On rigidity of locally symmetric spaces

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Chris Peters

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Introduction

A classical result due to Calabi and Vesentini [Cal-V] states that a *compact* locally symmetric space is rigid, provided all of its irreducible factors have dimension at least 2. This implies that such varieties (known to be algebraic) can be defined over a numberfield. This was first remarked by Shimura in [Sh]. For a modern variant of the proof see [Pe].

Faltings [F] remarked that one can show that the Kodaira-Spencer class for any "spread family" of the given variety is zero which suffices for rigidity. This is true without any restriction on the type of irreducible factors, and even for non-compact locally symmetric spaces. The proof uses first of all Mumford's theory of toroidal compactifications [A-Mu-R-T] of locally symmetric varieties together with the existence of "good" extensions of metric homogeneous vector bundles to these compactifications as shown in [Mu]. The second ingredient is a careful analysis of the extension of classical harmonic theory to a suitable L^2 version.

I show in this note that the same techniques can be used to extend the results of Calabi and Vesentini to the non-compact case. This is stated as Theorem 4.3.

Mumford's ideas are sketched in Sect. 1 and in Sect. 2 I have explained the basic L^2–techniques used by Faltings. This is done in some detail since the arguments in [F] are rather sketchy.

Thanks to Christopher Deninger for pointing out to the reference [F].

1 Poincaré growth and good metrics

In this section I recall some concepts and results from [Mu]. Let X be a smooth quasi-projective complex variety and let $\overline{X}$ be a "good" compactification: $\overline{X}$ is non-singular, projective and $\partial X := \overline{X} - X$ a normal crossing divisor. Hence, locally at a point of the boundary, coordinates $(z_1, \ldots, z_n)$ can be chosen such that the boundary is given by the equation $z_1 \cdots z_r = 0$ and ∂X can be covered by a collection of polydisks Δ^n on which X cuts out $(\Delta^*)^r \times \Delta^{n-r}$. Let $\| \|_P$ be the Poincaré norm on such a product. Any smooth p form, say η on X is said to have *Poincaré growth near the boundary*, if for all tangent vectors $\{t_1, \ldots, t_p\}$ at a point of $\Delta^n \cap X$, one has the estimate $|\eta(t_1, \cdots, t_p)|^2 \leq \text{Const.} \|t_1\|_P \cdots \|t_p\|_P$. This notion does not depend on choices. By [Mu, Prop. 1.1] such a form defines a current on X. Mumford calls a smooth form ω on X a *good* form if ω as well as $d\omega$ have Poincaré growth near the boundary.

Let (E, h) be a hermitian holomorphic vector bundle on X. Recall the following definition:

Definition 1.1. The *Chern connection* for (E, h) is the unique metric connection ∇_E on E whose $(0,1)$-part is the operator $\bar{\partial} : \mathscr{A}^0_X(E) \to \mathscr{A}^{0,1}_X(E)$ coming from the complex structure on E.

Assume that $E = \overline{E}|_X$ where $\overline{E}$ is a holomorphic vector bundle on $\overline{X}$.

Definition 1.2. The metric h is good relative to $\overline{E}$, if locally near the boundary for every frame of $\overline{E}$ the following holds:

1. the matrix entries h_{ij} of h, respectively h^{-1}_{ij} of h^{-1}, with respect to the frame grow at most logarithmically: in local coordinates $z_1, \ldots, z_n$ as above, $|h_{ij}|, |h^{-1}_{ij}| \leq \text{Const.} \cdot (\log |z_1 \cdots z_k|)^N$ for some integer N.

2. the entries of the connection matrix $\omega_h = \partial h \cdot h^{-1}$ for the Chern connection are good forms.

By [Mu, Prop. 1.3] there is at most one extension $\overline{E}$ of E such that h is good relative to that extension. Note also that the dual E^* carries a natural metric and this metric is good relative $(\overline{E})^*$.

