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Differential Geometry of the Mixed Hodge Metric

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Summary We investigate properties of the mixed Hodge metric of a mixed period domain. In particular, we calculate its curvature and the curvature of the Hodge bundles. We also consider when the pull back metric via a period map is Kähler. Several applications in cases of geometric interest are given, such as for normal functions and biextension bundles.

1 Introduction

1.1 Overview

Let $f : X \to S$ be a smooth, proper morphism between complex algebraic varieties. Then, by the work of Griffiths [Gr], the associated local system $\mathscr{H}_{\mathbb{Q}} = R^k f^* \mathbb{Q}_X$ underlies a variation of pure Hodge structure of weight k, which can be described by a *period map*

$$\varphi : S \to \Gamma \backslash D, \tag{1}$$

where Γ is the *monodromy group* of the family. In the case where the morphism $X \to S$ is no longer smooth and proper the resulting local system underlies a variation of (graded-polarized) mixed Hodge structure over a Zariski open subset of S [SZ]. As in the pure case considered by Griffiths, a variation of mixed Hodge structure can be described in terms of a period map which is formally analogous to (1) except that D is now a classifying space of graded polarized mixed Hodge structure [P1, U].

As we shall explain below, there is a natural metric on such D, induced by the *mixed Hodge metric* (5). Deligne's second order calculations involving this metric in the pure case [D1] can be extended to the mixed setting, as we show in this article. For instance, we find criteria as to when the induced Hodge metric on S is Kähler. We also compute the curvature tensor of this metric, with special emphasis on cases of interest in the study of algebraic cycles, archimedean heights and iterated integrals. The alternative approach [Ca-MS-P, Chap. 12] in the pure case based on the Maurer-Cartan form does not seem to generalize as we encounter incompatibilities between the metric and the complex structure as demonstrated in § 9.

1.2 The Pure Case

Returning to the pure case, we recall that D parametrizes Hodge structures of weight k on a reference fiber $H_{\mathbb{Q}}$ of $\mathscr{H}_{\mathbb{Q}}$ with given Hodge numbers $\{h^{p,q}\}$ and polarized by a non-degenerate bilinear form Q of parity $(-1)^k$. The monodromy group Γ is contained in the real Lie group $G_{\mathbb{R}} \subset GL(H_{\mathbb{R}})$ of automorphisms of the polarization Q.

In terms of differential geometry, the first key fact is that $G_{\mathbb{R}}$ acts transitively on D with compact isotropy, and hence D carries a $G_{\mathbb{R}}$ invariant metric. It is induced by the polarizing form Q as follows. Any Hodge filtration F on $H_{\mathbb{C}}$ then induces

$$h_F(x,y) := Q(C_F x, \bar{y}), x, y \in H_{\mathbb{C}}, \tag{2}$$

where $C_F|H^{p,q} = \mathrm{i}^{p-q}$ is the Weil-operator. This is a metric as a consequence of the two Riemann bilinear relations: the first, $Q(F^p, F^{k-p+1}) = 0$ states that the Hodge decomposition is h_F-orthogonal and the second states that h_F is a metric on each Hodge-component.

Next, by describing the Hodge structures parameterized by D in terms of the corresponding flags

$$F^p H_{\mathbb{C}} = \bigoplus_{a \geq p} H^{a,k-a}$$

we obtain an open embedding of D into the flag manifold $\check{D}$ consisting of decreasing filtrations $F^* H_{\mathbb{C}}$ such that $\dim F^p = \sum_{a \geq p} h^{a,k-a}$ which satisfy only the first Riemann bilinear relation. In particular, via this embedding, the set D inherits the structure of a complex manifold upon which the group $G_{\mathbb{R}}$ acts via biholomorphisms.

As a flag manifold, the tangent space at F to $\check{D}$ can be identified with a subspace of

$$\bigoplus_p \mathrm{Hom}(F^p, H_{\mathbb{C}}/F^p). \tag{3}$$

Via this identification, we say that a tangent vector is *(strictly) horizontal* if it is contained in the subspace

$$\bigoplus_p \mathrm{Hom}(F^p, F^{p-1}/F^p).$$

One of the basic results of [Gr] is that the period map associated to a smooth proper morphism $X \to S$ as above is holomorphic, horizontal and locally liftable.

Combining the previous two paragraphs, the metric (2) on V induces a functorial metric on (3) and hence induces a hermitian metric h on the analytic open subset D of the smooth variety $\check{D}$. In particular, since $G_{\mathbb{R}}$ acts transitively on D via biholomorphisms and

$$h_{g.F}(x,y) = h(g^{-1}x, g^{-1}y)$$

for all $g \in G_{\mathbb{R}}$ and $F \in D$, it follows that h is a $G_{\mathbb{R}}$-invariant metric on D.

By [GS, Theorem 9.1] the holomorphic sectional curvature of D along horizontal tangents is negative and bounded away from zero. In particular, as a consequence of this curvature estimate, if $S \subset \bar{S}$ is a smooth normal crossing compactification with unipotent monodromy near $p \in \bar{S} - S$, then by [Sc] the period map φ has at worst logarithmic singularities near p.

1.3 Mixed Domains

In the mixed case, period maps of geometric origin are holomorphic and satisfy the analogous horizontality condition ([U, SZ]). However, although there is a *natural Lie group* G (see § 2.1) which acts transitively on the classifying spaces of graded-polarized mixed Hodge structure, the isotropy group is no longer compact, and hence there is no G-invariant hermitian structure. In spite of this, A. Kaplan observed in [Ka] that one could construct a natural hermitian metric on D in the mixed case which was invariant under a pair of subgroups $G_{\mathbb{R}}$ and $\exp(\Lambda)$ of G which taken together act transitively on D. The subgroup $\exp(\Lambda)$ (see § 2.2) depends upon a choice of base point in D and intersects the group $G_{\mathbb{R}}$ non-trivially. Nonetheless, as we said before, by emulating the computations of Deligne in [D1], we are able to compute the curvature tensor of D in the mixed case (cf. §3).

Let us elaborate on this by defining the natural metric. A mixed Hodge structure (F, W) on V induces a unique functorial bigrading [D2], the *Deligne splitting*

$$V_{\mathbb{C}} = \bigoplus_{p,q} I^{p,q} \tag{4}$$

such that $F^p = \bigoplus_{a \geq p} I^{a,b}$, $W_k = \bigoplus_{a+b \leq k} I^{a,b}$ and

$$\bar{I}^{p,q} = I^{q,p} \mod \bigoplus_{a<q, b<p} I^{a,b}.$$

In the pure case a polarization induces a hermitian inner product for which the Hodge decomposition is orthogonal. In the mixed case we first declare the splitting (4) to be $h_{(F,W)}$–orthogonal and then define the metric on $I^{p,q}$ making use of the graded polarization $(\mathrm{Gr}\, h)_F$ as follows. The summand $I^{p,q}$ maps isomorphically onto the subspace $H^{p,q}$ of Gr^W_{p+q}. So on classes $[z]$ of elements $z \in I^{p,q} \subset W^{p+q}$ modulo W^{p+q-1} the metric $h_{F,W}$ can be defined by setting:

$$h_{(F,W)}(x, y) = (\mathrm{Gr}\, h)_F([x], [y]), \quad x, y \in I^{p,q}. \tag{5}$$

This is the *mixed Hodge metric* alluded to previously. By functoriality it induces Hodge metrics on $\mathrm{End}(V)$ (see (16)) and hence also on the Lie algebra of G. As in the pure case this induces a natural metric on the mixed period domain (see Definition 2.6). It is these metrics that form our principal subject of investigation of this paper.

Remark 1.1. A Mumford–Tate domain D_M classifies pure Hodge structures with extra Hodge tensors [GGK]. In analogy with the classifying spaces of pure Hodge structures, D_M is the orbit of a generic point $F \in D_M$ under the real points Mumford–Tate group of F. The analog for mixed Hodge structures are mixed Mumford–Tate domains, e.g. the mixed Shimura varieties of Pink and Milne. See Remark 2.4. All of the Lie theoretic calculations done in section 2, and hence all of the applications in the subsequent sections remain true for mixed Mumford–Tate domains.

1.4 Examples

To get an idea of the nature of these metrics in the mixed situation we give a few examples.

1. Consider the mixed Hodge structure on the cohomology of quasi-projective curves. So, let X be a compact Riemann surface of genus g and S be a finite set of points on X. Then, $F^1H^1(X-S,\mathbb{C})$ consists of holomorphic 1-forms Ω on $X-S$ with at worst simple poles along S, and the mixed Hodge metric is given by

$$||\Omega||^2 = 4\pi^2 \sum_{p\in S} |\mathrm{Res}_p(\Omega)|^2 + \sum_{j=1}^{g} \left| \int_X \Omega \wedge \bar{\varphi}_j \right|^2, \qquad (6)$$

where $\{\varphi_j\}$ is unitary frame for $H^{1,0}(X)$ with respect to the standard Hodge metric on $H^1(X,\mathbb{C})$.

To verify this, we recall that in terms of Green's functions, the subspace $I^{1,1}$ can be described as follows: If H is the space of real-valued harmonic functions on $X-S$ with at worst logarithmic singularities near the points of S, then

$$I^{1,1} \cap H^1(X-S,\mathbb{R}) = \left\{ \sqrt{-1}\cdot\partial(f) \mid f \in H \right\}. \qquad (7)$$

Indeed, the elements of $I^{1,1}$ will be meromorphic 1-forms with simple poles along S. The elements $\sqrt{-1}\cdot\partial(f)$ are also real cohomology classes since the imaginary part is exact.

Direct calculation using (7) and Stokes' theorem shows that $I^{1,1}$ consists of the elements in F^1 which pair to zero against $H^{0,1}$. Therefore, the terms $\int_X \Omega\wedge\bar{\varphi}_j$ appearing in (6) only compute the Hodge inner product for the component of Ω in $I^{1,0}$.

2. Recall that the dilogarithm [Ha1, §1] is the double integral

$$\ln_2(x) = \int_0^x w_1\cdot w_2, \quad w_1 = \frac{1}{2\pi\mathrm{i}}\cdot\frac{dz}{1-z}, \quad w_2 = \frac{1}{2\pi\mathrm{i}}\cdot\frac{dz}{z}.$$

For the corresponding variation of mixed Hodge structure arising from the mixed Hodge structure on $\pi_1(\mathbb{P}^1 - \{0, 1, \infty\}, x)$, the pull back metric is given by

$$\|\nabla_{d/dz}\|^2 = \left[\frac{1}{|z|^2} + \frac{1}{|z-1|^2}\right]. \tag{8}$$

For a proof, we refer to § 6.

3. Consider mixed Hodge structures whose Hodge numbers are $h^{0,0} = h^{-1,-1} = 1$. The corresponding classifying space is isomorphic to $\mathbb{C}$ with the Euclidean metric. In particular, the curvature is identically zero. Note that the corresponding extensions are parametrized by $\mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Z}(0), \mathbb{Z}(1)) = \mathbb{C}^*$: these are equivalence classes of mixed Hodge structures, but we are not considering these.

4. Let (X, ω) be a compact Kähler manifold of dimension n, and (F, W) denote the mixed Hodge structure on $V = \bigoplus_p H^p(X, \mathbb{C})$ defined by setting $I^{p,q} = H^{n-p,n-q}(X)$. For any $u \in H^{1,1}(X)$ let $N(u)$ denote the linear map on V defined by

$$N(u)v = u \wedge v \tag{9}$$

Then, $N(u)$ is of type $(-1, -1)$ with respect to (F, W). By the Hard–Lefschetz theorem, if u is a Kähler class the intersection pairing on X can be used to construct a graded-polarization of (F, W). In the language of [Sc, CKS] (F, W) is an example of a mixed Hodge structure polarized by N.

5. The period domain quotients $\Gamma\backslash D$ and their Mumford–Tate domain analogs can be partially compactified by adjoining boundary components consisting of nilpotent orbits [KU]. Via the theory of polarized mixed Hodge structures, such boundary components acquire mixed Hodge metrics.

Some properties the Hodge metric has in the pure case are no longer valid in the mixed situation. This is already clear from Example 3: we can not expect D to have holomorphic sectional curvature which is negative and bounded away from zero along horizontal directions. Nonetheless, period maps of variations of mixed Hodge structure of geometric origin satisfy a system of admissibility conditions which ensure that they have good asymptotic behavior. At the level of D-modules, this is exemplified by Saito's theory of mixed Hodge modules. At the level of classifying spaces, one has the analogs of Schmid's nilpotent orbit theorem [P2, Hay-P] and the SL_2-orbit theorem [KNU, P3].

1.5 Results

1. A mixed period domain D is an open subset of a homogeneous space for a complex Lie group $G_{\mathbb{C}}$, and hence we can identify $T_F(D)$ with a choice (22) of complement $\mathfrak{q}$ to the stabilizer of F in $\mathrm{Lie}(G_{\mathbb{C}})$. In analogy with Théorème

(5.16) of [D1], the holomorphic sectional curvature in the direction $u \in \mathfrak{q} \simeq T_F^{1,0}(D)$ is given by (cf. Theorem (3.4)):

$$R_\nabla(u,\bar{u}) = -\,[(\mathrm{ad}\,\bar{u}_+^*\,)_{\mathfrak{q}}, (\mathrm{ad}\,\bar{u}_+\,)_{\mathfrak{q}}] - \mathrm{ad}\,[u,\bar{u}]_0 \\ -\,(\mathrm{ad}\,([u,\bar{u}]_+ + [u,\bar{u}]_+^*)\,)_{\mathfrak{q}}$$

where the subscripts $_{\mathfrak{q}}$, $_0$, $_+$ denote projections onto various subalgebras of $\mathrm{Lie}(G_{\mathbb{C}})$, and $*$ is adjoint with respect to the mixed Hodge metric; the adjoint operation is meant to be preceded by the projection operator $_+$.

2. In the pure case it is well known [Gr2, Prop. 7.7] that the "top" Hodge bundle[1] $\mathcal{F}^n$ is positive in the differential geometric sense while the "dual" bundle $\mathcal{F}^0/\mathcal{F}^1$ is negative. In the mixed setting, the Chern form of the top Hodge bundle is non-negative, and positive wherever the $(-1,1)$-component of the derivative of the period map acts non-trivially on the top Hodge bundle. See Corollary 5.4.

3. By [Lu], the pseudo-metric obtained by pulling back the Hodge metric along a variation of *pure* Hodge structure is also Kähler, and so it is a natural question to ask when there are more instances where the pullback of the mixed Hodge metric along a mixed period map is Kähler. In §7, we answer this question in terms of a system of partial differential equations; in particular we prove:

   Theorem (c.f. Theorem 7.5)**.** *Let $\mathcal{V}$ be a variation of mixed Hodge structure with only two non-trivial weight graded-quotients Gr_a^W and Gr_b^W which are adjacent, i.e. $|a-b| = 1$. Then, the pullback of the mixed Hodge metric along the period map of $\mathcal{V}$ is pseudo-Kähler.*

   An example (cf. §6) of a variation of mixed Hodge structure of the type described at the end of the previous paragraph arises in *homotopy theory* as follows: Let X be a smooth complex projective variety and J_x be the kernel of the natural ring homomorphism $\mathbb{Z}\pi_1(X,x) \to \mathbb{Z}$. Then, the stalks J_x/J_x^3 underlie a variation of mixed Hodge structure with weights 1 and 2 and constant graded Hodge structure [Ha1]. We show:

   Proposition (c.f. Corollaries 6.7, 7.3)**.** *If the differential of the period map of J_x/J_x^3 is injective for a smooth complex projective variety X then the pull back metric is Kähler and its holomorphic sectional curvature of is non-positive.*

   Concerning the injectivity hypothesis, which is directly related to mixed Torelli theorems we note that these hold for compact curves [Ha1] as well as once punctured curves [Kae].

[1] In standard notation; it differs from the notation employed in [Gr2].

4. The curvature of a *Hodge–Tate domain* is identically zero:

   Proposition (c.f. Lemma 3.3 and Corollary 7.3)**.** *Suppose $h^{p,q} = 0$ unless $p = q$. Then the curvature of the mixed Hodge metric is identically zero, and pulls back to a Kähler pseudo-metric along any period map $\varphi : S \to \Gamma\backslash D$.*

   Consequently, a necessary condition for a period map $\varphi : S \to \Gamma\backslash D$ of Hodge-Tate type to have injective differential is that S support a Kähler metric of holomorphic sectional curvature ≤ 0. Important examples of such variations arise in the study of mixed Tate motives and polylogarithms [D3] and mirror symmetry [D4].

5. Let $X \to \Delta^r$ be a holomorphic family of compact Kähler manifolds of dimension n equipped with a choice of Kähler class common to every member of the family. Let $(F(s), W)$ be the corresponding variation of mixed Hodge structure defined by setting $I^{p,q} = H^{n-p,n-q}(X_s)$ as in Subsection 1.4.4. Suppose that $\lambda_1, \ldots, \lambda_k \in H^{1,1}(X_s, \mathbb{R})$ for all s (e.g. a set of Kähler classes common to all members of the family). Let $L_{\mathbb{C}}$ be the complex linear span of $\lambda_1, \ldots, \lambda_r$ and let $u : \Delta^r \to L_{\mathbb{C}}$ be a holomorphic function. Then, with N as in (9)

   $$(e^{\mathrm{i}N(u(s))} \cdot F(s), W), \tag{10}$$

   is a variation of mixed Hodge structure. The curvature of the corresponding classifying space is semi-negative along directions tangent to (10), and strictly negative wherever the period map of $F(s)$ has non-zero derivative. See Example (4.5). The resulting metric is also pseudo-Kähler, cf. Corollary (7.3).

6. Turning now to *algebraic cycles*, recall that by [Sa], a normal function is equivalent to an extension in the category of variations of mixed Hodge structure[2]

   $$0 \to \mathcal{H} \to \mathcal{V} \to \mathbb{Z}(0) \to 0. \tag{11}$$

   The classical example comes from the Abel-Jacobi map for degree zero divisors on a compact Riemann surface and its natural extension

   $$\mathrm{AJ} : \mathrm{CH}^k_{\mathrm{hom}}(Y) \to J^k(Y) \tag{12}$$

   to homologically trivial algebraic cycles on a smooth complex projective variety Y [Gr]. Application of this construction pointwise to a family of algebraic cycles $Z_s \subset Y_s$ yields the prototypical example of a *normal function*

   $$\nu : S \to J(\mathcal{H}) \tag{13}$$

   where $\mathcal{H}$ is the variation of pure Hodge structure attached to the family Y_s.

[2]Note: We have performed a Tate twist to make $\mathcal{H}$ have weight -1 here.

Proposition. *1. The pullback of the mixed Hodge metric along a normal function is a pseudo-Kähler (c.f. Example 7.6).*
2. In the case where the underlying variation of pure Hodge structure is constant (e.g. a family of cycles on a fixed smooth projective variety Y), the holomorphic sectional curvature is semi-negative (Corollary 6.3).

Using the polarization of $\mathcal{H}$, one can construct a natural *biextension* line bundle $B \to S$ whose fibers parametrize mixed Hodge structures with graded quotients

$$\mathrm{Gr}_0^W \cong \mathbb{Z}(0), \qquad \mathrm{Gr}_{-1}^W \cong \mathcal{H}_s, \qquad \mathrm{Gr}_{-2}^W \cong \mathbb{Z}(1)$$

and such that the extension between Gr_0^W and Gr_{-1}^W is determined by $\nu(s)$ and the extension from Gr_{-1}^W and Gr_{-2}^W is determined by the dual of $\nu(s)$.

As noted by Richard Hain, the biextension line bundle B carries a natural hermitian metric h which is based on measuring how far the mixed Hodge structure defined by $b \in B_s$ is from being split over $\mathbb{R}$. In [Hay-P], the first author and T. Hayama prove that for $B \to \Delta^{*r}$ arising from an admissible normal function with unipotent monodromy, the resulting biextension metric is of the form

$$h = e^{-\varphi} \tag{14}$$

with $\varphi \in L^1_{\mathrm{loc}}(\Delta^r)$, i.e. it defines a singular hermitian metric in the sense of [Dem] and hence can be used to compute the Chern current of the extension of $\bar{B}$ obtained by declaring the admissible variations of mixed Hodge structure to define the extending sections (cf. [Hay-P, BP2]). For this situation we show (§8):

Proposition. *Let S be a curve and let $\mathcal{B}$ be a variation of biextension type over S. Then the Chern form of the biextension metric* (14) *is the* $(1,1)$*–form*

$$-\frac{1}{2\pi \mathrm{i}}\partial\bar{\partial}h(s) = \frac{1}{2}[\gamma^{-1,0}, \bar{\gamma}^{-1,0}]\, ds \wedge \overline{ds},$$

where $\gamma^{-1,0}$ is the Hodge component of type $(-1,0)$ of $\varphi_*(d/ds)$ viewed as an element of $\mathfrak{g}_{\mathbb{C}}$. For self-dual variations this form is semi-negative.

Remark. This result was also obtained Richard Hain (§13, [Ha2]) by a different method.

We then deduce (see Cor. 8.3 for a precise statement):

Corollary. *Let $\mathcal{B}$ be a self-dual biextension over S with associated normal function ν. Then, the Chern form of the biextension metric vanishes along every curve in the zero locus of ν.*

The asymptotic behavior of the biextension metric is related to the Hodge conjecture: Let L be a very ample line bundle on a smooth complex projective variety X of dimension $2n$ and $\bar{P}$ be the space of hyperplane sections of X. Then, over the locus of smooth hyperplane sections $P \subset \bar{P}$, we have a natural variation of pure Hodge structure $\mathscr{H}$ of weight $2n-1$. Starting from a primitive integral, non-torsion Hodge class ζ of type (n,n) on X, we can then construct an associated normal function ν_ζ by taking a lift of ζ to Deligne cohomology. The Hodge conjecture is then equivalent [GG, BFNP] to the existence of singularities of the normal function ν_ζ (after passage to sufficiently ample L). In [BP2], it will be shown that the existence of singularities of ν_ζ is detected by the failure of the biextension metric to have a smooth extension to $\bar{P}$.

1.6 Structure

We start properly in §2 and summarize the basic properties of the classifying spaces of graded-polarized mixed Hodge structures following [P1] and compute the dependence of the bigrading (4) on $F \in D$ up to second order. Using these results, we then compute the curvature tensor and the holomorphic sectional curvature of D in §3–4.

In §5 and §8 we compute the curvature of the Hodge bundles and the biextension metrics using similar techniques. Likewise, in §7 we use the computations of §4 to determine when the pull back of the mixed Hodge metric along a period map is Kähler. In §6 we show how these calculations apply to particular situations of geometric interest.

In §9, we construct a classifying space D which is a reductive domain such that its natural complex structure is not compatible with the usual complex structure making the Hodge metric a hermitian equivariant metric. So the Chern connection for the Hodge metric is not the same as the one coming from the Maurer-Cartan form on $G_\mathbb{C}$. This makes the calculations in the mixed setting intrinsically more involved than in the pure case, even in the case of a split mixed domain.

In Appendix A we compute the Levi-Civita connection for the Hodge metric. In general it does not conserve the splitting of the complex tangent bundle into the holomorphic and anti-holomorphic parts which makes the formulas more complicated than the one for the Chern connection. Nevertheless in certain cases it simplifies which has in favorable cases consequences for the curvature and for geodesics (Cor. A.10).

