminaries on field extensions, algebraic matroids.
Let K be a field, and E be a finite set. We write $K[X_E] := K[X_e : e \in E]$ for the polynomial ring over K with a variable X_e for each element of E, and $K(X_E)$ for its field of fractions. For a polynomial $q \in K[X_E]$, let $\underline{q}$ denote the smallest set F so that $q \in K[X_F]$, i.e. $\underline{q}$ is the set of indices of variables which are mentioned in q.

Lemma 12. *Let $I \subseteq K[X_E]$ be an ideal, and let $q, r \in I$ be irreducible over K. If $\underline{q} \neq \underline{r}$ and $e \in \underline{q} \cap \underline{r}$, then there exists a nonzero polynomial $s \in I$ such that $e \notin \underline{s} \subseteq \underline{q} \cup \underline{r}$.*

If L is an extension field of K, and $x_e \in L$ for $e \in E$, then x_F is *algebraically dependent over K* if there is a nonzero polynomial $q \in K[X_F]$ so that $q(x) = 0$ (when variables and values are both indexed by E, then $q(x)$ arises by substituting X_e with x_e for all $e \in E$).

Theorem 5. *Let L/K be a field extension, let E be a finite set and let $x_e \in L$ for each $e \in E$. Let $\mathcal{C}$ be the set of inclusionwise minimal elements of*

$$\mathcal{A} := \{F \subseteq E : x_F \text{ is algebraically dependent over } K\} \setminus \{\emptyset\}.$$

Then $(E, \mathcal{C})$ is a matroid.

Proof. (MC0) and (MC1) hold for $\mathcal{C}$ as $\mathcal{C}$ is the set of inclusionwise minimal elements of an $\mathcal{A} \subseteq 2^E \setminus \{\emptyset\}$. We prove (MC2). Suppose $C, C' \in \mathcal{C}$ are distinct. Then there are polynomials $q, r \in K[X_E]$ so that $C = \underline{q}$ and $C' = \underline{r}$. If q is reducible, some factor q' of q will have $\emptyset \neq \underline{q'} \subseteq \underline{q}$, and then $\underline{q'} = \underline{q}$ by minimality of $C = \underline{q}$ in $\mathcal{A}$. Replacing q, r by such a factor if necessary, we may assume q, r are irreducible over K. Consider the homomorphism $h : K[X_E] \to L$ which maps $h : X_e \mapsto x_e$, and let $I := \ker(h)$. Then $q, r \in I$, and by Lemma 12, there exists a polynomial $s \in I$ so that

$$e \notin \underline{s} \subseteq \underline{q} \cup \underline{r} = C \cup C'.$$

Then $A := \underline{s} \in \mathcal{A}$, so that there is some $C'' \in \mathcal{C}$ with $C'' \subseteq A \subseteq C \cup C' \setminus \{e\}$, as required. $\qquad\square$

We denote the matroid of the elements $x \in L^E$ in the field extension L/K by $M(K, x)$.

Lemma 13. *Let L/K be a field extension, let $x \in L^E$, and let $h : K[X_E] \to L$ be the homomorphism which maps $h : X_e \mapsto x_e$. If C is a circuit of $M(K, x)$ and $q \in K[X_C]$, then $K[X_C] \cap \ker(h) = qK[X_C]$ if and only if q irreducible. Moreover, if $qK[X_C] = q'K[X_C]$ then $q = \alpha q'$ for some $\alpha \in K^\star$.*

We say that a polynomial q as in the lemma *decorates* the circuit C of $M(K, x)$.

5.2. **The space of derivations.** Let R be any ring. A *derivation* of R is a map $D : R \to R$ such that

(D0) $D(1) = 0$
(D1) $D(x + y) = D(x) + D(y)$
(D2) $D(xy) = D(x)y + xD(y)$

If $S \subseteq R$, then we say that a derivation D is an *S-derivation* if $D(s) = 0$ for all $s \in S$.

Consider a field extension L/K and $x \in L^E$, and let D be a K-derivation. For any polynomial $q \in K[X_E]$ so that $q(x) = 0$, we have $D(q(x)) = D(0) = 0$. Applying (D0), (D1), (D2) to expand $D(q(x))$ we obtain $\sum_{e \in E} \frac{\partial q}{\partial x_e} D(x_e) = 0$. Here $\frac{\partial q}{\partial x_e}$ denotes the formal derivative $\frac{\partial q}{\partial X_e}$ as evaluated in $X_E = x$. It follows that

$$(1) \qquad d(q) := \left(\frac{\partial q}{\partial x_e} : e \in E \right) \perp (D(x_e) : e \in E) =: D(x).$$

The following stronger statement is Theorem 5.1 of [Lan02]. In the statement of this theorem, q^D denotes the result of applying D to each coefficient of $q \in K[X_E]$.

Theorem 6. *Let L/K be a field extension, let $x \in L^E$. Let $h : K[X_E] \to L$ be the homomorphism such that $h(X_e) = x_e$, and let $q_1, \ldots, q_t$ be a set of generators for $\ker(h)$. Suppose D is a derivation of K. If $u \in L^E$ is such that for $i = 1, \ldots, t$*

$$0 = q_i^D(x) + \sum_e \frac{\partial q_i}{\partial x_e} u_e,$$

then there is one and only one derivation D^* of $K(x_E)$ coinciding with D on K, and such that $D^*(x_e) = u_e$ for every $e \in E$.

This theorem may be used to characterize $Der(K, x) := \{D(x) : D \text{ a } K\text{-derivation of } K(x_E)\}$.

Corollary 1. *Let L/K be a field extension, let $x \in L^E$. Then*

$$Der(K, x) = \{d(q) : q \text{ decorates a circuit of } M(K, x)\}^{\perp}.$$

Proof. The polynomials decorating the circuits of $M(K, x)$ generate the kernel of h as in the theorem. Apply the theorem to the trivial K-derivation D. Since D is trivial, we have $q^D(x) = 0$ for any decorating polynomial q. We obtain that D^* is a K-derivation of $K(x_E)$ if and only if $D^*(x) \perp d(q)$ for each polynomial q decorating a circuit of $M(K, x)$. $\qquad \square$

If $q \in K[X_E]$, then clearly $\underline{d(q)} \subseteq \underline{q}$, but equality need not hold if K has positive characteristic p. We then have $e \in \underline{q} \setminus \underline{d(q)}$ if and only if q can be written as a polynomial in X_e^p. The polynomial q is *separable* in X_e exactly if $e \in \underline{d(q)}$.

If k is any subfield of L and $y \in L$, then y is *separable* over k if there is a polynomial $q \in k[Y]$ which is separable in Y so that $q(y) = 0$. The *separable closure* of k in L is

$$k^{sep} := \{y \in L : y \text{ separable over } k\}.$$

As a consequence of Theorem 6, any derivation of k will extend uniquely to k^{sep}.

