

The geometry of hyperbolic lines in polar spaces

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The geometry of hyperbolic lines in polar spaces

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Abstract

In this paper we consider partial linear spaces induced on the point set of a polar space, but with as lines the hyperbolic lines of this polar space. We give some geometric characterizations of these and related spaces.

The results have applications in group theory, in the theory of Lie algebras and in graph theory.

1 Introduction

Classical polar spaces arise from symplectic, hermitian and pseudo-quadratic forms on a vector space V . They consist of those points and lines of the corresponding projective space $\mathbb{P}(V)$ on which the form vanishes. Actually, the results of Tits [32], Veldkamp [33], Buekenhout-Shult [5] and Johnson [25] imply that under some weak additional conditions (on non-degeneracy and rank) all these polar spaces can be characterized by the famous ‘one-or-all axiom’; see (2.1). In this paper we provide a characterization of the geometries that have the same point set as a polar space, but whose lines are the so-called *hyperbolic* lines of the polar space. Our results are inspired by and can be viewed as a geometric counterpart of Timmesfeld’s theory of transvection groups [30, 31]. Before stating our results, we introduce some notation and terminology.

A *partial linear space* is a pair (P, L) consisting of a set P of *points* and a set L of subsets of P of size at least 2 called *lines*, such that two points are in at most one line. A partial linear space is called *thick* if all lines contain at least 3 points. If two points p and q are on a line, then they are called *collinear*. This is denoted by $p \sim q$, while the unique line containing them is denoted by pq . By $p \perp q$ we denote that the points p and q are equal or non-collinear. (As we are interested in the geometry

of hyperbolic lines, this notation differs from the common notation that \perp denotes collinearity.) By definition, a point p is not collinear to itself. The set p^\perp , respectively p^\sim , denotes the set of all points non-collinear, respectively collinear to p . By (P, \perp) and (P, \sim) we denote the graphs with point set P and two points x, y being adjacent if and only if $x \perp y$ or $x \sim y$, respectively. The space Π is called *connected* if (P, \sim) is connected and *co-connected* if (P, \perp) is connected.

A *subspace* X of Π is a subset of P with the property that any line intersecting it in at least two points is completely contained in X . Since the intersection of subspaces is again a subspace, we can, for each subset Y of P , define the subspace $\langle Y \rangle$ *generated by* Y to be the intersection of all subspaces containing Y . A *plane* is a subspace generated by two intersecting lines.

A subspace X is called *transversal* if for each line $\ell \subseteq X$ we have that X is partitioned by the sets $x^\perp \cap X$ for $x \in \ell$. The subsets $x^\perp \cap X$ are called the *transversals* of X . The following result is the main theorem of this paper.

1.1. Main Theorem. *Let $\Pi = (P, L)$ be a connected and co-connected partial linear space satisfying the following:*

- (a) *all lines in L contain at least 4 points;*
- (b) *if x and y are points with $x^\perp \subseteq y^\perp$ then $x = y$;*
- (c) *every triple of points x, y and z with $x \sim y \sim z$ and $x \perp z$ generates a transversal plane;*
- (d) *for each point $x \in P$ and transversal plane π we have $x^\perp \cap \pi$ is empty, a point, a line, or a transversal of π .*

Then (P, \perp) is a non-degenerate polar graph of rank at least 2.

Moreover, if

- (e) *for all $h \in L$ we have $h = h^{\perp\perp}$,*

then (P, L) is the geometry of points and hyperbolic lines of the polar graph (P, \perp) .

In the geometry of points and hyperbolic lines of classical polar spaces defined over fields (see (2.7)) any two intersecting hyperbolic lines are contained in a subspace forming either a dual affine plane or a classical unital. The following result characterizes the geometry of hyperbolic lines of these spaces by this property. To state the result we need the following definition. A partial linear space $\Pi = (P, L)$ is called *planar*, if and only if every pair of intersecting lines is contained in a unique plane.

1.2. Theorem. *Let $\Pi = (P, L)$ be a planar, connected and co-connected partial linear space. Suppose the following hold in Π :*

- (a) *planes are either isomorphic to a dual affine plane or are linear;*

- (b) *in a linear plane no 4 lines intersect in 6 points;*
- (c) *for all points x and y we have that $x^\perp \subseteq y^\perp$ implies $x = y$;*
- (d) *for all $l \in L$ we have $|l| \geq 4$ and $l^{\perp\perp} = l$.*

Then Π is isomorphic to the geometry of hyperbolic lines in a non-degenerate polar space of rank at least 2.

The above theorem generalizes the characterizations of the geometry of hyperbolic lines in symplectic spaces as given by Hall [22] and Cuypers [9] to the geometry of hyperbolic lines in arbitrary polar spaces.

Notice that the classical unitals do satisfy condition (b) of the above theorem. This was already observed by O’Nan for finite fields (see [27]), but also holds true in the infinite case; see (2.7). The restriction that lines contain at least 4 points is necessary, as the Fischer spaces of 3-transposition groups with trivial O_2 do also satisfy the other conditions, see [4, 12, 11, 18]. In particular, the Fischer spaces of the sporadic groups Fi_{22} , Fi_{23} and Fi_{24} do satisfy the conditions (a)-(c) of the theorem but have 3 points per line.

Condition (b) is also necessary. Indeed, the generalized Fischer spaces of non-isotropic points and tangent lines in a unitary geometry over \mathbb{F}_4 are geometries satisfying (a), (c) and (d), but not (b); see [10, 14].

Combining the above results with [18, Section 4] we obtain the following result.

1.3. Theorem. *Let $\Pi = (P, L)$ be a planar and connected partial linear space of finite order $q \geq 2$. Suppose the following hold in Π :*

- (a) *all planes are finite and either isomorphic to a dual affine plane or a classical unital;*
- (b) *there exists a pair of non-collinear points;*
- (c) *for all $x, y \in P$ we have that $x^\perp \subseteq y^\perp$ or $x^\sim \cup \{x\} \subseteq y^\sim \cup \{y\}$ implies $x = y$;*
- (d) *if π is a linear plane of Π and x a point, then $x^\perp \cap \pi \neq \emptyset$.*

Then we have one of the following:

- (a) *q is a prime power and Π is isomorphic to the geometry of hyperbolic lines in a non-degenerate symplectic or unitary polar space over the field \mathbb{F}_q or \mathbb{F}_{q^2} , respectively.*
- (b) *Π is a symmetric Fischer space.*
- (c) *Π is an orthogonal Fischer space over \mathbb{F}_2 .*

(d) Π is an orthogonal Fischer space over \mathbb{F}_3 of dimension at most 7.

(e) Π is a sporadic Fischer space related to $\Omega^+(8, 2) : S_3$ or Fi_{22} .

The remainder of this paper is organized as follows. In the next section we discuss various examples appearing in the theorems. Then in Section 3 we start with the proof of Theorem 1.1. Here we prove Reduction Theorem 3.12 in which we show how a geometry satisfying some weaker conditions than as in the hypothesis of Theorem 1.1 reduces to a geometry satisfying the hypothesis of (1.1). The proof of Theorem 1.1 is then finished in Section 4. Section 5 is devoted to a proof of Theorem 1.2, while Theorem 1.3 will be proven in Section 6.

Some of the results of this paper have been around for several years and have been available in preprint-form. Since over the years Theorem 1.1 has found applications in group theory [16], graph theory [1, 21] and in the theory of Lie algebras [7, 24], we have decided to publish the results. In the final section of this paper we briefly discuss these applications of our results.

2 Hyperbolic lines in classical polar spaces

In this section we discuss some basic results on polar spaces and their hyperbolic lines and describe various examples of geometries occurring in the theorems from the first section.

2.1. Definitions. A *partial linear space* $\mathcal{S} = (\mathcal{P}, \mathcal{L})$ is called a *polar space* if it satisfies the so-called one-or-all axiom:

For each point p and line l of \mathcal{S} with $p \notin l$, the point p is collinear with either one or all points of l .

The *radical* $\text{Rad}(\mathcal{S})$ of a polar space \mathcal{S} is the set of points which are collinear to all points. A polar space is *non-degenerate*, if its radical is empty. A non-degenerate polar space in which a point p is either on a line l or just collinear with a unique point of l is also called a *generalized quadrangle*.

If $\mathcal{S} = (\mathcal{P}, \mathcal{L})$ is a polar space and $p \in \mathcal{P}$, then by p^\perp we denote the set of all points equal or collinear to p . If $p, q \in \mathcal{P}$ then $p \perp q$ denotes that $q \in p^\perp$. (Notice the difference with the definition of \perp as given in the introduction!) If X is a subset of \mathcal{P} , then $X^\perp = \bigcap_{x \in X} x^\perp$.

The following two results are well known to experts in the field, but for the sake of completeness we include them in this paper.