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2 Classifying Spaces

2.1 Homogeneous Structure

We begin this section by reviewing some material on classifying spaces of graded-polarized mixed Hodge structure [U] which appears in [P1, P2, P3]. Namely, in analogy with the pure case, given a graded-polarized mixed Hodge structure (F, W) with underlying real vector space $V_{\mathbb{R}}$, the associated classifying space D consists of all decreasing filtrations of $V_{\mathbb{C}}$ which pair with W to define a graded-polarized mixed Hodge structure with the same graded Hodge numbers as (F, W). The data for D is therefore

$$(V_{\mathbb{R}}, W_{\bullet}, \{Q_{\bullet}\}, h^{\bullet,\bullet})$$

where $W_{\bullet}$ is the weight filtration, $\{Q_{\bullet}\}$ are the graded-polarizations and $h^{\bullet,\bullet}$ are the graded Hodge numbers.

To continue, we recall that given a point $F \in D$ the associated bigrading (4) gives a functorial isomorphism $V_{\mathbb{C}} \cong \mathrm{Gr}^W$ which sends $I^{p,q}$ to $H^{p,q} \subseteq \mathrm{Gr}^W_{p+q}$ via the quotient map. The pullback of the standard Hodge metrics on Gr^W via this isomorphism then defines a mixed Hodge metric on $V_{\mathbb{C}}$ which makes the bigrading (4) orthogonal and satisfies

$$h_F(u, v) = \mathrm{i}^{p-q} Q_{p+q}([u], [\bar{v}])$$

if u, $v \in I^{p,q}$. By functoriality, the point $F \in D$ induces a mixed Hodge structure on $\mathrm{End}(V)$ with bigrading

$$\mathrm{End}(V_{\mathbb{C}}) = \bigoplus_{r,s} \mathrm{End}(V)^{r,s} \tag{15}$$

which is orthogonal with respect the associated metric

$$h_F(\alpha, \beta) = \mathrm{Tr}(\alpha\beta^*) \tag{16}$$

where β^* is the adjoint of β with respect to h.

Let $\mathrm{GL}(V_{\mathbb{C}})^W \subset \mathrm{GL}(V_{\mathbb{C}})$ denote the Lie group of complex linear automorphisms of $V_{\mathbb{C}}$ which preserve the weight filtration W. For $g \in \mathrm{GL}(V_{\mathbb{C}})^W$ we let $Gr(g)$ denote the induced linear map on Gr^W. Let $G_{\mathbb{C}}$ be the subgroup consisting of elements which induce complex automorphisms of the graded-polarizations of W, and $G_{\mathbb{R}} = G_{\mathbb{C}} \cap GL(V_{\mathbb{R}})$.

In the pure case, $G_{\mathbb{R}}$ acts transitively on the classifying space and $G_{\mathbb{C}}$ acts transitively on the compact dual. The mixed case is slightly more intricate: Let G denote the subgroup of elements of $G_{\mathbb{C}}$ which act by real transformations on Gr^W. Then,

$$G_{\mathbb{R}} \subset G \subset G_{\mathbb{C}}$$

and we have the following result:

Theorem 2.1 ([P1, §3]). *The classifying space D is a complex manifold upon which G acts transitively by biholomorphisms.*

Remark. Hertling [He] defines a period domain of polarized mixed Hodge structures on a fixed real vector space V equipped with a polarization Q and weight filtration induced by a nilpotent infinitesimal isometry N of (V,Q). The difference with our approach is that the latter domain is homogeneous under the subgroup of G consisting of elements commuting with N. So in a natural way it is a submanifold of our domain.

2.2 Hodge Metric on the Lie Algebra

Let $\mathfrak{g}_{\mathbb{R}} = \mathrm{Lie}(G_{\mathbb{R}})$ and $\mathfrak{g}_{\mathbb{C}} = \mathrm{Lie}(G_{\mathbb{C}})$. By functoriality, any point $F \in D$ induces a mixed Hodge structure on $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_{\mathbb{R}} \otimes \mathbb{C}$ with bigrading inherited from the one on $\mathrm{End}(V_{\mathbb{C}})$, i.e. $\mathfrak{g}^{r,s} = \mathfrak{g}_{\mathbb{C}} \cap \mathrm{End}(V)^{r,s}$. For future reference, we note that:

- $\mathfrak{g}_{\mathbb{C}} \cap \mathrm{End}(V)^{r,s} = 0$ if $r + s > 0$;
- $W_{-1}\,\mathrm{End}(V) \subset \mathfrak{g}_{\mathbb{C}}$.
- The orthogonal decomposition
$$\mathrm{End}(V_{\mathbb{C}}) = \mathfrak{g}_{\mathbb{C}} \oplus \mathfrak{g}_{\mathbb{C}}^{\perp} \tag{17}$$
induces a decomposition
$$\mathrm{End}(V)^{p,-p} = \mathfrak{g}_{\mathbb{C}}^{p,-p} \oplus (\mathfrak{g}_{\mathbb{C}}^{\perp})^{p,-p} \tag{18}$$
- Let $*$ denote ajoint with respect to the metric h_F. Then,
$$* : \mathrm{End}(V)^{p,q} \to \mathrm{End}(V)^{-p,-q}; \tag{19}$$
- By Lemma (2.14) below $\alpha \in \mathfrak{g}^{p,-p} \implies \alpha^* \in \mathfrak{g}^{-p,p}$.

Remark 2.2. In general, for a mixed Hodge structure which is not split over $\mathbb{R}$, the operations of taking adjoint with respect to the mixed Hodge metric and complex conjugate do not commute.

Let $\mathrm{Flag}(D)$ denote the flag variety containing D, i.e. the set of all complex flags of $V_{\mathbb{C}}$ with the same rank sequence as the flags parametrized by D. Then, since $G \subset G_{\mathbb{C}}$ acts transitively on D, it follows that the orbit of any point $F \in D$ under $G_{\mathbb{C}}$ gives a well defined "compact dual" $\check{D} \subset \mathrm{Flag}(D)$ upon which $G_{\mathbb{C}}$ acts transitively by homeomorphisms:
$$\check{D} = G_{\mathbb{C}}/G_{\mathbb{C}}^{F}. \tag{20}$$

Remark 2.3. As in the pure case, D is an open subset of $\check{D}$ with respect to the analytic topology. In the mixed case however, $\check{D}$ is usually not compact: in Example 1.4.3 one has $G = G_{\mathbb{C}}$ and hence $D = \check{D} = \mathbb{C} \subset \mathrm{Flag}(D) = \mathbb{P}^1$. One could consider the closure of $\check{D}$ in the ambient flag variety to obtain a compact object, but as the example shows, this need not be a homogeneous space for $G_{\mathbb{C}}$.

Remark 2.4. In analogy with the above, one defines the *mixed Mumford–Tate domains* as follows: Let (F, W) be a graded-polarized mixed Hodge structure with MT group M and M_{split} be the direct sum of the Mumford–Tate groups of the associated pure Hodge structures on Gr^W. Then, M is an extension of M_{split} by a unipotent group U. Let $\mathfrak{u}$ denote the Lie algebra of $U(\mathbb{C})$ viewed as a real Lie algebra and $\mathfrak{m}_{\mathbb{R}}$ denote the Lie algebra of $M(\mathbb{R})$. Let G_M denote the real Lie group with Lie algebra $\mathfrak{u} + \mathfrak{m}_{\mathbb{R}}$ viewed as a real subalgebra of $\mathrm{Lie}(M(\mathbb{C}))$. Then, the associated mixed Mumford–Tate domain D_M is the orbit of F under G_M.

The proof that D_M is a complex manifold is parallel to the proof for D: The compact dual $\check{D}_M$ is the complex homogeneous space defined by the orbit of F under $M(\mathbb{C})$, and hence it is sufficient to check that there exists a neighborhood O of $1 \in M(\mathbb{C})$ such that $O \cdot F \subset D_M$.

It follows that in subsequent calculations we may replace $\mathfrak{g}_{\mathbb{C}}$ by $\mathrm{Lie}(M(\mathbb{C}))$.

By the defining properties of the bigrading (4), it follows that

$$\mathfrak{g}_{\mathbb{C}}^F = \bigoplus_{r \geq 0} \mathfrak{g}^{r,s} \tag{21}$$

is the Lie algebra of the stabilizer of $F \in D$ with respect to the action of $G_{\mathbb{C}}$ on $\check{D}$. Accordingly,

$$\mathfrak{q}_F = \bigoplus_{r < 0} \mathfrak{g}^{r,s} \tag{22}$$

is a vector space complement to $\mathfrak{g}_{\mathbb{C}}^F$ in $\mathfrak{q}_{\mathbb{C}}$ and hence:

Lemma 2.5. *The map*

$$u \in \mathfrak{g}_{\mathbb{C}} \mapsto \gamma_*(d/dt)_0, \qquad \gamma(t) = e^{tu} \cdot F$$

determines an isomorphism between $\mathfrak{q}_F$ *and* $T_F^{\mathrm{hol}}(D)$.

The preceding Lemma gives a way to induce a hermitian metric on the tangent bundle $T(D)$:

Definition 2.6. The isomorphism (22) provides D with a metric, the *Hodge metric*.

For $F \in D$ let $\pi_{\mathfrak{q}}$ denote orthogonal projection $\mathrm{End}(V_{\mathbb{C}}) \to \mathfrak{g}_{\mathbb{C}}$. We note that the restriction of $\pi_{\mathfrak{q}}$ to $\mathfrak{g}_{\mathbb{C}}$ is just projection with respect to the decompostion

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_{\mathbb{C}}^F \oplus \mathfrak{q}_F. \tag{23}$$

Lemma 2.7. *Let $f \in \mathfrak{g}_{\mathbb{C}}^F$. Then,*

$$\pi_{\mathfrak{q}}\circ(\operatorname{ad} f)^n = \left(\pi_{\mathfrak{q}}\circ \operatorname{ad} f\right)^n \tag{24}$$

as linear operators on $\mathfrak{g}_{\mathbb{C}}$.

Proof: Induct on n, with the base case $n = 1$ a tautology. Observe that

$$(\operatorname{ad} f)^n u = v + w. \tag{25}$$

with $v \in \mathfrak{q}_F$ and $w \in \mathfrak{g}_{\mathbb{C}}^F$. Therefore, $(\operatorname{ad} f)^{n+1}u = [f,v] + [f,w]$ and hence

$$\pi_{\mathfrak{q}}((\operatorname{ad} f)^{n+1}u) = \pi_{\mathfrak{q}}[f,v]. \tag{26}$$

By equation (25), $v = \pi_{\mathfrak{q}}((\operatorname{ad} f)^n u)$ which is equal to $(\pi_{\mathfrak{q}}\circ \operatorname{ad} f)^n u$ by induction. Substituting this identity into (26) gives

$$\pi_{\mathfrak{q}}((\operatorname{ad} f)^{n+1}u) = (\pi_{\mathfrak{q}}\circ \operatorname{ad} f)^{n+1}u. \quad \square$$

Before stating the next result, we emphasize that unlike the pure case, the operation of taking adjoint with respect to the mixed Hodge metric does not preserve $\mathfrak{g}_{\mathbb{C}}$. Therefore, the statement and proof of the next result all occur in the Lie algebra $\operatorname{End}(V)$.

Corollary 2.8. *Let $f \in \mathfrak{g}_{\mathbb{C}}^F$ and v, $w \in \mathfrak{q}$. Then,*

$$h_F(v, \exp(\pi_{\mathfrak{q}}\circ \operatorname{ad} f)w) = h_F(\exp(\pi_{\mathfrak{q}}\circ \operatorname{ad} f^*)v, w) \tag{27}$$

Proof: It is sufficient to prove

$$h_F(v, (\pi_{\mathfrak{q}}\circ \operatorname{ad} f)^m\, w) = h_F((\pi_{\mathfrak{q}}\circ \operatorname{ad} f^*)^m\, v, w)$$

We induct on m. For $m = 1$ we have

$$h_F(v, \pi_{\mathfrak{q}}[f,w]) = h_F(v,[f,w]) = h_F([f^*,v],w) = h_F(\pi_{\mathfrak{q}}[f^*,v],w)$$

since $[f,w] = w' + w''$ with $w' \in \mathfrak{q}$ and $w'' \in \mathfrak{q}^{\perp}$, which justifies

$$h_F(v, \pi_{\mathfrak{q}}[f,w]) = h_F(v,[f,w]) = h_F([f^*,v],w)$$

Likewise, $[f^*,v] = v' + v''$ with $v' \in \mathfrak{q}$ and $v'' \in \mathfrak{q}^{\perp}$ and so

$$h_F([f^*,v],w) = h_F(\pi_{\mathfrak{q}}[f^*,v],w)$$

Since at each stage we project onto $\mathfrak{q}$, passage from m to $m+1$ follows from the formula for $m = 1$.

Define

$$\Lambda = \bigoplus_{r,s<0} \mathfrak{g}^{r,s} \tag{28}$$

and note that since the conjugation condition appearing in (4) can be recast as

$$\bar{\mathfrak{g}}^{p,q} \subset \mathfrak{g}^{q,p} + [\Lambda, \mathfrak{g}^{q,p}], \tag{29}$$

it follows that Λ has a real form

$$\Lambda_{\mathbb{R}} = \Lambda \cap \mathfrak{g}_{\mathbb{R}}. \tag{30}$$

Lemma 2.9 ([P1, Lemma 4.11])**.** *If $g \in G_{\mathbb{R}} \cup \exp(\Lambda)$ then*

$$g(I_F^{p,q}) = I_{g\cdot F}^{p,q}.$$

Recall that a mixed Hodge structure (F, W) is said to be *split over* $\mathbb{R}$ if

$$\overline{I^{p,q}} = I^{q,p}.$$

Those mixed Hodge structures make up a real analytic subvariety $D_{\mathbb{R}} \subset D$. To any given mixed Hodge structure (F, W), one associates a special split real mixed Hodge structure $\hat{F} = e_F \cdot F$ as follows.

Proposition 2.10 ([CKS, Prop. 2.20])**.** *Given a mixed Hodge structure there is a unique $\delta \in \Lambda_{\mathbb{R}}$ such that the spaces $\hat{I}^{p,q} = \exp(-\mathrm{i}\delta) I^{p,q}$ give the splitting of a split real mixed Hodge structure $\hat{F} = e_F \cdot F$, the* Deligne splitting.

A *splitting operation* is a particular type of fibration $D \to D_{\mathbb{R}}$ of D over the locus of split mixed Hodge structures (cf. Theorem (2.15) [P3]). Our calculations below use the following result due to Kaplan:

Theorem 2.11 ([Ka])**.** *Given a choice of splitting operation and choice of base point $F \in D$, for each element $g \in G$ exists a distinguished decomposition*

$$g = g_{\mathbb{R}} \exp(\lambda) f, \qquad \lambda \in \Lambda, \quad g_{\mathbb{R}} \in G_{\mathbb{R}}, \quad f \in \exp(W_{-1}\mathfrak{gl}(V_{\mathbb{C}})) \cap G^F.$$

Moreover, if the splitting operation is an analytic or C^∞ map, the map $(F, g) \mapsto (g_{\mathbb{R}}, e^\lambda, f)$ is analytic, respectively C^∞.

Using the identification of $T_F D$ with $\mathfrak{q}_F$ as given by Lemma 2.5, the mixed Hodge metric (16) induces a hermitian structure on D. In analogy with Lemma (2.9) and the fact that G acts by isometry on Gr^W it follows that

Lemma 2.12 ([Ka, P3])**.** *For any $g = g_{\mathbb{R}} e^\lambda$, $g_{\mathbb{R}} \in G_{\mathbb{R}}$, $\lambda \in \Lambda$, the mixed Hodge metric on $\mathfrak{g}_{\mathbb{C}}$ changes equivariantly:*

$$h_{g\cdot F}(\mathrm{Ad}\,(g)\,\alpha, \mathrm{Ad}\,g)\,\beta) = h_F(\alpha, \beta), \forall \alpha, \beta \in \mathfrak{g}.$$

and hence $g : T_F(D) \to T_{g\cdot F}(D)$ is an isometry.

Remark 2.13. (1) In [KNU, KNU2], the authors consider a different metric on D which is obtained by replacing the bigrading (4) attached to (F, W) by the bigrading attached to the *canonical* or *sl_2-splitting* of (F, W). They then twist this metric by a distance to the boundary function (§4,[KNU2]). In particular, although the resulting metric on D in invariant under $G_{\mathbb{R}}$, it is no longer true that $g \in \exp(\Lambda)$ induces an isometry from $T_F(D)$ to $T_{g \cdot F}(D)$. The metric of [KNU, KNU2] is not quasi-isometric to the metric considered in this paper except when D is pure. See [Hay-P] for details on the geometry of this metric.
(2) The previous Lemma implies that, understanding how the decomposition appearing in Theorem 2.11 depends on $F \in D$ up to second order is sufficient to compute the curvature of D (cf.[D1]).

For future use, we introduce the subalgebras

$$\mathfrak{n}_+ := \bigoplus_{a \geq 0, b < 0} \mathfrak{g}^{a,b}, \qquad \mathfrak{n}_- := \bigoplus_{a < 0, b \geq 0} \mathfrak{g}^{a,b}. \tag{31}$$

Then, recalling the definition (28) of Λ, we have a splitting

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{n}_+ \oplus \mathfrak{g}^{0,0} \oplus \mathfrak{n}_- \oplus \Lambda$$

and we let

$$\begin{aligned} \operatorname{End}(V_{\mathbb{C}}) \to \quad & \mathfrak{n}_+, \mathfrak{g}^{0,0}, \mathfrak{n}_-, \Lambda \\ u \mapsto \quad & u_+, u_0, u_-, u_\Lambda \end{aligned} \tag{32}$$

denote orthogonal projection from $\operatorname{End}(V_{\mathbb{C}})$ to $\mathfrak{g}_{\mathbb{C}}$ followed by projection onto the corresponding factor above.

We conclude this section with a formula for the adjoint operator $\alpha \mapsto \alpha^*$ with respect to the mixed Hodge metric.

Lemma 2.14. *Let* $\mathfrak{z} = \bigoplus_p \mathfrak{g}^{-p,p}$ *and denote*

$$\pi_{\mathfrak{z}} : \operatorname{End}(V_{\mathbb{C}}) \to \mathfrak{z} \tag{33}$$

the corresponding orthogonal projection. Then (with C_F the Weil operator of $\operatorname{Gr}^W V$*) we have*

$$\alpha \in \mathfrak{z} \implies \alpha^* = -\operatorname{Ad}(C_F)\, \pi_{\mathfrak{z}}(\bar{\alpha}).$$

Proof: In the pure case, the statement is well known. Since both sides belong to $\mathfrak{z}$, we only have to check that we get the correct formula on $\operatorname{Gr}^W_0(\mathfrak{g}_{\mathbb{C}})$. □

2.3 Second Order Calculations

In this subsection, we compute the second order behavior of the decomposition of $g = \exp(u)$ given in Theorem 2.11. The analogous results to first order appear in [P1].

Employing the notation[3] from (22) and (31) consider the following splitting

$$\mathfrak{g}_{\mathbb{C}} = \underbrace{\mathfrak{g}^{0,0} \oplus \mathfrak{n}_+}_{\mathfrak{g}^F_{\mathbb{C}}} \oplus \underbrace{\mathfrak{n}_- \oplus \Lambda}_{\mathfrak{q}} . \tag{34}$$

Since $\mathfrak{q}$ is a complement to $\mathfrak{g}^F_{\mathbb{C}}$, the map

$$u \in \mathfrak{q} \mapsto e^u \cdot F \tag{35}$$

restricts to biholomorphism of a neighborhood U of 0 in $\mathfrak{q}$ onto a neighborhood of F in D. Relative to this choice of coordinates, the identification of $\mathfrak{q}$ with $T_F(D)$ coincides with the one considered above (cf. (22)).

We need to compare this with the real structure on $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_{\mathbb{R}} \otimes \mathbb{C}$. As usual, we write

$$\alpha = \mathrm{Re}(\alpha) + \mathrm{i} \cdot \mathrm{Im}(\alpha), \quad \mathrm{Re}(\alpha) = \frac{1}{2}(\alpha + \bar{\alpha}), \quad \mathrm{i} \cdot \mathrm{Im}(\alpha) = \frac{1}{2}(\alpha - \bar{\alpha}).$$

Lemma ([P1, Theorem 4.6]). *Set*

$$\mathfrak{I}(\mathfrak{g}^{0,0}) := \left\{ \varphi \in \mathfrak{g}^{0,0} \mid \bar{\varphi}^{(0,0)} = -\varphi \right\} .$$

Then

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_{\mathbb{R}} \oplus \mathfrak{I}(\mathfrak{g}^{0,0}) \oplus \mathfrak{n}_+ \oplus \mathrm{i}\Lambda_{\mathbb{R}}. \tag{36}$$

Corollary 2.15 ([P1, Corollary 4.7]). *There exists a neighborhood of* $1 \in G_{\mathbb{C}}$ *such that every element* g *in this neighborhood can be written uniquely as*

$$g = g_{\mathbb{R}} \exp(\lambda) \exp(\varphi), \quad g_{\mathbb{R}} \in \mathfrak{g}_{\mathbb{R}},\ \lambda \in \mathrm{i}\Lambda_{\mathbb{R}}, \quad \varphi \in \mathfrak{g}^{0,0} \oplus \mathfrak{n}_+ \subset \mathfrak{g}^F_{\mathbb{C}},$$

where $\varphi^{0,0}$ *is purely imaginary.*

This implies that, possibly after shrinking U there are unique functions $\gamma, \lambda, \varphi : U \to \mathfrak{g}_{\mathbb{R}}, \mathrm{i}\Lambda_{\mathbb{R}}, \mathfrak{g}^F_{\mathbb{C}}$ respectively such that

$$\exp(u) = \underbrace{\exp(\gamma(u))}_{\text{in } G_{\mathbb{R}}} \cdot \exp(\lambda(u)) \cdot \underbrace{\exp(\varphi(u))}_{\text{in } G^F_{\mathbb{C}}} . \tag{37}$$

Now we introduce $g(u) = \exp(u) = g_{\mathbb{R}}(u) \cdot \exp(\lambda(u)) \cdot \exp(\varphi(u))$ as functions on $U \cap \mathfrak{q}$.

As a prelude to the next result, we recall that by the Campbell–Baker–Hausdorff formula we have

$$e^x e^y = e^{x+y+\frac{1}{2}[x,y]+\cdots}.$$

Alternatively, making the change of variables $u = -y$, $v = x + y$ this can be written as

$$e^{u+v} e^{-u} = e^{\psi(t_0, t_1, \dots)},$$

[3]We simplify notation by writing $\mathfrak{q}$ instead of $\mathfrak{q}_F$.

where $t_m = (\operatorname{ad} u)^m v$ and ψ is a universal Lie polynomial. In a later computation (see the proof of Lemma 6.1) we need more information, namely on the shape of the part linear in v:

$$\psi_1(u,v) = \sum_m \frac{1}{(m+1)!} t_m = \frac{e^{\operatorname{ad} u} - 1}{\operatorname{ad} u} v. \tag{38}$$

Proposition 2.16. *Let $F \in D$ and $u = u_- + u_\Lambda \in \mathfrak{n}_- \oplus \Lambda = \mathfrak{q}$. Then,*

$$\varphi(u) = -\bar{u}_+ + \frac{1}{2}[u,\bar{u}]_0 + [u,\bar{u}]_+ + \frac{1}{2}[\bar{u},\bar{u}_\Lambda]_+ + O^3(u,\bar{u})$$

where the subscripts mean the orthogonal projections onto $\mathfrak{g}^{0,0}$, Λ, $\mathfrak{n}_+$ respectively.