Corollary 2. *Let L/K be a field extension, let $x \in L^E$. Then $\dim Der(K, x)$ equals the rank of $M(K, x)$.*

Proof. B is a basis of $M(K, x)$ if and only $K(x_E)$ is algebraic over $K(x_B)$. Pick a basis B so that the index $[K(x_E) : K(x_B)^{sep}]$ is as small as possible. Then for each $e \in E \setminus B$, the circuit $C \subseteq B + e$ is decorated by a polynomial q which is separable in X_e. If not, q (being irreducible) is separable in some $f \in C - e \subseteq B$. Taking $B' := B + e - f$, we then have $K(x_B)^{sep} \subseteq K(x_{B'})^{sep}$, and the inclusion is strict since $x_e \notin K(x_B)^{sep}$ and $x_e \in K(x_{B'})^{sep}$. Then $[K(x_E) : K(x_{B'})^{sep}] < [K(x_E) : K(x_B)^{sep}]$, contradicting the choice of B.

Consider values $u_e \in K(x_E)$ satisfying the condition of Theorem 6. Observe that upon fixing u_f for each $f \in B$, the values of u_e for $e \in E \setminus B$ are determined by the relation $0 = \sum_e \frac{\partial q}{\partial x_e} u_e$, where q is the polynomial

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decorating $C \subseteq B + e$, since $\frac{\partial q}{\partial x_e} \neq 0$. Hence $\dim Der(K, x) \leq |B|$. On the other hand the derivations $(D_e := \partial/\partial x_e)_{e \in B}$ are independent, since $D_e(x_f) \neq 0$ if and only if $e = f$, for all $e, f \in B$. It follows that $\dim Der(K, x) \geq |B|$ as well, and hence $\dim Der(K, x) = |B| = r(M(K, x))$. $\square$

5.3. **The matroid of σ-derivatives.** Let $K \subseteq L$ be a field extension in positive characteristic p, let E be a finite set, let $x \in L^E$, and put $N := M(K, x)$. We will assume that L is algebraically closed, and we write $\sigma : L \to L$ for the Frobenius automorphism $\sigma : x \mapsto x^p$. In what follows, we will create a left L^σ-signature for N and a right L^σ-signature for N^*, aiming to showing orthogonality of these signatures. For brevity, we will not repeat our choice E, K, L, σ in the lemmas of this section.

For a vector $u \in \mathbb{N}^E$, write $x^u = \prod_{e \in E} x_e^{u_e}$. Let $q = \sum_u q_u x^u \in K[X_E]$, and put

$$m_e := \max\{m \in \mathbb{N} : p^m \text{ divides } u_e \text{ for all } u \text{ such that } q_u \neq 0\}.$$

Then let $\overline{q} \in K[Z_E]$ be the polynomial such that $q = \overline{q}\left(X_e^{p^{m_e}} : e \in E\right)$. The σ-derivative $d^\sigma(q) : E \to L^\sigma$ is defined as

$$d^\sigma(q) : e \mapsto \frac{\partial \overline{q}}{\partial z_e} T^{m_e}$$

where $z_e := x_e^{p^{m_e}}$ for each $e \in E$. Note that $\underline{d^\sigma(q)} = \overline{q} = q$, since by construction $\overline{q}$ is separable in each variable Z_e. Let

$$\mathcal{C}_x := \{\alpha \cdot d^\sigma(q) : q \text{ decorates a circuit } C \text{ of } N, \ \alpha \in (L^\sigma)^\star\}.$$

Lemma 14. Let $x \in L^E$. Then $\mathcal{C}_x$ is a left L^σ-signature of N.

Proof. We verify (C0), (C1), (C2) for $\mathcal{C}_x$. Clearly, (C0) and (C1) are true by construction. To see (C2), suppose $U, V \in \mathcal{C}_x$ are such that $\underline{U} \subseteq \underline{V}$. By definition of $\mathcal{C}_x$, we have $U = \alpha \cdot d^\sigma(q)$ and $V = \alpha' \cdot d^\sigma(q')$ where q decorates C and q' decorates C', so that $\underline{U} = C, \underline{V} = C'$ both are circuits of $M(K, x)$, and hence $\underline{U} = \underline{V}$. It follows that q and q' both decorate the same circuit C of $M(K, x)$. By Lemma 13, there is a $\beta \in K^\star$ so that $q' = \beta \cdot q$. Then

$$V = \alpha' \cdot d^\sigma(q') = \alpha' \cdot \beta \cdot d^\sigma(q) = \alpha' \cdot \beta \cdot \alpha^{-1} \cdot U,$$

as required. $\square$

On the dual side, for any K-derivation D of $K(x_E)^{sep}$ we define $D^\sigma(x) : E \to L^\sigma$ by setting

$$D^\sigma(x) : e \mapsto T^{m_e} D\left(x_e^{p^{-m_e}}\right),$$

where $m_e = \max\{m \in \mathbb{N} : x_e^{p^{-m_e}} \in K(x_E)^{sep}\}$. If C is a cocircuit of N, $H = E \setminus C$ is the complementary hyperplane, and D is a nonzero $K(x_H)$-derivation D of $K(x_E)^{sep}$, then $D(z) \neq 0$ for all $z \in K(x_E)$ such that $z^{p^{-1}} \notin K(x_E)$. Hence $\underline{D^\sigma} = H$. We define

$$\mathcal{D}_x := \{D^\sigma(x) \cdot \beta : D \text{ a } K(x_H)\text{-derivation of } K(x_E)^{sep}, \ D \neq 0, \ H \text{ hyperplane of } N, \ \beta \in (L^\sigma)^\star\}$$

Lemma 15. Let $x \in L^E$. Then $\mathcal{D}_x$ is a right L^σ-signature of N^*.

Proof. We verify (C0), (C1), (C2) for $\mathcal{D}_x$, noting that for a right signature we must reverse the order of multiplication in these axioms. As before, (C0) and (C1) are true by construction. We verify (C2). Let $U, V \in \mathcal{D}_x$ have $\underline{U} \subseteq \underline{V}$. Since both supports are cocircuits of N, we have $\underline{U} = C = \underline{V}$ for some cocircuit C of N, and with $H = E \setminus C$ there are nonzero $K(x_H)$-derivations D, D' of $K(x_E)^{sep}$ and $\beta, \beta' \in L^\sigma$ so that $U = D^\sigma(x) \cdot \beta$ and $V = (D')^\sigma(x) \cdot \beta'$. Since the set of $K(x_H)$-derivations of $K(x_E)^{sep}$ is a vector space of dimension 1, there is an $\alpha \in K(x_E)^{sep}$ so that $D' = D \cdot \alpha$. Then

$$V = (D')^\sigma(x) \cdot \beta' = D^\sigma(x) \cdot \alpha \cdot \beta' = U \cdot \beta^{-1} \cdot \alpha \cdot \beta',$$

as required. $\square$

Lemma 16. Let $x, y \in L^E$ and $n \in \mathbb{Z}^E$ be such that $y_e = x^{p^{n_e}}$ for all $e \in E$, and let $\rho : E \to L^\sigma$ be given by $\rho : e \mapsto T^{n_e}$. Then $\mathcal{C}_x = \mathcal{C}_y^\rho$ and $\mathcal{D}_y = \mathcal{D}_x^\rho$.