2.2. *If $\mathcal{S} = (\mathcal{P}, \mathcal{L})$ is a non-degenerate polar space and $x, y \in \mathcal{P}$ are points with $x^\perp \subseteq y^\perp$, then $x = y$.*

Proof. Suppose $x \neq y$ are points with $x^\perp \subseteq y^\perp$. Let u be a point not collinear with y . Then u is collinear to a point z on the line xy through x and y . Let v be a point not collinear to z . Then v is collinear to a point w on xy different from x, y and z . The point u is collinear to a point on vw , which we may assume to be v . Now y^\perp meets uv in a point r . This point is then also collinear with x and hence with z and w . But u is the only point on uv collinear with z and v the only point collinear with w and we find $u = r = v$ contradicting $u \neq v$. \square

A *hyperbolic line* of a polar space \mathcal{S} is a set of points of the form $\{x, y\}^{\perp\perp} := (\{x, y\}^\perp)^\perp$ where x, y are two non-collinear points.

2.3. *Suppose that \mathcal{S} is a thick non-degenerate polar space. Then for any two points z_1, z_2 of a hyperbolic line H we have $z_1 \not\perp z_2$ and $H = \{z_1, z_2\}^{\perp\perp}$.*

Proof. Suppose $x, y \in \mathcal{P}$ are non-collinear points and $H = \{x, y\}^{\perp\perp}$ is the hyperbolic line on x and y . Let $z \in H$ be different from x and y . Then we have $x^\perp \cap y^\perp \subseteq z^\perp$.

First we prove that $x \not\perp z$. Assume this is not true, so z is collinear to x . Every line ℓ on x contains a point $x' \in y^\perp$ different from x , which is then also in z^\perp . So the line ℓ contains two points collinear to z and hence is contained in z^\perp . But this implies that $x^\perp \subseteq z^\perp$, contradicting \mathcal{S} to be non-degenerate; see (2.2). So, indeed $x \not\perp z$.

As $x^\perp \cap y^\perp \subseteq z^\perp$ we have $x^\perp \cap y^\perp \subseteq x^\perp \cap z^\perp$. We will now prove these sets to be equal. Suppose the contrary and let $u \in x^\perp \cap z^\perp$ but not in y^\perp . Since x and z are not collinear, u is different from x . The line on x and u contains a point $v \neq u$ in y^\perp . But then $v \in x^\perp \cap y^\perp$ and hence also in z^\perp . But now we find that the points u and v and hence also x , which is on the line uv , are in z^\perp . So indeed, $x^\perp \cap y^\perp = x^\perp \cap z^\perp$, which implies that H equals $\{x, z\}^{\perp\perp}$, which is the hyperbolic line on x and z .

Now suppose z_1, z_2 are two points in H , then applying the above twice we find $H = \{x, z_1\}^{\perp\perp} = \{z_1, z_2\}^{\perp\perp}$ and $x \not\perp z_1 \not\perp z_2$. \square

Denote by \mathcal{H} the set of all hyperbolic lines of a non-degenerate polar space $\mathcal{S} = (\mathcal{P}, \mathcal{L})$. Then $(\mathcal{P}, \mathcal{H})$ is a partial linear space, called the *geometry of hyperbolic lines* of \mathcal{S} .

In the remainder of this section we describe the classical polar spaces and their hyperbolic lines.

2.4. Classical polar spaces. Let \mathbb{K} be a skew field with anti-automorphism σ , where $\sigma^2 = id$ and $\epsilon \in \mathbb{K}$. Assume that W is a right vector space over \mathbb{K} . As in Tits [32, (8.2.1), (8.1.2)], a (σ, ϵ) -hermitian form f on W is a mapping $f : W \times W \rightarrow \mathbb{K}$ such that for all $v, w, u \in W$ and $\lambda, \mu \in \mathbb{K}$ we have

- (a) $f(v + w, u) = f(v, u) + f(w, u)$ and $f(u, v + w) = f(u, v) + f(u, w)$;
- (b) $f(v\lambda, w\mu) = \lambda^\sigma f(v, w)\mu$;
- (c) $f(v, w) = f(w, v)^\sigma \epsilon$.

The form f is called *trace-valued* if and only if $f(v, v) \in \{t + t^\sigma \epsilon \mid t \in \mathbb{K}\}$ for all $v \in V$.

We set

$$\mathbb{K}_{\sigma, \epsilon} := \{c - \epsilon c^\sigma \mid c \in \mathbb{K}\}, \quad \mathbb{K}^{\sigma, \epsilon} := \{c \in \mathbb{K} \mid \epsilon c^\sigma = -c\}.$$

A *pseudo-quadratic form* (or (σ, ϵ) -quadratic form) q with associated trace-valued (σ, ϵ) -hermitian form $f : W \times W \rightarrow \mathbb{K}$ is a map $q : W \rightarrow \mathbb{K}/\mathbb{K}_{\sigma, \epsilon}$, with the property that

$$q(v\lambda) = \lambda^\sigma q(v)\lambda, \text{ and } q(v+w) = q(v) + q(w) + f(v, w) + \mathbb{K}_{\sigma, \epsilon}$$

for all $v, w \in W$ and $\lambda \in \mathbb{K}$.

Suppose W is endowed with a (σ, ϵ) -hermitian form f or pseudo-quadratic form q with associated (σ, ϵ) -hermitian form f . For any subset U of W , we set $U^\perp = \{w \in W \mid f(w, u) = 0 \text{ for all } u \in U\}$. The *radical* of f is the space $\text{Rad}(W, f) = W^\perp$. For any $U, V \subseteq W$ we denote by $U \perp V$ that $f(u, v) = 0$ for all $u \in U$ and $v \in V$.

A vector $w \in W$ with $f(w, w) = 0$ or $q(w) = 0$, respectively, is called *isotropic*. (Note that such a vector is also called *singular* in the literature.) We assume that there exists an isotropic vector, not contained in $\text{Rad}(W, f)$. A subspace U of W is called *isotropic* if $U \perp U$. In the case where W is endowed with a pseudo-quadratic form, we define the *isotropic radical* of (W, q) to be the subspace $R = \{r \in \text{Rad}(W, f) \mid q(r) = 0 + \mathbb{K}_{\sigma, \epsilon}\}$.

We call f *non-degenerate* if $\text{Rad}(W, f) = 0$ and q *non-degenerate* if $R = \{0\}$.

Fix a complement W_0 to R in W . Then $q|_{W_0}$ is non-degenerate. We set

$$\Phi_q := \{c \in \mathbb{K} \mid \text{there exists } r_c \in \text{Rad}(W_0, f) \text{ with } q(r_c) = c + \mathbb{K}_{\sigma, \epsilon}\}.$$

Then Φ_q is an additive subgroup of \mathbb{K} . For each $c \in \Phi_q$, there is a unique $r_c \in \text{Rad}(W_0, f)$ with $q(r_c) = c + \mathbb{K}_{\sigma, \epsilon}$. Further, $d^\sigma \Phi_q d = \Phi_q$ for $0 \neq d \in \mathbb{K}$ and $\mathbb{K}_{\sigma, \epsilon} \subseteq \Phi_q \subseteq \mathbb{K}^{\sigma, \epsilon}$. If $\text{Rad}(W_0, f) = 0$ (in particular if $\text{char } K \neq 2$), then $\Phi_q = \mathbb{K}_{\sigma, \epsilon}$.

By $\mathcal{S} = \mathcal{S}_f$ (or \mathcal{S}_q , respectively) we denote the polar space consisting of the point set \mathcal{P} of isotropic 1-spaces and line set \mathcal{L} of all isotropic 2-spaces of W with respect to f (or q , respectively). Any polar space arising in this way is called *classical*.

By the results of Tits [32, (8.1) and (8.2)], any non-degenerate classical polar space with non-empty line set can be obtained (up to isomorphism) as \mathcal{S}_f or \mathcal{S}_q with f or q as below.

- (I) $f : W \times W \rightarrow \mathbb{K}$ is $(\sigma, -1)$ -hermitian form and $\mathbb{K}_{\sigma, -1} = \mathbb{K}^{\sigma, -1}$;
- (II) $q : W \rightarrow \mathbb{K}$ is a quadratic form, i.e. a $(id, 1)$ -quadratic form and $1 \in \Phi_q$ or $\Phi_q = \{0\}$.
- (III) $q : W \rightarrow \mathbb{K}/\mathbb{K}_{\sigma, -1}$ is a pseudo-quadratic form with associated (trace-valued) $(\sigma, -1)$ -hermitian form $f : W \times W \rightarrow \mathbb{K}$. Moreover, $1 \in \Phi_q$.

2.5. Hyperbolic lines in classical polar spaces. Let \mathcal{S} be a non-degenerate classical polar space \mathcal{S}_f or \mathcal{S}_q of some non-degenerate (σ, ϵ) -hermitian form f or (pseudo-)quadratic form q , as in (2.4)(I)-(III). Assume that the points of \mathcal{S} generate W and that \mathcal{S} contains lines.