If h is a good metric on a vector bundle E relative to an extension $\overline{E}$, then, by definition any Chern form calculated from the Chern connection is good and by [Mu, Thm. 1.4], the class it represents, is the corresponding Chern class of $\overline{E}$.

2 Relevant L^2 harmonic theory

Let me continue with the set-up of the previous section. So (E,h) is a hermitian holomorphic vector bundle on X such that E is the restriction to X of a holomorphic vector bundle $\overline{E}$ on $\overline{X}$ with the property that h is good relative to $\overline{E}$. In addition, make the following, admittedly strong assumptions:

Assumption 2.1. 1. X carries a *complete* Kähler metric h_X whose $(1,1)$-form has Poincaré growth near ∂X (and hence its volume form has Poincaré growth).
2. Smooth sections of the bundle $\mathscr{A}^k_{\overline{X}}(\overline{E})$ of *complex* k–forms with values in $\overline{E}$ are bounded in the metric induced from h and h_X.

Let me recall how to introduces metrics on the spaces $A^k(\overline{E})$ of global *complex* k-forms with values in $\overline{E}$. On a fibre $\mathscr{A}^k_{X,x}(E)$ at $x \in X$ of the vector bundle $\mathscr{A}^k_X(E)$, one has a fiberwise metric induced by the metrics h and h_X:

$$h_x(\alpha\otimes s, \beta\otimes t) = h_X(\alpha,\beta)\,h(s,t), \quad \alpha,\beta \in \mathscr{A}^k_{X,x}, \, s,t \in E_x. \tag{1}$$

Assumption 2 means that for any two sections $\omega_i \in A^k(\overline{E})$, $i = 1,2$ the function $\{x \mapsto h_x(\omega_1,\omega_2)\}$ is bounded on X. Since by assumption 1, the volume form for h_X has Poincaré growth near ∂X it follows that the global inner product

$$\langle \omega_1, \omega_2\rangle = \int_X h_x(\omega_1,\omega_2)\cdot \text{vol. form w.r. to } h_X, \quad \omega_1,\omega_2 \in A^k_{\bar{X}}(\overline{E})$$

exists; in other words, one has an inclusion

$$A^k(\overline{E}) \hookrightarrow L^2(X, A^k(E)) = \{\text{square integrable } E\text{-valued } k \text{ forms}\}$$

and one can do harmonic theory for certain differential operators on these spaces. The particular operators here are those that are induced from the Chern connection $\nabla = \nabla_E$ (see Defn. 1.1), namely

$$\begin{aligned} \nabla : \mathscr{A}^k_X(E) &\to \mathscr{A}^{k+1}_X(E), \quad \nabla^{0,1} = \bar{\partial}, \\ \alpha\otimes s &\mapsto d\alpha\otimes s + (-1)^k \alpha\otimes\nabla s. \end{aligned}$$

The operator $\bar\partial$, extends in the distributional sense to an operator

$$\bar\partial : L^2(X, A^{0,q}(E)) \to L^2(X, A^{0,q+1}(E))$$

and since the metric on X is complete and $\bar\partial^2 = 0$, one can apply a result of Van Neumann (cf. [De, Sect. 12]) which says that there is a formal adjoint operator $\bar\partial^* : L^2(X, A^{0,q+1}(E)) \to L^2(X, A^{0,q}(E))$ in the sense of distributions. Moreover, the formal adjoint of $\bar\partial^*$ exists and equals $\bar\partial$. These adjoints, viewed as operators on the bundles $\mathscr{A}^{0,*}_X(E)$ coincide with the classical ones:

Lemma 2.2. *Let $*_E : \mathscr{A}^{p,q}_X(E) \to \mathscr{A}^{n-q,n-p}_X(E)$ be the fiber wise defined operator induced by the Hodge star-operator.*
1) The formal adjoint $\bar\partial^*$ is induced by

$$- *_E \nabla^{1,0} *_E : \mathscr{A}^{0,q+1}_X(E) \to \mathscr{A}^{0,q}_X(E).$$

2) The formal adjoint of $\nabla^{1,0}$ equals $(\nabla^{1,0})^* = - *_E \bar\partial *_E$.