Proof: For the linear approximation note that

$$u = \operatorname{Re}[2(u_-) - \bar{u}_\Lambda] - \mathrm{i}\operatorname{Im}(\bar{u}_\Lambda) - \bar{u}_+ \in \mathfrak{g}_\mathbb{R} \oplus \mathrm{i}\Lambda_\mathbb{R} \oplus \mathfrak{g}_\mathbb{C}^F$$

and that equation (37) yields the first degree approximation $u = \gamma_1(u)+\lambda_1(u)+\varphi_1(u)$ so that the result follows by uniqueness.

The computation proceeds by expanding the left hand side of

$$\exp(\lambda)\exp(\varphi)\exp(-u) = \exp(-\gamma) \in G_\mathbb{R}$$

using the Campbell–Baker–Hausdorff formula, and then using the fact that the right hand side is real. To first order the decomposition is

$$u = \gamma_1(u) + \lambda_1(u) + \varphi_1(u)$$

where

$$\begin{aligned}
\gamma_1(u) &= u + \bar{u} - \frac{1}{2}\pi_\Lambda(\bar{u}) - \frac{1}{2}\overline{\pi_\Lambda(\bar{u})} \\
\lambda_1(u) &= -\frac{1}{2}\pi_\Lambda(\bar{u}) + \frac{1}{2}\overline{\pi_\Lambda(\bar{u})} \\
\varphi_1(u) &= -\bar{u} + \pi_\Lambda(\bar{u})
\end{aligned}$$

where we have used π_Λ to denote projection to Λ for clarity regarding the order of complex conjugation, since these two operations do not commute.

The second degree approximation then yields that

$$\lambda_2 + \varphi_2 + \frac{1}{2}\left([\lambda_1, \varphi_1 - u] - [\varphi_1, u]\right) \text{ is real.}$$

The projection to $\mathfrak{n}_+$ equals $[\varphi_2]_+ + \frac{1}{2}([\lambda_1, \varphi_1 - u]_+ - [\varphi_1, u]_+)$. Since $\bar{\lambda}_1 = -\lambda_1$, the reality constraint implies that

$$\begin{aligned}
(\varphi_2)_+ &= -\frac{1}{2}\left\{[\lambda_1, \varphi_1 + \bar{\varphi}_1 - u - \bar{u}]_+ + [\bar{\varphi}_1, \bar{u}]_+ - [\varphi_1, u]_+\right\} \\
&= -\frac{1}{2}\left\{[\bar{\varphi}_1, \bar{u}]_+ - [\varphi_1, u]_+ + [\lambda_1, \varphi_1 + \bar{\varphi}_1 - u - \bar{u}]_+\right\}.
\end{aligned}$$

By the conjugation rules $\bar{\mathfrak{n}}^{\pm} \subset \bar{\mathfrak{n}}_{\mp} + \Lambda$, the fact that $\Lambda, \mathfrak{n}_+, \mathfrak{n}_-$ are subalgebras, and using $[\mathfrak{n}_{\pm}, \Lambda] \subset \mathfrak{n}_{\pm} + \Lambda$ this simplifies to

$$(\varphi_2)_+ = -\frac{1}{2}\left\{[\lambda_1, \varphi_1 - \bar{u}]_+ + [\bar{\varphi}_1, \bar{u}]_+ - [\varphi_1, u]_+\right\}.$$

Now set $\varphi_1 = -\bar{u} + \pi_\Lambda(\bar{u})$ so that $\varphi_1 - \bar{u} = -2\bar{u} \mod \Lambda$. The first term thus reads $\frac{1}{2}[2\lambda_1, \bar{u}]_+$, and since $\overline{\varphi_1} = -\overline{\pi_+\bar{u}}$, the second term becomes $\frac{1}{2}[\overline{\bar{u}_+}, \bar{u}]_+$ while the last simplifies to $-\frac{1}{2}[\bar{u}, u]_+$; in total we get

$$(\varphi_2)_+ = \frac{1}{2}[2\lambda_1 + \overline{\pi_+\bar{u}}, \bar{u}]_+ + \frac{1}{2}[\lambda_1, \bar{u}]_+.$$

Putting $2\lambda_1 = \overline{\pi_\Lambda \bar{u}} - \pi_\Lambda(\bar{u})$ so that $2\lambda_1 + \overline{\pi_+\bar{u}} = u - \pi_\Lambda(\bar{u})$ shows

$$(\varphi_2)_+ = \frac{1}{2}\left\{[u, \bar{u}]_+ - [\bar{u}_\Lambda, \bar{u}]_+ - [\bar{u}, u]_+\right\},$$

which is indeed equal to the stated expression for $(\varphi_2)_+$. Similarly we find for the $\mathfrak{g}^{0,0}$-component

$$(\varphi_2)_0 = \frac{1}{2}[u, \bar{u}]_0. \qquad \square$$

Corollary 2.17. *Let $F \in D$. Let*

$$h_{e^u \cdot F}(L_{e^u *}\alpha, L_{e^u *}\beta) = h_F(\exp H(u)\alpha, \beta), \quad \alpha, \beta \in \mathfrak{q}$$

denote the local form of the mixed Hodge metric on $T(D)$ relative to the choice of coordinates (35). Then, up to second order in[4] $(u, \bar{u})$

$$\begin{aligned} H(u) \;=\; & \underbrace{-(\mathrm{ad}\,(\bar{u})_+^*)_{\mathfrak{q}}}_{(1,0)\text{-term}} + \underbrace{-(\mathrm{ad}\,(\bar{u})_+)_{\mathfrak{q}}}_{(0,1)\text{-term}} \\ & + \underbrace{\frac{1}{2}\,(\mathrm{ad}\,[\bar{u}, \bar{u}_\Lambda]_+ + [\bar{u}, \bar{u}_\Lambda]_+^*)_{\mathfrak{q}}}_{(2,0)+(0,2)\text{-term}} \\ & + \underbrace{\left(\frac{1}{2}[(\mathrm{ad}\,(\bar{u})_+^*)_{\mathfrak{q}}, (\mathrm{ad}\,(\bar{u})_+)_{\mathfrak{q}}] + (\mathrm{ad}\,[u, \bar{u}]_0)_{\mathfrak{q}} + \mathrm{ad}\,[u, \bar{u}]_+ + \mathrm{ad}\,[u, \bar{u}]_+^*\right)_{\mathfrak{q}}}_{(1,1)\text{-term}}. \end{aligned}$$

Here, by "$A(x, y)$ is a (p, q)-term" we mean $A(tx, ty) = t^p \bar{t}^q A(x, y)$.

Proof: Let us first check the assertion about types. This follows directly from the the facts that ad and $\pi_{\mathfrak{q}}$ are $\mathbb{C}$-linear, while for any $\mathbb{C}$-linear operator A, one has $(tA)^* = \bar{t}A^*$ and $\overline{tA} = \bar{t}\bar{A}$.

[4] We write $x_{\mathfrak{q}}$ instead of $\pi_{\mathfrak{q}} x$ for clarity and if no confusion is likely.

Let us now start the calculations. By (37), we have

$$\begin{aligned} h_{e^u\cdot F}(L_{e^u *}\alpha, L_{e^u *}\beta) &= h_F(L_{\exp(\varphi(u)) *}\alpha, L_{\exp(\varphi(u)) *}\beta)) \\ &= h_F(\pi_{\mathfrak{q}} \operatorname{Ad}\exp(\varphi(u))\,\alpha, \pi_{\mathfrak{q}} \operatorname{Ad}\exp(\varphi(u))\,\beta)) \\ &= h_F(\pi_{\mathfrak{q}} \operatorname{Ad}\exp(\varphi(u))\,\alpha, \operatorname{Ad}\exp(\varphi(u))\,\beta)) \end{aligned} \tag{39}$$

since $\mathfrak{g}_{\mathbb{C}}^F$ and $\mathfrak{q}$ are orthogonal with respect to the mixed Hodge metric at F. Therefore,

$$\begin{aligned} h_{e^u\cdot F}(L_{e^u *}\alpha, L_{e^u *}\beta) &= h_F(\operatorname{Ad}\exp(\varphi(u))^*\,\pi_{\mathfrak{q}} \operatorname{Ad}\exp(\varphi(u))\,\alpha, \beta)) \\ &= h_F(\exp(\operatorname{ad}\varphi(u)^*\,)\pi_{\mathfrak{q}} \exp(\operatorname{ad}\varphi(u)\,)\alpha, \beta)) \\ &= h_F(\exp(\operatorname{ad}\varphi(u)^*\,)\exp(\pi_{\mathfrak{q}} \operatorname{ad}\varphi(u)\,)\alpha, \beta)) \end{aligned}$$

by equation (24). Likewise, although

$$\exp(\operatorname{ad}\varphi(u)^*\,)\exp(\pi_{\mathfrak{q}} \operatorname{ad}\varphi(u)\,)\alpha$$

is in general only an element of $\operatorname{End}(V_{\mathbb{C}})$, since we are pairing it against an element $\beta \in \mathfrak{q}$, it follows that

$$\begin{aligned} h_{e^u\cdot F}(L_{e^u *}\alpha, L_{e^u *}\beta) &= h_F(\pi_{\mathfrak{q}} \exp(\operatorname{ad}\varphi(u)^*\,)\exp(\pi_{\mathfrak{q}} \operatorname{ad}\varphi(u)\,)\alpha, \beta)) \\ &= h_F(\exp(\pi_{\mathfrak{q}} \operatorname{ad}\varphi(u)^*\,)\exp(\pi_{\mathfrak{q}} \operatorname{ad}\varphi(u)\,)\alpha, \beta)), \end{aligned}$$

where the last equality follows from (27). By the Baker–Campbell–Hausdorff formula, up to third order in $(u, \bar{u})$ the product of the exponents in the previous formula can be replaced by

$$\exp\left(\pi_{\mathfrak{q}} \operatorname{ad}\varphi(u)^* \; + \pi_{\mathfrak{q}} \operatorname{ad}\varphi(u) \; + \frac{1}{2}[\pi_{\mathfrak{q}} \operatorname{ad}\varphi(u)^*, \pi_{\mathfrak{q}} \operatorname{ad}\varphi(u)\,]\right).$$

So, we may assume that

$$H(u) = \pi_{\mathfrak{q}} \operatorname{ad}\varphi(u)^* \; + \pi_{\mathfrak{q}} \operatorname{ad}\varphi(u) \; + \frac{1}{2}[\pi_{\mathfrak{q}} \operatorname{ad}\varphi(u)^*, \pi_{\mathfrak{q}} \operatorname{ad}\varphi(u)\,].$$

To obtain the stated formula for $H(u)$, insert the formulas from Proposition 2.16 into the above equations and compute up to order 2 in u and $\bar{u}$. Use is made of the equality $[u, \bar{u}]_0^* = [u, \bar{u}]_0$ guaranteed by Lemma 2.14. □

3 Curvature of the Chern Connection

We begin this section by recalling that given a holomorphic vector bundle E equipped with a hermitian metric h, there exists a unique *Chern connection* ∇ on E which is compatible with both h and the complex structure $\bar{\partial}$. With respect to any local holomorphic framing of E, the connection form of ∇ is given by

$$\theta = \mathbf{h}^{-1}\partial\mathbf{h}, \tag{40}$$

where $\mathbf{h}$ is the transpose of the Gram–matrix of h with respect to the given frame. The curvature tensor is then

$$R_\nabla = \bar{\partial}\,\theta. \tag{41}$$

Theorem 3.1. *The connection 1-form of the mixed Hodge metric with respect to the trivialization of the tangent bundle given in Lemma 2.5 is*

$$\theta(\alpha) = -\left(\mathrm{ad}\,(\bar{\alpha})^*_+\right)_{\mathfrak{q}}$$

for $\alpha \in \mathfrak{q} \cong T_F(D)$.

Proof: By Corollary (2.17), this is the first order holomorphic term of $H(u)$. □

Lemma 3.2. *Let* (D,h) *be a complex hermitian manifold and let* $U \subset D$ *be a coordinate neighborhood centered at* $F \in D$ *and let* $\alpha,\beta \in T_F(U)\otimes\mathbb{C}$ *be of type* $(1,0)$*. In a local holomorphic frame, write the transpose Gram-matrix* $\mathbf{h}_U = (h(e^j,e^i)) = \exp H$ *for some function* H *with with values in the hermitian matrices and with* $H(0) = 0$*. Then at the origin one has*

$$R_\nabla(\alpha,\bar{\beta}) = -\partial_\alpha\partial_{\bar{\beta}}H + \frac{1}{2}\left[\partial_{\bar{\beta}}H, \partial_\alpha H\right].$$

Proof: Since the curvature is a tensor, its value on vector fields at a given point only depends on the fields at that point. Choose a complex surface $u : V \hookrightarrow U$, $V \subset \mathbb{C}^2$ a neighborhood of 0 (with coordinates (z,w)) and $u_*(d/dz)_0 = (\partial_\alpha)_0$, $u_*(d/dw)_0 = (\partial_\beta)_0$. Replace $\mathbf{h}$ by $\mathbf{h}\circ u$ and write it as

$$h = \exp(H) = I + H + \frac{1}{2}H^2 + O^3(z,\bar{z}).$$

Formulas (40),(41) tell us that the curvature at the origin equals

$$(\bar{\partial}h \wedge \partial h + \partial\bar{\partial}h)_0.$$

This 2-form evaluates on the pair of tangent vectors $(\partial_z, \partial_{\bar{w}})$ as

$$R_\nabla(\alpha,\bar{\beta}) = \partial_{\bar{w}}h\circ\partial_z h - \partial_z\partial_{\bar{w}}h. \tag{42}$$

Now use the Taylor expansion of h up to order 2 of which we give some relevant terms[5]:

$$\begin{aligned} h(z,\bar{z},w,\bar{w})_2 = {} & I + (\partial_z H)_0 z + (\partial_{\bar{w}}H)_0\bar{w} + \text{ linear terms involving } \bar{z}, w \\ & + \text{terms involving } z^2, w^2, \bar{z}^2, \bar{w}^2 + \\ & + \left(\partial_z\partial_{\bar{w}}\, H + \frac{1}{2}(\partial_z\, H)(\partial_{\bar{w}}\, H) + \frac{1}{2}(\partial_{\bar{w}}\, H)(\partial_z\, H)\right)_0 z\bar{w} \\ & + \text{terms involving } z\bar{z}, w\bar{z}, w\bar{w}. \end{aligned}$$

Now substitute in (42). □

As a first consequence, we have:

[5] Remember H is a matrix so that $\partial_z H$ and $\partial_{\bar{w}}H$ do not necessarily commute.

Lemma 3.3. *The submanifold* $\exp(\Lambda)\cdot F$ *of* D *is a flat submanifold with respect to the Hodge metric. In particular, the holomorphic sectional curvature in directions tangent to this submanifold is identically zero.*

Proof: If $\mathbf{f}$ is a unitary Hodge-frame for the mixed Hodge structure on V corresponding to F, then for all $g \in \exp(\Lambda)$, $(L_g)_*\mathbf{f}$ is a unitary Hodge frame at $g\cdot F$ and this gives a holomorphic unitary frame on the entire orbit. Hence the Chern connection is identically zero. This also follows immediately from the formula for the connection form given above. □

Theorem 3.4. *Let* D *be a period domain for mixed Hodge graded-polarized structures. Let* ∇ *be the Chern connection for the Hodge metric on the holomorphic tangent bundle* $T(D)$ *at* F. *Then for all tangent vectors* $u \in T_F^{1,0}(D) \simeq \mathfrak{q}$ *we have*

$$R_\nabla(u,\bar u) = -[(\operatorname{ad}\bar u_+^*)_{\mathfrak q},(\operatorname{ad}\bar u_+)_{\mathfrak q}] - \operatorname{ad}[u,\bar u]_0 - \left(\operatorname{ad}([u,\bar u]_+ + [u,\bar u]_+^*)\right)_{\mathfrak q}.$$

We use the following convention: for all $u \in \mathfrak g$ *we write* u_0^*, u_+^*, u_-^* *to mean: first project onto* $\mathfrak g^{0,0}$*, respectively* $\mathfrak n_+$, $\mathfrak n_-$ *and* then *take the adjoint.*

Proof: Apply the formula of Lemma (3.2). Proceeding as in the proof of that Lemma, choose a complex curve $u(z)$ tangent to $u \in T_F D$ and write $H(u(z)) = H(z,\bar z)$. We view the curve $u(z)$ as an element of $\mathfrak q$, i.e., in the preceding expression we replace u by zu and $\bar u$ by $\bar z\bar u$. Then from Corollary 2.17 we have $\partial_z H(0) = -(\operatorname{ad}(\bar u)_+^*)_{\mathfrak q}$, $\partial_{\bar z} H(0) = -(\operatorname{ad}(\bar u)_+)_{\mathfrak q}$ and

$$\begin{aligned}\partial_z\partial_{\bar z}H(0) \;&=\; \tfrac12[(\operatorname{ad}(\bar u)_+^*)_{\mathfrak q},(\operatorname{ad}(\bar u)_+)_{\mathfrak q}] + (\operatorname{ad}[u,\bar u]_0)_{\mathfrak q}\\ &\quad + (\operatorname{ad}[\bar u,u]_+ + [\bar u,u]_+^*)_{\mathfrak q}\,.\end{aligned}$$

Since at the point $F \in D$ we have $R_\nabla(u,\bar u) = -\partial_u\partial_{\bar u}H + \frac12[\partial_{\bar u}H,\partial_u H]$, the result follows. □

Remark 3.5. (1) Note that in the pure case this gives back $R_\nabla(u,\bar u) = -\operatorname{ad}[u,\bar u]_0$ as it should.
(2) By Remark 2.4, the formula for the curvature of a mixed Mumford–Tate domain is the same as the one for the mixed period domain.
(3) Exactly the same proof shows that the full curvature tensor, evaluated on pairs of tangent vectors $\{u,v\} \in T_F^{1,0}D$ is given by

$$\begin{aligned}R_\nabla(u,\bar v) \;&=\; -([(\operatorname{ad}\bar u_+^*)_{\mathfrak q},(\operatorname{ad}\bar v_+)_{\mathfrak q}])\\ &\quad -\tfrac12(\operatorname{ad}[u,\bar v]_0 + \operatorname{ad}[v,\bar u]_0)\\ &\quad + (\operatorname{ad}([\bar v,u]_+ + [\bar u,v]_+^*))_{\mathfrak q}\,.\end{aligned}$$

Alternatively, one may use (5.14.3) of [D1]. In that formula $R(u,v)$ stands for the curvature in any pair (u,v) of complex directions. So $R(u,v) = R_\nabla(u^{1,0},v^{0,1}) - R_\nabla(v^{1,0},u^{0,1})$.

4 Holomorphic Sectional Curvature in Horizontal Directions

Recall that the holomorphic sectional curvature is given by

$$R(u) := h(R_\nabla(u,\bar{u})u,u)/h(u,u)^2. \tag{43}$$

Our aim is to prove:

Theorem 4.1. *Let $u \in T_F(D)$ be a* horizontal vector *of unit length. Then $R(u) = A_1 + A_2 + A_3 + A_4$ where*

$$\begin{aligned} A_1 &= -\|[\bar{u}_+,u]_{\mathfrak{q}}\|^2, \\ A_2 &= \|[\bar{u}_+^*,u]_{\mathfrak{q}}\|^2, \\ A_3 &= -h([[u,\bar{u}]_0,u],u) \\ A_4 &= -h([[u,\bar{u}]_+,u]_{\mathfrak{q}},u) - h(u,[[u,\bar{u}]_+,u]_{\mathfrak{q}}). \end{aligned}$$

Each of these terms is real.

Proof: We start by stating the following two self-evident basis principles which can be used to simplify (43):

- Orthogonality: The decomposition $\mathfrak{g} = \bigoplus \mathfrak{g}^{p,q}$ is orthogonal for the Hodge metric;
- Jacobi identity: For all $X,Y,Z \in \mathrm{End}(V)$ we have
$$[X,[Y,Z]] = [[X,Y],Z] + [Y,[X,Z]].$$
- Metric conversion: The relation
$$h([X,Y],Z) = h(Y,[X^*,Z]) \tag{44}$$
implies
$$-h(\mathrm{ad}\,[X,X^*]\,Y,Y) = \|[X,Y]\|^2 - \|[X^*,Y]\|^2 \tag{45}$$

Theorem 3.4 and the previous rules imply:

$$\begin{aligned} h(R_\nabla(u,\bar{u})u,u) &= -h([(\mathrm{ad}\,(\bar{u}_+)^*\,)_{\mathfrak{q}},(\mathrm{ad}\,(\bar{u})_+\,)_{\mathfrak{q}}]u,u) - h((\mathrm{ad}\,[u,\bar{u}]_0\,)u,u) \\ &\quad - h((\mathrm{ad}\,[u,\bar{u}]_+\,)u,u) - h((\mathrm{ad}\,[u,\bar{u}]_+^*\,)_{\mathfrak{q}}u,u) \\ &= -\|[\bar{u}_+,u]_{\mathfrak{q}}\|^2 + \|[\bar{u}_+^*,u]_{\mathfrak{q}}\|^2 - h([[u,\bar{u}]_0,u],u) \\ &\quad - h([[u,\bar{u}]_+,u]_{\mathfrak{q}},u) - h(u,[[u,\bar{u}]_+,u]_{\mathfrak{q}}). \end{aligned}$$

This shows that $h(R_\nabla(u,\bar{u})u,u) = A_1 + A_2 + A_3 + A_4$ where the terms A_j are as stated. In particular, the terms A_1, A_2, A_4 are real. Metric conversion allows us to show that A_3 is real: since $[u,\bar{u}]_0 = [\alpha,\pi_3\bar{\alpha}] = [\alpha,\alpha^*]$ we find that

$$\begin{aligned} A_3 &= -h([[\alpha,\alpha^*],u],u)) \\ &= \|[\alpha,u]\|^2 - \|[\alpha^*,u]\|^2 \in \mathbb{R}. \end{aligned} \tag{46}$$

The next result gives the refinement of the curvature calculations with respect to the decomposition of a horizontal vector into its Hodge components:

Theorem 4.2. *For* $u = \sum_{j\leq 1} u^{-1,j} \in \mathfrak{g}_{\mathbb{C}}$ *set* [6]

$$\begin{aligned} \alpha = &\quad u^{-1,1}, \quad \beta = \quad u^{-1,0}, \qquad \lambda = \sum\nolimits_{j\geq 1} u^{-1,-j} \\ \bar{\alpha}_+ = &\quad \alpha^* + \epsilon, \quad \alpha^* = \pi_3\bar{\alpha}_+ = \bar{\alpha}_+^{1,-1}, \qquad \epsilon = \sum\nolimits_{j\geq 2} \bar{\alpha}_+^{0,-j}. \end{aligned}$$

Then,

$$\begin{aligned} A_1 &= -\left(\|[\bar{\beta}_+ + \epsilon, \alpha]\|^2 + \|[\bar{\beta}_+ + \epsilon, \beta]\|^2 + \|[\bar{\beta}_+ + \epsilon, \lambda]_{\mathfrak{q}}\|^2\right), \\ A_2 &= \|[\alpha, \beta]\|^2 + \|[\alpha, \lambda]\|^2 + \|[\bar{\beta}_+^*, \beta]_{\mathfrak{q}}\|^2 + \|[\bar{\beta}_+^* + \epsilon^*, \lambda]\|^2, \\ A_3 &= \|[\alpha, \beta]\|^2 + \|[\alpha, \lambda]\|^2 - \|[\alpha^*, \alpha]\|^2 - \|[\alpha^*, \beta]\|^2 - \|[\alpha^*, \lambda]\|^2, \\ A_4 &= -2\|[\alpha^*, \lambda]\|^2 - 2\|[\alpha^*, \beta]\|^2 + R(\alpha, \beta, \lambda), \end{aligned}$$

where

$$R(\alpha,\beta,\lambda) = -2\mathrm{Re}\left(h([[\lambda,\alpha^*],\lambda],\lambda) + h([[\alpha^*,\beta],\lambda],\lambda) + h([[\alpha^*,\lambda],\beta],\lambda)\right).$$

This last term vanishes if λ *has pure type.*
Moreover, in the $\mathbb{R}$*–split situation we have* $\bar{\alpha}^+ = \alpha^*$ *so that* $\epsilon = 0$.