Let v_1 and v_2 be two isotropic vectors such that $f(v_1, v_2) = 1$. Then $x = \langle v_1 \rangle$ and $y = \langle v_2 \rangle$ are two non-collinear points of \mathcal{S} . The 2-dimensional space $H := \langle v_1, v_2 \rangle$ is non-degenerate and $W = H \perp H^\perp$.

2.6. *The hyperbolic line $\{x, y\}^{\perp\perp}$ equals $\{\langle v \rangle \mid v \in \langle v_1, v_2 \rangle\} \cap \mathcal{S}_f$ or $\{\langle v \rangle \mid v \in \langle v_1, v_2 \rangle + \text{Rad}(W_0, f)\} \cap \mathcal{S}_q$.*

Proof. Suppose $z \in \{x, y\}^{\perp\perp}$. As \mathcal{S} is generated by x^\perp and y , (and similarly y^\perp and x), see [14], we see that $\langle \{x, y\}^\perp \rangle = H^\perp$. But that implies that $\{x, y\}^{\perp\perp}$ is contained in $H + \text{Rad}(W_0, f)$. \square

Assume that $\mathcal{S} = \mathcal{S}_f$ for some $(\sigma, -1)$ -hermitian form f as in (I). Then the isotropic 1-spaces inside H are $\langle v_2 \rangle$ and the spaces $\langle v_1\lambda + v_2 \rangle$ where $\lambda \in \mathbb{K}_{\sigma, -1} = \mathbb{K}^{\sigma, -1}$. Indeed, $f(v_1\lambda + v_2, v_1\lambda + v_2) = \lambda^\sigma - \lambda = 0$ if and only if $\lambda^\sigma = \lambda$.

In particular, as $1 \in \mathbb{K}_{\sigma, -1} \neq \{0\}$, the hyperbolic lines are thick. Denote by \mathcal{H} the set of hyperbolic lines of \mathcal{S} . Then $\Pi = (\mathcal{P}, \mathcal{H})$ is a partial linear space.

Now suppose $v_3 \in V \setminus H$ is an isotropic vector with $f(v_3, v_2) = 1$ and $f(v_1, v_3) = 0$. Then $v_2 + v_3$ is also an isotropic vector. Now consider the non-degenerate 2-spaces $H_1 = \langle v_1, v_2 + v_3 \rangle$ and $H_2 = \langle v_1\lambda + v_2, v_3\mu + v_2 \rangle$, where $\lambda \neq 0$ and $\mu \neq 0, 1$ are distinct elements in $\mathbb{K}_{\sigma, -1}$. The intersection $H_1 \cap H_2$ is the 1-space $\langle v_1\lambda(1 - \mu^{-1}) + v_2 + v_3 \rangle$. This is a point of \mathcal{S} if and only if $\lambda\mu^{-1} \in \mathbb{K}_{\sigma, \epsilon}$. So, if inside Π the three points $\langle v_1 \rangle$, $\langle v_2 \rangle$ and $\langle v_3 \rangle$ are in a subspace isomorphic to a dual affine plane, then for all λ and $\mu \neq 0$ in $\mathbb{K}_{\sigma, -1}$ we have $\lambda\mu^{-1} \in \mathbb{K}_{\sigma, -1}$. But that implies that $\mathbb{K}_{\sigma, -1}$ is a (commutative) field. Conversely, if $\mathbb{K}_{\sigma, -1}$ is a commutative field, then all points of the form $\langle v_1\lambda + v_2\mu + v_3\mu \rangle$ with $\lambda, \mu, \rho \in \mathbb{K}_{\sigma, -1}$ different from $\langle v_1 - v_3 \rangle$ form a subspace of Π isomorphic to a dual affine plane.

Now consider the case that v_1, v_2 and v_3 are three isotropic vectors generating a 3-space without isotropic lines.

Then without loss of generality we have $f(v_1, v_2) = 1 = f(v_1, v_3)$ and $f(v_2, v_3) = \alpha \neq 0$. Since there is no isotropic 2-space in $\langle v_1, v_2, v_3 \rangle$, we find that

$$f(v_1 + v_2, v_1\lambda + v_3) = 1 - \lambda + \alpha \neq 0$$

for all $\lambda \in \mathbb{K}_{\sigma, -1}$. This is equivalent with $\alpha \notin \mathbb{K}_{\sigma, -1} = \mathbb{K}^{\sigma, -1}$.

Consider an isotropic point $\langle v \rangle \subseteq \langle v_1, v_2 \rangle$ and an isotropic point $\langle w \rangle \subseteq \langle v_1, v_3 \rangle$. Then there are $\lambda, \mu \in \mathbb{K}_{\sigma, -1} \setminus \{0\}$ with (up to a scalar) $v = v_1 + v_2\lambda$ and $w = v_1 + v_3\mu$. The intersection point of $\langle v, w \rangle$ with $\langle v_2, v_3 \rangle$ is then the point $\langle v_2\lambda - v_3\mu \rangle$. Now we have

$$f(v_2\lambda - v_3\mu, v_2\lambda - v_3\mu) = -\lambda^\sigma\alpha\mu + \mu^\sigma\alpha^\sigma\lambda = (\lambda\alpha\mu)^\sigma - \lambda\alpha\mu.$$

Clearly, if $\mathbb{K}_{\sigma,-1} = \mathbb{K}^{\sigma,-1}$ is a field, then $\lambda\alpha\mu \notin \mathbb{K}^{\sigma,-1}$ and we find the intersection point not to be isotropic. This implies that in $\langle v_1, v_2, v_3 \rangle$ there are no 4 hyperbolic lines meeting in 6 isotropic points.

Now assume that $\mathcal{S} = \mathcal{S}_q$ for some non-degenerate quadratic or pseudo-quadratic form q with associated $(\sigma, -1)$ -hermitian form f . Suppose $\langle v_1 \rangle$ and $\langle v_2 \rangle$ are two isotropic points with $f(v_1, v_2) = 1$. Then the points on the hyperbolic line through these two points are, besides $\langle v_2 \rangle$, the unique isotropic points in each of the subspaces $\langle v_1\lambda + v_2 \rangle + \text{Rad}(W, f)$, where $\lambda \in \Phi_q$. So, hyperbolic lines of \mathcal{S} are thick if and only if the set Φ_q contains at least 2 elements. Suppose Φ_q contains at least 2 elements. Let \mathcal{H} be the set of all hyperbolic lines and set $\Pi := (\mathcal{P}, \mathcal{H})$. As above we can prove that any triple of points p, q, r with $p \sim q \sim r \perp p$ are in a subspace of Π isomorphic to a dual affine plane if and only if for all $\lambda, \mu \in \Phi_q$ we have $\lambda\mu^{-1} \in \Phi_q$. Since $1 \in \Phi_q$, we find Φ_q to be a commutative field. Moreover, the same arguments as used above also reveal that, provided Φ_q is a commutative field, in a subspace spanned by two intersecting hyperbolic lines but not containing an isotropic line, there are no 4 hyperbolic lines meeting in 6 isotropic points.

From the above we now easily deduce the following.

2.7. Proposition. *Let \mathcal{S} be a non-degenerate classical polar space \mathcal{S}_f or \mathcal{S}_q defined by a (σ, ϵ) -hermitian form f as in (2.4)(I) or a (pseudo-)quadratic form q with associated (σ, ϵ) -hermitian form f as in (2.4)(II) or (III), respectively.*

Suppose \mathcal{S} contains lines and let Π be the corresponding geometry of hyperbolic lines. Then Π is a connected and co-connected thick partial linear space satisfying the conditions (a)-(d) of Theorem 1.2 if and only if

- (a) *f is as in (2.4)(I) and $\mathbb{K}_{\sigma,\epsilon}$ is a commutative field of order at least 3;*
- (b) *q is as in (2.4)(II) or (III) and Φ_q is a commutative field of order at least 3.*

2.8. Exceptional polar spaces of rank 3. Besides the classical polar spaces there are many classes of generalized quadrangles and two more classes of polar spaces of rank 3; see [32]. We now consider the two additional classes of polar spaces of rank 3.

Let \mathbb{P} be a projective space of (projective) dimension 3. Then denote by P the set of lines of \mathbb{P} and by L the set of pencils of lines on a point inside a plane of \mathbb{P} . Then (P, L) is a polar space. If the space \mathbb{P} is defined over a field, then by the Klein correspondence we can identify this polar space with an orthogonal polar space as described above. However, if \mathbb{P} is defined over a proper skew field, then this provides us with a new polar space. If l and m are two skew lines of \mathbb{P} , then $\{l, m\}^\perp$ consists of all the lines of \mathbb{P} meeting both l and m . If n is a line that meets all lines in $\{l, m\}$, then it is straightforward to check that $n = l$ or m . Thus all hyperbolic lines of the polar space (P, L) are thin.