Proof. Since $\bar\partial = -(*_E \nabla^{1,0} *_E)^* = - *_E (\nabla^{1,0})^* *_E$, the second assertion follows from the first. The meaning of the first assertion is that for $\omega_1 \in A^{0,q}(\overline{E})$ and $\omega_2 \in A^{0,q+1}(\overline{E})$ one has

$$\langle \bar\partial \omega_1, \omega_2 \rangle = -\langle \omega_1, (*_E \nabla^{1,0} *_E)\omega_2 \rangle. \tag{2}$$

To show this, let me go through the classical calculation. First, using the metric contraction

$$\begin{aligned} h_E : A^k(E) \otimes A^\ell(E) &\to A^{k+\ell} \\ (\alpha \otimes s, \beta \otimes t) &\mapsto h_E(s,t) \alpha \wedge \bar\beta \end{aligned}$$

one observes the fundamental equaton

$$h_E(\varphi_1, *_E \varphi_2) = h_x(\varphi_1, \varphi_2) \cdot \text{ vol. form } dV, \quad x \in X, \quad \varphi_1, \varphi_2 \in A^k(E). \tag{3}$$

Next, the Chern connection being metric implies that for the forms restricted to X (denoted by the same symbols) one has

$$h_E(\nabla \omega_1, *_E \omega_2) + (-1)^k h_E(\omega_1, \nabla(*_E \omega_2)) = d h_E(\omega_1, *_E \omega_2),$$

and hence, using (3) and the relation $*_E \cdot *_E = (-1)^k$, one finds

$$\bar\partial h_E(\omega_1, *_E \omega_2) = \Big[h_x(\bar\partial \omega_1, \omega_2) + h_x(\omega_1, (*_E \nabla^{1,0} *_E)\omega_2) \Big] \cdot dV. \tag{4}$$

I claim that $\bar\partial h_E(\omega_1, *_E\omega_2)$ is bounded near ∂X and that it integrates over X to zero. Assume this for a moment. Since the first term on the right is bounded, the other is too. Hence after integration one obtains

$$0 = \langle \bar\partial\omega_1, \omega_2\rangle + \langle \omega_1, (*_E\nabla^{1,0}*_E)\,\omega_2\rangle$$

and the result follows.

It remains to show the assertion about $\bar\partial h_E(\omega_1, *_E\omega_2)$. Let U_δ be a tubular neighborhood of ∂X with radius δ. By Stokes' theorem,

$$\int_X \bar\partial h_E(\omega_1, *_E\omega_2) = \lim_{\delta\to 0}\int_{\partial U_\delta} h_E(\omega_1, *_E\omega_2) = 0. \tag{5}$$

The last equality follows since by (3) the integrand has Poincaré growth near the boundary and hence the integral tends to zero (compare the proof of [Mu, Prop 1.2]. $\square$

The Laplacian $\Delta_E := \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$ preserves $L^2(A^{0,q}(X))$ and the forms ω with $\Delta_E\omega = 0$ are by definition the *harmonic* forms. Reasoning as in the classical situation (cf. [De, Sect. 12]) one shows:

Corollary 2.3. *1. For all $\omega \in A^{0,q}_{\bar X}(\overline{E})$ one has*

$$\langle \Delta_E\omega, \omega\rangle = \langle \bar\partial\omega, \bar\partial\omega\rangle + \langle \bar\partial^*\omega, \bar\partial^*\omega\rangle.$$

Hence, in the distributional sense, one has $\Delta_E\omega = 0 \iff \bar\partial\omega = 0 = \bar\partial^*\omega$.
2. There is an orthogonal decomposition

$$L^2(X, \mathcal{A}^{0,q}_X(E)) = [\bar\partial A^{0,q-1}_X(E)]^{\mathrm{cl}} \oplus [\bar\partial^* A^{0,q+1}_X(E)]^{\mathrm{cl}} \oplus \mathsf{H}^{0,q}_{(2)}(E), \tag{6}$$

where the symbol $^{\mathrm{cl}}$ stands for "topological closure" and the symbol $\mathsf{H}_{(2)}$ stands for the harmonic L^2-forms, i.e. L^2-forms ω with $\Delta_E\omega = 0$ in the sense of distributions.

