Realization theory for LPV state-space representations with affine dependence

Realization Theory for LPV State-Space Representations with Affine Dependence

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***Abstract*—We present a Kalman-style realization theory for linear parameter-varying state-space representations whose matrices depend on the scheduling variables in an affine way (abbreviated as LPV-SSA). We show that minimality of LPV-SSAs is equivalent to observability and span-reachability rank conditions, and that minimal LPV-SSAs of the same input-output map are isomorphic. We present necessary and sufficient conditions for existence of an LPV-SSA in terms of the rank of a Hankel-matrix and a Ho-Kalman-like realization algorithm.**

I. Introduction

Linear parameter-varying (LPV) systems are linear systems where the coefficients are functions of a time-varying signal, the so-called *scheduling variable*. Control design and system identification of LPV systems is a popular topic [1]–[11].

Despite these advances, realization theory of LPV systems has not been completely solved. The goal of realization theory is to characterize LPV state-space representations which describe the same set of input-output trajectories, and to construct such an LPV state-space representation from a suitable representation of the input-output behavior.

Contribution: In this paper we present a realization theory for LPV state-space representations with affine static dependence of coefficients, abbreviated as *LPV-SSA*. We consider both the *discrete-time* (DT) and the *continuous-time* (CT) cases. We present a necessary and sufficient condition for an input-output map to admit a realization by an LPV-SSA. This condition involves the rank of a suitably defined Hankel-matrix. We show that a minimal LPV-SSA realization of an input-output map can be calculated from the Hankel-matrix using a Ho-Kalman-like realization algorithm. We show that minimality is equivalent to observability and span-reachability and that all minimal LPV-SSA realizations of the same input-output map are isomorphic. The latter isomorphism is linear and does not depend on the scheduling variable.

Motivation for realization theory: Realization theory can serve as a tool for analyzing system identification and model reduction algorithms and can be the cradle of new ones. This was the case for linear time-invariant (LTI) systems, where it was used for analysis of subspace identification algorithms [12], parametric identification algorithms [13] and for model reduction [14]. Realization theory is also useful as a theoretical tool for control theory. For LTI systems, it was essential in the proof of many basic results: internal model principle, Bounded Real Lemma, etc. We expect the same for LPV systems. In fact, the results of this paper were already used [15]–[18]. Besides, it is important to characterize equivalent LPV model representations, irrespectively of whether they are obtained via system identification or from the first principles.

Motivation for LPVS-SSAs: LPV-SSAs are popular models for control synthesis, model reduction and system identification. This popularity is due to the existence of efficient control synthesis algorithms for LPV-SSAs [1], [2]. In contrast, control synthesis methods for LPV models with a nonlinear and dynamic dependence on the scheduling variables tend to be computationally hard.

Related work: In [4], [19], realization theory was developed for LPV state-space representations where the system matrices depend on the parameters in a *meromorphic and dynamic way*, *i.e.*, the matrices are meromorphic functions of the scheduling variables and their derivatives (in continuous-time), or of the current and future values of the scheduling variables (discrete-time). The system theoretic transformations (passing from an input-output to a state-space representation, transforming a state-space representation to a minimal one, etc.) of [4], [19] introduce LPV models with a dynamic and nonlinear dependence on the parameters. In [20], using [21], realizability of LPV input-output model by LPV state-space representations with a *nonlinear* (hence not necessarily affine) and static dependence is studied. In contrast, we deal with the realizability of input-output maps and not of input-output equations, and we are interested in LPV state-space representations with affine and static dependence on the parameter. That is, [4], [19], [20] do not solve the realization problem for LPV-SSAs. Hankel-matrices and Markov-coefficients of LPV-SSAs appeared in [5], [7], [10], but in contrast to [5], [7], [10], in the current paper, these concepts are defined directly for input-output maps, and they are used to characterize existence of an LPV-SSA realization of an input-output map. Extended observability and reachability matrices were also presented in [4], [22], but their system-theoretic interpretation and their relationship with minimality were not explored. The problem studied in [11], namely, the existence of a stable LPV state-space representation which reproduces a given stable transfer function for each constant scheduling signal, is related to but different from the problem studied in this paper. This paper is an extended version of [23], [24], which present the results

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without detailed proofs and which deal only with the DT case. A version of the present paper is available in the report [25]. We use realization theory of linear switched systems [26]–[29] to prove the results of the present paper. Linear switched systems are a very special case of LPV-SSAs, but the latter is more general, hence [26]–[29] cannot be directly applied to LPV-SSA, for that several non-trivial steps are needed.

Outline: In Section II we present the definition of LPV-SSAs, input-output maps, equivalence and minimality. In Section III, the main results of the paper are presented. All proofs are collected in Appendix A.

II. Preliminaries

A. Notation and terminology

Let $\mathbb{N}$ be the set of all natural numbers including zero. For a finite set X, denote by $\mathcal{S}(X)$ the set of finite sequences generated by elements from X, *i.e.*, each $s \in \mathcal{S}(X)$ is of the form $s = \zeta_1\zeta_2\cdots\zeta_k$ with $\zeta_1, \zeta_2, \ldots, \zeta_k \in X$, $k \in \mathbb{N}$; $|s|$ denotes the length of the sequence s. For $s, r \in \mathcal{S}(X)$, $sr \in \mathcal{S}(X)$ denotes the concatenation of s and r. The symbol ε is used for the empty sequence and $|\varepsilon| = 0$ with $s\varepsilon = \varepsilon s = s$. Denote by $X^{\mathbb{N}}$ the set of all functions of the form $f : \mathbb{N} \to X$. Let $\mathbb{I}_{\tau_1}^{\tau_2} = \{s \in \mathbb{Z} \mid \tau_1 \leq s \leq \tau_2\}$ be an index set.

Let $\mathbb{T} = \mathbb{R}_0^+ = [0, +\infty)$ be the time axis in the *continuous-time* (CT) case and $\mathbb{T} = \mathbb{N}$ in the *discrete-time* (DT) case. Note that in both cases we exclude negative time instances. Denote by ξ *the differentiation operator* $\frac{d}{dt}$ *(in CT) and the forward time-shift operator q (in DT)*, *i.e.*, if $z : \mathbb{T} \to \mathbb{R}^n$, then $(\xi z)(t) = \frac{d}{dt}z(t)$, if $\mathbb{T} = \mathbb{R}_0^+$, and $(\xi z)(t) = z(t+1)$, if $\mathbb{T} = \mathbb{N}$. Denote by ξ^k the k-fold application of ξ, *i.e.*, for any $z : \mathbb{T} \to \mathbb{R}^n$, $\xi^0 z = z$, and $\xi^{k+1} z = \xi(\xi^k z)$ for all $k \in \mathbb{N}$.

A function $f : \mathbb{R}_0^+ \to \mathbb{R}^n$ is called *piecewise-continuous*, if f has finitely many points of discontinuity on any compact subinterval of $\mathbb{R}_0^+$ and, at any point of discontinuity, the left-hand and right-hand side limits of f exist and are finite. We denote by $\mathcal{C}_{\mathrm{p}}(\mathbb{R}_0^+, \mathbb{R}^n)$ the set of all *piecewise-continuous functions* of the above form. We denote by $\mathcal{C}_{\mathrm{d}}(\mathbb{R}_0^+, \mathbb{R}^n)$ the set of all differentiable functions of the form $f : \mathbb{R}_0^+ \to \mathbb{R}^n$.

B. System theoretic definitions

An LPV *state-space* (SS) representation with *affine* linear dependence on the *scheduling variable* (abbreviated as LPV-SSA) is a continuous-time (CT) or discrete-time (DT) state-space representation of the form

$$\Sigma \begin{cases} (\xi x)(t) = A(p(t))x(t) + B(p(t))u(t), \\ y(t) = C(p(t))x(t) + D(p(t))u(t), \end{cases} \quad (1)$$

where $x(t) \in \mathbb{X} = \mathbb{R}^{n_{\mathrm{x}}}$ is the state, $y(t) \in \mathbb{Y} = \mathbb{R}^{n_{\mathrm{y}}}$ is the output, $u(t) \in \mathbb{U} = \mathbb{R}^{n_{\mathrm{u}}}$is the input, and $p(t) \in \mathbb{P} \subseteq \mathbb{R}^{n_{\mathrm{p}}}$ is the value of the *scheduling variable* at time t, and A, B, C, D are matrix valued functions on $\mathbb{P}$ defined as

$$A(\mathbb{p}) = A_0 + \sum_{i=1}^{n_{\mathrm{p}}} A_i \mathbb{p}_i, \quad B(\mathbb{p}) = B_0 + \sum_{i=1}^{n_{\mathrm{p}}} B_i \mathbb{p}_i,$$
$$C(\mathbb{p}) = C_0 + \sum_{i=1}^{n_{\mathrm{p}}} C_i \mathbb{p}_i, \quad D(\mathbb{p}) = D_0 + \sum_{i=1}^{n_{\mathrm{p}}} D_i \mathbb{p}_i, \quad (2)$$

for every $\mathbb{p} = [\ \mathbb{p}_1 \ \ \mathbb{p}_2 \ \ \cdots \ \ \mathbb{p}_{n_{\mathrm{p}}}\ ]^\top \in \mathbb{P}$, with constant matrices $A_i \in \mathbb{R}^{n_{\mathrm{x}} \times n_{\mathrm{x}}}$, $B_i \in \mathbb{R}^{n_{\mathrm{x}} \times n_{\mathrm{u}}}$, $C_i \in \mathbb{R}^{n_{\mathrm{y}} \times n_{\mathrm{x}}}$ and $D_i \in \mathbb{R}^{n_{\mathrm{y}} \times n_{\mathrm{u}}}$ for all $i \in \mathbb{I}_0^{n_{\mathrm{p}}}$. Recall that $(\xi x)(t) = \frac{d}{dt}x(t)$ in CT, and $(\xi x)(t) = x(t+1)$ in DT. *It is assumed that* $\mathbb{P}$ *contains an affine basis of* $\mathbb{R}^{n_{\mathrm{p}}}$ (see [30] for the definition of an affine basis). In the sequel, we use the tuple

$$\Sigma = (\mathbb{P}, \{A_i, B_i, C_i, D_i\}_{i=0}^{n_{\mathrm{p}}})$$

to denote an LPV-SSA of the form (1) and use $\dim(\Sigma) = n_{\mathrm{x}}$ to denote its state dimension. Define $\mathcal{X} = \mathcal{C}_{\mathrm{d}}(\mathbb{R}_0^+, \mathbb{X})$, $\mathcal{Y} = \mathcal{C}_{\mathrm{p}}(\mathbb{R}_0^+, \mathbb{Y})$, $\mathcal{U} = \mathcal{C}_{\mathrm{p}}(\mathbb{R}_0^+, \mathbb{U})$, $\mathcal{P} = \mathcal{C}_{\mathrm{p}}(\mathbb{R}_0^+, \mathbb{P})$ in CT and $\mathcal{X} = \mathbb{X}^{\mathbb{N}}$, $\mathcal{Y} = \mathbb{Y}^{\mathbb{N}}$, $\mathcal{U} = \mathbb{U}^{\mathbb{N}}$, $\mathcal{P} = \mathbb{P}^{\mathbb{N}}$ in DT. By a solution of Σ we mean a tuple of trajectories $(x, y, u, p) \in (\mathcal{X}, \mathcal{Y}, \mathcal{U}, \mathcal{P})$ such that (1) holds for all $t \in \mathbb{T}$. For an initial state $x_{\mathrm{o}} \in \mathbb{X}$ define the *input-to-state map* $\mathfrak{X}_{\Sigma,x_{\mathrm{o}}}$ and the *input-output* map $\mathfrak{Y}_{\Sigma,x_{\mathrm{o}}}$ of Σ induced by x_{o} as

$$\mathfrak{X}_{\Sigma,x_{\mathrm{o}}} : \mathcal{U} \times \mathcal{P} \to \mathcal{X}, \quad \mathfrak{Y}_{\Sigma,x_{\mathrm{o}}} : \mathcal{U} \times \mathcal{P} \to \mathcal{Y}, \quad (3)$$

such that for any $(x, y, u, p) \in \mathcal{X} \times \mathcal{Y} \times \mathcal{U} \times \mathcal{P}$, $x = \mathfrak{X}_{\Sigma,x_{\mathrm{o}}}(u, p)$ and $y = \mathfrak{Y}_{\Sigma,x_{\mathrm{o}}}(u, p)$ holds if and only if (x, y, u, p) is a solution of (1) and $x(0) = x_{\mathrm{o}}$.