The second class of polar spaces of rank 3 are the exceptional polar spaces of type E_7 over some field k as described in [32, Section 9]. It has been shown in [30] that the automorphism group of this polar space contains a class of k -transvection groups whose elements are in one-to-one correspondence to the points of the polar space.

As follows from [16], the geometry of these k -transvection subgroups is a geometry satisfying the conditions of Theorem 1.1. In fact, in this geometry the subspace generated by three points $x \sim y \sim z \perp x$ is a dual affine plane. Unfortunately, it is not known to us what the structure of an arbitrary plane is.

We finish this section with a description of Fischer spaces, some of which appear in the conclusion of Theorem 1.3.

2.9. Fischer spaces. A *Fischer space* is a partial geometry $\Pi = (P, L)$ of order 2 (i.e., all lines contain $2 + 1 = 3$ points) in which any pair of intersecting lines is contained in a subspace isomorphic to a dual affine plane or an affine plane. Fischer spaces are closely related to groups generated by 3-transpositions. Indeed, if D is a normal set of 3-transpositions in a group G , that is a normal set of involutions with the property that de is of order at most 3 for any two elements $d, e \in D$, then the space with D as point set and as lines the triple of elements from D in the subgroups of the form $\langle d, e \rangle$ with $d, e \in D$ with de of order 3 is a Fischer space. On the other hand if Π is a Fischer space, then to each point $p \in P$ we can attach an involutory automorphism τ_p of Π which fixes p and all points not collinear to p and switches all points q, r with $\{p, q, r\}$ being a line. The set $\{\tau_p \mid p \in P\}$ is a normal set of 3-transpositions in the group $\text{Aut}(\Pi)$.

By exploiting this relationship all connected Fischer spaces containing two non-collinear points have been classified by Cuypers and Hall [12, 11]. We briefly describe the so-called irreducible Fischer spaces (see [12, 11]).

- (a) *Symmetric Fischer spaces.* Let Ω be a set and D the class of transpositions in $G = \text{Sym}(\Omega)$ the symmetric group on Ω . Then D is a normal set of 3-transpositions in G . The corresponding Fischer space is called a symmetric Fischer space.
- (b) *Symplectic and unitary Fischer spaces.* These are the geometries of hyperbolic lines of a non-degenerate symplectic polar space over the field \mathbb{F}_2 or a unitary polar space over \mathbb{F}_4 .
- (c) *Orthogonal Fischer spaces over \mathbb{F}_2 .* Let q be a non-degenerate quadratic form on a vector space V over \mathbb{F}_2 with trivial radical. Then the space whose point set is the set of all non-isotropic 1-spaces and whose lines are the elliptic lines is an orthogonal Fischer space over \mathbb{F}_2 .
- (d) *Orthogonal Fischer spaces over \mathbb{F}_3 .* Let q be a non-degenerate quadratic form on a vector space V over \mathbb{F}_3 . Take the space with as points the non-isotropic

1-spaces and as lines the sets of three points contained in a 2-space tangent to the quadric. Then any connected component of this geometry is called an orthogonal Fischer space over \mathbb{F}_3 .

- (e) *Sporadic Fischer spaces.* These are the Fischer spaces corresponding to the classes of 3-transpositions in a sporadic Fischer group Fi_{22} , Fi_{23} , Fi_{24} , $\Omega(8, 2) : \text{Sym}_3$ or $\Omega(8, 3) : \text{Sym}_3$.

3 A Reduction Theorem

In this section we start with a proof of Theorem 1.1. Actually we first consider a more general situation as in the following:

3.1. Setting. Let $\Pi = (P, L)$ be a connected partial linear space satisfying the following conditions.

- (a) All lines in L contain at least 4 points;
- (b) if x and y are points with $x^\perp \subseteq y^\perp$ then $x^\perp = y^\perp$;
- (c) there exist non-collinear points;
- (d) every triple of points $x, y, z \in P$ with $x \sim y \sim z$ and $x \perp z$ is contained in a transversal plane;
- (e) if π is a transversal plane of Π and p a point, then $p^\perp \cap \pi$ is empty, a point, a line, a transversal of π or π itself.

It is the purpose of this section to show that we can reduce the above setting to the setting of Theorem 1.1.

Let Π be as in (3.1).

3.2. If x is a point and l a line of Π , then x is collinear with no point on l , with all but one point on l or with all points of l .

Proof. Suppose x is collinear to some but not all points of the line l . Then we can find a point $y \in l \cap x^\sim$ and a point $z \in l \cap x^\perp$. The points x, y, z form a triangle and hence are contained in a transversal plane π of Π . Clearly $l \subseteq \pi$. Within X we see that the point x is collinear to all but one of the points of l . \square

The above shows that Π is a so-called Δ -space; see [23].

3.3. Let π be a transversal plane in Π . Then every transversal of π is a coclique of size at least 3.

Proof. See [17]. □

3.4. *Let $x \in P$ be a point then x^\perp and $x^\perp \setminus \{x\}$ are subspaces of Π .*

Proof. Let l be a line in L meeting x^\perp or $x^\perp \setminus \{x\}$ in at least two points. Then (3.2) implies that l is contained in $x^\perp \setminus \{x\}$. Thus x^\perp and $x^\perp \setminus \{x\}$ are subspaces of Π . □

3.5. *Every connected subspace of Π has diameter at most 2.*

Proof. Suppose v, x, y, z is a path of length 3 in the collinearity graph of Π . By (3.2) the points v and z are collinear to all but one of the points on the line $l = xy$ on x and y . Since l contains at least 3 points, there is a point $u \in l$ collinear to both v and z . This clearly implies that the diameter of any connected component is at most 2. □

3.6. *Let π be a transversal plane and l a line meeting π in a point x . If for some $y \in l \setminus \{x\}$ we have $y^\perp \cap \pi$ being a transversal of π , then for all $y \in l \setminus \{x\}$ the intersection $y^\perp \cap \pi$ is a transversal of π .*

If for some $y \in l \setminus \{x\}$ we have $y^\perp \cap \pi$ being a line of π , then for all $y \in l \setminus \{x\}$ the intersection $y^\perp \cap \pi$ is a line of π .

Proof. Suppose y and z are two points of $l \setminus \{x\}$ with $y^\perp \cap \pi$ being a transversal T_y of π . Let m be a line of π on x . Then $\langle m, l \rangle$ is a transversal plane. So m contains a point different from x which is non-collinear to z . As the above is true for every line m in π on x , we find that $z^\perp \cap \pi$ contains at least 2 points. But then $z^\perp \cap \pi$ is either a transversal or a line of π . Suppose we are in the latter case and $n = z^\perp \cap \pi$ is a line. Then this line n meets T_y in some point, u say. Inside the transversal plane π we see that all points of T_y , and hence in particular u , are collinear to x . However, u^\perp contains y and z and by (3.2) all points of l including x . This contradiction proves the first statement.

The second follows immediately from (3.1)(e) and the above. □

3.7. *Let x and y be two non-collinear points and suppose $z_1 \neq z_2$ are two collinear points both collinear to x and y .*

If there is a line ℓ in z_1^\perp meeting both z_2x and z_2y , then there is a line m in z_2^\perp meeting both z_1x and z_1y .

Proof. As z_1 is not collinear to all the points on yz_2 , we find that z_1, z_2 and y generate a transversal plane. In particular, there is a point $u \in z_1z_2$ which is not collinear to y .

As z_1^\perp meets the transversal plane $\langle x, y, z_2 \rangle$ in the line ℓ , the intersection of u^\perp with this plane is also a line; see (3.6). In particular, u is collinear with x . But, as the line xz_2 meets ℓ , we find that $\langle z_1, z_2, x \rangle$ is a transversal plane, and hence u^\perp meets xz_1 in a point different from x . This implies that u^\perp meets the transversal plane $\langle x, y, z_1 \rangle$ in a line. Now (3.6) implies that also z_2^\perp meets $\langle x, y, z_1 \rangle$ in a line. □

3.8. *The connected components of (P, \perp) have diameter at most 2.*

Proof. Let $a \perp b \perp c \perp d$ be a path of length 3 in (P, \perp) . Then $\langle a, c, d \rangle$ is a transversal plane. Pick a point x on ac different from a and c and a line on x meeting ad in a point y different from a and d . This line meets the transversal of $\langle a, c, d \rangle$ on c in a point e . Clearly, b is collinear to e, d and y . Now applying (3.7) yields that there is a line m meeting bd and be which is contained in y^\perp . Let f be the point on m in the transversal on d and e in $\langle b, d, e \rangle$. Then f^\perp contains dy and hence a . Thus we have found a path of length 2 from a to d . This shows that the diameter of the connected components of (P, \perp) is at most 2. \square

3.9. Corollary. *Connected components of (P, \perp) of diameter 2 are connected subspaces of Π .*

Proof. If x and y are two collinear points in a connected component of (P, \perp) , then by the above (3.8) we can find a point z non-collinear to both x and y . But it follows from (3.2) that all points on the line xy are in z^\perp . Hence xy is contained in the connected component of (P, \perp) containing x, y and z . This shows that this component is a subspace of Π .