To apply this, recall that by Dolbeault's theorem the cohomology group $H^k(\overline{X}, \overline{E})$ can be calculated as the k-th cohomology of the complex $\mathcal{A}^{0,*}_{\overline{X}}\overline{E})$.

Proposition 2.4 ([F, Lemma 2]). *Assume that $\overline{E}$ is a holomorphic vector bundle on $\overline{X}$ and that $(E = \overline{E}|_X, h)$ is a hermitian bundle on X such that h is good* *relative $\overline{E}$. If assumption 2.1 holds, then there is natural* injective *homomorphism*

$$j^*_{L^2} : H^k(\overline{X},\overline{E}) = H^k(\mathscr{A}^{0,*}_{\overline{X}}(\overline{E})) \to \mathsf{H}^{0,k}_{(2)}(X,E),$$

with target the space of E-valued harmonic square integrable $(0,k)$–forms.

Proof. The map $j^*_{L^2}$ is induced from orthogonal projection to $\mathsf{H}^k_{L^2}(E)$. The procedure is as follows. Pick $\alpha \in \mathscr{A}^{0,k}_{\overline{X}}(\overline{E})$ for which $\bar\partial\alpha = 0$ representing a given cohomology class $[\alpha] \in H^k(\overline{X},\overline{E})$. By assumption 2.1, $\beta = \alpha|X$ is an E- valued L^2-form whose orthogonal projection to the harmonic forms is $j^*_{L^2}\alpha$. One needs to verify independence of choices: since $\bar\partial\alpha = 0$, one has $\bar\partial\beta = 0$ in the sense of currents and so, another representative for α leads to a form which differs from β by a current of the form $\bar\partial\gamma$. Hence the harmonic projection is independent of choices.

To see that it is injective, suppose that the harmonic part of β vanishes. By (2) one has $\langle\beta,\bar\partial^*\varphi\rangle = \langle\bar\partial\beta,\varphi\rangle = 0$ and hence β belongs to the first summand of (6) so that

$$\beta = \lim_{j\to\infty} \bar\partial\gamma_j, \quad \gamma_j \in \mathscr{A}^{0,k-1}_{\overline{X}}(\overline{E}).$$

To test that this gives the zero class in $H^k(\overline{X},\overline{E})$, one uses the Serre duality pairing:

$$H^k(\overline{X},\overline{E}) \otimes H^{n-k}(\overline{X},\Omega^n_{\overline{X}} \otimes \overline{E}^*) \to H^{n,n}(\overline{X}) = \mathbf{C}$$

as induced by the pairing

$$\mathscr{A}^{0,k}_{\overline{X}}(\overline{E}) \otimes \mathscr{A}^{0,n-k}_{\overline{X}}(\Omega^n_{\overline{X}} \otimes \overline{E}^*) \to \mathscr{A}^{n,n}_{\overline{X}}.$$

To this end, consider for a closed $\beta' \in \mathscr{A}^{0,n-k}_{\overline{X}}(\Omega^n_{\overline{X}} \otimes \overline{E}^*)$. I claim that near ∂X it is bounded in norm. To see this let $s \in \Gamma(\overline{X},\Omega^n_{\overline{X}}(\overline{E}^*))$, then, with f a local equation for ∂X, the product $f\cdot s$ is a section in the unique extension $\Omega^n(\overline{X})(\log\partial X) \otimes \overline{E}^*$ on $\overline{X}$ of the bundle $\Omega^n_X \otimes E^*$ on X for which $h = h_X \otimes h_{E^*}$ is good. That this is the case will be shown later (Examples 4.2.1). In particular, since $h(f\cdot s, f\cdot s) = |f|^2 h(s,s)$ has logarithmic growth near ∂X it follows that $h(s,s)$ and hence also $h(\beta',\beta')$ must vanish near ∂X. Hence $\beta' \in L^2(A^{0,n-k}_X(\Omega^n_X \otimes E^*))$. The Serre pairing therefore is given by