We say that Σ is *span-reachable* from an initial state $x_{\mathrm{o}} \in \mathbb{X}$, if $\mathrm{Span}\{\mathfrak{X}_{\Sigma,x_{\mathrm{o}}}(u,p)(t) \mid (u,p) \in \mathcal{U} \times \mathcal{P}, t \in \mathbb{T}\} = \mathbb{X}$. We say that Σ is *observable*, if for any two initial states $\bar{x}_{\mathrm{o}}, \hat{x}_{\mathrm{o}} \in \mathbb{R}^{n_{\mathrm{x}}}$, $\mathfrak{Y}_{\Sigma,\hat{x}_{\mathrm{o}}} = \mathfrak{Y}_{\Sigma,\bar{x}_{\mathrm{o}}}$ implies $\hat{x}_{\mathrm{o}} = \bar{x}_{\mathrm{o}}$. That is, observability means that for any two distinct states of the system, the resulting outputs will be different for some input and scheduling signals. Let Σ of the form (1) and $\Sigma' = (\mathbb{P}, \{A_i', B_i', C_i', D_i'\}_{i=0}^{n_{\mathrm{p}}})$ with $\dim(\Sigma) = \dim(\Sigma') = n_{\mathrm{x}}$. A nonsingular matrix $T \in \mathbb{R}^{n_{\mathrm{x}} \times n_{\mathrm{x}}}$ is said to be an *isomorphism* from Σ to Σ', if

$$\forall i \in \mathbb{I}_0^{n_{\mathrm{p}}} : A_i' T = T A_i, \ B_i' = T B_i, \ C_i' T = C_i, \ D_i' = D_i.$$

We formalize the input-output behavior of LPV-SSAs as maps of the form

$$\mathfrak{F} : \mathcal{U} \times \mathcal{P} \to \mathcal{Y}. \quad (4)$$

While any input-output map of an LPV-SSA induced by some initial state is of the above form, the converse is not true. The LPV-SSA Σ is a *realization* of an input-output map $\mathfrak{F}$ of the form (4) from *the initial state* $x_{\mathrm{o}} \in \mathbb{X}$, if $\mathfrak{F} = \mathfrak{Y}_{\Sigma,x_{\mathrm{o}}}$. When the specific choice of the initial state is not of interest, we will say Σ is *realization* of $\mathfrak{F}$, if Σ is a realization of $\mathfrak{F}$ from some initial state x_{o}. An LPV-SSA Σ is a *minimal realization of* $\mathfrak{F}$ *from the initial state* x_{o}, if Σ is a realization of $\mathfrak{F}$ from the initial state x_{o}, and for every LPV-SSA Σ' which is a realization of $\mathfrak{F}$, $\dim(\Sigma) \leq \dim(\Sigma')$. Again, when the specific choice of the initial state is not of interest, we say that Σ is a *minimal realization* of $\mathfrak{F}$, if Σ is a minimal realization of $\mathfrak{F}$ from some initial state x_{o}.

In the sequel, we assume that $D_i = 0$ for all $i \in \mathbb{I}_0^{n_{\mathrm{p}}}$. Rewriting the results of the paper for the general case is an easy exercise.

III. Main results

A. Minimality

Theorem 1 (Minimal realizations)**.** *Assume that* $\mathfrak{F}$ *is an input-output map of the form* (4)*. Assume that the LPV-SSA* Σ *is a realization of* $\mathfrak{F}$ *from the initial state* x_{o}*. Then* Σ *is a minimal realization of* $\mathfrak{F}$ *from* x_{o}*, if and only if* Σ *is observable and span-reachable from* x_{o}*. Any two minimal LPV-SSA realizations of* $\mathfrak{F}$ *are isomorphic.*

The proof is presented in the Appendix. Note that Theorem 1 does not exclude the possibility that two LPV state-space representations of the same input-output map are related by a non-constant isomorphism, if these state-space representations are not minimal or they are not LPV-SSAs, see [31].

Similarly to the LTI case, but unlike for general LPV state-space representations [19], rank conditions for observability and reachability can be obtained to verify minimality for LPV-SSA. To this end, we recall the following definition from [5].

Definition 1 (Ext. reachability & observability matrices). *For an initial state x_o, the n-step extended reachability matrices $\mathcal{R}_n$ of Σ from x_o, $n \in \mathbb{N}$, are defined recursively as follows*

$$\mathcal{R}_0 = \left[\begin{array}{cccc} x_o & B_0 & \dots & B_{n_p} \end{array}\right], \tag{5a}$$

$$\mathcal{R}_{n+1} = \left[\begin{array}{cccc} \mathcal{R}_0 & A_0\mathcal{R}_n & \dots & A_{n_p}\mathcal{R}_n \end{array}\right]. \tag{5b}$$

The extended n-step observability matrices $\mathcal{O}_n$ of Σ, $n \in \mathbb{N}$, are defined recursively as follows

$$\begin{aligned} \mathcal{O}_0 &= \left[\begin{array}{ccc} C_0^\top & \dots & C_{n_p}^\top \end{array}\right]^\top, \\ \mathcal{O}_{n+1} &= \left[\begin{array}{cccc} \mathcal{O}_0^\top & A_0^\top\mathcal{O}_n^\top & \dots & A_{n_p}^\top\mathcal{O}_n^\top \end{array}\right]^\top. \end{aligned} \tag{6}$$

Theorem 2 (Rank conditions). *The LPV-SSA Σ is span-reachable from x_o, if and only if* $\operatorname{rank}\{\mathcal{R}_{n_x-1}\} = n_x$*, and Σ is observable, if and only if* $\operatorname{rank}\{\mathcal{O}_{n_x-1}\} = n_x$.

The proof is given in the Appendix. This theorem leads to the following *Kalman-decomposition* for LPV-SSAs. Consider an LPV-SSA Σ of the form (1) and an initial state $x_0 \in \mathbb{R}^{n_x}$. Choose a basis $\{b_i\}_{i=1}^{n_x} \subset \mathbb{R}^{n_x}$ such that $\mathrm{Span}\{b_1,\dots,b_r\} = \mathrm{Im}\{\mathcal{R}_{n_x-1}\}$ and $\mathrm{Span}\{b_{r_m+1},\dots,b_r\} = (\mathrm{Im}\{\mathcal{R}_{n_x-1}\} \cap \ker\{\mathcal{O}_{n_x-1}\})$ for some $r, r_m \geq 0$. Define $T = \begin{bmatrix} b_1 & b_2 & \dots & b_{n_x}\end{bmatrix}^{-1}$, and let $\hat{A}_i = TA_iT^{-1}$, $\hat{B}_i = TB_i$, $\hat{C}_i = C_iT^{-1}$, $i \in \mathbb{I}_0^{n_p}$, $\hat{x}_o = Tx_o$. Then

$$\begin{aligned} \hat{A}_i &= \begin{bmatrix} A_i^m & 0 & A_i'' \\ A_i' & \hat{A}' & A_i''' \\ 0 & 0 & A_i'''' \end{bmatrix},\ \hat{B}_i = \begin{bmatrix} B_i^m \\ B_i' \\ 0 \end{bmatrix},\ \hat{C}_i = \begin{bmatrix} (C_i^m)^\top \\ 0 \\ (C_i')^\top \end{bmatrix}^\top, \\ \hat{x}_o &= \begin{bmatrix} (x_o^m)^\top & \bar{x}_o^\top & 0 \end{bmatrix}^\top, \end{aligned} \tag{7}$$

where $A_i^m \in \mathbb{R}^{r_m \times r_m}, B_i^m \in \mathbb{R}^{r_m \times n_u}$, $C_i^m \in \mathbb{R}^{n_y \times r_m}$, $x_o^m \in \mathbb{R}^{r_m}$, $A_i''' \in \mathbb{R}^{(n-r)\times(n-r)}$, $B_i' \in \mathbb{R}^{(r-r_m)\times n_u}$, $C_i' \in \mathbb{R}^{n_y \times (n-r)}$. Clearly, $\hat{\Sigma} = (\mathbb{P}, \{\hat{A}_i, \hat{B}_i, \hat{C}_i, 0\}_{i=0}^{n_p})$ is isomorphic to Σ and can be viewed as the Kalman-decomposition of Σ.

Corollary 1. $\Sigma^m = (\mathbb{P}, \{A_i^m, B_i^m, C_i^m, 0\}_{i=0}^{n_p})$ *is a minimal realization of $\mathfrak{F} = \mathfrak{Y}_{\Sigma,x_0}$ from the initial state x_o^m.*

Example 1. *Consider an LPV-SSA Σ as in* (1)*, with $\mathbb{P} = \mathbb{R}$,*

$$A_0 = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}, B_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, C_0 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}^\top, D_0 = 0$$

$$A_1 = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}, B_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, C_1 = \begin{bmatrix} 2 \\ -2 \\ -2 \end{bmatrix}^\top, D_1 = 0.$$

Take $x_o = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^\top$. It is easy to see that $\operatorname{rank}\{\mathcal{R}_2\} = 2$ *and* $\operatorname{rank}\{\mathcal{O}_2\} = 1$*. If we set, $b_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^\top$, $b_2 = \begin{bmatrix} 2 & 1 & 1 \end{bmatrix}^\top$, $b_3 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^\top$, then $\{b_1, b_2\}$ span* $\mathrm{Im}\{\mathcal{R}_2\}$ *and b_2 spans* $\mathrm{Im}\{\mathcal{R}_2\} \cap \ker\{\mathcal{O}_2\}$*. If we apply the basis transformation $T = \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix}^{-1}$, then we obtain the matrices $\hat{A}_i = TA_iT^{-1}, \hat{B}_i = TB_i, \hat{C}_i = C_iT^{-1}$, $i = 0, 1$ and the vector $\hat{x}_o = Tx_o$ are of the form* (7)*, with $A_0^m = 2$, $A_1^m = 3$, $B_0^m = 1$, $B_1^m = -2$, $C_0^o = 1$, $C_1^m = 2$, $x_o^m = -1$. By Corollary 1, $\Sigma^m = (\mathbb{P}, \{A_i^m, B_i^m, C_i^m\}_{i=0}^{n_p})$ is a minimal realization of $\mathfrak{F} = \mathfrak{F}_{\Sigma,x_o}$ from x_o^m.*

Corollary 1 can be proven using Theorem 1 – 2 and arguments similar to the ones used for LTI systems, therefore it is omitted, see [25, page 14] for the proof. The matrix T and hence Σ^m can easily be computed from Σ, see [25] for the code. Note that for computing Σ^m, or checking the rank conditions of Theorem 2, it is sufficient to compute a basis of $\mathrm{Im}\{\mathcal{R}_{n_x-1}\}$ and $\ker\{\mathcal{O}_{n_x-1}\}$, which can be done in polynomial time w.r.t. n_p and n_x [18, Algorithm 1 – Algorithm 2].