It remains to prove connectedness in Π of a connected component C of (P, \perp) . Thus let x and y be two non-collinear points in C , and z a point collinear to both of them. We may assume that z is in a connected component of (P, \perp) different from C . Let u be a point in the transversal plane generated by x, y and z , different from x and y , but on the transversal coclique on x and y inside this plane. If $x^\perp \neq u^\perp$, then by condition (3.1)(c) there exists a point $v \in u^\perp$ collinear to x . The point v is in C and, by condition (3.1)(d), also collinear to y . Similarly $y^\perp \neq u^\perp$ implies that there is a point in C collinear to both x and y .

If $x^\perp = y^\perp = u^\perp$, then, as C has diameter 2, there is point $v \in C$ collinear to all three points x, y and u . In any case we have found a point in C collinear to both x and y . \square

3.10. *Suppose (P, \perp) contains a connected component of diameter 1. Then all components of (P, \perp) have diameter 1.*

Proof. Suppose C_1, C_2 are connected components of (P, \perp) with C_1 having diameter 1 and C_2 having diameter 2. Suppose x, y, z is a path of length 2 in C_2 and u is a point of C_1 . Fix a point p on xu different from x and u and a point q on uz different from z and u . The subspaces of Π spanned by u, y, x and by u, y, z are both transversal planes. Inside these planes we find a points $r \in py$ and $s \in qz$ not collinear with u . Clearly $r, s \in C_1$. So $r \perp s$. The subspace $\langle y, s, u \rangle$ of Π is a transversal plane. The intersection of r^\perp with $\langle y, s, u \rangle$ equals the transversal coclique of $\langle y, s, u \rangle$ containing u and s . Applying (3.6) to the line yr and plane $\langle y, q, u \rangle$ implies that $p^\perp \cap \langle y, q, u \rangle$ is also a transversal coclique. Applying (3.6) again but now to the line up and $\langle y, q, u \rangle$ shows that also x^\perp meets $\langle u, y, q \rangle$ in a transversal coclique. However, as y is in x^\perp , this transversal coclique is the unique transversal coclique containing y and z . This contradicts that x and z are collinear. \square

On the point set P of Π we can define the relation \equiv by

$$x \equiv y \Leftrightarrow x^\perp = y^\perp.$$

This relation is clearly an equivalence relation on P . The \equiv -equivalence class of a point x is denoted by $[x]$. For each line $l \in L$ define $[l]$ to be the set $\{[x] \mid x \in l\}$ and let L/\equiv be $\{[l] \mid l \in L\}$.

3.11. *Suppose $x, y \in P$. Then $x \perp y$ if and only if $x' \perp y'$ for all $x' \in [x]$ and $y' \in [y]$.*

Proof. Suppose $x \perp y$, $x' \in [x]$ and $y' \in [y]$. Then $y \in x^\perp = x'^\perp$. So $x' \in y'^\perp$ which equals y'^\perp . \square

The above Lemma 3.11 justifies the following definitions. If $x, y \in P$, then we write $[x] \perp [y]$ if and only if $x \perp y$. Moreover, $[x]^\perp$ denotes the set of all classes $[y]$ with $x \perp y$.

3.12. Reduction Theorem. $\Pi/\equiv := (P/\equiv, L/\equiv)$ is a connected partial linear space satisfying:

- (a) all lines contain at least 4 points;
- (b) every triple of points X, Y, Z with $X \sim Y \sim Z \perp X$ is contained in a transversal plane;
- (c) if X and Y are points with $X^\perp \subseteq Y^\perp$ then $X = Y$;
- (d) if π is a transversal plane of Π/\equiv and X a point, then $X^\perp \cap \pi$ is empty, a point, a line, a transversal of π or π itself.
- (e) $(P/\equiv, L/\equiv)$ is linear if and only if (P, \perp) has only components of diameter 1.

Proof. If p and q are collinear points, then $[p] \neq [q]$, so all elements of L/\equiv contain at least 4 points of P/\equiv . This proves (a) and connectedness of Π/\equiv .

Let X and Y be points of P/\equiv . Then $X \perp Y$ if and only if $x \perp y$ for some points $x \in X$ and $y \in Y$, if and only if $x \perp y$ for all points $x \in X$ and $y \in Y$. Thus if $X^\perp \subseteq Y^\perp$, then for each $x \in X$ and $y \in Y$ we have $x^\perp \subseteq y^\perp$, hence $x \equiv y$ and $X = Y$. This proves (c).

Now suppose X and Y are two points in P/\equiv on a line $[l]$, $l \in L$. Then, for any two points $x \in X$ and $y \in Y$ we have $x \sim y$. Fix $x \in X$ and $y \in Y$ and let z be a point on the line through x and y . We will show that $[z] \in [l]$. First assume that $x \in l \neq xy$. Then $\langle l, x, y \rangle$ is a transversal plane, and l contains a unique point $z' \in z^\perp$. Suppose $u \perp z'$. If $u \perp x$, then $u \perp a$ for all $a \in \langle l, y \rangle$. Thus $u \perp z$. If $u \sim x$, then there is a point on ux non-collinear to the unique point in $[y] \cap l$ and therefore also to y and all points in the transversal of $\langle l, y \rangle$ on y . But then (3.6) implies that $u^\perp \cap \langle l, y \rangle$ is also

a transversal, in particular the transversal on z' which contains z . Thus $u \perp z$. This shows that $z'^{\perp} \subseteq z^{\perp}$ and hence $z' \in [z]$.

Now suppose xy does not meet l . Then let x' be the point on l which is in $[x]$. Consider the line $x'y$. By the above there is a point $z' \in x'y \cap [z]$ and a point $z'' \in l \cap [z']$. But then $z'' \in [z] \cap l$. Hence, in any case $[l]$ contains $[z]$.

This proves that $(P/\equiv, L/\equiv)$ is a partial linear space. Furthermore, it shows that if X, Y, Z are three points with $X \sim Y \sim Z \perp X$ in $(P/\equiv, L/\equiv)$, then there are points $x \sim y \sim z \perp x$ in Π with $(X, Y, Z) = ([x], [y], [z])$. No two distinct points in $\langle x, y, z \rangle$ are \equiv -equivalent. Indeed, suppose $u \perp v$ are two distinct points in $\langle x, y, z \rangle$ with $u \equiv v$. Then $u \equiv v'$ for every point u' on the transversal of $\langle x, y, z \rangle$ containing u and v . So, without loss of generality, we can assume u to be on the line through y and x and v to be on the line through y and z . Moreover, since $[x] \neq [z]$, we find that $u \neq x$ and $v \neq z$. Let $w \in x^{\perp} \setminus z^{\perp}$. Then $w \notin u^{\perp}$ for otherwise it is in $x^{\perp} \cap u^{\perp} \cap v^{\perp}$, which is contained in z^{\perp} . So, $\langle w, y, x \rangle$ is a transversal plane containing a point $w' \neq u$ which is in u^{\perp} and thus also in v^{\perp} . So w'^{\perp} meets $\langle x, y, z \rangle$ in a transversal and, by (3.6), so does w^{\perp} , which contradicts that $w \notin z^{\perp}$. Hence $\{[u] \mid u \in \langle x, y, z \rangle\}$ is a subspace of $(P/\equiv, L/\equiv)$ isomorphic to $\langle x, y, z \rangle$ and hence a transversal plane. Furthermore, (d) follows now easily by the above.

Finally, $(P/\equiv, L/\equiv)$ is linear if and only if $x^{\perp} = y^{\perp}$ for all $x, y \in P$ with $x \perp y$. But that is equivalent with (P, \perp) having only components of diameter 1. Now (3.10) implies (e). \square

4 The geometry of hyperbolic lines

Now we focus on the proof of Theorem 1.1. So, assume that $\Pi = (P, L)$ is a partial linear space satisfying the hypothesis of that theorem.

4.1. *Let x be a point in P . The subspace $x^{\perp} \setminus \{x\}$ of Π is connected.*

Proof. Suppose y and z are two non-collinear points in $x^{\perp} \setminus \{x\}$. Since Π is connected and of diameter at most 2, there is a point u collinear with y and z . If $u \perp x$ then we are done, thus suppose that u and x are collinear. Then $\langle x, u, y \rangle$, $\langle x, u, z \rangle$ and $\langle y, u, z \rangle$ are transversal planes. By condition (1.1)(c) and our assumption that all \equiv -classes are of size 1, there is a point v collinear with y but not with x . If v and z are collinear, we are done. Assume $z \perp v$. Thus by condition (1.1)(d) the point z is the only point on the transversal coclique T on z in the transversal plane $\langle y, u, z \rangle$ not collinear with v .