$$(\beta,\beta') := \lim_{j\to\infty}\int_X \bar\partial\gamma_j \wedge \beta' = \lim_{j\to\infty}\lim_{\delta\to 0}\int_{\partial U_\delta} \gamma_j \wedge \beta',$$

where U_δ is a tubular neighborhood of ∂X whose radius is δ (the last equation follows from Stokes' theorem). Since β' tends to zero near ∂X, this integral vanishes. Consequently, the cohomology class of β is zero by Serre duality. $\square$

I want to finish this section by showing that the Nakano inequality [Na] still holds for E-values harmonic $(0,q)$-forms on X. To explain this, one needs some more notation. The Lefschetz operator L - which is wedging with the fundamental $(1,1)$–form for the metric h_X - preserves L^2–forms since the fundamental form has Poincaré growth near ∂X. Moreover, since L is real,

$$h_x(L\alpha,\beta)dV = h_E(L\alpha,\beta) = L\alpha \wedge \overline{*\beta} = \alpha \wedge \overline{*(*^{-1}L*\beta)}$$

and so $\Lambda = *^{-1}L*$ is the formal adjoint of L. Since $*$ is an isometry, one concludes that also Λ preserves the L^2–forms.

Lemma 2.5 (Nakano Inequality [Na]). *Let $\omega \in \mathsf{H}^{0,k}_{(2)}(X,E)$. With F_h the curvature of the metric connection on (E,h) and Λ the formal adjoint of the Lefschetz operator, one has the inequality*

$$\mathrm{i}\langle \Lambda F_h\omega,\omega\rangle \geq 0.$$

Proof. For simplicity, write $\nabla^{1,0} = \partial_E$ with adjoint ∂_E^* . One has the Kähler identity (see e.g. [De, Sect. 13])

$$\Lambda\bar{\partial} - \bar{\partial}\Lambda = -\mathrm{i}\partial_E^*,$$

which is derived in the L^2-setting as in the classical setting. Using this relation, $\bar{\partial}\omega = 0 = \bar{\partial}^*\omega$, as well as $F_h(\omega) = \bar{\partial}\partial\omega$, one calculates

$$\begin{aligned} 0 \leq \langle \partial_E\omega, \partial_E\omega\rangle = \langle \partial_E^*\partial_E\omega,\omega\rangle &= \mathrm{i}\langle \Lambda\bar{\partial}\partial_E\omega - \bar{\partial}\Lambda\partial_E\omega,\omega\rangle \\ &= \mathrm{i}\langle \Lambda F_h\omega,\omega\rangle - \mathrm{i}\langle \Lambda\partial_E, \bar{\partial}^*\omega\rangle \\ &= \mathrm{i}\langle \Lambda F_h\omega,\omega\rangle. \end{aligned}$$ $\square$

3 The Calabi-Vesentini method in the L^2–setting

In this section I shall indicate how the method used in [Cal-V, Sect. 7,8] to show vanishing of the groups $H^q(T_X)$ for X compact can be adapted step by step to the non-compact setting.