B. Existence of a realization

First, we define the *impulse response representation (IIR)* for an input-output map. To this end, we will use the following notation.

Notation 1. *For any sequence $s \in \mathcal{S}(\mathbb{I}_0^{n_p})$, for any signal $p \in \mathcal{C}_p(\mathbb{R}_0^+, \mathbb{P})$ (in CT) or $p \in (\mathbb{R}^{n_p})^\mathbb{N}$ (in DT), and for any time instance $\tau \in \mathbb{T}$, and real number $t \in [\tau, +\infty)$ (in CT) or integer $t \in \{\tau-1, \tau, \dots,\}$ (in DT), define the* sub-Markov dependence $(w_s \diamond p)(t,\tau)$ *recursively as follows: for the empty sequence, $s = \epsilon$,*

$$(w_\epsilon \diamond p)(t,\tau) = \begin{cases} 1 & \text{for CT or (DT and } t = \tau - 1) \\ 0 & \text{for DT and } t \neq \tau - 1 \end{cases}.$$

If $s = s'i$ for some $i \in \mathbb{I}_0^{n_p}$ and $s' \in \mathcal{S}(\mathbb{I}_0^{n_p})$, then

$$(w_s \diamond p)(t,\tau) = \begin{cases} \int_\tau^t p_i(\delta) \cdot (w_{s'} \diamond p)(\delta,\tau)\, d\delta & \text{for CT} \\ p_i(t)(w_{s'} \diamond p)(t-1,\tau) & \text{for DT} \end{cases},$$

where for all $\delta \in \mathbb{T}$, $p_i(\delta)$ denotes the i^{th} entry of the vector $p(\delta) \in \mathbb{R}^{n_p}$, if $i > 0$ and $p_i(\delta) = 1$ if $i = 0$.

Definition 2 (Impulse response representation). *Let $\mathfrak{F}$ be of the form* (4)*. Then $\mathfrak{F}$ is said to have an* impulse response representation *(IIR) if there exists a function*

$$\theta_\mathfrak{F} : \mathcal{S}(\mathbb{I}_0^{n_p}) \mapsto \mathbb{R}^{(n_p+1)n_y \times (n_u(n_p+1)+1)}, \tag{8}$$

such that,

(1) *there exist constants $K, R > 0$ such that*

$$\forall s \in \mathcal{S}(\mathbb{I}_0^{n_p}) : ||\theta_\mathfrak{F}(s)||_F \leq KR^{|s|} \tag{9}$$

where $\|.\|_F$ denotes the Frobenius norm;

(2) *for every $p \in \mathcal{P}$, there exist functions $g_\mathfrak{F} \diamond p : \mathbb{T} \to \mathbb{R}^{n_y}$ and $h_\mathfrak{F} \diamond p : \{(\tau,t) \in \mathbb{T} \times \mathbb{T} \mid \tau \leq t\} \to \mathbb{R}^{n_y \times n_u}$, such that for each $(u,p) \in \mathcal{U} \times \mathcal{P}$, $t \in \mathbb{T}$,*

$$\mathfrak{F}(u,p)(t) = (g_\mathfrak{F} \diamond p)(t) + \begin{cases} \int_0^t (h_\mathfrak{F} \diamond p)(\delta,t)u(\delta)\, d\delta & CT \\ \sum_{\delta=0}^{t-1} (h_\mathfrak{F} \diamond p)(\delta,t)u(\delta) & DT \end{cases} \tag{10}$$

Moreover, for any $i, j \in \mathbb{I}_0^{n_p}$, $s \in \mathcal{S}(\mathbb{I}_0^{n_p})$, let $\eta_{i,\mathfrak{F}}(s) \in \mathbb{R}^{n_y \times 1}$ and $\theta_{i,j,\mathfrak{F}}(s) \in \mathbb{R}^{n_y \times n_u}$ be such that

$$\theta_\mathfrak{F}(s) = \begin{bmatrix} \eta_{0,\mathfrak{F}}(s) & \theta_{0,0,\mathfrak{F}}(s) & \cdots & \theta_{0,n_p,\mathfrak{F}}(s) \\ \eta_{1,\mathfrak{F}}(s) & \theta_{1,0,\mathfrak{F}}(s) & \cdots & \theta_{1,n_p,\mathfrak{F}}(s) \\ \vdots & \vdots & \dots & \vdots \\ \eta_{n_p,\mathfrak{F}}(s) & \theta_{n_p,0,\mathfrak{F}}(s) & \cdots & \theta_{n_p,n_p,\mathfrak{F}}(s) \end{bmatrix}.$$

Then $g_{\mathfrak{F}} \diamond p$ and $h_{\mathfrak{F}} \diamond p$ can be expressed via $\theta_{\mathfrak{F}}$ as

$$(g_{\mathfrak{F}} \diamond p)(t) = \sum_{\substack{i \in \mathbb{I}_0^{n_\mathrm{p}},\\ s \in \mathcal{S}(\mathbb{I}_0^{n_\mathrm{p}})}} p_i(t) \eta_{i,\mathfrak{F}}(s) (w_s \diamond p)(\bar{t}, 0), \tag{11a}$$

$$(h_{\mathfrak{F}} \diamond p)(\delta, t) = \sum_{\substack{i,j \in \mathbb{I}_0^{n_\mathrm{p}},\\ s \in \mathcal{S}(\mathbb{I}_0^{n_\mathrm{p}})}} \theta_{i,j,\mathfrak{F}}(s) p_i(t) p_j(\delta) (w_s \diamond p)(\bar{t}, \hat{\delta}), \tag{11b}$$

where $\bar{t} = t$ and $\hat{\delta} = \delta$ in CT, and $\bar{t} = t - 1$ and $\hat{\delta} = \delta + 1$ in DT. The values of the function $\theta_{\mathfrak{F}}$ will be called the sub-Markov parameters *of $\mathfrak{F}$.*

Example 2. *In order to illustrate the notation above, consider the case when $n_\mathrm{p} = 1$. If $s = 0101$, $|s| = n = 4$, then, for DT, $(w_s \diamond p)(5, 2) = p(3)p(5)$, and for CT, $(w_s \diamond p)(5, 2) = \int_2^5 p(s_1)(\int_2^{s_1}(\int_2^{s_2}(s_3 - 2)p(s_3)ds_3)ds_2)ds_1$. In this case, for DT, $(h_{\mathfrak{F}} \diamond p)(2, 5)$ is of the form*

$$\theta_{0,0,\mathfrak{F}}(00) + p(4)\theta_{0,0,\mathfrak{F}}(01) + \cdots + p(2)p(5)p(3)p(4)\theta_{1,1,\mathfrak{F}}(11).$$

For CT, $(h_{\mathfrak{F}} \diamond p)(2, 5)$ is of the form

$$\theta_{0,0,\mathfrak{F}}(\epsilon) + \cdots + p(5)p(2)\theta_{1,1,\mathfrak{F}}(\epsilon) + \cdots + \\ + p(2)p(5)\theta_{1,1,\mathfrak{F}}(101)\int_2^5 p(s_1) \int_2^{s_1}\int_2^{s_2} p(s_3) ds_3 ds_2 ds_1 + \cdots.$$

That is, in DT, $(h_{\mathfrak{F}} \diamond p)(2, 5)$ is a polynomial of $p(2), p(3), p(4), p(5)$, while in CT, it is an infinite sum of iterated integrals.

Example 3. *Next, we illustrate how the sub-Markov parameters and the maps $(h_{\mathfrak{F}} \diamond p)$, $(g_{\mathfrak{F}} \diamond p)$ relate to $\mathfrak{F}$. Let $n_\mathrm{u} = n_\mathrm{y} = 1$ and let $\mathfrak{F}$ be an input-output map of the form* (4) *in CT defined as follows:*

$$\mathfrak{F}(u, p)(t) = (1 + 2p(t))e^{2t + 3\int_0^t p(s)ds} + \\ \int_0^t (1 + 2p(t))e^{2(t-\tau) + 2\int_\tau^t p(s)ds}(1 - 2p(\tau))u(\tau)d\tau$$

Then $\mathfrak{F}$ admits an IIR with

$$(h_{\mathfrak{F}} \diamond p)(\tau, t) = (1 + 2p(t))e^{2(t-\tau) + 3\int_\tau^t p(s)ds}(1 - 2p(\tau)),$$

$$(g_{\mathfrak{F}} \diamond p)(t) = (1 + 2p(t))e^{2t + 2\int_0^t p(s)ds},$$

and for every $s \in \mathcal{S}(\mathbb{I}_0^{n_\mathrm{p}})$ which contains k 0's and l 1's, $\theta_{\mathfrak{F}}(s) = 2^k 3^l \begin{bmatrix} -1 & 1 & -2 \\ -2 & 2 & -4 \end{bmatrix}$. The LPV-SSA Σ from Example 1 is a realization of $\mathfrak{F}$ from the state $x_\mathrm{o} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^\top$.

Note that in CT, the right-hand side of (11) is an infinite sum, which raises the question of its convergence.

Lemma 1. *Under the assumptions of Definition 2 in CT, the infinite sums in the right-hand sides of* (11) *are absolutely convergent.*

The proof of Lemma 1 is presented in Appendix. We can show that there is a one-to-one correspondence between input-output maps admitting an IIR and sub-Markov parameters.

Lemma 2 (Uniqueness of the IIR)**.** *If $\mathfrak{F}$ and $\hat{\mathfrak{F}}$ are two input-output maps which admit an IIR with sub-Markov parameters $\theta_{\mathfrak{F}}$ and $\theta_{\hat{\mathfrak{F}}}$ respectively, then*

$$\mathfrak{F} = \hat{\mathfrak{F}} \iff \theta_{\mathfrak{F}} = \theta_{\hat{\mathfrak{F}}}.$$

The proof of Lemma 2 is presented in the Appendix. It turns out that any input-output map which is realizable by an LPV-SSA admits an IIR, and the sub-Markov parameters can be expressed via the matrices of this LPV-SSA realization.

Lemma 3 (Existence of the IIR)**.** *The LPV-SSA Σ of the form* (1)*, with $D_i = 0$, $i \in \mathbb{I}_0^{n_\mathrm{p}}$, is a realization of an input-output map $\mathfrak{F}$, if and only if $\mathfrak{F}$ has an IIR and, for all $i, j \in \mathbb{I}_0^{n_\mathrm{p}}$, $s \in \mathcal{S}(\mathbb{I}_0^{n_\mathrm{p}})$,*

$$\eta_{i,\mathfrak{F}}(s) = C_i A_s x_\mathrm{o}, \quad \theta_{i,j,\mathfrak{F}}(s) = C_i A_s B_j\,, \tag{12}$$

where for $s = \epsilon$, A_s is the identity matrix, and for $s = s_1 s_2 \cdots s_n$, $s_1, \ldots, s_n \in \mathbb{I}_0^{n_\mathrm{p}}$, $n > 0$, $A_s = A_{s_n} A_{s_{n-1}} \cdots A_{s_1}$.