Again by condition (1.1)(c), there is a point w in x^{\perp} collinear with z . By the same reasoning as above we can assume that y is the unique point on T not collinear with w . Since T contains at least 3 points, there is a point on T collinear with v and w , and by condition (1.1)(d) not to x . But that means that we have found a path from y to z in the collinearity graph of Π that is completely inside $x^{\perp} \setminus \{x\}$. \square

The following lemma will be of great importance in constructing a polar space on the point set P .

4.2. *Let x and y be two non-collinear points and suppose z is collinear to both x and y . Then there is a transversal T on x and y that contains a point in z^\perp .*

Proof. Let x and y be two non-collinear points and z a point collinear to both of them. Then by (3.8) there is a point u in $x^\perp \cap z^\perp$.

First suppose that u and y are collinear. Let u' be a point on yu different from u and y . Then x is not collinear to u' . Moreover, there is a point z' on yz not collinear to u' . This point is different from y and hence collinear with x . Consider now line on z meeting xz' in a point different from x and z' . This line contains a point w in the transversal of x inside the plane $\langle x, y, z \rangle$. Notice that w and u' are collinear. As u'^\perp contains a point of the line xz' , we find $\langle u', w, z \rangle$ to be a transversal plane. So, there is a point p on the line zu' which is not collinear to w . Since x is collinear to z and not u' , it is collinear to p . By condition (3.1)(e), also y is collinear to p . So w^\perp meets the transversal plane generated by y, z and u in the line yp . Fix the point t of this line that is not collinear with z . Then t is collinear with x and y but not with z and w . Now (3.7) applied to x, y, t and z' implies that there is a line in $\langle x, y, t \rangle$ which is contained in z'^\perp . This line contains a point q which is in the transversal of x and y inside this plane. But, then q^\perp contains y and z' and hence also z . So we have found the point we are looking for.

Now assume that $u \perp y$. By the previous lemma there is a point v in x^\perp collinear with both u and y . If $v \perp z$ we are in the above situation with v instead of u . Thus we can assume that v and z are collinear. By the arguments in the preceding paragraph, but now with v instead of y , there is a transversal coclique on x and v containing a point u' in z^\perp . Since y and v are collinear but y and x are not, we have that y and u' are collinear. So we can apply the above with u' instead of u . \square

4.3. *Let x be a point and y and z two points in x^\perp different from x . If $x^\perp \cap y^\perp \subseteq x^\perp \cap z^\perp$, then $x^\perp \cap y^\perp = x^\perp \cap z^\perp$.*

Proof. Suppose $x^\perp \cap y^\perp$ is a proper subset of $x^\perp \cap z^\perp$. Then $y \perp z$.

By (4.1) and (3.5) there is a point $u \in x^\perp$ collinear to both y and z .

Now let v be a point in $x^\perp \cap z^\perp$ collinear with y . By the above Lemma 4.2, there is a transversal on z and v containing a unique point t (possibly equal to v) in u^\perp . This point t is in $x^\perp \cap z^\perp$, but collinear with y . Applying (4.2) one more time, we find a transversal on u and t containing a point s in y^\perp . Then s is in $x^\perp \cap y^\perp$, but s is collinear with z , since $z \perp t$ but $z \sim u$. A contradiction, which proves the lemma. \square

Now we are in a position to define a set \mathcal{L} on P such that (P, \mathcal{L}) is a polar space.

Let x and y be two distinct points of Π . Then define

$$(xy)^{\perp\perp} = \{z \in P \mid z \perp u \text{ for all } u \in x^\perp \cap y^\perp\}.$$

Now let \mathcal{L} be the set of all subsets $(xy)^{\perp\perp}$ of P , with x and y two non-collinear points in P . The elements of \mathcal{L} will be called *singular lines*. If $x \sim y$, then $(xy)^{\perp\perp}$ is called the *hyperbolic line* through x and y .

4.4. *Let x and y be two non-collinear points. Then for all distinct points s and t in $(xy)^{\perp\perp}$ we have $s \perp t$ and $(st)^{\perp\perp} = (xy)^{\perp\perp}$.*

Proof. Suppose $s \in (xy)^{\perp\perp}$ different from x . Then $s \perp x, y$. By (4.3) we have $x^\perp \cap y^\perp = x^\perp \cap s^\perp$. Hence $(xy)^{\perp\perp} = (xs)^{\perp\perp}$. Applying this argument one more time we see $(xs)^{\perp\perp} = (st)^{\perp\perp}$. \square

4.5. Proposition. *(P, \mathcal{L}) is a non-degenerate polar space of rank at least 2.*

Proof. By the previous lemma and the definition of singular lines we see that for any singular line l and point x we have $x^\perp \cap l$ is empty, consists of a single point or equals l . In view of the ‘one-or-all’ axiom and condition (c) of Theorem 1.1, we only have to show that this intersection is never empty. So fix two points y and z in l collinear with x . Then $l = (yz)^{\perp\perp}$ and by (4.2) there is a transversal on y and z containing a point in x^\perp . By condition (d) of Theorem 1.1 this transversal is contained in l . Thus x^\perp meets l non-trivially.

Since Π contains non-collinear points, we find (P, \mathcal{L}) to contain lines. This implies that (P, \mathcal{L}) has rank at least 2. \square

Proposition 4.5 finishes the proof of Theorem 1.1.

5 Planar spaces of hyperbolic lines

In this section we prove Theorem 1.2. So, let $\Pi = (P, L)$ be a partial linear space as in the hypothesis of (1.2).

5.1. *Let π be a linear plane of Π . Any three non-collinear points in π generate π .*

Proof. Let x, y and z be three non-collinear points in π . Then the lines xy and xz are distinct and contained in the planes π and $\langle x, y, z \rangle$. Since Π is planar, these planes are the same. \square

A dual affine plane is transversal.

5.2. *Let x be a point in Π and π a dual affine plane. Then x^\perp meets π in the empty set, a point, a line, the full plane or a transversal.*

Proof. As all planes of Π are Δ -spaces, so is Π . In particular, x^\perp is a subspace for any point $x \in P$. As a dual affine plane is generated by any three points not on a line or a transversal, we easily deduce that for any point x and dual affine plane π of Π we have that x^\perp meets π in the empty set a point, a line, the full plane, or part of a transversal.

Suppose that x^\perp meets π in at least two points of a transversal T . Suppose y and z are two points of $T \cap x^\perp$. If T is not contained in x^\perp , then there is a line l in π completely in x^\sim . So x and l generate a linear plane. Fix two points u and v on l but not in T . Let w be the intersection point of yu and zv . Let w' be the point on l not collinear with w . Since x is not collinear with y , there is a point y' on xu not collinear to w . As $x \perp z$, there is also a point z' on xv not collinear to w . By condition (b) of Theorem 1.2, the lines $w'y'$ and $w'z'$ are distinct. Hence $\langle x, l \rangle = \langle w', y', z' \rangle \subseteq w^\perp$, which is clearly a contradiction. Thus $T \subseteq x^\perp$. \square

With the help of Theorem 1.1 and Section 3 we obtain the following.

5.3. Proposition. *(P, \perp) is a non-degenerate polar graph of rank at least 2.*

Proof. By the above results Π satisfies the conditions of Theorem 1.1. This proves the result. \square

Theorem 1.2 is a direct consequence of the above result.

6 Hyperbolic lines in polar spaces over finite fields

The purpose of this section is to provide a proof of Theorem 1.3.

We start with the case that $\Pi = (P, L)$ is a partial linear space of order $q \geq 3$ satisfying the hypothesis of Theorem 1.3. Since classical unitals are linear spaces in which no 4 lines meet in 6 points, we find that Π does satisfy the conditions of (a)-(c) of Theorem 1.2. In particular, we can apply some of the results from the previous sections.

Indeed, by 5.2 we can apply the Reduction Theorem 3.12 and obtain the following.

6.1. Proposition *Suppose Π satisfies the conditions of Theorem 1.3. If $q \geq 3$, then every connected component of (P, \perp) is a non-degenerate polar graph.*

We will analyze this situation somewhat further. Assume that (P, \perp) is connected. Let \mathcal{L} be the set of singular lines. We first consider the case where (P, \mathcal{L}) is a generalized quadrangle. Since every point in P is on at least $q + 1 \geq 4$ singular lines and every singular line contains at least $q \geq 3$ points, all singular lines have the same number of points, and every point is on the same number (possibly infinite) of singular lines. Suppose the generalized quadrangle (P, \mathcal{L}) has order (s, t) , where $s, t \in \mathbb{N} \cup \{\infty\}$.

6.2. s and t are finite.

Proof. First we consider t . Suppose there are more than $q + 1$ singular lines on a point x . Then $x^\perp \setminus \{x\}$ contains a linear plane π say. Let y be a point on a singular line through x , then $y^\perp \cap \pi$ contains a point, z say, and $y \perp x, z$. Hence y is contained in the singular line through x and z . This implies that there are at most $|\pi|$ singular lines through x and t is finite.