Let (X,h) be a Kähler manifold and let T_X be the holomorphic tangent bundle. Suppose that the assumptions 2.1 hold. The metric h induces hermitian metrics on the bundles $\mathscr{A}_X^{p,q} = \wedge^p T_X^* \otimes \wedge^q \bar{T}_X^*$ of forms on X of type (p,q). The Chern connection on T_X is the standard Levi-Civita connection and its curvature is a global T_X–valued $(1,1)$–form:

$$F_h \in A_X^{1,1}(\mathrm{End}(T_X)).$$

Using the metric one has an identification $\bar{T}_X^* \simeq T_X$ and hence F_h induces an endomorphism of $T_X \otimes T_X$:

$$F_h \in T_X^* \otimes \bar{T}_X^* \otimes T_X^* \otimes T_X \simeq T_X^* \otimes T_X^* \otimes T_X \otimes T_X \simeq \mathrm{End}(T_X \otimes T_X).$$

One can show, using the Bianchi identity, that the resulting endomorphism vanishes on skew-symmetric tensors and hence induces

$$Q : S^2 T_X \to S^2 T_X, \quad R = 2\,\mathrm{Tr}(Q), \tag{7}$$

where the function R is the scalar curvature of the metric. The operator Q is self-adjoint and hence at each $x \in X$ it has real eigenvalues. Let λ_x be the smallest eigenvalue at x and suppose that

$$-\infty < \lambda := \int_{x\in X} \lambda_x < 0, \quad \lambda_x \text{ smallest eigenvalue of } Q_x. \tag{8}$$

The operator Q together with the metric h induces a Hermitian form h_Q on the bundles $\mathscr{A}^{0,q}(T_X)$, $q > 0$ as follows:

$$\begin{aligned} h_Q : (\wedge^q \bar{T}_X^* \otimes T_X) \otimes (\wedge^q \bar{T}_X^* \otimes T_X) &\simeq T_X \otimes T_X \otimes (\wedge^q \bar{T}_X^* \otimes \wedge^q \bar{T}_X^*) \\ &\xrightarrow{Q} T_X \otimes T_X \otimes (\wedge^q \bar{T}_X^* \otimes \wedge^q \bar{T}_X^*) \to \mathbf{C}, \end{aligned}$$

where the last map is induced from the hermitian metric h. If h is Kähler-Einstein, one has [Cal-V, Sect. 8]:

$$\mathrm{i}h_x(\Lambda F\omega, \omega) = \frac{R}{2n}\|\omega\|^2 - h_Q(\omega,\omega) \tag{9}$$

On the other hand, by [Cal-V, Lemma 3] one has the inequality

$$h_Q(\omega,\omega) \geq \frac{1}{2}(q+1)\lambda_x\{\omega\|^2. \tag{10}$$

In (loc. cit.) it is shown that first of all $R < 0$ implies $\lambda < 0$, and hence, combining (10) and (9) that

$$\mathrm{i}h_x(\Lambda F\omega,\omega) \le \left(\frac{R}{2n} - \frac{1}{2}(q+1)\lambda_x\right)\|\omega\|^2. \tag{11}$$

The above function is ≤ 0 whenever $\frac{R}{2n} - \frac{1}{2}(q+1)\lambda < 0$ and it is identically zero if and only if $\omega = 0$. Now contrast this with the version 2.5 of Nakano's Lemma which holds under the assumptions of Sect. 2. The conclusion is:

Proposition 3.1. *Suppose that the assumptions 2.1 hold for a quasi projective Kähler-Einstein manifold (X,h) and its holomorphic tangent bundle (T_X,h). Suppose also that $R < 0$, where R is the scalar curvature.*

Then for all integers q for which $q < \frac{R}{n\lambda} - 1$, one has $\mathsf{H}^{0,q}_{(2)}(X,T_X) = 0$.

Remark 3.2. The above proof has to be modified slightly for $q = 0$. In that case the term $h_Q(\omega,\omega)$ in (9) vanishes and since $R < 0$ the above argument directly shows that $\mathsf{H}^0_{(2)}(X,T_X) = 0$. This implies that $\bar{X}$ admits no vectorfields tangent to ∂X.