The proof of Lemma 3 is given in the Appendix. Finally, we can formulate the necessary and sufficient conditions for the existence of an LPV-SSA realization for a given input-output map in terms of rank conditions for the Hankel-matrix. To this end, define the *lexicographic ordering* $\prec$ on $\mathcal{S}(\mathbb{I}_0^{n_\mathrm{p}})$ as follows. For any $s, r \in \mathcal{S}(\mathbb{I}_0^{n_\mathrm{p}})$, $r \prec s$ holds if either **(i)** $|r| < |s|$ (smaller length), or **(ii)** $0 < |r| = |s| = n$, and $r = r_1 \cdots r_n$, $s = s_1 \cdots s_n$, $r_i, s_j \in \mathbb{I}_0^{n_\mathrm{p}}$, and for some $l \in \{1, \cdots, n\}$, $r_i = s_i$ for $i = 1, \ldots, l - 1$, and $r_l < s_l$ with the usual ordering of integers. Note that for all $s, r \in \mathcal{S}(\mathbb{I}_0^{n_\mathrm{p}})$, $s \prec sr$ if $r \neq \epsilon$. The elements of $\mathcal{S}(\mathbb{I}_0^{n_\mathrm{p}})$ can be arranged into a sequence of ordered elements

$$\epsilon = s^{(0)} \prec s^{(1)} \prec s^{(2)} \ldots \prec s^{(i)} \prec \ldots, \tag{13}$$

i.e., any $s \in \mathcal{S}(\mathbb{I}_0^{n_\mathrm{p}})$ arises as the $i + 1$th element $s^{(i)}$ of (13) for some $i \in \mathbb{N}$. Then, the so called Hankel-matrix of $\mathfrak{F}$ both in CT and DT can be defined as follows.

Definition 3 (Hankel matrix)**.** *Consider an input-output map $\mathfrak{F}$ which has an IIR in terms of Definition 2, with the sub-Markov parameter $\theta_{\mathfrak{F}}$. For integers $k, l \geq 0$, the Hankel-matrix $\mathcal{H}_{\mathfrak{F}}(k, l)$ of $\mathfrak{F}$ is defined as*

$$\mathcal{H}_{\mathfrak{F}}(k, l) = \begin{bmatrix} \theta_{\mathfrak{F}}(s^{(0)}s^{(0)}) & \theta_{\mathfrak{F}}(s^{(1)}s^{(0)}) & \cdots & \theta_{\mathfrak{F}}(s^{(l)}s^{(0)}) \\ \theta_{\mathfrak{F}}(s^{(0)}s^{(1)}) & \theta_{\mathfrak{F}}(s^{(1)}s^{(1)}) & \cdots & \theta_{\mathfrak{F}}(s^{(l)}s^{(1)}) \\ \vdots & \vdots & \cdots & \vdots \\ \theta_{\mathfrak{F}}(s^{(0)}s^{(k)}) & \theta_{\mathfrak{F}}(s^{(1)}s^{(k)}) & \cdots & \theta_{\mathfrak{F}}(s^{(l)}s^{(k)}) \end{bmatrix},$$

where $s^{(0)}, s^{(1)}, \ldots, s^{(\max\{k,l\})}$ are as in (13)*.*

That is, the $n_\mathrm{y}(n_\mathrm{p} + 1) \times (n_\mathrm{u}(n_\mathrm{p} + 1) + 1)$ block of $\mathcal{H}_{\mathfrak{F}}(k, l)$ in the block row i and block column j equals the Markov-parameter $\theta_{\mathfrak{F}}(s)$, where $s = s^{(j)}s^{(i)} \in \mathcal{S}(\mathbb{I}_0^{n_\mathrm{p}})$ is the concatenation of the sequences $s^{(i)}$ and $s^{(j)}$.

Now we formulate conditions for the existence of an LPV-SSA realization and the correctness of Ho-Kalman algorithm (see Algorithm 1). To this end, we use the following notation: consider the sequence (13), and for all $\mu \in \mathbb{N}$, let

$$M(\mu) = \max\{i \in \mathbb{N} \mid |s^{(i)}| \leq \mu\}, \tag{14}$$

i.e., $\{s^{(0)}, s^{(1)}, \ldots, s^{(M(\mu))}\}$ is precisely the set of all elements of $\mathcal{S}(\mathbb{I}_0^{n_\mathrm{p}})$ of length at most μ.

Theorem 3 (Existence)**.** *An input-output map $\mathfrak{F}$ has a LPV-SSA realization, if and only if $\mathfrak{F}$ has an IIR and*

$$\sup_{k,l \geq 0} \operatorname{rank}\{\mathcal{H}_{\mathfrak{F}}(k, l)\} = n_{\mathfrak{F}} < \infty. \tag{15}$$

Any minimal LPV-SSA realization of $\mathfrak{F}$ has a state dimension which equals $n_{\mathfrak{F}}$.

Let Σ and x_o be the LPV-SSA and the initial state respectively returned by Algorithm 1. If $m > n \geq 0$, and

$$\operatorname{rank} H_{\mathfrak{F}}(M(n), M(m-1)) = \sup_{k,l \geq 0} \operatorname{rank}\{H_{\mathfrak{F}}(k, l)\} < \infty, \tag{16}$$

then Σ is a minimal realization of $\mathfrak{F}$ from $x_{\rm o}$. The condition (16) *holds if there exists an LPV-SSA realization of $\mathfrak{F}$ of dimension at most $n-1$.*

The proof is given in the Appendix.

Algorithm 1 Ho-Kalman realization algorithm

Require: size parameters $n, m \in \mathbb{N}$ with $m > n \geq 0$, a Hankel matrix $\mathcal{H}_{\mathfrak{F}}(M(n), M(m))$ for an input-output map $\mathfrak{F}$, where $M(n), M(m)$ are as in (14).

1: Compute the *economical singular value decomposition* (SVD) of $\mathcal{H}_{\mathfrak{F}}(M(n), M(m))$
$$\mathcal{H}_{\mathfrak{F}}(M(n), M(m)) = USV^\top,$$
where $S \in \mathbb{R}^{n_{\rm x} \times n_{\rm x}}$ is a diagonal matrix with strictly positive elements on the diagonal, and $n_{\rm x}$=rank $\mathcal{H}_{\mathfrak{F}}(M(n), M(m))$.

2: Consider the decomposition
$$S^{\frac{1}{2}} V^\top = [\ R^{(s^{(0)})} \quad \dots \quad R^{(s^{M(m)})}\ ],$$
where each block $R^{(s^{(i)})}$ is $n_{\rm x} \times (n_{\rm u}(n_{\rm p}+1)+1)$. Define
$$\bar{\mathcal{R}} = \begin{bmatrix} R^{(s^{(0)})} & \dots & R^{(s^{(M(m-1))})} \end{bmatrix},$$
$$\tilde{\mathcal{R}}_i = [\ R^{(s^{(0)}i)} \quad \dots \quad R^{(s^{(M(m-1))}i)}\ ], \quad i \in \mathbb{I}_0^{n_{\rm p}}.$$
Not that for all $i \in \mathbb{I}_0^{n_{\rm p}}$, $\kappa \in \{0, \dots, M(m-1)\}$ there exists $j \in \{1, \dots, M(m)\}$ such that $s^{(\kappa)} i = s^{(j)}$.

3: Let $x_{\rm o} \in \mathbb{R}^{n_{\rm x}}$, $A_i \in \mathbb{R}^{n_{\rm x} \times n_{\rm x}}$, $B_i \in \mathbb{R}^{n_{\rm x} \times n_{\rm u}}$, $C_i \in \mathbb{R}^{n_{\rm y} \times n_{\rm x}}$, $i \in \mathbb{I}_0^{n_{\rm p}}$ be such that
$$A_i = \tilde{\mathcal{R}}_i (\bar{\mathcal{R}}^\top \bar{\mathcal{R}})^{-1} \bar{\mathcal{R}}^\top, \quad i \in \mathbb{I}_0^{n_{\rm p}},$$
$$[x_{\rm o} \ \ B_0 \ \cdots \ B_{n_{\rm p}}] = R^{(s^{(0)})},$$
$$\begin{bmatrix} C_0^\top & C_1^\top & \cdots & C_{n_{\rm p}}^\top \end{bmatrix}^\top : \text{first } n_{\rm y}(n_{\rm p}+1) \text{ rows of } US^{\frac{1}{2}}.$$

4: **return** : $\Sigma = (\mathbb{P}, \{A_i, B_i, C_i, 0\}_{i=0}^{n_{\rm p}})$ and $x_{\rm o}$.

Example 4. *Consider the input-output map $\mathfrak{F}$ from Example 3. For every $v \in \mathcal{S}(\mathbb{I}_0^{n_{\rm p}})$, $\theta_{\mathfrak{F}}(v) = 2^\alpha 3^\beta \theta_{\mathfrak{F}}(\epsilon)$, where α and β are the number of occurrences of* 0 *and* 1 *in v. From this it follows that all the columns of $H_{\mathfrak{F}}(n, m)$ are scalar multiples of the first column of $H_{\mathfrak{F}}(n, m)$. Algorithm 1 applied to to*
$$H_{\mathfrak{F}}(1,2) = \begin{bmatrix} -1 & 1 & -2 & -2 & 2 & -4 & -3 & 3 & -6 \\ -2 & 2 & -4 & -4 & 4 & -8 & -6 & 6 & -12 \end{bmatrix}$$
yields $A_0 = 2$, $A_1 = 3$, $C_0 = -2.0245$, $C_1 = -4.0491$, $x_{\rm o} = 0.4939$, $B_0 = -0.4939$, $B_1 = 0.9879$. It is easy to see that Σ returned by Algorithm 1 is a realization of $\mathfrak{F}$ from the initial state $x_{\rm o}$. In fact, the LPV-SSA returned by Algorithm 1 is isomorphic to the minimal LPV-SSA $\Sigma_{\rm m}$ from Example 4. The code implementing Algorithm 1 is available in [25].

Note that not all input-output maps of the form (4) have an IIR and nor an LPV-SSA realization. Even if an input-output map has a IIR, the rank condition of Theorem 3 may fail. In the latter case, the input-output map may have an LPV state-space realization, but with a more general form of coefficient dependence, *e.g.*, rational, dynamic, etc.

IV. Conclusions

We have presented a complete realization theory for LPV-SSAs, which mirrors the results for LTI, bilinear [32] and switched linear systems [26]–[28]. In fact, the latter was used to prove the results of this paper. Future research will be directed towards extending the obtained results to the stochastic case, and applying them to systems identification and model reduction of LPV-SSA representations and to control synthesis for LPV-SSAs.