Now consider s . Suppose there are more than $q + 1$ points on a singular line l . Fix distinct points x, y and z on l and let m be an ordinary line through y contained in x^\perp . Let n be a singular line on z different from l . Then n does not meet m and contains a point x' that is collinear to all points on m . Let π be the linear plane generated by x' and m . For every point p of $\pi \setminus \{y\}$ there is a unique point in $p^\perp \cap l$. Thus either l contains at most $|\pi|$ points and hence s is finite, or it contains a point y' different from y that is collinear with all points of π except for y . Suppose we are in the latter situation. Then let k be a singular line on y' different from l . Then again, for every point $p \in \pi$, there is a unique point in $k \cap p^\perp$. By assumption (1.3)(d), all points of k are collinear (in Π) to some point of π . So, $|\pi| \geq |k|$ and s is finite. \square

Next we show that the elements of L are the hyperbolic lines of the generalized quadrangle (P, \mathcal{L}) .

6.3. *Let x and y be two collinear points and X a set contained in $(xy)^{\perp\perp}$. Then $\langle X \rangle$ is a linear subspace contained in $(xy)^{\perp\perp}$.*

Proof. This is a direct consequence of the Delta space property of Π , and the fact that in a generalized quadrangle hyperbolic and singular lines meet in at most one point. \square

6.4. *The elements of L are the hyperbolic lines of (P, \mathcal{L}) .*

Proof. Let x and y be two collinear points, then by the above lemma xy is contained in the hyperbolic line $(xy)^{\perp\perp}$. Now suppose there is a point z in $(xy)^{\perp\perp}$ which is not in xy . Then the plane $\pi = \langle x, y, z \rangle$ is contained in the hyperbolic line on x and y . Let p and p' be distinct points in $x^\perp \cap y^\perp$. If l is a singular line on p (or p'), then for any point $q \neq p$ on l but not in π , we find, by condition (d) of Theorem 1.3, a point $r \in q^\perp \cap \pi \subseteq p^\perp$. So, $r \in l$. So, all singular lines on p (and also on p') meet π , which therefore has cardinality $t + 1$. By 1.4.2 of [26] we have that $s \geq t$. Thus on the singular line $(zp')^{\perp\perp}$ we can find a point z' not in π but collinear with all points on xy . Consider the plane $\pi' = \langle x, y, z' \rangle$. This is contained in p'^\perp and contains therefore at most $t + 1$ points. Now every point of π' is non collinear with a unique point on $(pz)^{\perp\perp}$ and every point of $(pz)^{\perp\perp}$ is non collinear to some point in π' . Since xy is contained in p^\perp this implies that $(pz)^{\perp\perp}$ contains at most $|\pi'| - q$ points. Hence $s + 1 \leq |\pi'| - q < t$, a contradiction. \square

6.5. Proposition. *Suppose (P, \perp) is connected. If (P, \mathcal{L}) has rank 2, then it is a classical symplectic or unitary generalized quadrangle defined over the field \mathbb{F}_q or \mathbb{F}_{q^2} , respectively.*

Proof. By the above (P, \mathcal{L}) is a finite generalized quadrangle in which two intersecting hyperbolic lines generate a linear subspace of Π or a dual affine plane. Now Theorem 5.6.8 of [26] proves the proposition. \square

Now suppose (P, \mathcal{L}) is a polar space of rank at least 3 and possibly infinite.

6.6. Proposition. *Suppose (P, \perp) is connected. Then (P, \mathcal{L}) is a classical symplectic or unitary polar space defined over the field \mathbb{F}_q or \mathbb{F}_{q^2} , respectively.*

Proof. If the rank r of (P, \mathcal{L}) is finite, then by considering the subspaces $x^\perp \cap y^\perp$ for two points x and y with $x \sim y$, we obtain a polar space of rank $r - 1$, which by the Delta space property of Π is also a subspace of Π . Thus by induction all singular lines contain either $q + 1$ points, or $q^2 + 1$ points. Now it follows easily from the classification of polar spaces of rank at least 3, see [25, 32], that Π is one of the spaces of the conclusion of Theorem 1.3.

If the rank of the polar space is infinite, then it is embeddable in a projective space such that hyperbolic lines are contained in projective lines, see [13, 25, 32]. Assume we have such an embedding of (P, \mathcal{L}) . By intersecting the polar space with an appropriate finite dimensional projective subspace we can get a non-degenerate polar subspace of finite rank which is also a subspace of Π . So again we find that singular lines are of size q or q^2 and L is the set of hyperbolic lines in (P, \mathcal{L}) and it follows from the classification of polar spaces of rank at least 3, (see [25, 32]), that Π is isomorphic to one of the spaces in the conclusion of Theorem 1.3. \square

To finish the proof of Theorem 1.3 in the case that $q \geq 3$ we still have to prove that the graph (P, \perp) is connected. But this is done in the next lemma.

6.7. *The graph (P, \perp) is connected.*

Proof. Suppose C is a connected component of (P, \perp) and x a point not in C . By the above results, C is the polar graph of a classical symplectic or unitary polar space. In particular singular lines of C contain $q + 1$ or $q^2 + 1$ points.

Fix a singular line l of C and a line m of Π on x meeting l in a point. If n is a second line on x meeting l , then $\langle m, n \rangle$ is a dual affine plane meeting l in a transversal coclique. As each dual affine plane is generated by any two of its lines, we find that there are $(|l| - 1)/(q - 1)$ dual affine planes on m meeting l in a transversal. But, that contradicts $q > 2$ and we can conclude that (P, \perp) is connected. \square

This completes the proof of Theorem 1.3 in the case that $q \geq 3$. It remains to consider the case that $q = 2$.

Suppose that $\Pi = (P, L)$ is a partial linear space satisfying the hypothesis of Theorem 1.3 and assume that all lines of L have exactly 3 points. In that case Π is a *Fischer space* and we can apply the results of Cuypers and Hall [12] and of [18]. The conditions in (1.3)(c) imply that Π is both θ - and τ -reduced in the terminology of [12]. Now we easily deduce from [18, Section 4] that Π is one of the spaces in the conclusion of Theorem 1.3.

7 Applications

In this final section we briefly discuss some applications of our results. The first is concerned with groups generated by so-called \mathbb{K} -transvections, \mathbb{K} being a field, the second with Lie algebras generated by special elements called extremal, and the final one on local recognition of graphs.

7.1. \mathbb{K} -Transvection groups. Suppose \mathbb{K} is a field. Let G be a group generated by a conjugacy class of Abelian subgroups Σ such that for any two elements $A, B \in \Sigma$ we have one of the following:

- (a) $\langle A, B \rangle$ is isomorphic to $(P)SL(2, \mathbb{K})$ and A and B are full unipotent subgroups of $\langle A, B \rangle$;
- (b) $[A, B] = 1$, notation $A \perp B$.

Then Σ is called a *class of \mathbb{K} -transvection (sub)groups* of G . Groups generated by \mathbb{K} -transvection groups have been studied by Aschbacher [2], Aschbacher and Hall [3], Fischer [19, 20], Steinbach [15, 28, 29] and others, and in particular by Timmesfeld [30, 31].

Our main result can be viewed as a geometric version of some of the results by Timmesfeld. This connection has been explored in [16].

Let G be a group generated by a conjugacy class of \mathbb{K} -transvection groups Σ . We construct a geometry $\Pi(\Sigma)$ with point set Σ . Lines are defined as follows. For each non-commuting pair $A, B \in \Sigma$ the line on A and B is the set of elements C from Σ with $C \leq \langle A, B \rangle$.

Let Σ be a class of \mathbb{K} -transvection subgroups of the group G . If \mathbb{K} contains at least 4 elements, then, as follows from [16], we are able to show that $\Pi(\Sigma)$ satisfies the following conditions as given in (3.1), which implies that we can invoke our main results, Theorem 3.12 and Theorem 1.1. This leads to the following result (for details, see [16]).

7.2. Theorem. *Let \mathbb{K} be a field and G be a group generated by a class Σ of \mathbb{K} -transvection subgroups. Suppose that \mathbb{K} contains at least 4 elements and that G has no nontrivial nilpotent normal subgroup. If Σ contains two commuting elements A and B with $C_\Sigma(A) \neq C_\Sigma(B)$, then (Σ, \perp) is the polar graph of a non-degenerate polar space of rank at least 2.*

The above result together with the work of Steinbach [29, 28] and Timmesfeld [30] provides a full classification of the groups G as in the hypothesis of the above theorem.

When we allow the field \mathbb{K} to have order 2 or 3, the situation is different. In that case we obtain a geometry in which 1.1 (d) might be violated. The additional geometries appearing in these cases are the so-called *generalized Fischer spaces*, see

[8, 16]. Combining the results of this paper, [16] and various results on 3-transposition groups leads to a classification of all groups generated by a class Σ of \mathbb{K} -transvection groups, \mathbb{K} an arbitrary field, having no nontrivial nilpotent normal subgroups. For details the reader is referred to [16].