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Proof. It suffices to prove the lemma for $n = \mathbf{1}_{e_0}$, where $e \in E$ is some fixed element. So then $y_e = x_e^p$ if $e = e_0$ and $y_e = x_e$ otherwise. Denote $N := M(K, x) = M(K, y)$.

Consider a circuit C of N, and suppose $U \in \mathcal{C}_x$ has $\underline{U} = C$. Then $U = \alpha \cdot d^\sigma(q)$ for some $q \in K[X_C]$ decorating C and an $\alpha \in L^\sigma$. Let $m \in \mathbb{N}^E$ be such that $q = \overline{q}\left(X^{p^{m_e}}\right)$. There are two cases to consider. If $m_{e_0} > 1$, then q is a polynomial in $X_{e_0}^p$, and substituting X_e with $Y_e^{1/p}$ if $e = e_0$ and Y_e otherwise in q gives a polynomial $q' \in K[Y_C]$. This polynomial q' is irreducible, for any factorization of q' would induce a factorization of q. Hence q' decorates C in $M(K, y)$. By construction of q', we have $\overline{q'} = \overline{q} \in K[Z_E]$, and $q' = \overline{q}\left(Y^{p^{m'_e}}\right)$, where $m = m' + n$. Hence

$$U = \alpha \cdot d^\sigma(q) = \alpha \cdot \left(\frac{\partial \overline{q}}{\partial z_e} T^{m_e}\right)_e = \alpha \cdot \left(\frac{\partial \overline{q}}{\partial z_e} T^{m'_e} \cdot \rho_e\right)_e = \alpha \cdot (d^\sigma(q'))^\rho \in \mathcal{C}_y^\rho.$$

If $m_{e_0} = 0$, then construct $q' \in K[Y_C]$ from q^p by substituting X_e with $Y_e^{1/p}$ if $e = e_0$ and Y_e otherwise. Again, any factorization of q' would induce a factorization of q, and hence q' decorates C. This time $\overline{q'} = \sigma(\overline{q})$, and $q' = \sigma(\overline{q})\left(Y^{p^{m'_e}}\right)$, where $m + \mathbf{1}_E = m' + n$. Hence

$$U = \alpha \cdot d^\sigma(q) = \alpha \cdot \left(T^{-1} \cdot \frac{\partial \sigma(\overline{q})}{\partial z_e} T^{m_e+1}\right)_e = \alpha \cdot T^{-1} \cdot \left(\frac{\partial \overline{q'}}{\partial z_e} T^{m'_e} \cdot \rho_e\right)_e = \alpha \cdot T^{-1} \cdot (d^\sigma(q'))^\rho \in \mathcal{C}_y^\rho,$$

hence also $U \in \mathcal{C}_y^\rho$. It follows that $\mathcal{C}_x \subseteq \mathcal{C}_y^\rho$, so that $\mathcal{C}_x = \mathcal{C}_y^\rho$ since both are L^σ-signatures of N.

Consider a hyperplane H of N, and let $V \in \mathcal{D}_y$. Then there is a nonzero $K(y_H)$-derivation D of $K(y_E)^{sep}$ and a $\beta \in L^\sigma$ so that $V = D^\sigma(y) \cdot \beta$. By definition of $D^\sigma(y)$, there is an $m \in \mathbb{N}^E$ so that $y^{p^{-m_e}} \in K(y_E)^{sep}$ and $D^\sigma(y_e) = T^{m_e} D\left(y^{p^{-m_e}}\right)$ for each $e \in E$. Again, there are two cases. If $m_{e_0} > 0$, then

$$x_{e_0}^{p^{-m_{e_0}+1}} = y_{e_0}^{p^{-m_{e_0}}} \in K(y_E)^{sep}.$$

Then D is a derivation of $K(x_E)^{sep}$ as well, and $D^\sigma(x_e) = T^{m'_e} D\left(x^{p^{-m'_e}}\right)$ where $m' = m - n$. Hence

$$V = D^\sigma(y) \cdot \beta = \left(T^{m_e} D\left(y^{p^{-m_e}}\right)\right)_e \cdot \beta = \left(\rho_e T^{m'_e} D\left(x^{p^{-m'_e}}\right)\right)_e \cdot \beta = (D^\sigma(x) \cdot \beta)^\rho \in \mathcal{D}_x^\rho.$$

If $m_{e_0} = 0$, then $x_e \notin K(y_E)^{sep}$. Then $D' : z \mapsto D(z^p)^{(1/p)}$ is a derivation of $K(x_E)^{sep} \subseteq (K(y_E)^{sep})^{(1/p)}$, and taking $m' = m - n + \mathbf{1}_E$ we have $(D')^\sigma(x_e) = T^{m'_e} D'\left(x^{p^{-m'_e}}\right)$. Hence

$$V = D^\sigma(y) \cdot \beta = \left(T^{m_e} D\left(y^{p^{-m_e}}\right)\right)_e \cdot \beta = \left(\rho_e T^{m'_e} D'\left(x^{p^{-m'_e}}\right) \cdot T^{-1}\right)_e \cdot \beta = ((D')_x^\sigma \cdot T^{-1} \cdot \beta)^\rho \in \mathcal{D}_x^\rho.$$

and hence $V \in \mathcal{D}_x^\rho$. It follows that $\mathcal{D}_y \subseteq \mathcal{D}_x^\rho$, so that $\mathcal{D}_y = \mathcal{D}_x^\rho$ since both are L^σ-signatures of N^*. $\qquad\square$

Lemma 17. *Let $x \in L^E$. Then $\mathcal{C}_x \perp \mathcal{D}_x$.*

Proof. Using Lemma 6, it is equivalent to prove that $\mathcal{C}_x^\rho \perp \mathcal{D}_x^{\rho^{-1}}$. We will invoke Lemma 16 to simplify the argument.

Let $U \in \mathcal{C}_x$ and $V \in \mathcal{D}_x$. Then $U = \alpha \cdot d^\sigma(q)$ for some circuit C of $M(K, x)$ and $\alpha \in (L^\sigma)^\star$, and $V = D^\sigma(x) \cdot \beta$ for some $K(x_H)$-derivation D, where H is a hyperplane of $M(K, x)$ and $\beta \in (L^\sigma)^\star$. It is our object to prove that $U \perp V$, so that we may assume without loss of generality that $\alpha = \beta = 1$.