Appendix

A. Proof of Lemma 1, Lemma 2 and Lemma 3

In order to present the proofs of these results for the CT case, we need the following slight generalization of *generating series* from [32], [33]: we define a *generating series* as a function $c : \mathcal{S}(\mathbb{I}_0^{n_{\rm p}}) \to \mathbb{R}^{n_r \times n_l}$ for some integers $n_l, n_r > 0$, such that there exist $K, R > 0$ which satisfy $\forall v \in \mathcal{S}(\mathbb{I}_0^{n_{\rm p}})$: $||c(v)||_F \leq K R^{|v|}$. Here, $||.||_F$ denotes the Frobenius norm for matrices. Note that c is a generating series according to the above definition, if and only if each entry of c is a generating series in the sense of [32]. From [32], [33] it then follows that the infinite sum $\sum_{v \in \mathcal{S}(\mathbb{I}_0^{n_{\rm p}})} c(v)(w_v \diamond p)(t, 0)$ is absolutely convergent in the usual topology of matrices. We can then define the function $F_c : \mathcal{C}_p(\mathbb{R}_0^+, \mathbb{R}^{n_p}) \to \mathcal{C}_p(\mathbb{R}_0^+, \mathbb{R}^{n_r \times n_l})$ as $F_c(u)(t) = \sum_{v \in \mathcal{S}(\mathbb{I}_0^{n_{\rm p}})} c(v)(w_v \diamond p)(t, 0)$.

We extend the notion of generating series to the DT case. A function $c : \mathcal{S}(\mathbb{I}_0^{n_{\rm p}}) \to \mathbb{R}^{n_r \times n_l}$ is called a generating series, and we define the function $F_c : (\mathbb{R}^{n_p})^{\mathbb{N}} \to \mathcal{Y} = Y^{\mathbb{N}}$ such that $F_c(p)(t) = \sum_{v \in \mathcal{S}(\mathbb{I}_0^{n_{\rm p}})} c(v)(w_v \diamond p)(t-1, 0)$.

Note that $\eta_{i,\mathfrak{F}}, \theta_{i,j,\mathfrak{F}}$, $i, j \in \mathbb{I}_0^{n_{\rm p}}$, from Definition 2 can be viewed as generating series and hence the functions $F_{\theta_{i,j,\mathfrak{F}}}$ $F_{\eta_{i,\mathfrak{F}}}$ are well defined on $\mathcal{P}$.

Proof of Lemma 1: Notice that $\sum_{s \in \mathcal{S}(\mathbb{I}_0^{n_{\rm p}})} \eta_{i,\mathfrak{F}}(s)(w_s \diamond p)(t, 0) = F_{\eta_{i,\mathfrak{F}}}(p)(t)$ and $\sum_{s \in \mathcal{S}(\mathbb{I}_0^{n_{\rm p}})} \theta_{i,j,\mathfrak{F}}(s)(w_s \diamond p)(t, \tau) = F_{\theta_{i,j,\mathfrak{F}}}(q_\tau(p))(t-\tau)$, $\forall \delta \in \mathbb{T} : q_\tau(p)(\delta) = p(\tau + \delta)$ and hence by [32], [33] these infinite sums are absolutely convergent. ■

For the proof of Lemma 2, we need the following extension of the results of [34], [35].

Lemma 4. *Assume that $\mathbb{P} \subseteq \mathbb{R}^{n_{\rm p}}$ contains an affine basis of $\mathbb{R}^{n_{\rm p}}$. Then for any two generating series c_1, c_2,*
$$\{\forall p \in \mathcal{P} : F_{c_1}(p) = F_{c_2}(p)\} \implies c_1 = c_2.$$

Proof: Assume that $\forall p \in \mathcal{P} : F_{c_1}(p) = F_{c_2}(p)$. Note that $c_i(\epsilon) = F_{c_i}(p)(0)$, $i = 1, 2$ and hence $c_1(\epsilon) = c_2(\epsilon)$. It is left to show that $c_1(v) = c_2(v)$ for all $v \in \mathcal{S}(\mathbb{I}_0^{n_{\rm p}})$, $|v| > 0$. To this end, for any integer $k > 0$ define the function $G_{i,k}$
$$G_{i,k}(\mathbb{p}^1, \dots, \mathbb{p}^k) = \sum_{q_1 \cdots q_k \in \mathbb{I}_0^{n_{\rm p}}} c_i(q_1 \cdots q_k) \mathbb{p}_{1,q_1} \cdots \mathbb{p}_{k,q_k},$$
$i = 1, 2$, where $\mathbb{p}_{l,0} = 1$ and $\mathbb{p}^l = \begin{bmatrix} \mathbb{p}_{l,1} & \dots & \mathbb{p}_{l,n_p} \end{bmatrix}^\top \in \mathbb{R}^{n_{\rm p}}$, $l = 1, \dots, k$. We show that $G_{1,k}$ and $G_{2,k}$ are equal on $\mathbb{P}^k$. For DT, this follows from $F_{c_i}(p)(k) = G_{i,k}(p(0), \dots, p(k-1))$ for all $p \in \mathcal{P}$, $k > 0$. For CT, consider a piecewise-constant $p \in \mathcal{P}$, such that there exists $0 < t_1, \cdots, t_k \in \mathbb{R}$, and $\mathbb{p}^1, \dots, \mathbb{p}^k \in \mathbb{P}$. such that $p(s) = \mathbb{p}^i \in \mathbb{P}$, $s \in [\sum_{j=1}^{i-1} t_i, \sum_{j=1}^{i} t_i)$, $i = 1, \dots, k$. From [34, Lemma 2.1] it follows that for fixed $\{\mathbb{p}_i\}_{i=1}^k$, $F_{c_i}(p)(\sum_{j=1}^k t_j)$ is an analytic function of $t_1, \dots, t_k$ and
$$G_{i,k}(\mathbb{p}^1, \dots, \mathbb{p}^k) = \frac{\partial^k F_{c_i}(p)(\sum_{j=1}^k t_j)}{\partial t_1 \dots \partial t_k} |_{t_1 = \cdots = t_k = 0} \quad (17)$$
for $i = 1, 2$. If $\forall p \in \mathcal{P} : F_{c_1}(p) = F_{c_2}(p)$, then the left-hand sides of (17) are the same for $i = 1, 2$, and hence by (17), $G_{1,k}$ and $G_{2,k}$ are equal on $\mathbb{P}^k$ for all $k > 0$.

To conclude, we show that if $G_{1,k}$ and $G_{2,k}$ are equal on $\mathbb{P}^k$ for all $k > 0$, then $c_1 = c_2$. Notice that $c_i(q_1 \cdots q_k) =$

$G_{i,k}(e^{q_1}\cdots e^{q_k})$ for all $q_1,\ldots,q_k \in \mathbb{I}_0^{n_\mathrm{p}}$, where $e^0 = 0$ and e^i is the ith standard basis vector of $\mathbb{R}^{n_\mathrm{p}}$, i.e., all entries of e^i are zero, except the ith entry, which equals 1. Consider an affine basis $\mathbb{B} = \{b^0,\ldots,b^{n_\mathrm{p}}\} \subseteq \mathbb{P}$ of $\mathbb{R}^{n_p}$. For all $i \in \mathbb{I}_0^{n_\mathrm{p}}$, there exist $\lambda_{i,j} \in \mathbb{R}$, $j \in \mathbb{I}_0^{n_\mathrm{p}}$ such that $\sum_{j=0}^{n_\mathrm{p}} \lambda_{i,j} = 1$ and $e^i = \sum_{j=0}^{n_\mathrm{p}} \lambda_{i,j} b^j$. In particular, $e^i_k = \sum_{j=0}^{n_\mathrm{p}} \lambda_{i,j} b^j_k$ for all $k \in \mathbb{I}_0^{n_\mathrm{p}}$, where for $k > 0$, e^i_k, b^j_k denote the kth entry of respectively e^i and b^j, and $e^j_0 = b^j_0 = 1$. Hence, $c_i(q_1\cdots q_k) = G_{i,k}(e^{q_1},\ldots,e^{q_k}) = \sum_{l_1=0}^{n_\mathrm{p}}\cdots\sum_{l_k=0}^{n_\mathrm{p}} \lambda_{q_1,l_1}\cdots\lambda_{q_k,l_k} G_{i,k}(b^{l_1},\ldots,b^{l_k})$ for $i = 1,2$ and all $q_1,\ldots,q_k \in \mathbb{I}_0^{n_\mathrm{p}}$. Since for all $l_1,\ldots,l_k \in \mathbb{I}_0^{n_\mathrm{p}}$, $G_{1,k}(b^{l_1},\ldots,b^{l_k}) = G_{2,k}(b^{l_1},\ldots,b^{l_k})$, as $b^{l_1},\ldots,b^{l_k} \in \mathbb{P}$, it then follows that $c_1(q_1\cdots q_k) = c_2(q_1\cdots q_k)$. ■

Proof of Lemma 2: The direction $\theta_{\mathfrak{F}} = \theta_{\hat{\mathfrak{F}}} \implies \hat{\mathfrak{F}} = \mathfrak{F}$ is trivial. Therefore, we concentrate on proving that $\hat{\mathfrak{F}} = \mathfrak{F}$ implies $\theta_{\mathfrak{F}} = \theta_{\hat{\mathfrak{F}}}$. To this end , notice that for all $p \in \mathcal{P}$,

$$
\begin{aligned}
(g_{\mathfrak{F}} \diamond p)(t) &= \sum_{i\in\mathbb{I}_0^{n_\mathrm{p}}} p_i(t) F_{\eta_{i,\mathfrak{F}}}(p)(t), \\
(h_{\mathfrak{F}} \circ p)(\tau,t) &= \sum_{q,r\in\mathbb{I}_0^{n_\mathrm{p}}} p_r(\tau)p_q(t) F_{\theta_{q,r,\mathfrak{F}}}(q_{\tau^+}(p))(t-\tau^+),
\end{aligned} \tag{18}
$$

$t,\tau \in \mathbb{T}$, $\tau \le t$, where $\tau^+ = \tau+1$ for DT and $\tau^+ = \tau$ for CT, and $(q_{\tau^+}p)(t) = p(t+\tau^+)$, $\forall t \in \mathbb{T}$.

If $\mathfrak{F} = \hat{\mathfrak{F}}$, then $g_{\mathfrak{F}} \diamond p = \mathfrak{F}(0,p) = \hat{\mathfrak{F}}(0,p) = g_{\hat{\mathfrak{F}}} \diamond p$ for all $p \in \mathcal{P}$ Using this and (10) it then follows that $\mathfrak{F} = \hat{\mathfrak{F}}$ implies that for all $u \in \mathcal{U}$, $p \in \mathcal{P}$, and $t \in \mathbb{T}$, $\int_0^t (h_{\mathfrak{F}} \diamond p)(\delta,t)u(\delta)d\delta = \int_0^t (h_{\hat{\mathfrak{F}}} \diamond p)(\delta,t)u(\delta)d\delta$ for CT, and $\sum_{\delta=0}^{t-1}(h_{\mathfrak{F}} \diamond p)(\delta,t)u(\delta) = \sum_{\delta=0}^{t-1}(h_{\hat{\mathfrak{F}}} \diamond p)(\delta,t)u(\delta)$ for DT.

For DT, one can choose u such that $u(\delta) = e_j$, $j = 1,\ldots,n_\mathrm{u}$, where e_j is the jth standard basis vector of $\mathbb{R}^{n_\mathrm{u}}$ (i.e., all entries of e_j equal zero, except the jth entry which equals 1) for some $\delta \in [0,t-1]$ and $u(s) = 0$ for all $s \neq \delta \in [0,t-1]$. By choosing $\delta = 0,1,\ldots,t-1$, $j = 1,2,\ldots,n_\mathrm{u}$ successively, from $\sum_{s=0}^{t-1}(h_{\mathfrak{F}} \diamond p)(s,t)u(s) = \sum_{s=0}^{t-1}(h_{\mathfrak{F}} \diamond p)(s,t)u(s)$ it follows that $(h_{\mathfrak{F}} \diamond p)(\delta,t) = (h_{\hat{\mathfrak{F}}} \diamond p)(\delta,t)$ for all $\delta \in [0,t]$.