Proof: **The term** A_3**.** Inserting $u = \alpha + \beta + \lambda$ in (46) immediately gives the A_3-term.
The terms A_1, A_2**.** We start by noting that $\bar{u}_+ = \bar{\alpha}_+ + \bar{\beta}_+ = \alpha^* + \epsilon + \bar{\beta}_+$ and so (note the precedence of the operators!) $\bar{u}_+^* = \alpha + \epsilon^* + \bar{\beta}_+^*$. Accordingly,

$$[\bar{u}_+, u]_{\mathfrak{q}} = [\alpha^* + \bar{\beta}_+ + \epsilon, u]_{\mathfrak{q}}, \qquad [\bar{u}_+^*, u]_{\mathfrak{q}} = [\alpha + \bar{\beta}_+^* + \epsilon^*, u].$$

The first expression gives

$$\begin{aligned} A_1 &= -\|[\bar{\beta}_+ + \epsilon, u]\|^2 \\ &= -\left(\|[\bar{\beta}_+ + \epsilon, \alpha]\|^2 + \|[\bar{\beta}_+ + \epsilon, \beta]\|^2 + \|[\bar{\beta}_+ + \epsilon, \lambda]\|^2\right). \end{aligned}$$

by orthogonality. The second expression expands as:

$$[\bar{u}_+^*, u]_{\mathfrak{q}} = [\alpha, u] + [\bar{\beta}_+^* + \epsilon^*, \alpha]_{\mathfrak{q}} + [\bar{\beta}_+^* + \epsilon^*, \beta + \lambda]_{\mathfrak{q}}$$

For weight reasons, $[\bar{\beta}_+^*, \alpha]_{\mathfrak{q}} = 0$ and $[\epsilon^*, \alpha]_{\mathfrak{q}} = [\epsilon^*, \beta]_{\mathfrak{q}} = 0$. Therefore, by orthogonality:

$$A_2 = \|[\bar{u}^*, u]_{\mathfrak{q}}\|^2 = \|[\alpha, \beta]\|^2 + \|[\alpha, \lambda]\|^2 + \|[\bar{\beta}_+^*, \beta]_{\mathfrak{q}}\|^2 + \|[\bar{\beta}_+^*, \lambda]_{\mathfrak{q}}\|^2 + \|[\epsilon^*, \lambda]_{\mathfrak{q}}\|^2.$$

The term A_4**.** To calculate A_4, we observe that

$$[u, \bar{u}]_+ = [\beta, \bar{\alpha}_+] + [\lambda, \bar{\alpha}_+] = [\beta, \alpha^* + \epsilon] + [\lambda, \alpha^* + \epsilon].$$

[6] Recall the notation (33).

So $h([[u,\bar{u}]_+,u],u) = h([[\beta,\alpha^*+\epsilon],u],u)+h([[\lambda,\alpha^*+\epsilon],u],u)$ and we consider each term separately. For the first term, note that $[[\beta,\epsilon],u]$ as well as $[[\lambda,\epsilon],u]$ belong to $\bigoplus_{j\geq 1}\mathfrak{g}^{-2,-j}$ and hence are both orthogonal to u and we can discard these terms. Moreover, $[\beta,\alpha^*]\in\mathfrak{g}^{0,-1}$ and so, by orthogonality,

$$h([[\beta,\alpha^*],u],u) = h([[\beta,\alpha^*],\alpha],\beta) + h([[\beta,\alpha^*],\beta],\lambda) + h([[\beta,\alpha^*],\lambda],\lambda).$$

Since $-h([\alpha,[\beta,\alpha^*]],\beta) = -h([\beta,\alpha^*],[\alpha^*,\beta]) = \|[\alpha^*,\beta]\|^2$ we find for the first term

$$h([[\beta,\alpha^*],u],u) = \|[\alpha^*,\beta]\|^2 + h([[\beta,\alpha^*],\beta],\lambda) + h([[\beta,\alpha^*],\lambda],\lambda).$$

Note that $[\lambda,\alpha^*]\in\bigoplus_{j\geq 0}\mathfrak{g}^{0,-2-j}$ so that by orthogonality,

$$h([[\lambda,\alpha^*],u],u) = h([[\lambda,\alpha^*],\lambda],\beta) + h([[\lambda,\alpha^*],\alpha+\lambda],\lambda).$$

The second term thus simplifies to

$$\begin{aligned} h([[\lambda,\alpha^*],\alpha],\lambda) + h([[\lambda,\alpha^*],\lambda],\lambda) &= -h([\alpha,[\lambda,\alpha^*]],\lambda) + h([[\lambda,\alpha^*],\lambda],\lambda) \\ &= -h([\lambda,\alpha^*],[\alpha^*,\lambda]) + h([[\lambda,\alpha^*],\lambda],\lambda) \\ &= \|[\alpha^*,\lambda]\|^2 + h([[\lambda,\alpha^*],\lambda],\lambda). \end{aligned}$$

It follows that

$$\begin{aligned} A_4 = {}&-2\|[\alpha^*,\lambda]\|^2 - 2\|[\alpha^*,\beta]\|^2 \\ &- \operatorname{Re}\left(h([[\lambda,\alpha^*],\lambda],\lambda) + h([[\alpha^*,\beta],\lambda],\lambda) + h([[\alpha^*,\lambda],\beta],\lambda)\right). \end{aligned}$$ □

Remark 4.3. We claim that ϵ and the Deligne splitting δ of (F,W) are related as follows:

$$\epsilon = [-2\mathrm{i}\delta,\bar{\alpha}]_+.$$

To see this, apply the Deligne splitting:

$$\alpha = \mathrm{Ad}\,(e^{\mathrm{i}\delta})\alpha^\ddagger$$

where $\alpha^\ddagger$ is type $(-1,1)$ at the split mixed Hodge structure $(\hat{F},W)$ defined by $\hat{F} = e^{-\mathrm{i}\delta}F$. At that point the complex conjugate and the adjoint of $\alpha^\ddagger$ coincide. Therefore,

$$\begin{aligned} \alpha^* &= \mathrm{Ad}\,(e^{\mathrm{i}\delta})[\alpha^{\ddagger *}]_{\hat{F}} = \mathrm{Ad}\,(e^{\mathrm{i}\delta})[\overline{\alpha^\ddagger}]_{\hat{F}} \\ \bar{\alpha} &= \mathrm{Ad}\,(e^{-\mathrm{i}\delta})[\overline{\alpha^\ddagger}]_{\hat{F}} = \mathrm{Ad}\,(e^{\mathrm{i}\delta})[\mathrm{Ad}\,(e^{-2\mathrm{i}\delta})\overline{\alpha^\ddagger}]_{\hat{F}}. \end{aligned}$$

Consequently,

$$\begin{aligned} \epsilon &= (\alpha^* - \bar{\alpha})_+ \\ &= \mathrm{Ad}\,(e^{\mathrm{i}\delta})((\mathrm{Ad}\,(e^{-2\mathrm{i}\delta}) - 1)\overline{\alpha^\ddagger})_{+,\hat{F}} \\ &= \mathrm{Ad}\,(e^{\mathrm{i}\delta})[-2i\delta,\bar{\alpha}^\ddagger]_{+,\hat{F}} \\ &= [-2\mathrm{i}\delta,\mathrm{Ad}\,(e^{\mathrm{i}\delta})\overline{\alpha^\ddagger}]_+ \\ &= [-2\mathrm{i}\delta,\mathrm{Ad}\,(e^{2\mathrm{i}\delta})\bar{\alpha}]_+ \\ &= [-2\mathrm{i}\delta,\mathrm{Ad}\,(e^{2\mathrm{i}\delta})\bar{\alpha}]_+ \\ &= [-2\mathrm{i}\delta,\bar{\alpha}]_+. \end{aligned}$$

We shall now discuss particular cases.

Corollary 4.4. *The holomorphic sectional curvature along a horizontal direction $u = \alpha + \lambda$ with α of type $(-1,1)$ and $\lambda \in \Lambda$ equals*

$$R(u) = \frac{2\|[\alpha,\lambda]\|^2 + f(u,\epsilon) - 3\|[\alpha^*,\lambda]\|^2 - \|[\alpha,\alpha^*]\|^2 - \mathrm{Re}(h([[\lambda,\alpha^*],\lambda],\lambda))}{(\|\alpha\|^2 + \|\lambda\|^2)^2},$$

where $f(u,\epsilon) = -\left(\|[\alpha,\epsilon]\|^2 + \|[\lambda,\epsilon]\|^2\right) + \|[\lambda,\epsilon^*]\|^2$. In particular:

- *$R(u) \leq 0$ if $[\alpha,\lambda] = 0 = [\lambda,\epsilon^*]$ and λ is of pure type $(-1,-k)$ for some $k < 0$ (since $[[\lambda,\alpha^*],\lambda]$ and λ have different types), and $R(u) < 0$ as soon as $\alpha \neq 0$.*
- *$R(u) > 0$ if $[\alpha^*,\lambda] = 0 = [u,\epsilon]$ provided $2\|[\alpha,\lambda]\|^2 + \|[\lambda,\epsilon^*]\|^2 > \|[\alpha^*,\alpha]\|^2$.*

Example 4.5. Let us return to the setting of the variation of mixed Hodge structure (10) arising from a variation of Kähler moduli along a family of compact Kähler manifolds. The original variation $F(s)$ of a direct sum of pure Hodge structures that can be expressed locally as

$$F(s) = e^{\Gamma(s)} \cdot F(0)$$

where $\Gamma : \Delta^r \to \mathfrak{q}$ vanishes at 0 and takes values in $\mathfrak{g}^{-1,1} \oplus \mathfrak{g}^{-2,2} \oplus \cdots$. The requirement that each γ_j be of type $(-1,-1)$ for all $F(s)$ implies that

$$\mathrm{Ad}\,(e^{-\Gamma(s)})\lambda_j = e^{-\,\mathrm{ad}\,\Gamma(s)}\lambda_j$$

is horizontal at $F(0)$ for all s. Via differentiation along a holomorphic arc through $s = 0$, this fact implies that $[\Gamma'(0),\gamma_j] = 0$ since $\Gamma'(0) \in \mathfrak{g}^{-1,1}$ and $\Gamma(0) = 0$.

The local normal form of the variation (10) is therefore

$$\tilde{F}(s) = e^{\mathrm{i}N(u(s))}e^{\Gamma(s)} \cdot F(0)$$

where $u(s)$ takes values in the complex linear span $L_{\mathbb{C}}$ of $\gamma_1,\ldots,\gamma_k$. Accordingly, the derivative of $(\tilde{F}(s),W)$ at $s = 0$ is

$$\xi = \xi^{-1,-1} + \xi^{-1,1}, \qquad \xi^{-1,-1} = \mathrm{i}N(u'(0)), \qquad \xi^{-1,1} = \Gamma'(0)$$

where $[\xi^{-1,-1},\xi^{-1,1}] = 0$.

Recall the statement of Theorem 4.2 for the definition of ϵ. We show that it vanishes in this situation. First observe that since the mixed Hodge structures $(F(s),W)$ are all split over $\mathbb{R}$, the element δ attached to $(\tilde{F}(0),W)$ is defined by the equation

$$e^{-\mathrm{i}N(\overline{u(0)})} \cdot Y_{(F(0),W)} = e^{-2\mathrm{i}\delta}e^{\mathrm{i}N(u(0))} \cdot Y_{(F(0),W)}.$$

Since δ commutes with all (p,p)-morphisms of $(\tilde{F}(0),W)$, it follows from the previous equation that $\delta = N(\mathrm{Re}(u(0)))$. Accordingly, δ is real and belongs to $L_{\mathbb{C}}$ and so

$$[\bar{\Gamma}'(0),\delta] = \overline{[\Gamma'(0),\delta]} = 0.$$

By Remark (4.3), it follows that indeed $\epsilon = 0$. Corollary 4.4 then implies:

$$R(\xi) \le 0 \text{ and } < 0 \text{ if } \xi \neq 0.$$

Corollary 4.6. *The holomorphic sectional curvature along a horizontal direction $u = \alpha + \beta$ with α type $(-1,1)$ and β type $(-1,0)$ is*

$$\begin{aligned} R(u) &= \frac{-n(\alpha,\beta) + p(\alpha,\beta)}{(\|\alpha\|^2 + \|\beta\|^2)^2}, \\ n(\alpha,\beta) &:= \|[\alpha^* + \epsilon, \alpha]\|^2 + \|[\epsilon,\beta]\|^2 + 3\|[\alpha^*,\beta]\|^2 + \|[\alpha,\bar{\beta}_+]\|^2 + \|[\bar{\beta}_+,\beta]\|^2, \\ p(\alpha,\beta) &:= \|[\alpha,\beta]\|^2 + \|[\bar{\beta}_+^*,\beta]_{\mathfrak{q}}\|^2. \end{aligned}$$

In particular, if $\alpha = 0$, $[\beta,\bar{\beta}_+] = 0 = [\epsilon,\beta]$ (which is the case if $W_{-1}\mathfrak{g}_{\mathbb{C}}$ is abelian) we have $R(u) \ge 0$.

Next, we look at a unipotent variation of mixed Hodge structure in the sense of Hain and Zucker [Ha-Z]. These are the variations where the pure Hodge structures on the graded quotients are constant so that $\alpha = u^{-1,1} = 0$ and hence $\epsilon = 0$. This situation occurs in two well known geometric examples:

- The VMHS on J_x/J_x^3, $x \in X$ where X is a smooth complex projective variety;
- The VMHS attached to a family of homologically trivial algebraic cycles moving in a fixed variety X.

Corollary 4.7. *For the curvature coming from a unipotent variation we have*

$$R(u) = \frac{-\|[\bar{\beta}_+,\beta]\|^2 - \|[\bar{\beta}_+,\lambda]\|^2 + \|[\bar{\beta}_+^*,\beta]_{\mathfrak{q}}\|^2 + \|[\bar{\beta}_+^*,\lambda]_{\mathfrak{q}}\|^2}{(\|\beta\|^2 + \|\lambda\|^2)^2}.$$

5 Curvature of Hodge Bundles

5.1 Hodge Bundles over Mixed Period Domains

In this subsection, we compute the curvature of the Hodge bundles over the classifying space D using the methods of § 2.3. Since the Hodge bundles of a variation of mixed Hodge structure $\mathcal{V} \to S$ are obtained by pulling back the Hodge bundles of D along local liftings of the period map, this furnishes a computation of the curvature of the Hodge bundles of a variation of mixed Hodge structure.

Let $F \in D$ and $\mathfrak{q}$ be the associated nilpotent subalgebra (22) and U be a neighborhood of zero in $\mathfrak{q}$ such that the map $u \to e^u \cdot F$ is a biholomorphism onto a neighborhood of F. Then, we obtain a local holomorphic framing for the bundle $\mathcal{F}^p$ over U via the sections $\alpha(u) = e^u\alpha$ for fixed $\alpha \in F^p$. Let $\beta(u) = e^u\beta$ be another such section of $\mathcal{F}^p$ over U, and L_g denote the linear action of $g \in GL(V_{\mathbb{C}})$

on $V_{\mathbb{C}}$. Let Π denote orthogonal projection from $V_{\mathbb{C}}$ to F^p. Then, as in § 2.3 by (37), the metric is

$$\begin{aligned} h_{e^u.F}(\alpha(u),\beta(u)) &= h_F(L_{\exp(\varphi(u))}\alpha, L_{\exp(\varphi(u))}\beta) \\ &= h_F(\Pi\circ L_{\exp(\varphi(u))}\alpha, L_{\exp(\varphi(u))}\beta) \\ &= h_F(L_{\exp(\varphi(u)^*)}\Pi\circ L_{\exp(\varphi(u))}\alpha, \beta) \\ &= h_F(\Pi\circ L_{\exp(\varphi(u)^*)}\Pi\circ L_{\exp(\varphi(u))}\alpha, \beta). \end{aligned}$$

In analogy with § 2.2, we have the identity

$$\Pi\circ L_{\exp(\varphi(u))} = L_{\exp(\Pi\circ\varphi(u))},$$

since $\varphi(u)$ belongs to the subalgebra preserving F^p. The identity

$$\Pi\circ L_{\exp(\varphi(u)^*)} = L_{\exp(\Pi\circ\varphi(u)^*)}$$

is also straightforward because $\varphi(u)$ is a sum of components of Hodge type (a,b) with $a \geq 0$. As such $\varphi(u)^*$ is a sum of components of Hodge type $(-a,-b)$ with $-a \leq 0$, and hence there is no way for the action of $\varphi(u)^*$ to move a vector of Hodge type (c,d) with $c < p$ back into F^p.

Accordingly, by the universal nature of the Campbell–Baker–Hausdorff formula, the only difference between the computation of the curvature of $\mathscr{F}^p$ and the curvature of $T(D)$ is that for the former we are use the linear action $GL(V_{\mathbb{C}})$ and $\mathfrak{gl}(V_{\mathbb{C}})$ and project orthogonally to F^p whereas in the later we use the adjoint action and project orthogonally to $\mathfrak{q}$. So, with Π the orthogonal projection from $V_{\mathbb{C}}$ to F^p for $u, v \in T_F^{\mathrm{hol}}(D)$ we find

$$\begin{aligned} R_\nabla(u,\bar{v}) = -([\Pi\circ(\bar{u}_+^*), \Pi\circ(\bar{v}_+)]) \\ - \frac{1}{2}\left(\Pi\circ([u,\bar{v}]_0) + \Pi\circ([v,\bar{u}]_0)\right) \\ + \Pi\circ\left([\bar{v},u]_+ + [\bar{u},v]_+^*\right). \end{aligned}$$

Taking account of the fact that the terms with subscript + (without an adjoint) and subscript 0 always preserve F^p this simplifies and we get:

Corollary 5.1. *Let Π denote orthogonal projection from $V_{\mathbb{C}}$ to F^p. Then, the curvature of the Hodge bundle $\mathscr{F}^p$ over D in the directions $u, v \in T_F^{\mathrm{hol}}(D)$ is*

$$\begin{aligned} R_\nabla(u,\bar{v}) = -([\Pi\circ(\bar{u}_+^*), \bar{v}_+]) \\ - \frac{1}{2}\left([u,\bar{v}]_0 + [v,\bar{u}]_0\right) \\ + \left([\bar{v},u]_+ + \Pi\circ[\bar{u},v]_+^*\right). \end{aligned}$$

The computation of the curvature of the quotient bundle $\mathscr{F}^p/\mathscr{F}^{p+1}$ proceeds along the same lines as the computation of the curvature of $\mathscr{F}^p$. However, in this case the corresponding projection operator Π' sends $V_{\mathbb{C}}$ to

$$\mathscr{F}^p/\mathscr{F}^{p+1} \cong U^p := \bigoplus_q \mathcal{I}^{p,q}_{(F,W)}.$$

The identity

$$\Pi'\circ L_{\exp(\varphi(u))} = L_{\exp(\Pi'\circ\varphi(u))}$$

results from the fact that elements of $\mathfrak{g}_{\mathbb{C}}^F$ have Hodge components of type (a,b) with $a \geq 0$ and such an element moves U^p to U^{p+a}. A similar argument works for $\Pi'\circ\varphi(u)^*$.

Corollary 5.2. *Let Π' denote orthogonal projection from $V_{\mathbb{C}}$ to U^p at F. Then, the curvature of the Hodge bundle $\mathcal{F}^p/\mathcal{F}^{p+1}$ over D in the directions u, $v \in T_F^{\mathrm{hol}}(D)$ is*

$$\begin{aligned} R_\nabla(u,\bar v) &= -([\Pi'\circ(\bar u_+^*), \Pi'\circ(\bar v_+)]) \\ &\quad -\tfrac{1}{2}\left(\Pi'\circ([u,\bar v]_0) + \Pi'\circ([v,\bar u]_0)\right) \\ &\quad +\Pi'\circ\left([\bar v,u]_+ + [\bar u,v]_+^*\right). \end{aligned}$$

Taking account of the fact that the terms with subscript 0 preserve U^p it follows that

$$\begin{aligned} R_\nabla(u,\bar v) &= -([\Pi'\circ(\bar u_+^*), \Pi'\circ(\bar v_+)]) \\ &\quad -\tfrac{1}{2}\left([u,\bar v]_0 + [v,\bar u]_0\right) \\ &\quad +\Pi'\circ\left([\bar v,u]_+ + [\bar u,v]_+^*\right). \end{aligned}$$

5.2 First Chern Forms and Positivity

Let us calculate the first Chern form of the Hodge bundles $\mathcal{U}^p$ over a disk $\Delta : f \to D$ with local coordinate s. Set $f(s) = F_s$ and $u = f_*(d/ds)_{F_s}$. We also let

$$u^{(p)} : \mathcal{U}^p \to \mathcal{U}^{p-1}, \quad u^p = \alpha^{(p)} + \beta^{(p)} + \lambda^{(p)}$$

be the restriction of u to $\mathcal{U}^p$ and $\alpha^{(p)}, \beta^{(p)}$ and $\lambda^{(p)}$ the decomposition into types $(-1,1), (-1,0)$, respectively $\sum_{k\geq 1}(-1,-k)$. Then we have

Lemma 5.3. *The first Chern form $c_1(\mathcal{U}^p)$ involves only the components $\alpha^{(p)}$ of u of type $(-1,1)$ and locally can be written*

$$c_1(\mathcal{U}^p) = \frac{1}{2\pi \mathrm{i}}\left(|||\alpha^{(p)}|||_{F_s} - |||\alpha^{(p+1)}|||_{F_s}\right) ds\wedge d\bar s.$$

Proof: We have to calculate $\operatorname{Tr} R_\nabla(u,\bar u)$ using Cor. 5.2. Let us write $u = \alpha+\beta+\lambda$ as before. Since $\Pi'\circ(\bar u_+) = \bar\beta_+$, we find

$$\begin{aligned} \left[\Pi'\circ(\bar u_+^*), \Pi'\circ(\bar u_+)\right] &= [\bar\beta_+^*, \bar\beta_+] && (47) \\ [u,\bar u]_0 &= [\alpha,\alpha^*] && (48) \\ \Pi'\circ[\bar u,u]_+ &= [\alpha^*, \beta+\lambda]. && (49) \end{aligned}$$

The first two terms preserve the bi-degree but this is not the case for (49). So, computing traces, we can discard it. The vanishing of the trace of $[\bar\beta_+^*, \bar\beta_+]$ follows from the standard calculation

$$\operatorname{Tr}([A^*,A]) = \operatorname{Tr}(A^*A) - \operatorname{Tr}(AA^*) = \operatorname{Tr}(AA^*) - \operatorname{Tr}(AA^*) = 0$$

with $A = \bar{\beta}_+ \in \mathrm{End}(\mathcal{U}^p)$. On the other hand, since α maps $\mathcal{U}^p$ to $\mathcal{U}^{p-1}$ this argument does not apply (48), and so

$$\begin{aligned}\mathrm{Tr}\, R_\nabla(u,\bar{u}) &= -\,\mathrm{Tr}[\bar{\beta}_+^*,\beta]\,|\mathcal{U}^p - \mathrm{Tr}[\alpha,\alpha^*]\,|\mathcal{U}^p \\ &= |||\alpha^{(p)}|||_{F_s} - |||\alpha^{(p+1)}|||_{F_s}. \quad \square\end{aligned}$$

Corollary 5.4. *The "top" Hodge bundle, say $\mathcal{U}^n \simeq \mathcal{F}^n$ (which is a holomorphic sub bundle of the total bundle) has a non-negative Chern form:*

$$c_1(\mathcal{U}^n) = \frac{\mathrm{i}}{2\pi}\left(|||\alpha^{(n)}|||_{F_s}\right) ds \wedge d\bar{s} \;\geq\; 0.$$

As in [Gr2, Prop. 7.15] one deduces form Lemma 5.3 also:

Corollary 5.5. *Let $\mathcal{E}^p := \mathcal{F}^p/\mathcal{F}^{p+1}$ and put*

$$K(\mathcal{F}^\bullet) := \bigotimes_p (\det(\mathcal{E}^p))^{\otimes p}.$$

Then the first Chern form of $K(\mathcal{F}^\bullet)$ is non-negative and is zero precisely in the horizontal directions $(-1,k)$ with $k \leq 0$.