By Lemma 16, we may assume that $V \in L^E$, and writing $U_e = T^{m_e} a_e$ with $a_e \in L$, that

$$\min\{m_e : e \in \underline{U}\} = \min\{m_e : e \in \underline{U} \cap \underline{V}\} = 0.$$

Then $D^\sigma(x) = D(x)$, and $d^\sigma(q)_e = d(q)_e$ for all $e \in \underline{U}$ so that $m_e = 0$, so that

$$\sum_{e \in E} U_e V_e = \sum_{e \in \underline{U} \cap \underline{V}} d^\sigma(q)_e \cdot D^\sigma(x_e) = \sum_{e \in \underline{U} \cap \underline{V}, m_e = 0} d^\sigma(q)_e \cdot D^\sigma(x_e) = \sum_{e \in \underline{U} \cap \underline{V}} d(q)_e \cdot D(x_e) = 0,$$

as the hypersum of any elements of L^σ is determined by the terms cT^m with m minimal, and $d(q) \perp D(x)$. $\quad\square$

Theorem 7. *Let $K \subseteq L$ be a field extension in positive characteristic p, let E be a finite set, let $x \in L^E$, and assume that L is algebraically closed. Then $M := (E, \mathcal{C}_x)$ is a left L^σ-matroid, and $M^* = (E, \mathcal{D}_x)$.*

Proof. Let $N := M(K,x)$. By the lemma's of this section, $\mathcal{C}_x$ is a left L^σ-signature of N, $\mathcal{D}_x$ is a right L^σ-signature of N^*, and $\mathcal{C} \perp_3 \mathcal{D}$. Using Theorem 2, it follows that $M := (E, \mathcal{C}_x)$ is a left L^σ-matroid, and $M^* = (E, \mathcal{D}_x)$. $\qquad\square$

We call the left L^σ-matroid $M^\sigma(K,x) := (E, \mathcal{C}_x)$ the *matroid of σ-derivatives*, and its dual $(E, \mathcal{D}_x)$ the *matroid of σ-differentials*, since each element e of the ground set represents a differential $d(x_e)$. By construction, the matroid underlying $M^\sigma(K,x)$ is $M(K,x)$, but $M^\sigma(K,x)$ captures further information about K, x.

Recall the hyperfield homomophism $\zeta : L^\sigma \to \mathbb{Z}_{\min}$.

Lemma 18. *Let $K \subseteq L$ be a field extension characteristic $p > 0$, let $x \in L^E$ and assume that $L = K(x_E)$. Let $M = M^\sigma(K,x)$. Then*

$$\zeta([Fa, Fb]_M) = log_p \frac{[L : K(x_{Fb})^{sep}]}{[L : K(x_{Fa})^{sep}]}$$

for all bases Fa, Fb of M.

Proof. Let q be the polynomial decorating the circuit C so that $a, b \in C \subseteq Fab$. Suppose that $d^\sigma(q)_a = vT^i$ and $d^\sigma(q)_b = wT^j$. Then we have

$$p^i[L : K(x_{Fb})^{sep}] = [L : K(x_{Fab})^{sep}] = p^j[L : K(x_{Fa})^{sep}].$$

Also, $\zeta([Fa, Fb]_M) = -\zeta(vT^i) + \zeta(wT^j) = -i + j$. The lemma follows. $\qquad\square$

By a theorem of Cartwright [Car17], the Lindström valuation ν of K, x is determined by

$$\nu(B) = log_p[K(x_E) : K(x_B)^{sep}]$$

for each basis B of $M(K,x)$. It follows that ν is a Grassmann-Plucker function for $\zeta_* M^\sigma(K,x)$.

Lemma 19. *Let $K \subseteq L$ be a field extension characteristic $p > 0$, and let $x \in L^E$. Then $Der(K,x)$ is spanned by the cocircuits of $M^\sigma(K,x)_0$.*

5.4. **Matroids over $K(T, \sigma)$.** If K is a field of characteristic p and σ is the Frobenius map, then the elements the Ore ring $K[T, \sigma]$ naturally correspond to *P-polynomials*. Consider the map $K[T, \sigma] \to K[Z]$ given by

$$\widehat{\sum_j a_j T^j} = \sum_j a_j Z^{p^j}.$$

Then for any $a, b \in K[T, \sigma]$, we have $\widehat{(a+b)}(Z) = \widehat{a}(Z) + \widehat{b}(Z)$ and $\widehat{ab}(Z) = \widehat{a}(\widehat{b}(Z))$.

For the remainder of this section we consider a fixed field K, an extension field L of K and a transcendence base $z_1, \ldots, z_d$ of L over K. Given this context, there is a natural embedding $\psi : K(T, \sigma)^d \to L$, which sends vectors $v \in K(T, \sigma)^d$ to P-polynomials in L as follows:

$$\psi : v \mapsto \sum_{i=1}^d \widehat{v_i}(z_i).$$

Lemma 20 (Lindström[Lin88]). *Let $V \subseteq K(T, \sigma)^d$ be a finite set of vectors. Then V is left linearly dependent over $K(T, \sigma)$ if and only if $\{\psi(v) : v \in V\}$ is algebraically dependent over K.*

Let E be a finite set and let $v_e \in K[T, \sigma]^d$ for each $e \in E$. Let $M(v)$ be the left $K(T, \sigma)$-matroid which is linearly represented by the vectors v_e. With $x_e := \psi(v_e)$ for all $e \in E$, we have $M(K,x) = \underline{M(v)}$ by Lindströms lemma. We show that in this context, the matroid of σ-derivatives $M^\sigma(K,x)$ may also be constructed directly from $M(v)$. Recall the skew field homomorphism $\mu : K(T, \sigma) \to K^\sigma$ from section 4.2, which maps $\mu : \sum_i a_i T^i \mapsto a_k T^i$, where $k = \min\{i : a_i \neq 0\}$. Let $\mu' : K(T, \sigma) \to L^\sigma$ be given by $\mu'(a) = \mu(a)$.

Lemma 21. *Let $E \subseteq K[T, \sigma]^d$ be a finite set, and let $x_e := \psi(e)$ for all $e \in E$. Then $M^\sigma(K,x) = \mu'_* M(v)$.*

Proof. By Lindströms Lemma, we have $M(K,x) = \underline{M(v)}$, so that $M^\sigma(K,x)$ and $M(v)$ have the same underlying matroid. It therefore suffices to show that for each circuit U of $M(v)$, the vector $\mu'_* U = (\mu'(U_e))_e$ is a circuit of $M^\sigma(K,x)$.