For CT, from [36, Theorem 9.3,Chapter 11] it follows that $\int_0^t (h_{\mathfrak{F}} \diamond p)(\delta,t)u(\delta)d\delta = \int_0^t (h_{\hat{\mathfrak{F}}} \diamond p)(\delta,t)u(\delta)d\delta$ for all $u \in \mathcal{U}$ implies that $(h_{\mathfrak{F}} \diamond p)(\delta,t) = (\hat{h}_{\mathfrak{F}} \diamond p)(\delta,t)$ for almost all $\delta \in [0,t]$ and all $t \in \mathbb{R}_+$. By [33, Lemma 2.2] $F_{\theta_{i,j,\mathfrak{F}}}$, $F_{\theta_{i,j,\hat{\mathfrak{F}}}}$ are continuous functions. Hence, if p is continuous at 0 from the right, then by (18), $(h_{\mathfrak{F}} \diamond p)(\delta,t)$, $(h_{\mathfrak{F}} \diamond p)(\delta,t)$ are continuous at $\delta = 0$ from the right, and therefore $(h_{\mathfrak{F}} \diamond p)(\delta,t) = (h_{\mathfrak{F}} \diamond p)(\delta,t)$ for almost all $\delta \in [0,t]$ implies $(h_{\mathfrak{F}} \diamond p)(0,t) = (h_{\mathfrak{F}} \diamond p)(0,t)$.

That is, if $\mathfrak{F} = \hat{\mathfrak{F}}$, then, for all $p \in \mathcal{P}$, such that in CT p is continuous at 0 from the right, and for all $t \in \mathbb{T}$,

$$
(h_{\mathfrak{F}} \diamond p)(0,t) = (h_{\hat{\mathfrak{F}}} \diamond p)(0,t), \;\; (g_{\mathfrak{F}} \diamond p)(t) = (g_{\hat{\mathfrak{F}}} \diamond p)(t) \tag{19}
$$

Fix $p \in \mathcal{P}$, $t \in \mathbb{T}$ and define

$$
\begin{aligned}
G_{p,t}(x) &= \sum_{i=0}^{n_\mathrm{p}} x_i (F_{\eta_{i,\mathfrak{F}}}(p)(t) - F_{\eta_{i,\hat{\mathfrak{F}}}}(p)(t)) \\
H_{p,t}(x,\bar{x}) &= \sum_{q,r=0}^{n_\mathrm{p}} x_r \hat{x}_q (F_{\theta_{q,r,\mathfrak{F}}}(p)(t) - F_{\theta_{q,r,\hat{\mathfrak{F}}}}(p)(t)),
\end{aligned}
$$

for $x = \begin{bmatrix} x_1 & \ldots & x_{n_\mathrm{p}} \end{bmatrix}^\top \in \mathbb{R}^{n_\mathrm{p}}$, $\bar{x} = \begin{bmatrix} \bar{x}_1 & \ldots & \bar{x}_{n_\mathrm{p}} \end{bmatrix}^\top \in \mathbb{R}^{n_\mathrm{p}}$, and $x_0 = \bar{x}_0 = 1$. We will show that (19) implies that

$$
\forall p \in \mathcal{P}, \forall t \in \mathbb{T}, \forall b,\hat{b} \in \mathbb{P} : G_{p,t}(b) = 0, H_{p,t}(b,\hat{b}) = 0. \tag{20}
$$

Assume that (20) holds for all $b,\hat{b} \in \mathbb{P}$ and for any $p \in \mathcal{P}$. Let $v_0,\ldots,v_{n_\mathrm{p}}$ be elements of $\mathbb{P}$ which form an affine basis of $\mathbb{R}^{n_\mathrm{p}}$. Then for any $x \in \mathbb{R}^{n_\mathrm{p}}$, $\bar{x} \in \mathbb{R}^{n_\mathrm{p}}$ there exist $\lambda_j,\mu_j \in \mathbb{R}$, $j \in \mathbb{I}_0^{n_\mathrm{p}}$, such that $\sum_{j=0}^{n_\mathrm{p}} \lambda_j = 1$, $\sum_{j=0}^{n_\mathrm{p}} \mu_j = 1$ and $x = \sum_{j=0}^{n_\mathrm{p}} \lambda_j v_j$, $\bar{x} = \sum_{j=0}^{n_\mathrm{p}} \mu_j v_j$. Since $v_0,\ldots,v_{n_\mathrm{p}}$ belong to $\mathbb{P}$, then by (20), $G_{p,t}(v_{j_1}) = 0, H_{p,t}(v_{j_1},v_{j_2}) = 0$, for all $j_1,j_2 \in \mathbb{I}_0^{n_\mathrm{p}}$. Hence, by a direct calculation it follows that $G_{p,t}(x) = G_{p,t}(\sum_{j=0}^{n_\mathrm{p}} \lambda_j v_j) = \sum_{j=0}^{n_\mathrm{p}} \lambda_j G_{p,t}(v_j) = 0$ and $H_{p,t}(x,\bar{x}) = H_{p,t}(\sum_{j=0}^{n_\mathrm{p}} \lambda_j v_j, \sum_{j=0}^{n_\mathrm{p}} \mu_j v_j) = \sum_{j_1,j_2=0}^{n_\mathrm{p}} \lambda_{j_1}\mu_{j_2} H_{p,t}(v_{j_1},v_{j_2}) = 0$. Since $x,\bar{x}$ are arbitrary, it then follows that $H_{p,t} = 0$, $G_{p,t} = 0$, and the latter implies that $F_{\eta_{i,\mathfrak{F}}}(p)(t) = F_{\eta_{i,\hat{\mathfrak{F}}}}(p)(t)$, $F_{\theta_{i,k,\mathfrak{F}}}(p)(t) = F_{\theta_{i,k,\hat{\mathfrak{F}}}}(p)(t)$ for all $i,j \in \mathbb{I}_0^{n_\mathrm{p}}$. Since p and t are arbitrary, by Lemma 4, $\eta_{i,\mathfrak{F}} = \eta_{i,\hat{\mathfrak{F}}}$, $\theta_{i,k,\mathfrak{F}} = \theta_{i,k,\hat{\mathfrak{F}}}$ for all $i,k \in \mathbb{I}_0^{n_\mathrm{p}}$, *i.e.*, $\theta_{\mathfrak{F}} = \theta_{\hat{\mathfrak{F}}}$

We finish the proof by proving that (19) implies (20). In the DT case, consider any $p \in \mathcal{P}$ and $t \in \mathbb{T}$. Fix any $b \in \mathbb{P}$ and define $\hat{p} \in \mathcal{P}$ by $\hat{p}(t) = b$ and $p(s) = \hat{p}(s)$ for $s = 0,\ldots,t-1$. Notice that $F_c(p)(t) = F_c(\hat{p})(t)$ for any convergent series c. From (18) it then follows that $(g_{\mathfrak{F}} \diamond \hat{p})(t) = (g_{\hat{\mathfrak{F}}} \diamond \hat{p})(t)$ which implies $G_{p,t}(b) = 0$ for all $b \in \mathbb{P}$. For any $b,\hat{b} \in \mathbb{P}$ define $\hat{p} \in \mathcal{P}$ as $\hat{p}(0) = \hat{b}$, $\hat{p}(t+1) = b$ and $\hat{p}(s) = p(s-1)$ for all $s = 1,\ldots,t$. Notice that for any convergent series c, $F_c(p)(t) = F_c(q_1(\hat{p}))(t)$, where $q_1(\hat{p})(\delta) = \hat{p}(\delta+1)$, $\delta \in \mathbb{T}$. Hence, from (18) and $(h_{\mathfrak{F}} \diamond \hat{p})(0,t+1) = (h_{\hat{\mathfrak{F}}} \diamond \hat{p})(0,t+1)$ it follows that $\forall b,\hat{b} \in \mathbb{P} : H_{p,t}(b,\hat{b}) = 0$.

For the CT case, for any $p \in \mathcal{P}$ and any $b,\hat{b} \in \mathbb{P}$, define $\hat{p}_n \in \mathcal{P}$ such that for all $n \in \mathbb{N}$, $n > 1$, $\hat{p}_n(s) = \hat{b}$, if $s \in [0,\frac{1}{n})$, $\hat{p}_n(s) = p(s)$, if $s \in [\frac{1}{n}, t-\frac{1}{n})$ and $\hat{p}_n(s) = b$ if $s \in [t-\frac{1}{n},+\infty)$. From (18) it follows that $H_{\hat{p}_n,t}(b) = (h_{\mathfrak{F}} \diamond \hat{p}_n)(0,t) - (h_{\hat{\mathfrak{F}}} \diamond \hat{p}_n)(0,t)$ and $G_{\hat{p}_n,t}(b) = (g_{\mathfrak{F}} \diamond \hat{p}_n)(t) - (g_{\hat{\mathfrak{F}}} \diamond \hat{p}_n)(t)$. Notice that $\hat{p}_n$ is continuous at zero from the right. Hence, $(g_{\mathfrak{F}} \diamond \hat{p}_n)(t) = (g_{\hat{\mathfrak{F}}} \diamond \hat{p}_n)(t)$ and $(h_{\mathfrak{F}} \diamond \hat{p_n})(0,t) = (h_{\hat{\mathfrak{F}}} \diamond \hat{p}_n)(0,t)$. Hence, $H_{\hat{p}_n,t}(b,\hat{b}) = 0$ and $G_{\hat{p}_n,t}(b) = 0$. Note that the restriction of $\hat{p}_n|_{[0,t]}$ converges to $p|_{[0,t]}$ in $L^1([0,t],\mathbb{R}^{n_\mathrm{p}})$. From [33, Lemma 2.2], $\lim_{n\to\infty} F_c(\hat{p}_n)(t) = F_c(p)(t)$ for any convergent series c. Therefore, $H_{p,t}(b,\hat{b}) = \lim_{n\to\infty} H_{\hat{p}_n,t}(b,\hat{b}) = 0$ and $G_{p,t}(b) = \lim_{n\to\infty} G_{\hat{p}_n,t}(b) = 0$. ■