Let us now consider the curvature form itself.

Example 5.6. Consider the case with two adjacent weights $0 \subset W_0 \subset W_1 = V$. Split the top Hodge bundle as $\mathcal{F}^n = \mathcal{J}^{n,-n} \oplus \mathcal{J}^{n,-n+1}$ and decompose the curvature matrix accordingly

$$R(u,\bar{u}) = \begin{pmatrix} \alpha^*\circ\alpha + \bar{\beta}\circ\bar{\beta}^* & \alpha^*\circ\beta \\ -\beta^*\circ\alpha & \alpha^*\circ\alpha - \bar{\beta}^*\circ\bar{\beta} \end{pmatrix}, \; u = \alpha + \beta.$$

We see that for $v \in V_{\mathbb{C}}$, $\|R(v)(u,\bar{u})\|_F = {}^{\mathsf{T}}\bar{v}R(u,\bar{u})v \geq 0$ if $u = \alpha$, but $\|R(v)(\beta,\bar{\beta})\|_F = \|\bar{\beta}^*(v^{(-n)})\|_F - \|\bar{\beta}(v^{(-n+1)})\|_F$ which need not be ≥ 0.

From the preceding example it follows that we can expect positive curvature at most in the α-direction. In fact, this is true:

Proposition 5.7. *The "top" Hodge bundle, say $\mathcal{U}^n \simeq \mathcal{F}^n$ has a positive curvature in the α-directions and has identically zero curvature in the λ-directions.*

Proof: We note the diagonal terms in the curvature form involve $\alpha^{(q)}\circ(\alpha^{(q)})^*$ acting on $\mathcal{J}^{n,q}$. Let r be the minimal q with $\mathcal{J}^{n,q} \neq 0$ and consider the splitting $\mathcal{U}^n = \mathcal{J}^{n,r} \oplus \mathcal{J}^{n,r+1} \oplus \mathcal{J}^{n,>r+1}$. Assume $\beta = 0$. The matrix of the curvature form splits accordingly:

$$R(u,\bar{u}) = \begin{pmatrix} \alpha^*\circ\alpha & 0 & \alpha^*\circ\lambda \\ 0 & \alpha^*\circ\alpha & 0 \\ -\lambda^*\circ\alpha & 0 & \alpha^*\circ\alpha \end{pmatrix}, \; u = \alpha + \lambda.$$

So with $v \in \mathcal{U}^n$ one finds for $u = \alpha + \lambda$:

$$R(v)(u,\bar{u}) = \|\alpha(v)\|_F^2 \geq 0$$

with equality if $\alpha(v) = 0$. $\square$

Here is an example of a variation where $\beta = 0$:

Example 5.8. Consider *higher normal functions associated to motivic cohomology* $H^p_{\mathcal{M}}(q)$, see [BPS]. Indeed, these give extension of $R^{p-1}\pi_*\mathbb{Z}(q)$ with $p-2q-1 < 0$ where $\pi : X \to S$ is a smooth projective family.
Assume moreover that the cohomology $H^{p-1}(X_t)$ of the fibres X_t is such that the non-zero Hodge numbers are $h^{p-1-q,q}, \cdots h^{q,p-1,q}$ (i.e. the Hodge structure has level $= p-1-2q$). With $n = 2q+1-p$ the non-zero Hodge numbers of the mixed variation are, besides $h^{0,0}$ indeed precisely $h^{-n,0}, \ldots, h^{0,-n}$. Here $\beta = 0$ while $\lambda \neq 0$.

5.3 Variations of Mixed Hodge Structure

We want to stress that, although the above calculations are done on the period domain, they apply also for variations of mixed Hodge structure: the Hodge bundles simply pull back and so does the Hodge metric. What remains to be done is to identify the actions of u, v when these are tangent to period maps.

To do this and also as a check on the preceding calculations, we shall now compute the curvature of the Hodge bundles of a variation of mixed Hodge structure starting from Griffiths computation for a *variation of pure Hodge structure* $\mathcal{H}$. To this end, we recall that the Gauss–Manin connection ∇ of $\mathcal{H}$ decomposes as

$$\nabla = \bar\theta_0 + \underbrace{\bar\partial_0 + \partial_0}_{D} + \theta_0,$$

where $\bar\partial_0$ and ∂_0 are conjugate differential operators of type $(0,1)$ and $(1,0)$ respectively which preserve the Hodge bundles $\mathcal{H}^{p,q}$, while θ_0 is an endomorphism valued 1-form which sends $\mathcal{H}^{p,q}$ to $\mathcal{H}^{p-1,q+1} \otimes \mathcal{E}^{1,0}$ and $\bar\theta_0$ is the complex conjugate of θ_0. The connection $D = \bar\partial_0 + \partial_0$ is hermitian with respect to the Hodge metric:

$$dh(u,v) = h((\bar\partial_0 + \partial_0)u, v) + h(u, (\bar\partial_0 + \partial_0)v).$$

In particular, since $\bar\partial_0$ coincides with the induced action of the $(0,1)$-part of the Gauss–Manin connection acting on

$$\mathcal{H}^{p,q} \cong \mathcal{F}^p / \mathcal{F}^{p+1},$$

it follows that D is the *Chern connection*, i.e., the hermitian holomorphic connection of the system of Hodge bundles attached to $\mathcal{H}$. Expanding out

$$(\bar\theta_0 + \bar\partial_0 + \partial_0 + \theta_0)^2 = 0$$

and decomposing with respect to Hodge types shows that

$$R_D = -(\theta_0 \wedge \bar\theta_0 + \bar\theta_0 \wedge \theta_0).$$

If d/ds is a holomorphic vector field on S, the value u of $\theta_0(f_*(d/ds))$ at zero belongs to $\mathfrak{g}^{-1,1}$ and $R_D(u, \bar u) = -[u, \bar u]$ which checks with the previous calculation.

To compute the curvature of the Hodge bundles $\mathcal{F}^p/\mathcal{F}^{p+1}$ of a variation of *mixed* Hodge structure, $\mathcal{V} \to S$ we consider the C^∞-subbundles $\mathcal{U}^p$ obtained by pulling back $\mathcal{U}^p \to D$ along the variation, i.e.

$$\mathcal{I}^{p,q}(s) = I^{p,q}_{(\mathcal{F}(s),\mathcal{W})}, \qquad \mathcal{U}^p = \bigoplus_q \mathcal{I}^{p,q}.$$

By [P1], the Gauss–Manin connection of $\mathcal{V}$ decomposes as

$$\nabla = \tau_0 + \bar{\partial} + \partial + \theta$$

where $\bar{\partial}$ and ∂ are differential operators of type $(0,1)$ and $(1,0)$ which preserve $\mathcal{U}^p$ whereas $\theta : \mathcal{U}^p \to \mathcal{U}^{p-1} \otimes \mathcal{E}^{1,0}$ and $\tau_0 : \mathcal{U}^p \to \mathcal{U}^{p+1} \otimes \mathcal{E}^{0,1}$. One has

$$\mathcal{I}^{p,q} \xrightarrow{\tau_0} (\mathcal{I}^{p+1,q-1} \otimes \mathcal{E}^{0,1}_S),$$
$$\mathcal{I}^{p,q} \xrightarrow{\theta=(\theta_0,\theta_-)} (\mathcal{I}^{p-1,q+1} \otimes \mathcal{E}^{1,0}_S) \oplus (\oplus_{k\geq 2}\mathcal{I}^{p-1,q+k} \otimes \mathcal{E}^{1,0}_S).$$

Similarly

$$\mathcal{I}^{p,q} \xrightarrow{\partial} \mathcal{I}^{p,q} \otimes \mathcal{E}^{1,0}_S,$$
$$\mathcal{I}^{p,q} \xrightarrow{\bar{\partial}=(\bar{\partial}_0,\tau_-)} (\mathcal{I}^{p,q} \otimes \mathcal{E}^{0,1}_S) \oplus (\oplus_{k\geq 1}\mathcal{I}^{p,q-k} \otimes \mathcal{E}^{0,1}_S).$$

To unify notation, we also write $\partial = \partial_0$. Then, we have

$$\nabla = \tau_0 + \tau_- + \bar{\partial}_0 + \partial_0 + \theta_- + \theta_0$$

In particular, relative to the C^∞ isomorphism of Gr^W_k with

$$\mathcal{E}_k := \bigoplus_{p+q=k} \mathcal{I}^{p,q}$$

the induced action of ∇ on Gr^W_k coincides with the action of

$$D_0 = \tau_0 + \bar{\partial}_0 + \partial_0 + \theta_0$$

on $\mathcal{E}_k$. Given that the mixed Hodge metric is just the pullback of the Hodge metric on Gr^W_k via the isomorphism with $\mathcal{E}_k$, it follows that $\bar{\partial}_0 + \partial_0$ is a hermitian connection on $\mathcal{U}^p$. In particular, since the induced holomorphic structure on $\mathcal{U}^p$ is given by $\bar{\partial}$ and by the adjoint property, it follows that

$$D = \underbrace{\tau_- + \bar{\partial}_0}_{\bar{\partial}} + \partial_0 - \tau_-^* \tag{50}$$

is the Chern connection of $\mathcal{U}^p$ relative to the mixed Hodge metric. Thus,

$$R_D = R_{(\bar{\partial}+\partial_0)-\tau_-^*} = R_{(\bar{\partial}+\partial_0)} - (\bar{\partial} + \partial_0)\tau_-^* + \tau_-^* \wedge \tau_-^*.$$

To simplify this, observe that τ_-^* is a differential form of type (1,0), so we must have

$$-\partial\tau_-^* + \tau_-^* \wedge \tau_-^* = 0$$

in order to get a differential form of type $(1,1)$. Therefore,

$$R_D = R_{(\bar\partial+\partial_0)} - \bar\partial\tau_-^*.$$

Expanding out

$$\nabla^2 = (\tau_0 + \bar\partial + \partial_0 + \theta)^2 = 0,$$

it follows that

$$R_{(\bar\partial+\partial_0)} = -(\theta \wedge \tau_0 + \tau_0 \wedge \theta) \tag{51}$$

and hence

$$R_D = -(\theta \wedge \tau_0 + \tau_0 \wedge \theta) - \bar\partial\tau_-^*.$$

To continue, we note that

$$\bar\partial\tau_-^* = (\bar\partial_0 + \tau_-)\tau_-^* = \bar\partial_0\tau_-^* + \tau_- \wedge \tau_-^* + \tau_-^* \wedge \tau_-$$

and so

$$R_D = -(\theta \wedge \tau_0 + \tau_0 \wedge \theta) - (\tau_- \wedge \tau_-^* + \tau_-^* \wedge \tau_-) - \bar\partial_0\tau_-^*. \tag{52}$$

To finish the calculation, we differentiate the identity

$$h(\tau_-(\sigma_1), \sigma_2) = h(\sigma_1, \tau_-^*(\sigma_2))$$

and take the $(1,1)$ part to obtain

$$\begin{aligned} &h((\partial_0\tau_-)(\sigma_1) + \tau_-(\partial_0\sigma_1), \sigma_2) + h(\tau_-(\sigma_1), \bar\partial_0\sigma_2) \\ &\qquad = h(\partial_0\sigma_1, \tau_-^*(\sigma_2)) + h(\sigma_1, (\bar\partial_0\tau_-^*)(\sigma_2) + \tau_-^*(\bar\partial_0\sigma_2)). \end{aligned}$$

Using the properties of the adjoint, this simplifies to

$$\bar\partial_0\tau_-^* = (\partial_0\tau_-)^*.$$

It remains to compute $\partial_0\tau_- = \partial\tau_-$. To do this, first observe that

$$R_{\bar\partial+\partial} = R_{\bar\partial_0+\partial_0+\tau_-} = R_{\bar\partial_0+\partial_0} + (\bar\partial_0 + \partial)\tau_- + \tau_- \wedge \tau_-.$$

Now note that equation (51) implies that $R_{\bar\partial+\partial}$ is of type $(1,1)$, and hence

$$R_{\bar\partial+\partial_0} = R_{\bar\partial_0+\partial_0} + \partial\tau_-,$$

since $R_{\bar\partial_0+\partial_0}$ is also of type $(1,1)$ as the curvature of hermitian holomorphic connection for h and $\bar\partial_0$. Moreover, since $\bar\partial_0 + \partial_0$ preserves the bigrading by $\mathcal{J}^{p,q}$ whereas $\partial\tau_-$ lowers weights, it follows from (51) that

$$\partial\tau_- = -(\theta_- \wedge \tau_0 + \tau_0 \wedge \theta_-).$$

Corollary 5.9. *The curvature of the Hodge bundles of a variation of mixed Hodge structure $\mathcal{V} \to S$ is*

$$R_D = -(\theta \wedge \tau_0 + \tau_0 \wedge \theta) - (\theta_- \wedge \tau_0 + \tau_0 \wedge \theta_-)^* - (\tau_- \wedge \tau_-^* + \tau_-^* \wedge \tau_-).$$

Let us compare the above results with the ones obtained on the period domain.

Proposition 5.10. *Let $\theta(\xi) = u$. then the action of $R_D(\xi, \bar{\xi})$ on $\mathcal{U}^p$ agrees with the action of $R_\nabla(u, \bar{u})$ on U^p from Corollary (5.2). More precisely, the four terms in the expression for $R_\nabla(u, \bar{u})$ compare as follows*

$$\begin{aligned} [\Pi' \circ (\bar{u}_+^*), \Pi' \circ (\bar{u}_+)] &= (\theta \wedge \tau_0 + \tau_0 \wedge \theta)(\xi, \bar{\xi}) \\ -[u, \bar{u}]_0 &= -(\theta_0 \wedge \tau_0 + \tau_0 \wedge \theta_0)(\xi, \bar{\xi}) \\ -\Pi' \circ [u, \bar{u}]_+ &= -(\theta_- \wedge \tau_0 + \tau_0 \wedge \theta_-)(\xi, \bar{\xi}), \\ -\Pi' \circ [u, \bar{u}]_+^* &= -(\theta_- \wedge \tau_0 + \tau_0 \wedge \theta_-)^*(\xi, \bar{\xi}). \end{aligned}$$

Proof: Recall that for vector valued A of type $(1,0)$ and B of type $(0,1)$ we have

$$(A \wedge B + B \wedge A)(\xi, \bar{\xi}) = [A(\xi), B(\bar{\xi})].$$

A check of Hodge types shows that $\tau_-(\xi) = \Pi' \circ (\bar{u})_+$ and hence

$$-(\tau_- \wedge \tau_-^* + \tau_-^* \wedge \tau_-)(\xi, \bar{\xi}) = -[\Pi' \circ (\bar{u})_+^*, \Pi' \circ (\bar{u})_+]$$

which is the first term of $R_\nabla(u, \bar{u})$. The partial term

$$-(\theta_0 \wedge \tau_0 + \tau_0 \wedge \theta_0)(\xi, \bar{\xi}) = -[u, \bar{u}]_0$$

is extracted from $-(\theta \wedge \tau_0 + \tau_0 \wedge \theta)$. What remains of this term,

$$-(\theta_- \wedge \tau_0 + \tau_0 \wedge \theta_-),$$

computes $-\Pi' \circ [u, \bar{u}]_+$. □

6 Special Case: $W_{-1}\mathfrak{g}_{\mathbb{C}}$ is Abelian

Negative Curvature

Consider a period map

$$F : \Delta \to D, \quad s \mapsto F(s).$$

One lets $\pi_{\mathfrak{q}}^{F(s)}$ denote projection onto $\mathfrak{q}_{F(s)}$ via the decomposition

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_{\mathbb{C}}^{F(s)} \oplus \mathfrak{q}_{F(s)}.$$

The following expression for the pushforward vector field d/ds on Δ is needed below:

Lemma 6.1. *We have*

$$F_*\left(\frac{d}{ds}\right) = \pi_{\mathfrak{q}}^{F(s)}\psi_1\left(\Gamma(s), \left(\frac{d\Gamma}{ds}\right)\right), \tag{53}$$

where we recall (38) *that* $\psi_1(u,v) = \dfrac{e^{\operatorname{ad} u} - 1}{\operatorname{ad} u - 1}v$.

Proof: By Lemma 2.5 we have $F(s) = e^{\Gamma(s)} \cdot F(0)$ and thus

$$\begin{aligned} F(s) &= e^{\Gamma(s)}e^{-\Gamma(p)}F(p) \\ &= e^{\Gamma(p)+[\Gamma(s)-\Gamma(p)]}e^{-\Gamma(p)}F(p). \end{aligned}$$

The Campbell-Baker-Hausdorff formalism (38) shows that

$$e^{\Gamma(p)+[\Gamma(s)-\Gamma(p)]}e^{-\Gamma(p)} = e^{\psi_1(\Gamma(p),\Gamma(s)-\Gamma(p))}.$$

Since $\Gamma(s) - \Gamma(p) = (s-p)\frac{d\Gamma}{ds}(p) + O((s-p)^2)$, we have

$$e^{\psi_1(\Gamma(p),\Gamma(s)-\Gamma(p)} = e^{\psi_1(\Gamma(p),\frac{d\Gamma}{ds}(p))(s-p)+O((s-p)^2)}.$$

So, for a given test function ζ at $F(p)$, we have

$$\begin{aligned} F_*\left(\frac{d}{ds}\right)_p \zeta &= \left(\frac{d}{ds}\right)_p \zeta(e^{\Gamma(s)} \cdot F(0)) \\ &= \left(\frac{d}{ds}\right)_p \zeta(e^{(s-p)\psi_1(\Gamma(p),\frac{d\Gamma}{ds}(p))} \cdot F(p)). \end{aligned}$$

The result then follows applying again Lemma 2.5 but now for the identification of $T_{F(p)}D$ and $\mathfrak{q}^{F(p)}$ (in loc. cit. take $t = s - p$ and $u = \frac{d\Gamma}{ds}(p)$). □

Proposition 6.2. *Let*

$$F : \Delta \to D, \quad s \mapsto F(s),$$

be the period map of a unipotent variation of mixed Hodge structure (i.e. the induced variations on Gr^W *are constant) and suppose further that* $W_{-1}\mathfrak{g}_{\mathbb{C}}$ *is abelian. Then the holomorphic sectional curvature of the pull back metric is* ≤ 0.

Proof: We have seen in Corollary 4.6 that the holomorphic sectional curvature of the Hodge metric on D at $F(0)$ is semi-positive. However, when we pull back a metric, the curvature gets an extra term which is ≤ 0. We shall show that due to the fact that $W_{-1}\mathfrak{g}_{\mathbb{C}}$ is abelian, the pull back metric gains sufficient negativity to compensate positivity.

By the choice of coordinates (35), we can write the period map in the local normal form

$$F(s) = e^{\Gamma(s)} \cdot F(0),$$

where $\Gamma(s)$ is a holomorphic function taking values in the intersection of $W_{-1}\mathfrak{g}_{\mathbb{C}}$ and $\mathfrak{q} = \mathfrak{q}_{F(0)}$, i.e. $\Gamma(s) \in \mathfrak{g}^{-1,0}$. Then $\overline{\Gamma(s)} \in \mathfrak{g}^{0,-1} + \Lambda$ and Kaplan's decomposition (Theorem 2.11) in this situation simplifies to

$$e^{\Gamma(s)} = \underbrace{e^{\Gamma(s)+\bar{\Gamma}(s)}}_{g_{\mathbb{R}}(s)} \cdot \underbrace{e^{-\pi_{\Lambda}(\bar{\Gamma}(s))}}_{e^{\lambda(s)}} \cdot \underbrace{e^{-\pi_{+}(\bar{\Gamma}(s))}}_{f(s)} \tag{54}$$

thanks to the fact that $W_{-1}\mathfrak{g}_{\mathbb{C}}$ is abelian.

The relation (53) becomes

$$F_{*}\left(\frac{d}{ds}\right) = \pi_{\mathfrak{q}}^{F(s)}\left(\frac{d\Gamma}{ds}\right), \tag{55}$$

since $\psi_1(\Gamma(p), \frac{d\Gamma}{ds}(p)) = \frac{d\Gamma}{ds}(p)$: indeed, in our case $\Gamma(p)$ and $\frac{d\Gamma}{ds}(p)$ commute. Next we need to replace $\pi_{\mathfrak{q}}^{F(s)}$ by an expression involving $\pi_{\mathfrak{q}} = \pi_{\mathfrak{q}}^{F(0)}$ since we want to calculate the Hodge metric at $F(0)$. Now note that $\operatorname{Ad} g_{\mathbb{R}}(s) \cdot e^{\lambda(s)}$ maps $\operatorname{End}(V)^{i,j}_{F(0)}$ to $\operatorname{End}(V)^{i,j}_{F(s)}$ and since $\pi_{\mathfrak{q}}^{F(s)}$ is defined in terms of projections onto such components,

$$\begin{aligned}
\pi_{\mathfrak{q}}^{F(s)} &= \operatorname{Ad} g_{\mathbb{R}}(s) \cdot \operatorname{Ad} e^{\lambda(s)} \circ \pi_{\mathfrak{q}} \circ \operatorname{Ad} e^{-\lambda(s)} \cdot \operatorname{Ad} g_{\mathbb{R}}^{-1}(s)\\
&= \operatorname{Ad} g_{\mathbb{R}}(s) \cdot \operatorname{Ad} e^{\lambda(s)} \circ \pi_{\mathfrak{q}} \circ \operatorname{Ad} e^{\varphi(s)} \cdot \operatorname{Ad} e^{-\Gamma(s)}.
\end{aligned}$$

Remark that (54) shows that $\varphi(s) = -\pi_{+}(\bar{\Gamma}(s)) \in \mathfrak{g}^{0,-1}$. Using all of this, again by commutativity, (55) becomes

$$F_{*}\left(\frac{d}{ds}\right) = \operatorname{Ad} g_{\mathbb{R}}(s) \cdot \operatorname{Ad} e^{\lambda(s)} \left(\frac{d\Gamma}{ds}\right). \tag{56}$$

Note that $\operatorname{Ad} g_{\mathbb{R}}(s) \cdot \operatorname{Ad} e^{\lambda(s)}$ acts by isometries and so

$$\begin{aligned}
h(s) &:= \left\| (F_{*}\left(\frac{d}{ds}\right) \right\|_{F(s)}\\
&= \left\| \left(\frac{d\Gamma}{ds}\right) \right\|_{F(0)}.
\end{aligned}$$

The function $\xi(s) = \dfrac{d\Gamma}{ds}$ is a holomorphic function and so $\dfrac{\partial \xi(s)}{\partial \bar{s}} = 0$. Put $\dot{\xi} = \dfrac{d\xi(s)}{ds}$ and $h_o = h_{F(0)}$. Then, the curvature of the pullback metric is:

$$\begin{aligned}
K &= -\frac{1}{h}\frac{\partial^2}{\partial s \partial \bar{s}} \log h = -\frac{1}{h_o(\xi,\xi)}\frac{\partial^2}{\partial s \partial \bar{s}} \log h_o(\xi,\xi)\\
&= -\frac{1}{h_o(\xi,\xi)}\frac{\partial}{\partial s}\left(\frac{h_o(\xi,\dot{\xi})}{h_o(\xi,\xi)}\right)\\
&= -\frac{1}{h_o(\xi,\xi)}\frac{h_o(\dot{\xi},\dot{\xi})h_o(\xi,\xi) - h_o(\dot{\xi},\xi)h_o(\xi,\dot{\xi})}{h_o(\xi,\xi)^2}\\
&= \frac{|h_o(\dot{\xi},\xi)|^2 - h_o(\dot{\xi},\dot{\xi})h_o(\xi,\xi)}{h^3(\xi,\xi)} \leq 0,
\end{aligned}$$

where the last step follows from the Cauchy-Schwarz inequality for $h_o(\dot{\xi},\xi)$. □

Remark. The proof shows that the Gaussian curvature of the pullback is negative wherever ξ and $\dot{\xi}$ are linearly independent.