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So consider a circuit $U \in K(T,\sigma)^d$ of $M(v)$. By definition, U is a left linear dependence $\sum_e U_e v_e = 0$, of minimal support. Assume first that $U \in K[T,\sigma]^E$. Then the entries U_e are formal polynomials in T, and we may define

$$q_U := \sum_{e \in \underline{U}} \widehat{U_e}(X_e) \in K[X_E].$$

Since U is a left linear dependence, we have $(\sum_e U_e v_e)_i = 0$ for $i = 1, \ldots, d$, and hence

$$q_U(x) = \sum_e \widehat{U_e}(x_e) = \sum_e \widehat{U_e}\left(\sum_i \widehat{(v_e)_i}\right) = \sum_e \sum_i \widehat{(U_e v_e)_i} = \sum_i \left(\widehat{\sum_e U_e v_e}\right)_i = 0.$$

Hence, the polynomial q_U decorates the circuit $\underline{U}$ of $M(K,x)$. We have $(d^\sigma q)_e = \mu'(\widehat{U_e})$ for each e, and hence $\mu'_* U = d^\sigma q$ is a circuit of $M^\sigma(K,x)$.

In case $U \notin K[T,\sigma]^E$, then there is a $c \in K[T,\sigma]$ so that $cU \in K[T,\sigma]^E$. Then $\mu_*(cU)$ is a circuit of $M^\sigma(K,x)$ and hence by the circuit axiom (C1), the vector $\mu'_* U = \mu'(c^{-1})\mu_*(cU)$ is a circuit of $M^\sigma(K,x)$. $\square$

Example. Consider the following vectors from $K(T,\sigma)^2$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ T^3 \end{bmatrix}, v_2 = \begin{bmatrix} T^2+T \\ T^2 \end{bmatrix}, v_4 = \begin{bmatrix} 1 \\ T^4 + aT \end{bmatrix}.$$

Each pair of these vectors is linearly independent over $K(T,\sigma)$, and hence $\underline{M(v)} \cong U_{2,4}$. From $x_e := \psi(v_e)$ we obtain

$$x_1 = z_1, \; x_2 = z_2^{p^3}, \; x_3 = z_1^{p^2} + z_1^p + z_2^{p^2}, \; x_4 = z_1 + z_2^{p^4} + a z_2^p.$$

where $L = K(z_1, z_2)$ has transcendence degree 2 over K. The circuit $U = (T^3 + T^2, 1, -T, 0)$ gives rise to an algebraic relation $q_U = X_1^{p^3} + X_1^{p^2} + X_2 - X_3^p$, so that $d^\sigma(q_U) = (T^2, 1, -T, 0) = \mu_* U$. The K-derivation $D = \frac{d}{dz_2}$ gives

$$D^\sigma(x) = (0, T^3, T^2, aT) = \mu_* V,$$

where $V = (0, T^3, T^2, T^4 + aT)$ is a cocircuit of $M(v)$.

In $M^\sigma(K,x)$, we have the cross ratio $cr(1,2,3,4) = [13,14] \cdot [24,23] = \left(-a^{-1}T\right) \cdot \left(-T^{-1}\right) = a^{-1} \in L^\sigma$.

6. Flocks

6.1. Preliminaries on matroid flocks.
In [BDP18], Bollen, Draisma, and the present author defined a *matroid flock of rank d on E* as a map M which assigns a matroid M_α on E of rank d to each $\alpha \in \mathbb{Z}^E$, satisfying the following two axioms.

(MF1) $M_\alpha/i = M_{\alpha+\mathbf{1}_e} \setminus i$ for all $\alpha \in \mathbb{Z}^E$ and $e \in E$.
(MF2) $M_\alpha = M_{\alpha+\mathbf{1}_E}$ for all $\alpha \in \mathbb{Z}^E$.

Here, $\mathbf{1}_e$ denotes the unit vector in $\mathbb{R}^E$ with a 1 in the e-th position, and $\mathbf{1}_E$ the all-one vector in $\mathbb{R}^E$. More generally we write $\mathbf{1}_F := \sum_{e \in F} \mathbf{1}_e$ for the incidence vector of any $F \subseteq E$.

Matroid flocks are cryptomorphic to *valuated matroids*. Using the definition of the boundary matroid from Section 4.3, and noting that valuated matroids are essentially $\mathbb{Z}_{\min}$-matroids, we will now paraphrase this characterization, Theorem 7 of [BDP18]. Let $\mathbb{M}(E,r)$ denote the collection of matroids of rank r on E.

Theorem 8. *Let* $\mathcal{M} : \mathbb{Z}^E \to \mathbb{M}(E,r)$. *The following are equivalent:*
  (1) $\mathcal{M}$ *is a matroid flock.*
  (2) *there is a* $\mathbb{Z}_{\min}$*-matroid* N *so that* $\mathcal{M} : \alpha \mapsto (N^{-\alpha})_0$.

In what follows, we generalize this theorem to one that characterizes H^σ-matroids in terms of H^σ-flocks. In the proof, we will use one further lemma from [BDP18]. For any $\mathbb{R}_{\min}$-matroid N on E, let

$$C(N, \beta) := \left\{ \alpha \in \mathbb{R}^E : \left(N^{-\alpha}\right)_0 \succeq \left(N^{-\beta}\right)_0 \right\}.$$

We will regard any $\mathbb{Z}_{\min}$-matroid as an $\mathbb{R}_{\min}$-matroid in the natural way.

Lemma 22. *Let* N *be a* $\mathbb{Z}_{\min}$*-matroid on* E *with valuation* ν, *and let* $\beta \in \mathbb{Z}^E$. *Then*

$$C(N, \beta) = \left\{ \alpha \in \mathbb{R}^E : \alpha_e - \alpha_f \geq \nu(B) - \nu(B - e + f) \text{ for all bases } B \text{ of } \left(N^{-\beta}\right)_0, e \in B, f \in E \setminus B \right\}.$$

6.2. H^σ-**flocks and matroids over** H^σ. Let H be a skew hyperfield, let $r \in \mathbb{N}$, and let E be a finite set. Let $\mathbb{M}_H(E, r)$ denote the collection of left H-matroids of rank r on E. Consider an automorphism σ of H. An H^σ-*flock of rank r on E* is a map $\mathcal{F} : \mathbb{Z}^E \to \mathbb{M}_H(E, r)$, with the following properties:

(F1) $\mathcal{F}_{\alpha + \mathbf{1}_e} \setminus e = \mathcal{F}_\alpha / e$ for all $\alpha \in \mathbb{Z}^E$ and $e \in E$.
(F2) $\mathcal{F}_{\alpha + \mathbf{1}_E} = \sigma_* \mathcal{F}_\alpha$ for all $\alpha \in \mathbb{Z}^E$.

We generalize Theorem 8, which characterizes $\mathbb{K}^{id}$-flocks (matroid flocks) as cryptomorphic to $\mathbb{K}^{id}$-matroids ($\mathbb{Z}_{\min}$-matroids). In the proof of this generalization, we will use Theorem 8 itself as a stepping stone. Recall that $\tau : \mathbb{Z}_{\min} \to H^\sigma$ is the homomorphism $\tau : i \mapsto T^i$.