Proof of Lemma 3: For any $(u,p) \in \mathcal{U}\times\mathcal{P}$ and for any $t \in \mathbb{T}, 0 \le \tau \le t$, define

$$
(h_{\mathfrak{Y}_{\Sigma,x_\mathrm{o}}} \diamond p)(t,\tau) = \begin{cases} C(p(t))\Phi(t,\tau)B(p(\tau)) & \text{CT} \\ C(p(t))\Phi(t-1,\tau+1)B(p(\tau)) & \text{DT} \end{cases},
$$

$$
(g_{\mathfrak{Y}_{\Sigma,x_\mathrm{o}}} \diamond p)(t) = C(p(t))\Phi(t,0)x_\mathrm{o}.
$$

where $\Phi(t,\tau)$ is the fundamental matrix of $A(p(t))$, *i.e.*, $\xi\Phi(t,\tau) = A(p(t))\Phi(t,\tau)$, $\Phi(\tau,\tau) = I_{n_x}$. It is then easy to see that for all $u \in \mathcal{U}$, $p \in \mathcal{P}$, $\mathfrak{Y}_{\Sigma,x_\mathrm{o}}(u,p)$ $(h_{\mathfrak{Y}_{\Sigma,x_\mathrm{o}}} \diamond p)$, $(g_{\mathfrak{Y}_{\Sigma,x_\mathrm{o}}} \diamond p)$ satisfy (10). We show that for all $p \in \mathcal{P}$, $(h_{\mathfrak{Y}_{\Sigma,x_\mathrm{o}}} \diamond p)$, $(g_{\mathfrak{Y}_{\Sigma,x_\mathrm{o}}} \diamond p)$ satisfy the other conditions of an IIR representation. To this end, consider the bilinear system

$$
\begin{aligned}
(\xi\eta)(\delta) &= A_0\eta(\delta) + \sum_{i=1}^{n_\mathrm{p}} (A_i\eta(\delta))w_i(\delta), \;\; \eta(0) = \eta_0, \\
y(\delta) &= C(p(t))\eta(\delta).
\end{aligned} \tag{21}
$$

Let the initial state η_0 of (21) be the ith column of $B(p(\tau))$. Notice that the ith column of $(h_{\mathfrak{Y}_{\Sigma,x_\mathrm{o}}} \diamond p)(t,\tau)$ is the output of (21) a time $t-\tau$ for $w(\delta) = p(\delta+\tau)$, $\delta \in \mathbb{T}$ in CT, and it is the output of (21) at time $t-\tau-1$ for $w(\delta)=p(\delta+\tau+1)$, $\delta \in \mathbb{T}$ in DT. Similarly, if we set $\eta_0 = x_\mathrm{o}$, then $(g_{\mathfrak{Y}_{\Sigma,x_\mathrm{o}}} \diamond p)(t)$ is the output at time t of (21) for $w = p$. Hence, by [32], [35],

$$(h_{\mathfrak{Y}_{\Sigma,x_\mathrm{o}}} \diamond p)(\tau,t) = \sum_{s \in \mathcal{S}(\mathbb{I}_0^{n_\mathrm{p}})} c(s)(w_s \diamond p)(\tau^+, t^-)$$

$$(g_{\mathfrak{Y}_{\Sigma,x_\mathrm{o}}} \diamond p)(t) = \sum_{s \in \mathcal{S}(\mathbb{I}_0^{n_\mathrm{p}})} c_0(s)(w_s \diamond p)(0, t^-),$$

where $\tau^+ = \tau$, $t^- = t$ in CT, and $\tau^+ = \tau+1$, $t^- = t-1$ in DT, and for all $s \in \mathcal{S}(\mathbb{I}_0^{n_\mathrm{p}})$, $c(s)=\sum_{r,q \in \mathbb{I}_0^{n_\mathrm{p}}} p_r(t)p_q(\tau)C_rA_sB_q$, and $c_0(s) = \sum_{q \in \mathbb{I}_0^{n_\mathrm{p}}} p_q(t)C_qA_sx_\mathrm{o}$. Let us define $\theta_{\mathfrak{Y}_{\Sigma,x_\mathrm{o}}}$ as $\theta_{\mathfrak{Y}_{\Sigma,x_\mathrm{o}}}(s)=\begin{bmatrix} C_1^\top & \dots & C_{n_\mathrm{p}}^\top \end{bmatrix}^\top A_s \begin{bmatrix} x_\mathrm{o} & B_1 & \dots & B_{n_\mathrm{p}} \end{bmatrix}$, for all $s \in \mathcal{S}(\mathbb{I}_0^{n_\mathrm{p}})$. Then it is easy to see that for all $p \in \mathcal{P}$, $\theta_{\mathfrak{Y}_{\Sigma,x_\mathrm{o}}}$ $(h_{\mathfrak{Y}_{\Sigma,x_\mathrm{o}}} \diamond p)$, $(g_{\mathfrak{Y}_{\Sigma,x_\mathrm{o}}} \diamond p)$ satisfy (11). Finally, if define $\alpha = \max\{||C_q||_F \mid q \in \mathbb{I}_0^{n_\mathrm{p}}\} \cup \{||x_\mathrm{o}||||B_q||_F \mid q \in \mathbb{I}_0^{n_\mathrm{p}}\}$ and $K = \alpha^2\sqrt{n_\mathrm{p}(n_\mathrm{p}+1)}$, $R = \max_{q \in \mathbb{I}_0^{n_\mathrm{p}}} ||A_q||_F$, then $\theta_{\mathfrak{Y}_{\Sigma,x_\mathrm{o}}}$ satisfies (9). That is, $\mathfrak{Y}_{\Sigma,x_\mathrm{o}}$ has a IIR.

Assume that Σ is a realization of $\mathfrak{F}$. Then $\mathfrak{Y}_{\Sigma,x_\mathrm{o}} = \mathfrak{F}$ for some initial state x_o of Σ. From Lemma 2, $\theta_{\mathfrak{Y}_{\Sigma,x_\mathrm{o}}} = \theta_{\mathfrak{F}}$ and hence $\theta_{\mathfrak{F}}$ satisfies (12). Conversely, assume that $\theta_{\mathfrak{F}}$ satisfies (12). Then $\theta_{\mathfrak{F}} = \theta_{\mathfrak{Y}_{\Sigma,x_\mathrm{o}}}$ and thus by Lemma 2 $\mathfrak{F}_{\Sigma,x_\mathrm{o}} = \mathfrak{F}$, *i.e.*, Σ is a realization of $\mathfrak{F}$. ■

B. Proofs of Theorem 1 – Theorem 3

We start by establishing the relationship between the LPV-SSAs and linear switched state-space representations (*abbreviated by LSS-SS*), which, in combination with the results of [26]–[28], will be used to prove Theorem 1 – Theorem 3. To this end, we introduce the following notation.

Notation 2. *For $i = 1, \dots, n_\mathrm{p}$, let e^i be the ith standard basis vector of $\mathbb{R}^{n_\mathrm{p}}$, i.e., all entries of e^i are zero, except the ith entry, which equals* 1. *Denote $\mathbb{P}_{sw} = \{0, e^1, \cdots, e^{n_\mathrm{p}}\}$ and let $\mathcal{P}_{sw}$ either $\mathcal{C}_p(\mathbb{R}_+, \mathbb{P}_{sw})$ (CT) or $\mathbb{P}_{sw}^{\mathbb{N}}$ (DT).*

Note that LSS-SSs can be viewed as a subclass of the LPV-SSAs for which the space of scheduling variables is $\mathbb{P}_{sw}$. The notions of realization, minimality, observability, span-reachability and isomorphism for LSS-SSs from [27], [28] are special cases of the corresponding concepts for LPV-SSAs, if LSS-SSs are viewed as LPV-SSAs. For each map $\mathfrak{F}$ of the form (4) admitting an IIR, the *associated switched input-output map* $\mathfrak{S}(\mathfrak{F}) : \mathcal{U} \times \mathcal{P}_{sw} \to \mathcal{Y}$ is defined as follows. Let $\theta_{\mathfrak{F}}$ be the sub-Markov parameter of $\mathfrak{F}$, which is unique by Lemma 2. For each $p \in \mathcal{P}_{sw}$, define $h_{\mathfrak{S}(\mathfrak{F})} \diamond p : \mathbb{T} \to \mathbb{R}^{n_\mathrm{y}}$ and $g_{\mathfrak{S}(\mathfrak{F})} \diamond p : \{(\tau,t) \in \mathbb{T} \times \mathbb{T} \mid \tau \le t\} \to \mathbb{R}^{n_\mathrm{y}}$ as

$$\begin{aligned}(g_{\mathfrak{S}(\mathfrak{F})} \diamond p)(t) &= \sum_{\substack{i \in \mathbb{I}_0^{n_\mathrm{p}},\\ s \in \mathcal{S}(\mathbb{I}_0^{n_\mathrm{p}})}} p_i(t)\eta_{i,\mathfrak{F}}(s)(w_s \diamond p)(\bar{t}, 0),\\ (h_{\mathfrak{S}(\mathfrak{F})} \diamond p)(\delta, t) &= \sum_{\substack{i,j \in \mathbb{I}_0^{n_\mathrm{p}},\\ s \in \mathcal{S}(\mathbb{I}_0^{n_\mathrm{p}})}} \theta_{i,j,\mathfrak{F}}(s)p_i(t)p_j(\delta)(w_s \diamond p)(\bar{t}, \hat{\delta}),\end{aligned} \tag{22}$$

where $\bar{t} = t$ and $\hat{\delta} = \delta$, in CT, and $\bar{t} = t-1$ and $\hat{\delta} = \delta+1$ in DT. In DT the right-hand sides of (22) are finite sums. For CT, by applying Lemma 1 to $\mathbb{P} = \mathbb{P}_{sw}$ it follows that the right-hand sides of (22) are absolutely convergent series. Hence, $(h_{\mathfrak{S}(\mathfrak{F}} \diamond p)$ and $(g_{\mathfrak{S}(\mathfrak{F}} \diamond p)$ are well defined for all $p \in \mathcal{P}_{sw}$. For any $(u,p) \in \mathcal{U} \times \mathcal{P}_{sw}$ and $t \in \mathbb{T}$, we define

$$\mathfrak{S}(\mathfrak{F})(u,p)(t) = (g_{\mathfrak{S}(\mathfrak{F})} \diamond p)(t) + \begin{cases} \int_0^t (h_{\mathfrak{S}(\mathfrak{F})} \diamond p)(\delta,t)u(\delta)d\delta & \text{CT} \\ \sum_{\delta=0}^{t-1} (h_{\mathfrak{S}(\mathfrak{F})} \diamond p)(\delta,t)u(\delta) & \text{DT}. \end{cases}$$

It is the easy to see that $\mathfrak{S}(\mathfrak{F})$ has an IIR and the corresponding sub-Markov parameter $\theta_{\mathfrak{S}(\mathfrak{F})}$ equals $\theta_{\mathfrak{F}}$. Moreover, $\mathfrak{S}(\mathfrak{F})$ is uniquely determined by $\mathfrak{F}$.

Let Σ be an LPV-SSAs $\left(\mathbb{P}, \{(A_i, B_i, C_i, 0)\}_{q=0}^{n_\mathrm{p}}\right)$ Then, the *LSS-SS* $\mathfrak{S}(\Sigma)$ *associated with* Σ is the defined as the LSS-SS $\mathfrak{S}(\Sigma) = \left(\mathbb{P}_{sw}, \{(A_i, B_i, C_i, 0)\}_{q=0}^{n_\mathrm{p}}\right)$.

Theorem 4. *Let Σ be an LPV-SSA and $\mathfrak{F}$ an input-output map admitting IIR.*

(1) For every initial state $x \in \mathbb{X}$ of Σ, $\mathfrak{S}(\mathfrak{Y}_{\Sigma,x}) = \mathfrak{Y}_{\mathfrak{S}(\Sigma),x}$.

(2) Σ is a realization of $\mathfrak{F}$ from the initial state x_o if and only if $\mathfrak{S}(\Sigma)$ is a realization of $\mathfrak{S}(\mathfrak{F})$ from the initial state x_o.

(3) $\dim \mathfrak{S}(\Sigma) = \dim \Sigma$.