In particular, Proposition (6.2) yields:

Corollary 6.3. *Let $\Delta \to D$ be a period map associated to a normal function with fixed underlying Hodge structure. Then the holomorphic sectional curvature of the pull back of the Hodge metric is semi-negative.*

Remark 6.4. Via isomorphism $\mathrm{Ext}^1_{\mathrm{MHS}}(A,B) \cong \mathrm{Ext}^1(\mathbb{Z}(0), B \otimes A^\vee)$, the observation of the previous paragraph also applies to families of cycles on a fixed variety X and the VMHS on J_x/J_x^3 of a smooth projective variety.

Another Application: Mixed Hodge Structures and Fundamental Groups

We treat this in some detail with an eye towards a reader less acquainted with this material.

Let X be a smooth complex algebraic variety, and $\mathbb{Z}\pi_1(X,x)$ be the group ring consisting of all finite, formal $\mathbb{Z}$-linear combinations of elements of $\pi_1(X,x)$. The augmentation ideal J_x is defined to be the kernel of the ring homomorphism

$$\epsilon : \mathbb{Z}\pi_1(X,x) \to \mathbb{Z}$$

which maps each element $g \in \pi_1(X,x)$ to $1 \in \mathbb{Z}$. By the work of Morgan [M], the quotients J_x/J_x^k carry functorial mixed Hodge structures constructed from the minimal model of the de Rham algebra of X. We follow Hain's alternative approach [Ha1]; the mixed Hodge structure on J_x/J_x^k can be described using so called iterated integrals as follows: The iterated integral on $\theta_1,\dots,\theta_r \in \mathscr{E}^1(X)$,

$$\int \theta_1 \cdots \theta_r$$

assigns to each smooth path $\gamma : [0,1] \to X$ the integral of $\theta_1 \cdots \theta_r$ over the standard simplex in $\mathbb{R}^r$, i.e.

$$\int_\gamma \theta_1 \cdots \theta_r = \int_{0 \le t_1 \le \cdots \le t_r \le 1} \theta_1(\gamma_*(d/dt_1)) \cdots \theta_r(\gamma_*(d/dt_r)) dt_1 \cdots dt_r.$$

Such an iterated integral is said to have length r. The spaces $\mathrm{Hom}_{\mathbb{Z}}(J_x/J_x^{s+1},\mathbb{C})$ can be described as spaces of certain linear combinations of iterated integrals of lengths $\le s$, the so called *homotopy functionals*. We only need their description for $s = 2$:

Theorem 6.5 ([Ha1, Prop. 3.1.]). *The iterated integral*

$$\int \theta + \sum_{j,k} a_{jk} \int \theta_j \theta_k \qquad (57)$$

is a homotopy functional if and only if $\theta_1, \ldots, \theta_r$ *are closed and*

$$d\theta + \sum_{jk} a_{jk}\theta_j \wedge \theta_k = 0. \tag{58}$$

The mixed Hodge structure (F, W) on $\mathrm{Hom}_{\mathbb{Z}}(J_x/J_x^{s+1}, \mathbb{C})$ is described on the level of iterated integrals as follows. Such a sum belongs F^p if and only if each integrand $\theta_1 \cdots \theta_k$ contains at least p terms $\theta_j \in \Omega^1(X)$. As for the weight filtration, α belongs to W_k if and only if α is representable by a sum of iterated integrals of length $\leq k$ plus the number of logarithmic terms dz_j/z_j in the integrand.

Suppose next that $H^1(X)$ has pure weight $\ell = 1$ or $\ell = 2$. The first happens for X projective, the second for instance when the compactification of X is $\mathbb{P}^1$. In these situations, following [Ha1, §6], the dual of J_x/J_x^3 is an extension of pure Hodge structures. To explain the result, note that the cup-product pairing $H^1(X) \otimes H^1(X) \to H^2(X)$ is a morphism of pure Hodge structures. It follows that

$$K := \ker\left[H^1(X) \otimes H^1(X) \to H^2(X)\right]$$

carries a pure Hodge structure of weight 2ℓ. Theorem 6.5 now implies:

Theorem 6.6. *The mixed Hodge structure on* $\mathrm{Hom}_{\mathbb{Z}}(J/J^3, \mathbb{C})$ *is the extension of pure Hodge structures of weight* ℓ *and* 2ℓ *given by*

$$0 \to H^1(X) \to \mathrm{Hom}_{\mathbb{Z}}(J/J^3, \mathbb{C}) \xrightarrow{p} K \to 0.$$

Explicitly, the iterated integral $\int \theta + \sum_{j,k} a_{jk} \int \theta_j\theta_k$ *is mapped by* p *to* $\sum a_{jk}[\theta_j] \otimes [\theta_k]$ *which, by construction, belongs to* K. *The kernel of* p *can be identified with the the length one homotopy integrals* $\int \theta$ *, i.e. those with* $d\theta = 0$. *Hence* $\ker p \simeq H^1(X)$. *It follows that the graded pieces have a natural polarization coming from the one on* $H^1(X)$ *and which is given by these identifications.*

In particular, the above implies that if X is smooth projective, the graded polarized mixed Hodge structure on $\mathrm{Hom}_{\mathbb{Z}}(J/J^3, \mathbb{C})$ has two adjacent weights and so if we now leave X fixed but vary the base point, we get a family of mixed Hodge structures over X for which $W_{-1}\mathfrak{g}_{\mathbb{C}}$ is abelian and by Proposition 6.2 we conclude:

Corollary 6.7. *Let* X *be a smooth complex projective variety, and suppose that the differential of the period map of* J_x/J_x^3 *is injective. Then the holomorphic sectional curvature of* X *is* ≤ 0.

Complements: Flat Structure and the Hodge Metric

1. The *flat structure* given by the local system attached to J/J^3 may be described as follows: Fix a point $x_o \in X$ and let U be a simply connected open subset containing x_o. Given a point $x \in U$ let $\gamma : [0,1] \to U$ be a smooth path connecting x_o to x. Then, conjugation

$$\alpha \mapsto \gamma\alpha\gamma^{-1} \tag{59}$$

defines an isomorphism $\pi_1(X,x) \to \pi_1(X,x_0)$ which is independent of γ since U is simply connected. Trivializing $(J/J^3)^*$ using (59), we then obtain the period map via the change of base point formula (see [Ha1, Remark 6.6]):

$$\int_{\gamma\alpha\gamma^{-1}} \theta_1\theta_2 = \int_\alpha \theta_1\theta_2 + \left(\int_\gamma \theta_1\right)\left(\int_\alpha \theta_2\right) - \left(\int_\gamma \theta_2\right)\left(\int_\alpha \theta_1\right) \qquad (60)$$

one then obtains the following result via differentiation:

Lemma 6.8. *The flat connection ∇ of $(J/J^3)^*$ operates on iterated integrals via the following rules:*

$$\nabla_\xi\left(\int \theta_1\theta_2\right) = \theta_1(\xi)\left(\int \theta_2\right) - \theta_2(\xi)\left(\int \theta_1\right)$$

and $\nabla_\xi(\int \theta) = 0$.

As a check of the formula for ∇ given in Lemma 6.8, note that by Theorem 6.5 that the iterated integral (57) appears in $(J/J^3)^*$ only if θ_j and θ_k is closed for all j, k an equation (58) holds. Therefore,

$$\nabla^2\left(\int \theta + \sum_{j,k} a_{jk} \int \theta_j\theta_k\right) = \sum a_{ij}\left(d\theta_j \int \theta_k - d\theta_j \int \theta_k\right) = 0$$

because $d\theta_j = 0$. Likewise, direct calculation using Lemma (6.8) shows that the Hodge filtration $\mathcal{F}$ of $(J/J^3)^*$ is holomorphic and horizontal with respect to ∇, and the weight filtration W is flat.

2. By way of illustration we shall prove the correctness of the expression (8) for *the mixed Hodge metric* as announced in the introduction. First of all (for $X = \mathbb{P}^1 - \{0,1,\infty\}$)

$$\nabla \int \frac{dz}{z}\cdot\frac{dz}{1-z} = \frac{dz}{z}\int\frac{dz}{z-1} - \frac{dz}{z-1}\int\frac{dz}{z},$$

and, secondly, from the above discussion it follows that

$$\left\|\int \frac{dz}{z-1}\right\|^2 = h([\frac{dz}{z-1}],[\frac{dz}{z-1}]) = (4\pi)^2.$$

where h is the Hodge metric on $H^1(X)$ (and similarly for $\|\int \frac{dz}{z}\|^2$).

3. As a further illustration, let us calculate *the mixed Hodge metric* when we specialize the preceding to a compact Riemann surface X of genus $g > 1$. Let $\theta_1, \dots, \theta_g$ be an unitary basis of $H^{1,0}(X)$ with respect to the Hodge metric. Then, up to a scalar, the metric on X obtained by pulling back the mixed Hodge metric via the period map of $(J/J^3)^*$ is given by

$$\|d/dz\|^2 = \sum_{j=1}^{g} \|\theta_j(d/dz)\|^2.$$

This follows directly from Lemma (6.8) and the discussion on the mixed Hodge structure on $(J/J^3)^*$ we just gave.

Remark. The above description of the mixed Hodge metric can be generalized in a straightforward manner to any smooth complex projective variety.

7 The Kähler Condition

We recall some facts about Kähler metrics. Let h be a hermitian metric on a complex manifold M. Given any system of local holomorphic coordinates $(z_1, \ldots, z_m)$ on M, the associated fundamental 2-form Ω is given by the formula

$$\Omega = -\frac{\sqrt{-1}}{2} \sum_{j,k} h_{jk} dz_j \wedge d\bar{z}_k, \qquad h_{jk} = h\left(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_k}\right). \tag{61}$$

This form is a globally defined $(1,1)$-form and by definition h is Kähler if and only if $d\Omega = 0$.

An equivalent condition can be given in terms of the torsion tensor for the associated Chern connection ∇_h on the holomorphic tangent bundle. Recall that the *torsion tensor* for any linear connection ∇ on the tangent bundle is defined by the formula

$$T_\nabla(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y],$$

where X and Y a local smooth vector fields. The Kähler condition is equivalent to $T_{\nabla_h} = 0$. see [Ko, Prop. I.7.19].

Proposition 7.1. *A hermitian metric h as above with Chern connection $\nabla = \nabla_h$ is Kähler if and only if for local holomorphic vector fields X, Y on M one has*

$$\nabla_X Y - \nabla_Y X - [X,Y] = 0.$$

Proof: The torsion is a tensor, i.e. bilinear over $C^\infty(M)$ and since all local vector fields are $C^\infty(M)$-linear combinations of the holomorphic coordinate vector fields and their complex conjugates, it suffices to test whether $T(X,Y) = 0$ with X and Y locally holomorphic or anti-holomorphic. If X and Y have different types one has $[X,Y] = 0$ [7] and hence the torsion vanishes on such pairs (X,Y). Since $T(\bar{X},\bar{Y}) = \overline{T(X,Y)}$, to show that the torsion vanishes, one therefore may restrict to pairs (X,Y) of local holomorphic vector fields. So $T = 0$ precisely if T vanishes on pairs of vector fields belonging to a holomorphic local frame for the holomorphic tangent bundle. □

Let Δ^m a polydisk at $0 \in \mathbb{C}^m$ with coordinates $(s_1, \ldots, s_m)$ and let $F : \Delta^m \to D$ be a holomorphic, horizontal map. Let $\mathfrak{q}$ be the subalgebra (22) attached to $F(0)$. Recalling the local biholomorphism (35) mapping a neighborhood of $0 \in \mathfrak{q}$ to a neighborhood of $F(0)$ in D, locally we can write as in [P1]

$$F(s) = e^{\Gamma(s)} \cdot F(0)$$

[7] Clearly, if X, Y are local holomorphic coordinate vector fields $[X,\bar{Y}] = 0$ and an easy calculation shows that $[fX, \bar{g}\bar{Y}] = 0$ whenever f, g are local holomorphic functions and X, Y holomorphic fields with $[X,\bar{Y}] = 0$.

for a unique $\mathfrak{q}$-valued *holomorphic* function Γ which vanishes at 0.

Theorem 7.2. *Let $h = F^*(h_D)$ denote the pullback of the mixed Hodge metric h_D to S. Set $\xi_j = \frac{\partial\Gamma}{\partial s_j}(0)$. Then h is Kähler if and only if for all j, k, ℓ one has*

$$h(\xi_j, \pi_{\mathfrak{q}}[\pi_+(\bar{\xi}_\ell), \xi_k]) - h(\xi_\ell, \pi_{\mathfrak{q}}[\pi_+(\bar{\xi}_j), \xi_k]) = 0. \tag{62}$$

Proof: First, remark that by Theorem 3.1 one has

$$\nabla_{\xi_j}\xi_\ell = -\pi_{\mathfrak{q}}[\pi_+(\overline{\xi_j})^*, \xi_\ell].$$

Since

$$\begin{aligned} h(\pi_{\mathfrak{q}}[\pi_+(\overline{\xi_j})^*, \xi_\ell], \xi_k) &= h([\pi_+(\overline{\xi_j})^*, \xi_\ell], \xi_k) \\ &= h(\xi_\ell, [\pi_+(\overline{\xi_j}), \xi_k]) \\ &= h(\xi_\ell, \pi_{\mathfrak{q}}[\pi_+(\overline{\xi_j}), \xi_k]), \end{aligned}$$

formula (62) for all j, k, ℓ is equivalent to

$$\nabla_{\xi_j}\xi_\ell - \nabla_{\xi_\ell}\xi_j = 0 \quad \text{for all } \ell, j$$

and hence, by the second condition from Prop. 7.1 we only have to show that the bracket $[\xi_j, \xi_\ell]$ vanishes.

To see this, recall that period maps are *horizontal*, i.e. all tangents to the image $F(s)$ of a period map belong to $U^{-1}_{F(s)} = \bigoplus_q I^{-1,q}_{F(s)}$. Working this out means

$$e^{-\operatorname{ad}\Gamma(s)} \frac{\partial}{\partial s_j} e^{\operatorname{ad}\Gamma(s)} \in U^{-1}_{F(0)}$$

and as in the proof of [P1, Theorem 6.9] this is equivalent to the commutativity relation

$$[\xi_j, \xi_\ell] = \left[\frac{\partial\Gamma}{\partial s_j}(0), \frac{\partial\Gamma}{\partial s_\ell}(0)\right] = 0. \qquad \square$$

Corollary 7.3. *The pullback of the mixed Hodge metric along an immersion is Kähler in the following cases:*

(a) *Variations of pure Hodge structure (Lu's result [Lu]);*

(b) *Hodge–Tate variations;*

(c) *The variations of mixed Hodge structure attached to J_x/J_x^3 for a smooth complex projective variety;*

(d) *The variations from § 1.4. Example 4 arising from the commuting deformations of the complex and Kähler structure of a compact Kähler manifold.*

Proof: In case (a), the derivatives of Γ at zero are of type $(-1,1)$ and so for all ℓ, j

$$[\pi_+(\overline{\xi_\ell}), \xi_j] = [\pi_+(\overline{d\Gamma/ds_\ell(0)}), d\Gamma/ds_j(0)] \tag{63}$$

is type $(0,0)$ which is annihilated by $\pi_{\mathfrak{q}}$.
In case (b), $\pi_+(\overline{d\Gamma}) = 0$.
In case (c) the bracket (63) is of type $(-1,-1)$ which is zero due to the short length of the weight filtration.
In case (d), the bracket (63) has terms of type $(0,0)$ and $(0,-2)$, both of which are annihilated by $\pi_{\mathfrak{q}}$.

Remark 7.4. In case (d) one can also show that the the holomorphic sectional curvature will be ≤ 0.

Theorem 7.5. *Let $\mathcal{V}$ be a variation of mixed Hodge structure with only two non-trivial weight graded-quotients Gr^W_a and Gr^W_b which are adjacent, i.e. $|a-b| = 1$. Then, the pullback of the mixed Hodge metric along the period map of $\mathcal{V}$ is pseudo-Kähler.*

Proof: We shall prove the symmetry relation (62) which in our situation due to the short nature of the weight filtration reduces to

$$h(\xi_j, [\bar{\xi}_\ell, \xi_k]) - h(\xi_\ell, [\bar{\xi}_j, \xi_k]) = 0. \tag{64}$$

Without loss of generality, we can assume that ξ_j, ξ_k, ξ_ℓ are of pure Hodge type. Inspection of the possibilities shows that the only non-trivial case is when $X = \xi_j$ and $Y = \xi_\ell$ are type $(-1,0)$ and $Z = \xi_k$ is type $(-1,1)$. Since by Lemma 2.14 we have $Z^* = -\bar{Z}$ in this case, the formula (44) and the fact that h is hermitian gives

$$\begin{aligned} h(X, [\bar{Y}, Z]) &= h([X, \bar{Z}], \bar{Y})) \\ &= h(Y, [\bar{X}, Z]), \end{aligned}$$

which is (64). □

Example 7.6. In particular, Theorem 7.5 applies to the tautological variations of Hodge structure over the moduli spaces $\mathcal{M}_{g,n}$ and more generally, to families of pairs (X_s, Y_s) of a smooth projective variety X_s and a smooth hypersurface $Y_s \subset X_s$ as well as a family of normal functions (11) over a curve S with $\mathcal{H}$ fixed and whose period map is an immersion.

8 The Biextension Line Bundle

Recall from the introduction that in this special case for the graded Hodge numbers we have $h^{-1,-1} = 1$ and all other $h^{p,q} = 0$ unless $p + q = -1$; the mixed Hodge structure is described as a biextension

$$\begin{aligned} &0 \to \mathrm{Gr}^W_{-1} \to W_0/W_{-2} \to \mathrm{Gr}^W_0 = \mathbb{Z}(0) \to 0 \\ &0 \to \mathrm{Gr}^W_{-2} = \mathbb{Z}(1) \to W_{-1} \to \mathrm{Gr}^W_{-1} \to 0. \end{aligned} \tag{65}$$

As explained below, a family of such mixed Hodge structures over a parameter space S comes with a biextension metric $h_{\text{biext}}(s)$. Its Chern form will be shown to be semi-positive along any curve, provided the biextension is self-dual: see Theorem. 8.2.

The point in this section is that the mixed Hodge structure is in general *not* split and that the metric h_{biext} can be found by comparing the given mixed Hodge structure (F, W) on the real vector space W_0 to its Deligne splitting $(e^{-\mathrm{i}\delta_{F,W}} F, W)$ where we recall from [CKS, Prop. 2.20] that

$$\delta_{F,W} = \frac{1}{2}\,\mathrm{Im}\, Y_{F,W} = \frac{1}{4\mathrm{i}}(Y_{F,W} - \bar{Y}_{F,W}) \in \Lambda_{F,W} \cap \mathfrak{g}_{\mathbb{R}}. \tag{66}$$

Here $Y_{F,W} \in \mathrm{End}(V_{\mathbb{C}})$ equals multiplication by $p + q$ on Deligne's $I^{p,q}(V)$.

Since $\mathrm{Gr}^W_{-2} \simeq \mathbb{R}$ and similarly for Gr^W_0, fixing bases, the map $\delta_{F,W}$ can then be viewed as a real number δ, depending on (F, W). By [Hay-P, §5], there exists a further real number λ depending only on W such that the positive number $h(F, W) = e^{-2\pi\delta/\lambda}$ depends only on the equivalence class of the extension.

Let us apply this in our setting of a family $(\mathcal{F}, W)$ of biextensions over a complex curve S. Then

$$h_{\text{biext}}(s) := h(F_s, W) = e^{\frac{-2\pi\delta_{F_s,W}}{\lambda}} \tag{67}$$

turns out to be a hermitian metric on S.

As before we write

$$F(s) = e^{\Gamma(s)} \cdot F, \tag{68}$$

where $F = F(0)$ and $\Gamma(s)$ is a holomorphic function on a coordinate patch in S with values in $\mathfrak{q}$. This is the main result we are after:

Theorem 8.1. *Let S be a curve and let $\mathcal{F}$ be a variation of biextension type over S with local normal form* (68). *Let $\gamma^{-1,0}$ be the Hodge component of type* $(-1,0)$ *of* $\Gamma'(0)$.