Theorem 9. *Let $\mathcal{F} : \mathbb{Z}^E \to \mathbb{M}_H(E, r)$. The following are equivalent:*

(1) $\mathcal{F}$ *is an H^σ-flock.*
(2) *there is a left H^σ-matroid M so that $\mathcal{F} : \alpha \mapsto \left(M^{\tau(-\alpha)}\right)_0$.*

Proof. (2)$\Rightarrow$(1): Assume (2). Let $N := \zeta_* M$. Then N is a $\mathbb{Z}_{\min}$-matroid, and therefore by Theorem 8, the map

$$\underline{\mathcal{F}} : \alpha \mapsto \left(N^{-\alpha}\right)_0 = \left(M^{\tau(-\alpha)}\right)_0$$

is a matroid flock. We verify the two H^σ-flock axioms (F1) and (F2).

(F1): Without loss of generality, $\alpha = 0$. We have $\mathcal{F}_0 = M_0$, and $\mathcal{F}_{\mathbf{1}_e} = (M^\rho)_0$, where $\rho = \tau(-\mathbf{1}_e)$. As $\underline{\mathcal{F}}$ is a matroid flock, we have

$$\underline{\mathcal{F}_0} \setminus e = \underline{M_0} \setminus e = \underline{(M^\rho)_0}/e = \underline{\mathcal{F}_{\mathbf{1}_e}}/e.$$

To show more strongly that $\mathcal{F}_0 \setminus e = \mathcal{F}_{\mathbf{1}_e}/e$, it remains to show that also $[.]_{\mathcal{F}_0 \setminus e} = [.]_{M_0 \setminus e} = [.]_{(M^\rho)_0/e} = [.]_{\mathcal{F}_{\mathbf{1}_e}/e}$. If e is not a coloop of M_0, then for each $(B, B') \in A_{M_0 \setminus e}$ we have

$$[B, B']_{M_0 \setminus e} = [B, B']_M = [B, B']_{M^\rho} = [B, B']_{(M^\rho)_0/e}.$$

If e is a coloop of M_0, then $M_0 \setminus e = M_0/e$, and for each $(B, B') \in A_{M_0 \setminus e}$ we have

$$[B, B']_{M_0 \setminus e} = [B + e, B' + e]_M = [B + e, B' + e]_{M^\rho} = [B, B']_{(M^\rho)_0/e}.$$

In either case, $[.]_{M_0} = [.]_{(M^\rho)_0}$, so that $\mathcal{F}_0 = M_0 = (M^\rho)_0 = \mathcal{F}_{\mathbf{1}_e}$, as required.

(F2): Without loss of generality $\alpha = 0$. Then $\mathcal{F}_0 = M$, and $\mathcal{F}_{\mathbf{1}_E} = (M^\rho)_0$, where $\rho := \tau(-\mathbf{1}_E) : e \mapsto T^{-1}$. For each $(B, B') \in A_{M_0}$, we have

$$[B, B']_{M^\rho} = T[B, B']_M T^{-1} = \sigma([B, B']_M).$$

If X is a circuit of M, then $T(X_e T^{-1} : e \in E) = (\sigma(X_e) : e \in E) = \sigma(X)$ is a circuit of M^ρ. Hence

$$\mathcal{F}_{\mathbf{1}_E} = (M^\rho)_0 = \sigma_*(M_0) = \sigma_* \mathcal{F}_0,$$

as required.

(1)$\Rightarrow$(2): Suppose (1). Then $\underline{\mathcal{F}} : \alpha \mapsto \underline{\mathcal{F}_\alpha}$ is a matroid flock. Hence by Theorem 8, there is a $\mathbb{Z}_{\min}$-matroid N so that

$$\underline{\mathcal{F}_\alpha} = \left(N^{-\alpha}\right)_0$$

If $M = (E, \mathcal{C})$ is a left H^σ-matroid so that $\mathcal{F}_\alpha = \left(M^{\tau(-\alpha)}\right)_0$, then the left quasi-Plucker coordinates $[.] = [.]_\mathcal{C}$ are a map $[.] : A_N \to H^\sigma$ so that $[.]_{\mathcal{F}_\alpha} = [.]^{\tau(-\alpha)}$ for all $\alpha \in \mathbb{Z}^E$. That is, for each $\alpha \in \mathbb{Z}^E$

$$(2) \qquad\qquad [Fa, Fb]_{\mathcal{F}_\alpha} = T^{\alpha_a}[Fa, Fb]T^{-\alpha_b}$$

whenever Fa, Fb are adjacent bases of $\underline{\mathcal{F}_\alpha}$. Conversely, if $[.]$ are left quasi-Plucker coordinates for N satisfying these requirements, then $M := (E, \mathcal{C}_{[.]})$ satisfies (2): then $\mathcal{F}_\alpha = \left(M^{\tau(-\alpha)}\right)_0$, as on either side of the equation, the matroids have the same underlying matroid and the same quasi-Plucker coordinates.

We first prove the existence of such a map $[.] : A_N \to H^\sigma$, satisfying (2) for each α. So fix adjacent bases Fa, Fb of N. We must argue that for each two $\alpha, \beta \in \mathbb{Z}^E$ so that Fa, Fb are both bases of $\mathcal{F}_\alpha$ and $\mathcal{F}_\beta$, we have

$$(3) \qquad\qquad T^{-\alpha_a}[Fa, Fb]_{\mathcal{F}_\alpha} T^{\alpha_b} = T^{-\beta_a}[Fa, Fb]_{\mathcal{F}_\beta} T^{\alpha_b}.$$

By (F2), we may assume that $\alpha \leq \beta$. We prove (3) by induction on $\sum_e (\beta_e - \alpha_e)$. Let $e \in E$ be such that $\alpha_e < \beta_e$. If $e \in F$, then with $F' := F - e$, $F'a, F'b$ are adjacent bases of $\mathcal{F}_\alpha / e = \mathcal{F}_{\alpha + \mathbf{1}_e} \setminus e$, and hence

$$[Fa, Fb]_{\mathcal{F}_\alpha} = [F'a, F'b]_{\mathcal{F}_\alpha / e} = [F'a, F'b]_{\mathcal{F}_{\alpha + \mathbf{1}_e} \setminus e} = [Fa, Fb]_{\mathcal{F}_\beta}.$$

Taking $\alpha' = \alpha + \mathbf{1}_e$ and using the induction hypothesis on the pair α', β, we obtain

$$T^{-\alpha_a}[Fa, Fb]_{\mathcal{F}_\alpha} T^{\alpha_b} = T^{-\alpha'_a}[Fa, Fb]_{\mathcal{F}_{\alpha'}} T^{\alpha'_b} = T^{-\beta_a}[Fa, Fb]_{\mathcal{F}_\beta} T^{\beta_b}.$$