(4) Two LPV-SSAs Σ_1 and Σ_2 are isomorphic if and only if $\mathfrak{S}(\Sigma_1)$ is isomorphic to $\mathfrak{S}(\Sigma_2)$.

(5) Σ is span-reachable from x_o if and only if $\mathfrak{S}(\Sigma)$ is span-reachable from x_o. Σ is observable if and only if $\mathfrak{S}(\Sigma)$ is observable.

Proof: **Proof (1)** From Lemma 3 it follows that for all $s \in \mathcal{S}(\mathbb{I}_0^{n_\mathrm{p}})$, $\theta_{\mathfrak{S}(\mathfrak{Y}_{\Sigma,x})}(s) = \theta_{\mathfrak{Y}_{\Sigma,\mathrm{x}}}(s) = \theta_{\mathfrak{Y}_{\mathfrak{S}(\Sigma),x}}(s) = \widetilde{C}A_s\widetilde{B}$, $\widetilde{C} = \begin{bmatrix} C_1^\top, & \dots, & C_{n_\mathrm{p}}^\top \end{bmatrix}^\top$, $\widetilde{B} = \begin{bmatrix} x_\mathrm{o}, & B_1, & \dots, & B_{n_\mathrm{p}} \end{bmatrix}$. By Lemma 2, $\theta_{\mathfrak{S}(\mathfrak{Y}_{\Sigma,x})} = \theta_{\mathfrak{Y}_{\mathfrak{S}(\Sigma),x}}$ implies $\mathfrak{S}(\mathfrak{Y}_{\Sigma,x}) = \mathfrak{Y}_{\mathfrak{S}(\Sigma),x}$.

Proof of (2) Σ is a realization of $\mathfrak{F}$ from the initial state x_o if and only if $\mathfrak{Y}_{\Sigma,x_\mathrm{o}} = \mathfrak{F}$. Note that $\theta_{\mathfrak{F}} = \theta_{\mathfrak{S}(\mathfrak{F})}$, and $\theta_{\mathfrak{Y}_{\Sigma,x_\mathrm{o}}} = \theta_{\mathfrak{S}(\mathfrak{Y}_{\Sigma,x_\mathrm{o}})}$. Hence, by Lemma 2, $\mathfrak{Y}_{\Sigma,x_\mathrm{o}} = \mathfrak{F}$ is equivalent to $\mathfrak{S}(\mathfrak{Y}_{\Sigma,x_\mathrm{o}}) = \mathfrak{S}(\mathfrak{F})$. From Part **(1)** of the current theorem, $\mathfrak{S}(\mathfrak{Y}_{\Sigma,x_\mathrm{o}}) = \mathfrak{S}(\mathfrak{F})$ is equivalent to $\mathfrak{Y}_{\mathfrak{S}(\Sigma),x_\mathrm{o}} = \mathfrak{S}(\mathfrak{F})$, and the latter is equivalent to $\mathfrak{S}(\Sigma)$ being a realization of $\mathfrak{S}(\mathfrak{F})$.

Proof of (3) and (4). Trivial.

Proof of (5). First we show that Σ is span-reachable from x_o if and only if $\mathfrak{S}(\Sigma)$ is span-reachable from x_o.

To this end, consider the input-to-state map $\mathfrak{X}_{\Sigma,x_\mathrm{o}}$ of Σ. Span-reachability of Σ is equivalent to $\forall \nu \in \mathbb{R}^{n_\mathrm{x}}$: $(\nu^T\mathfrak{X}_{\Sigma,x_\mathrm{o}} = 0 \iff \nu = 0)$. For every $\nu \in \mathbb{R}^{n_\mathrm{x}}$, consider the function $\mathfrak{F}_\nu(u,p) = \nu^T\mathrm{X}_{\Sigma,x_\mathrm{o}}(u,p)$. It is clear that the LPV-SSA Σ_ν, $\Sigma_\nu = (\mathbb{P}, \{A_i, B_i, \nu\}_{i=0}^{n_\mathrm{p}})$, is a realization of $\mathfrak{F}_\nu$ from the initial state x_o. It is easy to see that $\mathfrak{F}_\nu = 0$ if and only if $\theta_{\mathfrak{F}_\nu} = \theta_{\mathfrak{S}(\mathfrak{F}_\nu)} = 0$ and hence $\mathfrak{S}(\mathfrak{F}_\nu) = 0 \iff \mathfrak{F}_\nu = 0$. But from Part **(1)** of the current theorem, $\mathfrak{S}(\mathfrak{F}_\nu) = \nu^T\mathfrak{X}_{\mathfrak{S}(\Sigma),x_\mathrm{o}}$. Hence, $\forall \nu \in \mathbb{R}^{n_\mathrm{x}} : (\nu^T\mathfrak{X}_{\Sigma,x_\mathrm{o}} = 0 \iff \nu = 0)$ is equivalent to $\forall \nu \in \mathbb{R}^{n_\mathrm{x}} : (\nu^T\mathfrak{X}_{\mathfrak{S}(\Sigma),x_\mathrm{o}} = 0 \iff \nu = 0)$. The latter is equivalent to span-reachability of $\mathfrak{S}(\Sigma)$ from x_o.

Next, we show that Σ is observable if and only if $\mathfrak{S}(\Sigma)$ is observable. From **Part (1)** of the theorem, for any $x \in \mathbb{R}^{n_\mathrm{x}}$, $\mathfrak{S}(\mathfrak{Y}_{\Sigma,x}) = \mathfrak{Y}_{\mathfrak{S}(\Sigma),x}$. If $\mathfrak{Y}_{\Sigma,x_1} = \mathfrak{Y}_{\Sigma,x_2}$, then $\mathfrak{Y}_{\mathfrak{S}(\Sigma),x_1} = \mathfrak{S}(\mathfrak{Y}_{\Sigma,x_1}) = \mathfrak{S}(\mathfrak{Y}_{\Sigma,x_2}) = \mathfrak{Y}_{\mathfrak{S}(\Sigma),x_2}$. Conversely, assume $\mathfrak{Y}_{\mathfrak{S}(\Sigma),x_1} = \mathfrak{Y}_{\mathfrak{S}(\Sigma),x_2}$. Then $\theta_{\mathfrak{Y}_{\mathfrak{S}(\Sigma),x_1}} = \theta_{\mathfrak{Y}_{\mathfrak{S}(\Sigma),x_2}}$. Note that for $i = 1,2$, $\theta_{\mathfrak{Y}_{\Sigma,x_i}} = \theta_{\mathfrak{Y}_{\mathfrak{S}(\Sigma),x_i}}$. Hence, from Lemma 2, $\mathfrak{Y}_{\Sigma,x_1} = \mathfrak{Y}_{\Sigma,x_2}$. ■

Let $\mathcal{H} = (\mathbb{P}_{sw}, (A_i, B_i, C_i, 0)_{i=0}^{n_\mathrm{p}})$ be a LSS-SS. Define the *LPV-SSA associated with* $\mathcal{H}$ as $\mathfrak{L}(\mathcal{H}) = (\mathbb{P}, (A_i, B_i, C_i, 0)_{i=0}^{n_\mathrm{p}})$. It is easy to see that $\mathfrak{S}(\mathfrak{L}(\mathcal{H})) = \mathcal{H}$. Then from Theorem 4 we can deduce the following.

Corollary 2. *If H is an LSS-SS, then $\mathcal{H}$ is a realization of $\mathfrak{S}(\mathfrak{F})$ from the initial state x_{o}, if and only if $\mathfrak{L}(\mathcal{H})$ is a realization of $\mathfrak{F}$ from x_{o}. An LPV-SSA Σ is minimal realization of $\mathfrak{F}$ from the initial state x_{o} if and only if the LSS-SS $\mathfrak{S}(\Sigma)$ is minimal realization of $\mathfrak{S}(\mathfrak{F})$ from x_{o}.*

Proof of Corollary 2: By Theorem 4, $H = \mathfrak{S}(\mathfrak{L}(H))$ is a realization of $\mathfrak{S}(\mathfrak{F})$ from x_{o} if and only if $\mathfrak{L}(H)$ is a realization of $\mathfrak{F}$ from x_{o}. Assume that Σ is a minimal realization of $\mathfrak{F}$ from x_{o}. By Theorem 4, $\mathfrak{S}(\Sigma)$ is a realization of $\mathfrak{S}(\mathfrak{F})$ from x_{o}. Assume that $\mathcal{H}^{'}$ is an LSS-SS and $\mathcal{H}^{'}$ is a realization of $\mathfrak{S}(\mathfrak{F})$. It then follows that $\Sigma^{'} = \mathfrak{L}(\mathcal{H}^{'})$ is a realization of $\mathfrak{F}$. Since Σ is a minimal realization of $\mathfrak{F}$, $\dim \mathfrak{S}(\Sigma) = \dim \Sigma \leq \dim \Sigma^{'} = \dim \mathcal{H}^{'}$. Conversely, assume that $\mathfrak{S}(\Sigma)$ is minimal realization of $\mathfrak{S}(\mathfrak{F})$ from x_{o}. Assume that $\Sigma^{'}$ is an LPV-SSA realization of $\mathcal{F}$. From Theorem 4 it follows that $\mathfrak{S}(\Sigma^{'})$ is a realization of $\mathfrak{S}(\mathcal{F})$, and by minimality of $\mathfrak{S}(\Sigma)$ $\dim \Sigma = \dim \mathfrak{S}(\Sigma) \leq \dim \mathfrak{S}(\Sigma^{'}) = \dim \Sigma^{'}$. ■

Proof of Theorem 1: Follows from Corollary 2, Theorem 4, [27, Theorem 3] (DT), [28, Theorem 3] (CT). ■

Proof of Theorem 2: From [27, Theorem 4] for DT and [28, Proposition 1] for CT, it follows that $\mathrm{rank}\{\mathcal{R}_{n_{\mathrm{x}}-1}\} = n_{\mathrm{x}}$ if equivalent to $\mathfrak{S}(\Sigma)$ being span-reachable from x_0. From [27, Theorem 4] for DT and [28, Theorem 2] for CT, $\mathrm{rank}\{\mathcal{O}_{n_{\mathrm{x}}-1}\} = n_{\mathrm{x}}$ is equivalent to observability of $\mathfrak{S}(\Sigma)$. The statement of the theorem follows now from Part (5) of Theorem 4. ■

Proof of Theorem 3: Notice that $\theta_{\mathfrak{F}}(s) = \theta_{\mathfrak{S}(\mathfrak{F})}(s)$, $s \in \mathcal{S}(\mathbb{I}_0^{n_{\mathrm{p}}})$, and when applied to LSS-SSs, the sub-Markov parameters from Definition 2 coincide with the Markov-parameters of [27], [28]. Notice that $H_{\mathfrak{F}}(n,m) = H_{\mathfrak{S}(\mathfrak{F})}(n,m)$ and that the former definition of the Hankel-matrix coincides with the one for LSS-SSs (see [27, Definition 13], [28, Definition 21]). Note that Algorithm 1 applied to $H_{\mathfrak{F}}(n,m) = H_{\mathfrak{S}(\mathfrak{F})}(n,m)$, $m > n$, coincides with [26, Algorithm 1] (CT) [27, Algorithm 1] (DT). The statement of the theorem follows from Theorem 4, Corollary 2, from [27, Theorem 4,6] (DT) and [28, Theorem 6], [26, Theorem 4] (CT). ■

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