The Chern form of the biextension metric (67) *is the* $(1,1)$*–form*

$$\begin{aligned} -\frac{1}{2\pi\mathrm{i}}\partial\bar{\partial}\, h_{\text{biext}}(s) =& \mathrm{i}\frac{\partial^2\ \delta(s)}{\partial s \partial \bar{s}} ds \wedge \overline{ds} \\ =& \frac{1}{2}[\gamma^{-1,0}, \bar{\gamma}^{-1,0}]\, ds \wedge \overline{ds}. \end{aligned} \tag{69}$$

Proof: Let

$$e^{\Gamma(s)} = g_{\mathbb{R}}(s) e^{\lambda(s)} f(s) \tag{70}$$

as usual. Then, by Lemma 2.9 we have $Y(s) = g_{\mathbb{R}}(s)e^{\lambda(s)}Y$, where $Y = Y_{(F,W)}$. If we set $f(s) = e^{\varphi(s)}$, using (66), we get

$$\frac{\partial^2}{\partial \bar{s} \partial s}\delta(s) = \frac{1}{2}\,\mathrm{Im}\,\frac{\partial^2}{\partial s \partial \bar{s}} \underbrace{e^{\Gamma(s)} e^{-\varphi(s)}}_{d(s)} \cdot Y. \tag{71}$$

Since $\Gamma(s)$ is holomorphic, we have

$$\frac{\partial}{\partial \bar{s}} d(s) \cdot Y = \operatorname{Ad}(e^{\Gamma(s)}) \left(\frac{\partial}{\partial \bar{s}} e^{-\operatorname{ad}\varphi(s)} \cdot Y \right)$$

and so

$$\begin{aligned} \frac{\partial^2}{\partial s \partial \bar{s}} d(s) \cdot Y &= \left(\frac{\partial}{\partial s} e^{\operatorname{ad}\Gamma(s)} \right) \left(\frac{\partial}{\partial \bar{s}} e^{-\operatorname{ad}\varphi(s)} Y \right) \\ &\quad + \operatorname{Ad} e^{\Gamma(s)} \left(\frac{\partial^2 e^{-\operatorname{ad}\varphi(s)}}{\partial s \partial \bar{s}} Y \right). \end{aligned} \tag{72}$$

We now consider the Taylor expansion (note that $\varphi(0) = 0$)

$$\varphi(s) = \varphi_{01} s + \varphi_{10} \bar{s} + \sum_{j,k} \varphi_{jk}\, s^j \bar{s}^k + O^3(s, \bar{s}).$$

By Lemma 2.16, we also know

$$\varphi_{10} = 0, \tag{73}$$
$$\varphi_{01} = -(\overline{\Gamma'(0)})_+, \tag{74}$$
$$\begin{aligned} \varphi_{11} &= [\gamma, \bar{\gamma}]_0 + [\gamma, \bar{\gamma}]_+ \\ &= [\gamma^{-1,1}, \bar{\gamma}^{-1,1}]_0 + [\gamma^{-1,1}, \bar{\gamma}^{-1,0}]. \end{aligned} \tag{75}$$

Formula (73) shows that the term with $s\bar{s}$ in the Taylor expansion of

$$\frac{\partial^2}{\partial s \partial \bar{s}} e^{-\operatorname{ad}\varphi(s)} Y$$

is just $-[\varphi_{11}, Y]$. Together with equation (72) it follows that

$$\left. \frac{\partial^2}{\partial s \partial \bar{s}} d(s) \cdot Y \right|_0 = -[\Gamma'(0), [\varphi_{01}, Y]] - [\varphi_{11}, Y] \tag{76}$$

Eqn. (74) states that $\varphi_{0,1} = -\overline{\Gamma'(0)}_+$. Let $\gamma = \Gamma'(0)$. By horizontality and the short length of the weight filtration,

$$\gamma = \gamma^{-1,1} + \gamma^{-1,0} + \gamma^{-1,-1}.$$

Moreover, since (F, W) is a biextension

$$\bar{\gamma}^{-1,1} \in \mathfrak{g}^{1,-1}, \qquad \bar{\gamma}^{-1,0} \in \mathfrak{g}^{0,-1}, \qquad \bar{\gamma}^{-1,-1} \in \mathfrak{g}^{-1,-1}$$

Therefore,

$$-\varphi_{01} = (\overline{\Gamma'(0)})_+ = \bar{\gamma}^{-1,1} + \bar{\gamma}^{-1,0}.$$

In particular, since $\operatorname{ad} Y$ acts as multiplication by $a + b$ on $\mathfrak{g}^{a,b}$ it follows that

$$\begin{aligned} -[\Gamma'(0), [\varphi_{01}, Y]] &= [\gamma, [\bar{\gamma}^{-1,1} + \bar{\gamma}^{-1,0}, Y]] = [\gamma, \bar{\gamma}^{-1,0}] \\ &= [\gamma^{-1,1}, \bar{\gamma}^{-1,0}] + [\gamma^{-1,0}, \bar{\gamma}^{-1,0}]. \end{aligned} \tag{77}$$

Finally, using (75),

$$\varphi_{11} = [\gamma,\bar{\gamma}]_0 + [\gamma,\bar{\gamma}]_+ \\ = [\gamma^{-1,1},\bar{\gamma}^{-1,1}]_0 + [\gamma^{-1,1},\bar{\gamma}^{-1,0}],$$

so that

$$[\varphi_{11},Y] = -[\gamma^{-1,1},\bar{\gamma}^{-1,0}]. \tag{78}$$

Combining Eqns. (76)–(78), we have:

$$\frac{\partial^2}{\partial s\partial\bar{s}}d(s)\cdot Y\bigg|_0 = [\gamma^{-1,1},\bar{\gamma}^{-1,0}] + [\gamma^{-1,0},\bar{\gamma}^{-1,0}] + [\bar{\gamma}^{-1,1},\gamma^{-1,0}]. \tag{79}$$

The result then follows from (71). □

So far, we have not assumed anything special about the biextension variation $\mathcal{F}$. Of special interest in connection with the Hodge conjecture is the case where the two *normal functions* appearing in (65) are self-dual with respect to the polarization Q on $H := \mathrm{Gr}^W_{-1}$.

Theorem 8.2. *Let h be the Hodge metric on Gr^W_{-1} and let $\mathcal{F}$ be a self-dual biextension over a curve S with local normal form at a disk (Δ,s) at $s_0 \in S$ given by $F(s) = e^{\Gamma(s)}$. Choose a lift $e(0) \in I^{0,0}_F$ of $1 \in \mathbb{Z}(0)$ and let*

$$\gamma = \Gamma'(0) \in \mathrm{End}(W_0)_{\mathbb{C}}, \quad t := \gamma^{-1,0}(e(0)) \in I^{-1,0}_F,$$

where $\gamma^{-1,0}$ is the Hodge component of type $(-1,0)$ of $\Gamma'(0)$. Let $\nu \in \mathrm{Ext}^1_{VMHS}(\mathbb{Z}(0),\mathrm{Gr}^W_{-1}\mathcal{F})$ and its dual be the two normal functions associated to the biextension and let $\delta(s)$ be the Deligne δ-splitting of $\mathcal{F}_s$. Then

1. *the value of the infinitesimal invariant $\partial\nu$ for the normal function ν at s_0 can be identified with t.*

2. $$\frac{\partial^2}{\partial s\partial\bar{s}}\delta(s)\bigg|_0 (e(0)) = h(t,t) \in \mathbb{R}_{\geq 0}, \quad t = \gamma^{-1,0}(e(0)). \tag{80}$$

3. *The Chern form of the Hodge metric is semi-negative.*

Proof: 1. The point here is that $\gamma^{1,0} \in \mathrm{Hom}(I^{0,0}_F, I^{-1,0}_F)$ is the derivative at s_0 of the period map for the normal function ν which, from the set-up gets identified with t.
3. Follows from Theorem 8.1 and 2.
2. Recall (69). We have

$$\begin{aligned}\frac{1}{2\mathrm{i}}[\gamma^{-1,0},\bar{\gamma}^{-1,0}]e(0) &= -\frac{1}{2\mathrm{i}}\left(\gamma^{-1,0}(\bar{\gamma}^{-1,0}(e(0))) - \bar{\gamma}^{-1,0}(\gamma^{-1,0}(e(0)))\right) \\ &= -\frac{1}{2\mathrm{i}}\left(\gamma^{-1,0}(\bar{t}) - \overline{\gamma^{-1,0}(\bar{t})}\right) \\ &= -\mathrm{Im}(\gamma^{-1,0}(\bar{t})).\end{aligned}$$

Next, we express self-duality. Observe that the derivative of the period map of the dual extension ν^* can be expressed as a functional on W_{-1}: it is zero on W_{-2} and self-duality means precisely that on $H^* = \mathrm{Hom}(H, \mathbb{Z}(1))$ it restricts to the functional[8]

$$\beta = Q(s, -) \in H^* \mapsto -Q(s,t) \in \mathbb{C}.$$

This formula implies that, tracing through the identifications, one has $\gamma^{-1,0}(\bar{t}) = -Q(\bar{t}, \gamma^{-1,0}e(0)) = -Q(\bar{t}, t) = Q(t, \bar{t})$ and hence:

$$\frac{1}{2\mathrm{i}}[\gamma^{-1,0}, \bar{\gamma}^{-1,0}]e(0) \quad = \quad -\operatorname{Im}(Q(t, \bar{t})).$$

Since $h(t,t) = Q(-\mathrm{i}t, \bar{t}) = -\mathrm{i}Q(t, \bar{t})$ is real, we get indeed $\frac{1}{2\mathrm{i}}[\gamma^{-1,0}, \bar{\gamma}^{-1,0}]e(0) = h(t,t) \in \mathbb{R}$. □

Corollary 8.3. *If $\mathcal{V}$ is a variation of biextension type over a curve S with self-dual extension data, then $\delta(s)$ is a subharmonic function which vanishes exactly at the points $s \in S$ for which the infinitesimal invariants of the associated normal functions vanish.*

9 Reductive Domains And Complex Structures

In this section we consider special classifying domains: the reductive ones. Recall that a homogeneous space $D = G/H$ with G a real Lie-group acting from the left on D is *reductive* if the Lie algebra $\mathfrak{h} = \mathrm{Lie}(H)$ has a vector space complement $\mathfrak{n}$ which is $\operatorname{ad} H$-invariant:

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n}, \quad [\mathfrak{h}, \mathfrak{n}] \subset \mathfrak{n}. \tag{81}$$

Note that this implies that $\mathfrak{n}$ is the tangent space at the canonical base point of $D = G/H$; moreover, the tangent bundle is the G-equivariant bundle associated to the adjoint representation of H on $\mathfrak{n}$.

9.1 Domains for Pure Hodge Structures

These are reductive: in this situation $\mathfrak{n}_{\mathbb{C}} := \mathfrak{n}_+ \oplus \mathfrak{n}_-$ (see (31)) is the complexification of $\mathfrak{n} := \mathfrak{n}_{\mathbb{C}} \cap \mathfrak{g}$ and this is the desired complement.

Let us recall from [Ca-MS-P, Chap. 12] how the connection form for the metric connection (the one for the Hodge metric) can be obtained. Start with the Maurer-Cartan form ω_G on G. It is a $\mathfrak{g}$-valued 1-form on G. Decompose ω_G according to the reductive splitting. Then $\omega = \omega^{\mathfrak{h}}$, the $\mathfrak{h}$–valued part, is a connection form for the principal bundle $p : G \to G/H = D$. Let $\rho : H \to \mathrm{GL}(E)$ be a (differentiable) representation and let $[E] = G \times_\rho E$ be the associated vector bundle. It has an induced connection which can be described as follows. Locally over any open

[8]For simplicity we have discarded the Tate twist.

$U \subset D$ over which p has a section $s : U \to G$, the bundle $[E]$ gets trivialized and the corresponding connection form then is $s^*(\dot\rho \circ \omega)$, where $\dot\rho : \mathfrak{h} \to \operatorname{End} E$ is the derivative of ρ.

In the special case where $E = T_F D$ this leads to a canonical connection ∇_D on the holomorphic tangent bundle of D. If D is a period domain this canonical connection is the Chern connection for the Hodge metric.

From this description the curvature can then directly be calculated:

Theorem ([Ca-MS-P, Cor. 11.3.16]). *Let D be a period domain for pure polarized Hodge structures and let $\alpha, \beta \in \mathfrak{n} = T_F D$. Then $R_D \in A_D^{1,1}(\operatorname{End} \mathfrak{n})$, the curvature form of the canonical connection ∇_D on the holomorphic tangent bundle of D evaluates at F as:*

$$R_D(\alpha, \bar\beta) = -\operatorname{ad}[\alpha, \bar\beta]^{\mathfrak{h}}.$$

Remark 9.1. The above proof for the pure case makes crucial use of the compatibility of the complex structure of D and reductive structure: First, one needs the complex structure coming from the inclusion $D = G/G^F \subset \check{D} = G_{\mathbb{C}}/G^F_{\mathbb{C}}$ to see that the Maurer-Cartan form is the real part of a holomorphic form, the Maurer-Cartan form on $G_{\mathbb{C}}$ and hence ω is the real part of a holomorphic form. Next, one uses that the complex structure J on $\mathfrak{n}$ is such that $\mathfrak{n}_\pm \subset \mathfrak{n}_{\mathbb{C}}$ is the eigenspace for J with eigenvalue $\pm$i and one makes the identification

$$T_F D = (\mathfrak{n}, J) \simeq \mathfrak{n}_-.$$

In the mixed case there are situations where the domain is reductive, but the complex structure then does not behave as in the pure case, as we now show.

9.2 Differential Geometry of Reductive Domains

Let $D = G/V$ be a reductive homogeneous space and a choice $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n}$ of a reductive splitting. Let us recall some major results from [No]. The G-invariant connections on $T(D)$ are in one two one correspondence to bilinear $\operatorname{ad} H$–invariant functions

$$\alpha : \mathfrak{n} \times \mathfrak{n} \to \mathfrak{n}.$$

A given such connection ∇ corresponds to

$$\alpha(X, Y) := \nabla_X \tilde{Y},$$

where $\tilde{Y}$ is the vector field on D obtained from $Y \in T_o(D)$ by left G-translation ($o \in D$ is the coset of $1 \in G$). The Maurer-Cartan induced connection ∇^D on $T(D)$ is the one for which α is identically zero. In loc. cit. it is called the *canonical affine connection of the second kind*.

Suppose that we have a V–invariant metric g on $\mathfrak{n}$. This gives G–equivariant metric on D, likewise denoted g. By [No, Theorem 13.1] a G-invariant connection ∇ on $T(D)$ is metric with respect to g if and only if

$$\nabla_X \tilde{Y} = \frac{1}{2}[X, Y]^{\mathfrak{n}} + U(X, Y), \tag{82}$$

where $U : \mathfrak{n} \times \mathfrak{n} \to \mathfrak{n}$ is the $\mathbb{R}$–bilinear form which is determined by the formula

$$2g(U(X,Y),Z) = g([Z,X]^{\mathfrak{n}},Y) + g(X,[Y,Z]^{\mathfrak{n}}). \tag{83}$$

Moreover, the connection is free of torsion if and only if U is a symmetric form. For the Maurer-Cartan induced connection the left hand side of (82) vanishes and so it is metric, precisely when

$$U(X,Y) = -\frac{1}{2}[X,Y]^{\mathfrak{n}}. \tag{84}$$

So this can only be without torsion if $[X,Y]^{\mathfrak{n}} = 0$. In fact, By [No, Theorem 10.3] its torsion is given by

$$T(X,Y) = -[X,Y]^{\mathfrak{n}}. \tag{85}$$

So, the canonical connection in general differs from the Levi-Civita connection.

Remark 9.2. 1) We extend the above connections to the complex tangent bundle $T_{\mathbb{C}}(D)$. The same considerations then hold provided g and U are replaced by their $\mathbb{C}$–bilinear extensions.
2) Note that in general only the thus extended canonical connection preserves the decomposition $T_{\mathbb{C}}(D) = T^{1,0}D \oplus T^{0,1}D$ into the holomorphic and anti-holomorphic tangent bundles. For the Levi-Civita connection this holds if the metric is Kähler.

9.3 Split Domains

Mixed domains are seldom reductive, and, even if they are, we shall see that the complex structure does not satisfy the compatibility required by Remark 9.1.

Examples 9.3. 1. Suppose $\Lambda = 0$. Then equation (34) implies that $\mathfrak{n} = \mathfrak{n}_{\mathbb{C}} \cap \mathfrak{g}_{\mathbb{R}}$ is the desired complement. Note that in the pure case this equals also $\mathfrak{n}_{\mathbb{C}} \cap \mathfrak{g}$. This difference will influence the curvature calculations. Domains with $\Lambda = 0$ are called *split domains* because they parametrize split mixed Hodge structures. We investigate these below in more detail.
2. We consider the general mixed situation. Let D^{split} be the subdomain of D parametrizing split mixed Hodge structures[9]. This domain can be identified with $G_{\mathbb{R}}/G_{\mathbb{R}}^F$, where F is a fixed split mixed Hodge structure. Note that $\mathfrak{n}_{\mathbb{C}} \oplus \Lambda$ has a real structure which makes D^{split} a reductive domain for the splitting

$$\mathfrak{g}_{\mathbb{R}} = \underbrace{\mathfrak{g}^{0,0} \cap \mathfrak{g}_{\mathbb{R}}}_{\text{Lie}(G_{\mathbb{R}}^F)} \oplus (\mathfrak{n}_{\mathbb{C}} \oplus \Lambda)_{\mathbb{R}}.$$

In general D^{split} only has the structure of a differentiable manifold.
3. In general the group $G_{\mathbb{R}}$ does not act transitively on D. But there is another

[9]This has been called $D_{\mathbb{R}}$ in § 2.

natural subgroup of G which does act transitively. To explain this, introduce (for $r < 0$):

$$G_r^W \quad := \quad \{g \in G \mid \text{ for all } k \text{ the restriction } g|(W_k/W_{k+r}) \text{ is real.}\}$$

Note that G_{-2}^W contains $\exp(\Lambda)$ as well as $G_{\mathbb{R}}$ and hence it acts transitively on D. Under the minimal condition

$$\mathrm{Lie}(G_{-2}^W) = \mathfrak{g}_{\mathbb{R}} \oplus \mathrm{i}\Lambda$$

we clearly get a reductive splitting

$$\mathrm{Lie}(G_{-2}^W) = \mathfrak{g}^{0,0} \cap \mathfrak{g}_{\mathbb{R}} \oplus [(\mathfrak{n}_{\mathbb{C}} \oplus \Lambda)_{\mathbb{R}} \oplus \mathrm{i}\Lambda_{\mathbb{R}}]\,.$$

Domains which satisfy this condition are called *close to splitting*. An example is provided by the so-called type II domains from [P3].

Note that in general $(\mathfrak{n}_{\mathbb{C}} \oplus \Lambda)_{\mathbb{R}}$ does not admit a complex structure: $\dim \Lambda$ can be odd!

9.4 Two Step Filtrations

This case has been treated in detail in [U, § 2]. The domains in question are examples of split domains, and hence they are reductive. The mixed Hodge structures they parametrize indeed split over $\mathbb{R}$ since the associated weight filtration has only two consecutive steps, say $0 = W_0 \subset W_1 \subset W_2 = H$.

Assume that we are given two polarizations on W_1 and Gr_2^W, both denoted Q. One can choose an adapted (real) basis for H which

- restricts to a Q–symplectic basis $(a_1, \ldots, a_g, b_1, \ldots, b_g)$ for W_1;
- the remainder of the basis $(c_1, \ldots, c_k, c_1', \ldots, c_k', d_1, \ldots, d_\ell)$ projects to a basis for Gr_2^W diagonalizing Q, i.e. $Q = \mathrm{diag}(-\mathbf{1}_{2k}, \mathbf{1}_\ell)$.

Then

$$G = \left\{ \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \mid A \in \mathrm{Sp}(g;\mathbb{R}),\ C \in \mathrm{O}(2k,\ell),\ B \in \mathbb{C}^{2g\times(2k+\ell)} \right\},$$

reflecting the Levi-decomposition. More invariantly, the two matrices A and C on the diagonal give the semi-simple part G^{ss} while the matrices B give the unipotent radical

$$G^{\mathrm{un}} \simeq \mathrm{Hom}_{\mathbb{C}}(\mathrm{Gr}_2^W, W_1).$$

Here, the isomorphism (via the exponential map) in fact identifies G^{un} with its Lie-algebra:

$$\mathfrak{g}^{\mathrm{un}} = \mathrm{Hom}_{\mathbb{C}}(\mathrm{Gr}_2^W, W_1), \tag{86}$$

the endomorphisms in $\mathfrak{g}$ which lower the weight by one step.

The real group $G_{\mathbb{R}}$ consists of the group given by similar matrices, except that now the matrices B are taken to be real. In particular

$$\mathfrak{g}^{\mathrm{un}}_{\mathbb{R}} = \mathfrak{g}^{\mathrm{un}} \cap \mathfrak{g}_{\mathbb{R}} = \mathrm{Hom}_{\mathbb{R}}(\mathrm{Gr}^W_2, W_1). \tag{87}$$

Next, fix the Hodge flag $F = \{F^2 \subset F^1 \subset F^0 = H_{\mathbb{C}}\}$ which has the following adapted unitary basis

$$\left.\begin{aligned} &(\underbrace{\underbrace{f_1, \ldots, f_k}_{F^2}, d_1, \ldots, d_\ell, f'_1, \ldots, f'_g}_{F^1}, \bar{f}'_1, \ldots, \bar{f}'_g, \bar{f}_1, \ldots, \bar{f}_k), \\ &f_k := \frac{1}{\sqrt{2}}[c_k - \mathrm{i}c'_k], \quad f'_k = \frac{1}{\sqrt{2}}[a_k - \mathrm{i}b_k]. \end{aligned}\right\} \tag{88}$$

The group G^F consists of the subgroup of G with $A = \begin{pmatrix} U & -V \\ V & U \end{pmatrix}$, $U + \mathrm{i}V \in \mathrm{U}(g)$, $C \in \mathrm{O}(2k) \times \mathrm{O}(\ell)$ and the matrices B are of the form

$$\left\{ \begin{pmatrix} B' \\ -\mathrm{i}B' \end{pmatrix} \mid B' \in \mathbb{C}^{g \times (2k+\ell)} \right\}.$$

Note that $G^{\mathrm{ss}}/G^F \cap G^{\mathrm{ss}} = D_1 \times D_2$, the product of the domain $D_1 \simeq \mathbf{H}_g$, parametrizing weight 1 Hodge polarized structures with $h^{1,0} = g$ and D_2 parametrizing weight 2 polarized Hodge structures with $h^{2,0} = k, h^{1,1} = \ell$. The natural projection

$$G/G^F \to G^{\mathrm{ss}}/G^F \cap G^{\mathrm{ss}} = D_1 \times D_2 \tag{89}$$

is a holomorphic bundle with fiber associated to the adjoint representation of $G^{\mathrm{ss}} \cap G^F$ on $\mathfrak{g}^{\mathrm{un}}/\mathfrak{g}^{\mathrm{un}} \cap \mathfrak{g}^F$. Explicitly, this action is

$$g \cdot [B] = [ABC^{-1}], \quad g = \begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix}.$$

The fiber of (89) over F is the affine space consisting of the extension data of (W_1, F) by (Gr^W_2, F) on which G^{un} acts transitively as the group of translations. The group G^{ss} acts on this fiber bundle by holomorphic transformations from the left: $g \in G^{\mathrm{ss}}$ sends the fiber over F biholomorphically to the fiber over $g \cdot F$.

To obtain a reductive decomposition $\mathfrak{g}_{\mathbb{R}} = \mathfrak{h} \oplus \mathfrak{n}$, set

$$\mathfrak{h} := \mathfrak{g}^{0,0} \cap \mathfrak{g}, \quad \mathfrak{n} = \mathfrak{n}^{\mathrm{ss}} \oplus \mathfrak{g}^{\mathrm{un}}_{\mathbb{R}}, \quad \mathfrak{n}^{\mathrm{ss}} = \left(\oplus_{p \neq 0} \mathfrak{g}^{p,-p}\right) \cap \mathfrak{g}. \tag{90}$$

Let us study the metric properties of the Hodge metric h and its Chern connection ∇_h. It is invariant under the Hodge metric and so is determined by Eqn. (82).