If $e \notin Fab$, then Fa, Fb are both bases of $\mathcal{F}_\beta \setminus e = \mathcal{F}_{\beta - \mathbf{1}_e} / e$, and hence

$$[Fa, Fb]_{\mathcal{F}_{\beta - \mathbf{1}_e}} = [Fa, Fb]_{\mathcal{F}_{\beta - \mathbf{1}_e} / e} = [Fa, Fb]_{\mathcal{F}_\beta \setminus e} = [Fa, Fb]_{\mathcal{F}_\beta}.$$

Taking $\beta' = \beta - \mathbf{1}_e$ and again using induction, we have

$$T^{-\alpha_a}[Fa, Fb]_{\mathcal{F}_\alpha} T^{\alpha_b} = T^{-\beta'_a}[Fa, Fb]_{\mathcal{F}_{\beta'}} T^{\beta'_b} = T^{-\beta_a}[Fa, Fb]_{\mathcal{F}_\beta} T^{\beta_b}.$$

Thus we have reduced to the case when $\alpha_e = \beta_e$ for all e other than a, b. By Lemma 22, we have

$$C(N, \beta) = \left\{ \alpha \in \mathbb{R}^E : \alpha_i - \alpha_j \geq \nu(B) - \nu(B - e + f) \text{ for all bases } B \text{ of } \left(N^{-\beta}\right)_0, e \in B, f \in E \setminus B \right\}.$$

Since Fa, Fb are bases of of both $\mathcal{F}_\alpha$ and $\mathcal{F}_\beta$, it follows that $\alpha_a - \alpha_b \geq \nu(B) - \nu(B - e + f) = \beta_a - \beta_b$. Reversing α and β in this argument, we also have $\beta_a - \beta_b \geq \nu(B) - \nu(B - e + f) = \alpha_a - \alpha_b$, so that $\alpha_a - \beta_a = \alpha_b - \beta_b$. It follows that $\beta - \alpha = k\mathbf{1}_{ab}$. Consider the special case that $\beta - \alpha = \mathbf{1}_{ab}$, and let $G := E \setminus ab$. We have

$$\mathcal{F}_\alpha / G = \mathcal{F}_{\alpha - \mathbf{1}_G} \setminus G = \mathcal{F}_{\alpha + \mathbf{1}_{ab} - \mathbf{1}_E} \setminus G = \sigma_* \mathcal{F}_{\alpha + \mathbf{1}_{ab}} \setminus G.$$

Then

$$T^{-\alpha_a}[Fa, Fb]_{\mathcal{F}_\alpha} T^{\alpha_b} = T^{-\alpha_a}[a, b]_{\mathcal{F}_\alpha / G} T^{\alpha_b} = T^{-\beta_a}[a, b]_{\mathcal{F}_{\alpha + \mathbf{1}_{ab}} \setminus G} T^{\beta_b} = T^{-\beta_a}[Fa, Fb]_{\mathcal{F}_\beta} T^{\beta_b}.$$

In general if $\beta - \alpha = k\mathbf{1}_{ab}$ with $k > 1$, then $\alpha' := \alpha + \mathbf{1}_{ab} \leq \beta$ and $\alpha' \in C(N, \beta)$, so that Fa, Fb are bases of $\mathcal{F}_{\alpha'}$. The general case then follows by induction on k:

$$T^{-\alpha_a}[Fa, Fb]_{\mathcal{F}_\alpha} T^{\alpha_b} = T^{-\alpha'_a}[Fa, Fb]_{\mathcal{F}_{\alpha'}} T^{\alpha'_b} = T^{-\beta_a}[Fa, Fb]_{\mathcal{F}_\beta} T^{\beta_b}.$$

We have established that there exists a map $[.] : A_N \to H^\sigma$, satisfying (2) for each α.

Next, we show that $[.]$ are left quasi-Plucker coordinates. Consider (P3), say. Suppose Fac, Fad, Fbc, Fbd are bases of N, but Fab or Fcd are not. Then there exists an $\alpha \in \mathbb{Z}^E$ so that Fac, Fad, Fbc, Fbd are bases of $\underline{\mathcal{F}}_\alpha$. By (P3) for $\mathcal{F}_\alpha$, we have

$$[Fac, Fbc] = T^{-\alpha_a}[Fac, Fbc]_{\mathcal{F}_\alpha} T^{\alpha_b} = T^{-\alpha_a}[Fad, Fbd]_{\mathcal{F}_\alpha} T^{\alpha_b} = [Fad, Fbd].$$

To show (P0), (P1), (P2) it similarly suffices to that all bases in question are present in $\mathcal{F}_\alpha$ for some $\alpha \in \mathbb{Z}^E$.

To show (P4), consider F, a, b, c, d so that $\mathcal{B}' := \{Fac, Fad, Fbc, Fbd, Fab, Fcd\}$ are all bases of N. We need to show that

$$1 \in [Fbd, Fab] \cdot [Fac, Fcd] \boxplus [Fad, Fab] \cdot [Fbc, Fcd].$$

Let ν be the valuation associated with N, so $\zeta([B, B']) = \nu(B) - \nu(B')$ for all adjacent bases B, B' of N. By Theorem 4, we have $\infty \in (\nu(Fab) + \nu(Fcd)) \boxplus (\nu(Fac) + \nu(Fbd)) \boxplus (\nu(Fad) + \nu(Fbc))$ in $\mathbb{Z}_{\min}$. That is, the minimum of the three numbers

$$\nu(Fab) + \nu(Fcd), \nu(Fac) + \nu(Fbd), \nu(Fad) + \nu(Fbc)$$

is attained at least 2 times. There are four cases to consider. If $\nu(Fab) + \nu(Fcd) = \nu(Fac) + \nu(Fbd) = \nu(Fad) + \nu(Fbc)$, then there exists an α so that $\mathcal{B}' \subseteq \mathcal{F}_\alpha$, and then (P4) holds as it holds in $\mathcal{F}_\alpha$. If $\nu(Fab) + \nu(Fcd) = \nu(Fac) + \nu(Fbd) < \nu(Fad) + \nu(Fbc)$, then $[Fbd, Fab] \cdot [Fac, Fcd] = 1$ as there exists an $\alpha \in \mathbb{Z}^E$ so that Fbd, Fab, Fac, Fcd are bases of $\mathcal{F}_\alpha$, and Fad or Fbc are not. Also,

$$\zeta([Fbd, Fab] \cdot [Fac, Fcd]) = \nu(Fbd) - \nu(Fab) + \nu(Fac) - \nu(Fcd) = 0$$

and

$$\zeta([Fad, Fab] \cdot [Fbc, Fcd]) = \nu(Fad) - \nu(Fab) + \nu(Fbc) - \nu(Fcd) > 0$$

so that

$$1 = [Fbd, Fab] \cdot [Fac, Fcd] \in [Fbd, Fab] \cdot [Fac, Fcd] \boxplus [Fad, Fab] \cdot [Fbc, Fcd].$$

The case when $\nu(Fab) + \nu(Fcd) = \nu(Fad) + \nu(Fbc) < \nu(Fac) + \nu(Fbd)$ is similar.