7.3. Extremal elements in Lie algebras. Another situation in which one can apply our main result Theorem 1.1 is the following. Let \mathcal{L} be a Lie algebra over the field \mathbb{K} . An element $x \in \mathcal{L}$ is called *extremal* if $[x, [x, \mathcal{L}]] \subseteq \mathbb{K}x$. Extremal elements in Lie algebras are studied by Cohen et al. [6] and Cohen and Ivanyos [7]. In [6] it has been shown that for any two distinct extremal elements $x, y \in \mathcal{L}$ we have one of the following:

- (a) $[x, y] = 0$;
- (b) $[x, y]$ is an extremal element;
- (c) $\langle x, y \rangle$ is isomorphic to $\mathfrak{sl}(2, \mathbb{K})$.

If case (b) does not occur, then the geometry with as points the 1-dimensional subspace $\mathbb{K}x$ of \mathcal{L} and as lines the sets of points contained in an $\mathfrak{sl}(2, \mathbb{K})$ -subalgebra generated by two extremal elements is again a geometry satisfying the conditions of the Setting 3.1; see [24]. Theorem 3.12 and Theorem 1.1 can now be invoked to obtain an important step in the classification of Lie algebras generated by extremal elements.

7.4. Local recognition of graphs. Our final application is concerned with local recognition of graphs. Let Γ be a graph and x a vertex of Γ . By Γ_x we denote the subgraph of Γ induced on the set of neighbors of x .

Consider the graph $\Gamma = \Gamma(\mathcal{S})$ with as vertices the set the hyperbolic lines of a non-degenerate classical polar space \mathcal{S} , two vertices being adjacent if they are perpendicular to each other. Then for each vertex x of Γ the induced subgraph Γ_x is isomorphic to $\Gamma(\mathcal{T})$, where \mathcal{T} is the polar subspace x^\perp of \mathcal{S} .

For example, if \mathcal{S} is a non-degenerate symplectic polar space of rank n , then \mathcal{T} is a symplectic space of rank $n - 1$. It has been proved by Gramlich [21] that in case $n > 5$ the graph Γ is the unique connected graph such that for each vertex x the graph Γ_x is isomorphic to $\Gamma(\mathcal{T})$. In the proof of that result, Gramlich uses the main result of [9] to identify $\Gamma(\mathcal{S})$. The Theorems 1.1 and 1.2 provide tools to extend Gramlich's methods to obtain similar local recognition results on the graphs of hyperbolic lines of arbitrary polar spaces. This approach has been applied successfully in [1].

References

- [1] C. Altmann and R. Gramlich, On the hyperbolic unitary geometry, *J. Algebraic Combin.* **31** (2010), 547-583. \uparrow 4, 21

- [2] M. Aschbacher, A characterization of the unitary and symplectic groups over fields of characteristic at least 5, *Pac. J. Math.* **47** (1973), 5-26. ↑ 20
- [3] M. Aschbacher and M. Hall Jr., Groups generated by a class of elements of order 3, *J. of Algebra* **24** (1973), 591-612. ↑ 20
- [4] F. Buekenhout, *La géométrie des groupes de Fischer*, unpublished notes, Free University of Brussels, 1974. ↑ 3
- [5] F. Buekenhout and E. Shult, On the foundations of polar geometry. *Geom. Dedicata* **3** (1974), 155-170. ↑ 1
- [6] A.M. Cohen, A. Steinbach, R. Ushirobira, D. Wales, Lie algebras generated by extremal elements, *J. Algebra* **236** (2001), no. 1, 122-154. ↑ 21
- [7] A.M. Cohen, G. Ivanyos, Root filtration spaces from Lie algebras and abstract root groups. *J. Algebra* **300** (2006), no. 2, 433-454. ↑ 4, 21
- [8] H. Cuypers, On a generalization of Fischer spaces, *Geom. Dedicata* **34** (1990), 67-87. ↑ 20
- [9] H. Cuypers, Symplectic geometries, transvection groups and modules, *J. of Comb. Th. A*, **65** (1994), 39-59. ↑ 3, 21
- [10] H. Cuypers, Generalized Fischer spaces, in *Finite geometry and combinatorics* (Deinze, 1992), 121-129, London Math. Soc. Lecture Note Ser., **191**, Cambridge Univ. Press, Cambridge, 1993. ↑ 3
- [11] H. Cuypers and J.I. Hall, The 3-transposition groups with trivial center, *J. Algebra* **178** (1995), no. 1, 149-193. ↑ 3, 9
- [12] H. Cuypers and J.I. Hall, The classification of 3-transposition groups with trivial center, in *Groups, Combinatorics and Geometry*, proceedings of the L.M.S symposium on Groups and Combinatorics, Durham, 1990, eds M. Liebeck and J. Saxl, L.M.S. Lecture Notes Ser. **165**, 121-138, Cambr. Univ. Press, 1992. ↑ 3, 9, 19
- [13] H. Cuypers, P. Johnson and A. Pasini, On the embeddability of polar spaces, *Geom. Dedicata* **44** (1992), 349-358. ↑ 19
- [14] H. Cuypers and E.E. Shult, On the classification of generalized Fischer spaces, *Geom. Dedicata* **34** (1990), 89-99. ↑ 3, 7
- [15] H. Cuypers, A. Steinbach, Linear transvection groups and embedded polar spaces, *Invent. Math.* **137** (1999), no. 1, 169-198. ↑ 20

- [16] H. Cuypers, The geometry of k -transvections groups, *J. Algebra* **300** (2006), no. 2, 455-471. ↑ 4, 9, 20, 21
- [17] H. Cuypers, The geometry of secants in embedded polar spaces, *European J. Combin.* **28** (2007), no. 5, 1455-1472. ↑ 11
- [18] H. Cuypers, M. Horn, J. in 't panhuis and S. Shpectorov, *Lie algebras and 3-transpositions*, submitted. ↑ 3, 19
- [19] B. Fischer, *Finite groups generated by 3-transpositions*, University of Warwick lecture notes, 1969. ↑ 20
- [20] B. Fischer, Finite groups generated by 3-transpositions I, *Inv. Math.* **13** (1971), 232-246. ↑ 20
- [21] R. Gramlich, On the hyperbolic symplectic geometry, *J. Combin. Theory Ser. A* **105** (2004), no. 1, 97-110. ↑ 4, 21
- [22] J.I. Hall, The hyperbolic lines of finite symplectic spaces, *J. Combin. Theory Ser. A* **47** (1988), no. 2, 284-298. ↑ 3
- [23] D. Higman, Admissible graphs, in *Finite geometries* Proceedings of a Conference in honor of T. G. Ostrom held at Washington State University, Pullman, Wash., April 7-11, 1981. Edited by Norman L. Johnson, Michael J. Kallaher and Calvin T. Long, Pullman, Wash. (1981), pp. 211-222, Lecture Notes in Pure and Appl. Math., 82, Dekker, New York, 1983. ↑ 10
- [24] J. in 't panhuis, *Lie algebras, extremal elements, and geometries*, Thesis, Eindhoven University of Technology, 2009.
<http://alexandria.tue.nl/extra2/200612932.pdf> ↑ 4, 21
- [25] P. Johnson, Polar spaces of arbitrary rank, *Geom. Dedicata* **35** (1990), 229-250. ↑ 1, 19
- [26] S. Payne and J. Thas, *Finite generalized quadrangles*, Pitman Publ., 1984. ↑ 18
- [27] M. O'Nan, Automorphisms of unitary block designs, *J. Algebra* **20** (1972), 495-511. ↑ 3
- [28] A. Steinbach, Gruppen, die von k -Transvectionen erzeugt werden, *Mitt. Math. Sem. Univ. Giessen* **211** (1992), 27-48. ↑ 20
- [29] A. Steinbach, Generalized quadrangles arising from groups generated by abstract transvection groups in *Groups and geometries* (Siena, 1996), 189-199, Trends Math., Birkhäuser, Basel, 1998. ↑ 20

- [30] F.G. Timmesfeld, Groups generated by k -transvections, *Inv. Math.* **100** (1990), 169-206. ↑ 1, 9, 20
- [31] F.G. Timmesfeld, *Abstract root subgroups and simple groups of Lie-type*, Monographs Math. **95**. Birkhäuser, Basel 2001. ↑ 1, 20
- [32] J. Tits, *Buildings of spherical type and finite BN-pairs*, Lecture Notes in Math. **386**, Springer Verlag, 1974. ↑ 1, 5, 6, 8, 9, 19
- [33] F. Veldkamp, Polar geometry, I, II, III, IV, V. *Nederl. Akad. Wetensch. Proc. Ser. A* **62** and **63** (1959), 512-551, 207-212.

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