Lemma 9.4. *The canonical connection ∇^D on the complex tangent bundle $T_{\mathbb{C}}(D)$ of $D = G/G^F$ given by the reductive decomposition* (90) *is distinct from the (extended) Chern connection ∇_h on $T_{\mathbb{C}}(D)$.*

Proof: Both connections are metric for the Hodge metric and so they are both given by the formula (82). In particular, for $X, Y \in \mathfrak{g}_{\mathbb{C}}$ we have

$$\nabla^D_X \tilde{Y} = U(X,Y).$$

Let us calculate $U(X,X)$, $X \in \mathfrak{g}^{-1,0}$ with the aid of (83) where (cf. Remark 9.2) g is the complex bilinear extension of the real part of the Hodge metric on $\mathfrak{g}_{\mathbb{C}}$. We then see that for $Z \in \mathfrak{g}^{-1,1} \oplus \mathfrak{g}^{1,-1}$, we get

$$\begin{aligned} 2g(U(X,X),Z) &= g([Z,X],X) + g(X,[\bar{X},Z]) \\ &= h([Z,X],\bar{X}) + h([\bar{X},Z],\bar{X}) \\ &= -h(Z,[X^*,\bar{X}]) + h(Z,[\bar{X}^*,X]) \\ &= g(Z,[X,\bar{X}^*] - [\bar{X},X^*]) \end{aligned}$$

where the third line follows from (44). Hence $U(X,X) = \frac{1}{2}([X,\bar{X}^*]-[\bar{X},X^*])$ which does not always vanish. Indeed in the basis (88) the tangent vector X corresponds to a matrix with $A = C = 0$ and B arbitrary, while $\bar{X}^*$ is the transpose conjugate so that $U(X,X) = \begin{pmatrix} \mathrm{Im}\, B^{\mathsf{T}}B & 0 \\ 0 & -\mathrm{Im}\, {}^{\mathsf{T}}BB \end{pmatrix}$. Now compare this with what happens for ∇_h. Eqn. (84) tells us that we must have $U(\nabla^h)(X,X) = -\frac{1}{2}[X,X] = 0$. Indeed, the canonical connection has $\nabla^D_X \tilde{Y} = \alpha(X,Y) = 0$. This shows that $\nabla^D \neq \nabla^h$. □

As to the complex structure we have:

Lemma 9.5. *The complex structure compatible with the reductive structure is not the one coming from the embedding $G/G^F \subset G_{\mathbb{C}}/G^F_{\mathbb{C}}$.*

Proof: Write

$$\begin{aligned} \mathfrak{g}^{\mathrm{un}} &= \underbrace{\mathfrak{g}_F^{0,-1} \oplus \mathfrak{g}_F^{1,-2}}_{\mathfrak{g}^{\mathrm{un}}_{F,+}} \oplus \underbrace{\mathfrak{g}_F^{-1,0} \oplus \mathfrak{g}_F^{-2,1}}_{\mathfrak{g}^{\mathrm{un}}_{F,-}} \\ \mathfrak{n}^{\mathrm{ss}} &= [\underbrace{\mathfrak{g}^{-2,2} \oplus \mathfrak{g}^{-1,1}}_{\mathfrak{n}^{\mathrm{ss}}_{F,-}} \oplus \underbrace{\mathfrak{g}^{1,-1} \oplus \mathfrak{g}^{2,-2}}_{\mathfrak{n}^{\mathrm{ss}}_{F,+}}] \cap \mathfrak{g}. \end{aligned}$$

Since $\mathfrak{g}^{\mathrm{un}}_{F,+} = \mathfrak{g}^F \cap \mathfrak{g}^{\mathrm{un}}_F$, the tangent space at F to

$$D^{\mathrm{un}} := G^{\mathrm{un}}/G^F \cap G^{\mathrm{un}}$$

gets identified with

$$\begin{aligned} T_F D^{\mathrm{un}} &= \mathfrak{g}^{\mathrm{un}}/\mathfrak{g}^{\mathrm{un}}_{F,+} \\ &= \mathfrak{g}^{\mathrm{un}}_{F,-}, \end{aligned}$$

a space of complex dimension $g(2k+\ell)$. The complex structure comes from the standard complex structure J on $\mathfrak{g}^{\mathrm{un}}$, since $T_F D^{\mathrm{un}}$ is a quotient thereof.[10] Next, note that

$$\mathfrak{g}^F_{\mathbb{R}} \cap \mathfrak{g}^{\mathrm{un}} = 0$$

[10] I.e., J is multiplication by i.

and so

$$G^{\mathrm{un}}_{\mathbb{R}}/G^F \cap G^{\mathrm{un}}_{\mathbb{R}} = \mathfrak{g}^{\mathrm{un}}_{\mathbb{R}} = \mathrm{Hom}_{\mathbb{R}}(\mathrm{Gr}^W_2, W_1)$$

and this space gets a complex structure thanks to the weight 1 Hodge structure induced by F on W_1. It is induced by a complex structure J^F_1 whose complexification on $\mathfrak{g}^{\mathrm{un}}$ has eigenvalues as in the following table:

	$I_F^{2,0}$	$I_F^{1,1}$	$I_F^{0,2}$
$I_F^{1,0}$	i	i	i
$I_F^{0,1}$	−i	−i	−i

One deduces that the complex structure $(\mathfrak{g}^{\mathrm{un}}_{\mathbb{R}}, J^F_1)$ is not isomorphic to the complex structure $(\mathfrak{g}^{\mathrm{un}}_{\mathbb{R}}, J^F)$

The complex structure J^F coming from $G/G^F \subset G_{\mathbb{C}}/G^F_{\mathbb{C}}$ identifies the holomorphic tangent space at F as follows:

$$T_F D = \mathfrak{g}/\mathfrak{g}^F = \mathfrak{n}^{\mathrm{ss}}_{F,-} \oplus \mathfrak{g}^{\mathrm{un}}_{F,-} \simeq (\mathfrak{n}^{\mathrm{ss}}, J^F) \oplus (\mathfrak{g}^{\mathrm{un}}_{\mathbb{R}}, J^F).$$

The natural complex structure J^F on $\mathfrak{n}^{\mathrm{ss}}$ comes from the one inducing the complex structure on the base $D_1 \times D_2$ of the fiber bundle (89).

Taking the same complex structure on $\mathfrak{n}^{\mathrm{ss}}$ but the other on $\mathfrak{g}^{\mathrm{un}}_{\mathbb{R}}$ leads to a different holomorphic tangent space

$$(T_F D, J^F_1) = (\mathfrak{n}^{\mathrm{ss}}, J^F) \oplus (\mathfrak{g}^{\mathrm{un}}_{\mathbb{R}}, J^F_1);$$

it is a complex structure on $\mathfrak{n}$ whose ±i–eigenspaces inside $\mathfrak{n} \otimes \mathbb{C}$ are given by

$$\mathfrak{n}_{F,+} = \mathfrak{n}^{\mathrm{ss}}_{+,F} \oplus \mathrm{Hom}_{\mathbb{C}}(\mathrm{Gr}^W_2 \otimes \mathbb{C}, I^{0,1}_F)$$

respectively

$$\mathfrak{n}_{F,-} = \mathfrak{n}^{\mathrm{ss}}_{-,F} \oplus \mathrm{Hom}_{\mathbb{C}}(\mathrm{Gr}^W_2 \otimes \mathbb{C}, I^{1,0}_F).$$

Finally, note that the isomorphism

$$(T_F D, J^F_1) = (\mathfrak{n}, J^F_1) \simeq \mathfrak{n}_{F,-}.$$

gives $T_F D$ the complex structure which is required in the standard curvature calculations for reductive domains, as explained above. However, as we have seen, this structure is not the one which comes from the embedding $D = G/G^F \hookrightarrow G_{\mathbb{C}}/G^F_{\mathbb{C}}$.

□

Remark. 1. Clearly, J^F_1 and J^F commute.
2. Consider the surjective morphism

$$G_{\mathbb{R}}/G^F_{\mathbb{R}} \to G^{\mathrm{ss}}/G^F_{\mathbb{R}} \cap G^{\mathrm{ss}} = D_1 \times D_2.$$

It is a real-analytic complex vector bundle associated to the $G^{\mathrm{ss}} \cap G^F$–representation space $\mathfrak{g}^{\mathrm{un}}_{\mathbb{R}}$. This is also a holomorphic fiber bundle: if $U \in \mathrm{U}(g)$ and $V \in [\mathrm{O}(2k) \times \mathrm{O}(\ell)]$, the action on $\varphi \in \mathrm{Hom}_{\mathbb{R}}(\mathrm{Gr}^W_2, W_1)$ is given by $\varphi \mapsto U \circ \varphi \circ V^{-1}$ and hence is

J_1^F-complex. However, the action of G^{ss} on this bundle is no longer holomorphic: $g = (U, V) \in \mathrm{Sp}(g) \times \mathrm{O}(2k, \ell)$ sends φ in the fiber over F to $U \circ \varphi$ in the fiber over $g \cdot F$ and since U and J_1^F only commute when $U \in \mathrm{U}(g)$ this is *not* a J_1^F-complex-linear isomorphism. Since in our situation $G/G^F \simeq G_{\mathbb{R}}/G_{\mathbb{R}}^F$, this also confirms that the two complex structures are distinct.

A The Levi-Civita Connection

Suppose that M is a complex manifold and $\mathfrak{X}_M^{1,0}$ and $\mathfrak{X}_M^{0,1}$ denote the sheaves of complex vector fields of type $(1,0)$ and $(0,1)$ respectively. Then, the conjugation action $u \mapsto u_c$ defined by

$$u_c \cdot f = \overline{u \cdot \overline{f}}$$

defines an isomorphism of sheaves $\mathfrak{X}_M^{1,0} \xrightarrow{\sim} \mathfrak{X}_M^{0,1}$ as modules over the sheaf $C^\infty(M, \mathbb{R})$ of real valued smooth functions on M. It restricts to a conjugate linear morphism between the sheaves of holomorphic and anti-holomorphic vector fields on M.

Lemma A.1. *Let $\mathfrak{X}_M$ denote the sheaf of C^∞ real vector fields on M. Then,*

$$\begin{array}{ccc} \mathfrak{X}_M^{1,0} & \to & \mathfrak{X}_M \\ u & \mapsto & u_r := u + u_c \end{array}$$

defines a linear isomorphism over $C^\infty(M, \mathbb{R})$. Moreover, if x and y are holomorphic vector fields, then

$$[x_r, y_r] = [x, y]_r. \tag{91}$$

Proof: If $z_j = x_j + \sqrt{-1}y_j$ is a system of holomorphic coordinates on an open subset U of M then

$$\left(\frac{\partial}{\partial z_j}\right)_r = \frac{\partial}{\partial x_j}, \qquad \left(\sqrt{-1}\frac{\partial}{\partial z_j}\right)_r = \frac{\partial}{\partial y_j}$$

and hence the stated morphism induces an isomorphism over any holomorphic coordinate chart. Using partitions of unity, it then follows that it is a global isomorphism, $\mathfrak{X}_M^{1,0} \xrightarrow{\cong} \mathfrak{X}_M$. Since holomorphic and anti-holomorphic vector fields commute, (91) follows. □

Let g be a Riemannian metric on the underlying C^∞-manifold of M. Then, the associated Levi-Civita connection ∇^{LC} is determined by the Koszul formula:

$$\begin{aligned} 2g(\nabla^{\mathrm{LC}}_X Y, Z) = Xg(Y,Z) + Yg(X,Z) - Zg(X,Y) \\ + g([X,Y],Z) - g([X,Z],Y) - g([Y,Z],X) \end{aligned} \tag{92}$$

In particular, if h is a hermitian metric on M given as a pairing of sections of $\mathfrak{X}_M^{1,0}$ we obtain an associated Riemannian pairing on sections of $\mathfrak{X}_M$ by the rule

$$g(u_r, v_r) = \mathrm{Re}\, h(u,v) \tag{93}$$

By the above remarks, in order determine the Levi-Civita connection of the metric (93) it is sufficient to evaluate the expression (92) on vector fields $X = x_r$, $Y = y_r$ and $Z = z_r$ with x, y and z holomorphic vector fields on M. Unraveling the above, for holomorphic vector fields u, v and w we have:

$$\left.\begin{aligned} w_r \cdot g(u_r, v_r) &= w \cdot \mathrm{Re}\, h(u,v) + w_c \cdot \mathrm{Re}\, h(u,v) \\ &= (1/2)w \cdot (h(u,v) + h(v,u)) + (1/2)\overline{w \cdot (h(u,v) + h(v,u))} \\ &= \mathrm{Re}\,(w \cdot (h(u,v) + h(v,u))). \end{aligned}\right\} \tag{94}$$

Lemma A.2. *The Levi-Civita connection ∇^{LC} of the Riemannian metric* (93) *underlying a hermitian metric h on a complex manifold M is determined by the formula:*

$$2g(\nabla^{\mathrm{LC}}_{x_r} y_r, z_r) = \mathrm{Re}\,(x\cdot(h(y,z)+h(z,y)) + y\cdot(h(x,z)+h(z,x)) \\ -z\cdot(h(x,y)+h(y,x)) + h([x,y],z) - h([x,z],y) - h([y,z],x))\,,$$

where x_r, y_r and z_r arise from underlying holomorphic vector fields x, y and z.

Proof: The right hand side of the Koszul formula (92) for the Levi–Civita connection is the sum of the terms

$$x_r\cdot g(y_r,z_r) + y_r\cdot g(x_r,z_r) - z_r\cdot g(x_r,y_r) \\ = \mathrm{Re}\ (x\cdot(h(y,z)+h(z,y)) + y\cdot(h(x,z)+h(z,x)) - z\cdot(h(x,y)+h(y,x)))$$

and

$$g([x_r,y_r],z_r) - g([x_r,z_r],y_r) - g([y_r,z_r],x_r) \\ = \mathrm{Re}\ (h([x,y],z) - h([x,z],y) - h([y,z],x))\,. \quad \square$$

We want to apply this formula in the case of the mixed Hodge metric and holomorphic vector fields of the form

$$\tilde{\alpha}(e^u\cdot F) = L_{e^u *}\alpha$$

where $\alpha\in\mathfrak{q}$ acts as the derivation

$$\alpha\cdot f = \frac{d}{dz} f(e^{z\alpha}\cdot F)\Big|_{z=0}$$

on germs of functions at F and $u\mapsto e^u\cdot F$ gives a biholomorphism from a neighborhood of 0 in $\mathfrak{q}$ to a neighborhood of F in D.

Lemma A.3. *Let $\alpha,\beta,\gamma\in\mathfrak{q}$. Then*[11]*,*

$$\tilde{\alpha}\cdot h(\tilde{\beta},\tilde{\gamma})\big|_F = -h_F(\beta,[\pi_+(\bar{\alpha}),\gamma]).$$

Proof: We have

$$\tilde{\alpha}\cdot h(\tilde{\beta},\tilde{\gamma})\big|_F = \frac{d}{dz} h_{e^{z\alpha}\cdot F}(\tilde{\beta},\tilde{\gamma})\Big|_{z=0} = \frac{d}{dz} h_F(L_{f(z)*}\beta, L_{f(z)*}\gamma)\Big|_{z=0},$$

where $f(z) = \exp(-\bar{z}\pi_+(\bar{\alpha}) + O^2(z,\bar{z}))$. Therefore,

$$\tilde{\alpha}\cdot h(\tilde{\beta},\tilde{\gamma})\big|_F = -h_F(\beta,[\pi_+(\bar{\alpha}),\gamma]). \quad \square$$

[11]compare with Cor. 2.17

Theorem A.4. *For x_r, y_r and z_r arising from $\tilde{x}$, $\tilde{y}$, $\tilde{z}$ we have*

$$\begin{aligned}2g(\nabla^{\mathrm{LC}}_{x_r} y_r, z_r) = &-\mathrm{Re}(h_F(y,[\pi_+(\bar{x}),z]) + h_F(z,[\pi_+(\bar{x}),y]))\\ &-\mathrm{Re}(h_F(x,[\pi_+(\bar{y}),z]) + h_F(z,[\pi_+(\bar{y}),x]))\\ &+\mathrm{Re}(h_F(x,[\pi_+(\bar{z}),y]) + h_F(y,[\pi_+(\bar{z}),x]))\\ &+\mathrm{Re}(h_F([x,y]-[x^*,y]-[y^*,x],z)).\end{aligned}$$

Corollary A.5. *If $\tilde{x}$ and $\tilde{y}$ arise from $x, y \in \mathfrak{g}^{-p,-q}_{(F,W)}$, $p, q > 0$ by left translation, then for the corresponding vector fields x_r, y_r we have*

$$\nabla^{\mathrm{LC}}_{x_r} y_r = \frac{1}{2}[x,y]_r.$$

Proof: Let z_r arise from $\tilde{z}$ as above. The first two lines in the formula of Theorem A.4 vanish since $\pi_+(\bar{x}) = \pi_+(\bar{y}) = 0$ because $x, y \in \Lambda_F$. As for the third line of the formula for ∇, we note that $\pi_+(\bar{z})$ can never have a component of type $(0,0)$ and hence $[\pi_+(\bar{z}),x]$ is orthogonal to y and $[\pi_+(\bar{z}),y]$ is orthogonal to x. So, only the last line of the formula of Theorem A.4 survives which gives

$$2\nabla^{\mathrm{LC}}_{x_r} y_r = [x,y]_r - \pi_{\mathfrak{q}}([x^*,y]+[y^*,x])_r.$$

The last term then vanishes since $[x^*,y]+[y^*,x]$ has type $(0,0)$. □

Lemma A.6. *Let $x, y, z \in \mathfrak{q}_F$. Put $t := [y^*,x]+[x^*,y]$. Then*

$$\mathrm{Re}\, h_F(t,\pi_+(\bar{z})) = \mathrm{Re}\, h_F(\overline{\pi_+(t)^*}^{\,*}, z).$$

If (F,W) is split over $\mathbb{R}$ then

$$\overline{\pi_+(t)^*}^{\,*} = \pi_-(\bar{t}) = \pi_-([\bar{y}^*,\bar{x}]+[\bar{x}^*,\bar{y}]).$$

Proof: Since $h_F(u,v) = \overline{h_F(v,u)}$ we have

$$\begin{aligned}\mathrm{Re}\, h_F(t,\pi_+(\bar{z}))) &= \mathrm{Re}\, h_F(\pi_+(\bar{z}),t)\\ &= \mathrm{Re}\, h_F(\bar{z},\pi_+(t))\\ &= \mathrm{Re}\ \mathrm{Tr}(\bar{z}\circ(\pi_+(t))^*)\\ &= \mathrm{Re}\ \mathrm{Tr}(z\circ\overline{\pi_+(t))^*})\\ &= \mathrm{Re}\, h_F(z,\overline{\pi_+(t))^*}^{\,*})\\ &= \mathrm{Re}\, h_F(\overline{\pi_+(t))^*}^{\,*},z).\end{aligned}$$

In the split case, $*$ and complex conjugation commute and hence $\overline{\pi_+(t)^*}^{\,*} = \overline{\pi_+ t} = \pi_-(\bar{t})$ and the second assertion follows. □

Theorem A.7. *If (F,W) is split over $\mathbb{R}$ then $2\nabla^{\mathrm{LC}}_{x_r} y_r$ at F is the real derivation defined by*

$$\begin{aligned}&-\pi_{\mathfrak{q}}([\pi_+(\bar{x})^*,y]+[\pi_+(\bar{x}),y]+[\pi_+(\bar{y})^*,x]+[\pi_+(\bar{y}),x])\\ &\quad+\pi_-([\bar{y}^*,\bar{x}]+[\bar{x}^*,\bar{y}])+\pi_{\mathfrak{q}}([x,y]-[x^*,y]-[y^*,x]).\end{aligned}$$

Proof: Applying Lemma A.6 to Theorem A.4 we have

$$\begin{aligned}2g(\nabla^{\mathrm{LC}}_{x_r} y_r, z_r) = &- \mathrm{Re}(h_F(y,[\pi_+(\bar{x}),z]) + h_F(z,[\pi_+(\bar{x}),y])) \\ &- \mathrm{Re}(h_F(x,[\pi_+(\bar{y}),z]) + h_F(z,[\pi_+(\bar{y}),x])) \\ &+ \mathrm{Re}(h_F(\pi_-([\bar{y}^*,\bar{x}]),z) + h_F(\pi_-([\bar{x}^*,\bar{y}]),z)) \\ &+ \mathrm{Re}(h_F([x,y],z) - h_F([x,z],y) - h_F([y,z],x))\end{aligned}$$

which becomes

$$\begin{aligned}2g(\nabla^{\mathrm{LC}}_{x_r} y_r, z_r) = &- \mathrm{Re}(h_F([\pi_+(\bar{x})^*,y] + [\pi_+(\bar{x}),y],z) \\ &- \mathrm{Re}(h_F([\pi_+(\bar{y})^*,x] + [\pi_+(\bar{y}),x],z) \\ &+ \mathrm{Re}(h_F(\pi_-([\bar{y}^*,\bar{x}]) + \pi_-([\bar{x}^*,\bar{y}]),z) \\ &+ \mathrm{Re}(h_F([x,y] - [x^*,y] - [y^*,x],z)). \quad \square\end{aligned}$$

Corollary A.8. *Assume (F,W) is split over $\mathbb{R}$. Let x_r and y_r be vector fields arising from $\tilde{x}$, $\tilde{y}$ with x and y of type $(-1,1)$. Then,*

$$\nabla^{\mathrm{LC}}_{x_r} y_r = \frac{1}{2}[x,y]_r.$$

Proof: In this case, by Lemma 2.14 $x^* = \bar{x}$, $y^* = \bar{y}$ and so $\pi_+(\bar{x})^* = x$ and $\pi_+(\bar{y})^* = y$. We also note that $[\bar{x},y]$ and $[\bar{y},x]$ project to zero in $\mathfrak{q} = \mathfrak{n}_- \oplus \Lambda$. Therefore, the formula of Theorem A.7 reduces to the stated form. □

Corollary A.9. *Assume that W has only two weight graded quotients which are adjacent and let x_r and y_r arise from $\tilde{x}$ and $\tilde{y}$ with x and y of type $(-1,0)$. Then,*

$$2\nabla^{\mathrm{LC}}_{x_r} y_r = -[\bar{x}^*,y]_r - [\bar{y}^*,x]_r$$

Proof: For u and v of type $(-1,0)$ in this setting we have $\pi_+(\bar{u}) = \bar{u}$ and $[u,v] = [\bar{u},v] = 0$. Likewise, $[v^*,u]$ and $[\bar{v}^*,\bar{u}]$ are type $(0,0)$ while $[\bar{v}^*,u]$ is type $(-1,1)$. Consequently, the formula of Theorem A.7 reduces to the stated form. □

Let us apply this to flow curves

$$\gamma_x : \quad t \longmapsto \exp(tx)\cdot F, \quad x \in \mathfrak{q}_F$$

and set

$$x(t_0) := \left.\frac{d\gamma_x}{dt}\right|_{t_0} \in \mathfrak{q}.$$

Corollary A.10. (1) *For $x,y \in \Lambda_F$ of the same type we have*

$$\nabla^{\mathrm{LC}}_{x(t)_r} y(t)_r = \frac{1}{2}[x(t),y(t)]_r.$$

(2) *The flow curve γ_x is a geodesic. This is in particular the case when γ_x is the image under a period map.*

(3) *Suppose that $x,y,z \in \Lambda_F$ have the same type and commute. Then the Riemann curvature*

$$R(x_r,y_r)z_r = \nabla^{\mathrm{LC}}_{x}\nabla^{\mathrm{LC}}_{y_r} z_r - \nabla^{\mathrm{LC}}_{y_r}\nabla^{\mathrm{LC}}_{x_r} z_r - \nabla^{\mathrm{LC}}_{[x_r,y_r]} z_r$$

vanishes.

Proof: Under the flow the type need not be preserved. However, an application of Lemma 2.9 shows that the types are preserves when we start with $x \in \mathfrak{g}_F^{-p,-q}$ with $p, q > 0$. Then (1) follows from Cor. A.5. In particular, this vanishes for $x = y$. By definition the curve γ_x then is a geodesic. The formula for the Riemann curvature implies (3). □

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