If $\nu(Fab) + \nu(Fcd) > \nu(Fac) + \nu(bd) = \nu(ad) + \nu(bc)$, then $[Fac, Fad] = [Fbc, Fbd]$ and $[Fac, Fbc] = [Fad, Fbd]$ as before. Then $[Fbd, Fab] \cdot [Fac, Fcd] = -[Fad, Fab] \cdot [Fbc, Fcd]$, and $\zeta([Fbd, Fab] \cdot [Fac, Fcd]) = \zeta([Fad, Fab] \cdot [Fbc, Fcd]) < 0$, so that

$$1 \in [Fbd, Fab] \cdot [Fac, Fcd] \boxplus [Fad, Fab] \cdot [Fbc, Fcd],$$

as required. $\qquad\qquad\square$

7. Final words

7.1. Cross ratios.
In Section 3.9, we defined cross ratios and listed several properties. It is not clear to what extent these properties define matroids over skew hyperfields. There may not be coordinates which correspond with given cross ratios even if $N = U_{2,4}$ and H is commutative, but we think this might be the only obstacle, in the following sense.

Let N be a matroid. We say that a map $cr : CR_N \to H$ is *consistent* if there exists quasi-Plucker coordinates [.] for N such that $cr(F, a, b, c, d) = [Fac, Fad][Fbd, Fbc]$.

Conjecture 1. *Let N be a matroid, and let H be a skew hyperfield such that $1 = -1$ if N has a Fano minor. Let $cr : C_N \to H$ satisfy (CR0), (CR1), (CR2) and (CR3). The following are equivalent:*
  (1) *cr is consistent; and*
  (2) *the restriction of cr to $C_{N'}$ is consistent for each $U_{2,4}$-minor N' of N.*

The special case of this conjecture where $H = \mathbb{S}$ is a theorem of Gelfand, Rybnikov, and Stone [GRS95], and if H is commutative the conjecture follows from the work of Delucchi, Hoessly, Saini [DHS18].

7.2. The skew hyperfield of monomials.
If H is a field and σ is the identity, then $H(T, \sigma, \max)$ is commutative and equals the *hyperfield of monomials* described by Viro in [Vir10]. Viro notes that the role of $\mathbb{Z}$ in his definition can be replaced by any linearly ordered group $(\Gamma, \cdot, <)$. This seems to apply also to our construction. Consider a skew hyperfield H, and automorphism σ_i of H for each $i \in \Gamma$ so that $\sigma_{i \cdot j} = \sigma_i \circ \sigma_j$ for all $i, j \in \Gamma$. Then we can define a hyperfield

$$H \rtimes_\sigma \Gamma_{\max} := (H^\star \times \Gamma \cup \{0\}, 1, 0, \odot, \boxplus)$$

with $1 := (1, 1)$, multiplication given by $0 \odot x = x \odot 0 = 0$ and $(a, i) \odot (b, j) := (a\sigma_i(b), i \cdot j)$ for all $a, b \in H^\star$ and $i, j \in \Gamma$, and addition given by $0 \boxplus x = x \boxplus 0 = 0$ and

$$(a, i) \boxplus (b, j) := \begin{cases} \{(a, i)\} & \text{if } i > j \\ \{(b, j)\} & \text{if } i < j \\ (a + b) \times \{i\} & \text{if } i = j \text{ and } a \neq -b \\ (a + b) \times \{i\} \cup H^\star \times \{k \in \Gamma, k < i\} & \text{if } i = j \text{ and } a = -b \end{cases}$$

where $+$ is the hyperaddition of H.

This skew hyperfield resembles the *extended tropical hyperring* of [AGG14], but it is different when adding $(a, i) \boxplus (b, j)$ in the case that $i = j$ and $a \neq -b$. With trivial automorphisms $\sigma_i = id$ we have $\mathcal{T}(\mathbb{R}) = \mathbb{S} \rtimes \mathbb{R}_{\max}$ and $\mathcal{T}(\mathbb{C}) = \Phi \rtimes \mathbb{R}_{\max}$. Here $\mathcal{T}(\mathbb{R})$ and $\mathcal{T}(\mathbb{C})$ are Viro's tropical reals and tropical complex numbers, and Φ is the tropical phase hyperfield.

7.3. Groebner bases in positive characteristic.
In Section 5, we considered a field K of positive characteristic p, an extension field L and elements $x \in L$ for $e \in E$. The results of this section highlight that $K(x_E)$ has a certain robustness against applications of the Frobenius map $\sigma : x \mapsto x^p$ to the individual elements x_e. If $y_e = x_e^{p^{m_e}}$, then for any irreducible $q \in K[X_E]$ so that $q(x) = 0$ there is an irreducible $q' \in K[Y_e]$ so that $q'(y) = 0$, and $q^{p^n}(X_e : e \in E) = q'(X^{p^{m_e}} : e \in E)$. That is, irrespective of such Frobenius actions, the irreducible polynomial relations are always just a variation of the same polynomial $\overline{q} \in K[Z_E]$.

In the light of this invariance, it seems inappropriate that of a Groebner basis would change more than superficially when substituting a variable X by X^p, or that the steps taken by the Buchberger algorithm would turn out truly different. We imagine a variant which is indifferent to such changes.

To make the Buchberger algorithm ignore substitutions such as the above, we may no longer distinguish between a polynomial q and its power q^{p^k}. The monomial order $\preceq$ on $\mathbb{N}^E$ must ignore powers of p. That is, for any $u, v \in \mathbb{N}^E$ we must have

$$u \preceq v \text{ if and only if } u' \preceq v'$$

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where $u_e = u'_e p^{val_p(u_e)}$, $v_e = v' p^{val_p(v_e)}$ for each $e \in E$. The monomial order could otherwise be lexicographic, based on a linear order $<$ of E. When using q with leading monomial X^u to reduce r with leading monomial X^v, we must first replace q with a p^k-th power to ensure that $val_p(u_e) = val_p(v_e)$, where $e = \max\{f \in E : u_e \neq 0, v_e \neq 0\}$. Here the maximum is taken with respect to the chosen order of E.

We are not aware of any such variant of the Buchberger algorithm in the literature, but we think this could be the more efficient way to decide independence of sets in algebraic matroids.

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(Rudi Pendavingh) Department of Mathematics and Computer Science, Eindhoven University of Technology, P.O. Box 513, 5600 MB Eindhoven, The Netherlands

E-mail address: r.a.pendavingh@tue.nl