4 Application to locally symmetric varieties of hermitian type

Let G be a reductive $\mathbf{Q}$–algebraic group of hermitian type, i.e. for $K \subset G(\mathbf{R})$ maximal compact, $D = G(\mathbf{R})/K$ is a bounded symmetric domain. Fix some neat arithmetic subgroup $\Gamma \subset G(\mathbf{Q})$ and let $X = \Gamma\backslash D$ be the corresponding locally symmetric manifold. It is quasi-projective and by [A-Mu-R-T] admits a smooth toroidal compactification $\overline{X}$ with boundary a normal crossing divisor ∂X.

Let $\rho : G \to \mathrm{GL}(E)$ be a finite dimensional complex algebraic representation with $\tilde{E}_\rho$ the corresponding holomorphic vector bundle on D and E_ρ the bundle it defines on X. Fix also a G–equivariant hermitian metric $\tilde{h}$ on $\tilde{E}_\rho$ (which exists since the isotropy group of the $G(\mathbf{R})$–action on D is the compact group K) and write h for the induced metric on E_ρ. By [Mu, Thm. 3.1.], there is a unique extension of E_ρ to an algebraic vectorbundle $\overline{E}_\rho$ on $\overline{X}$ with the property that the metric h is a called good metric for the bundle E_ρ relative to $\overline{E}_\rho$.

For what follows it is important to observe:

Lemma 4.1. *The metric* $(1,1)$*-form* ω_{h_X} *of a Kähler-Einstein metric* h_X *has Poincaré growth near* ∂X.

Proof. The Kähler-Einstein condition means that

$$\omega_{h_X} = -k \cdot \mathrm{i}\partial\bar{\partial}\log(\det h_X),$$

for some positive real constant k. Up to some positive constant, the right hand side can be identified with the first Chern form for the canonical line bundle Ω^n_X with respect to the metric induced by h_X. Since this metric is $G(\mathbf{R})$-equivariant, it is good in Mumford's sense and so ω_{h_X} is also good. □

Clearly, if this is to be useful in applications, given a bundle (with some $G(\mathbf{R})$–equivariant hermitian metric), one needs to get hold of the extension making the metric good.

Examples 4.2. 1. Let $E = \Omega^p_X$. Then $\overline{E} = \Omega^p_{\overline{X}}(\log \partial X)$, the bundle of p-forms with at most log-poles along ∂X. This is not trivial. See [Mu, Prop. 3.4.] where this is shown for $p = 1$. Since $\Omega^p_{\overline{X}}(\log \partial X) = \bigwedge^p \Omega^1_{\overline{X}}(\log \partial X)$ this implies the result for all p. In particular, smooth sections of Ω^1 are bounded near ∂X. Indeed, if $f = 0$ is a local equation for ∂X and ω a smooth section of Ω^1_X, then $f \cdot \omega$ is a smooth section of $\Omega^1_X(\log \partial X)$. Then $\|f \cdot \omega\|^2 = \|f\|^2\|\omega\|^2$ and since $\|\omega\|^2 \leq C(\log\|f\|)^N$, $\|f \cdot \omega\|^2$ is bounded. A similar argument holds for smooth sections of Ω^p_X and hence for sections of $\mathcal{A}^{p,q}_X$.
2. One has $\overline{T}_X = T_{\overline{X}}(-\log \partial X)$, the bundle of holomorphic vector fields on $\overline{X}$ which are tangent to the boundary ∂X, since this is the dual of the bundle $\Omega^1_{\overline{X}}(\log \partial X)$. Any smooth section of this bundle is bounded near the boundary: its normal component tends to zero and the Poincaré growth of the metric implies (by compactness of ∂X) that tangential component remains bounded.
3. These two remarks show that the holomorphic tangent bundle T_X satisfies assumption 2.1 2.

I can finally state the main result:

Theorem 4.3. *Let $(\overline{X}, \partial X)$ as before, e.g. $X = \Gamma\backslash D$, $D = G(\mathbf{R})/K$ hermitian symmetric, Γ a neat arithmetic subgroup of $G(\mathbf{Q})$ and $\overline{X}$ a good toroidal compactification with boundary ∂D. Let R be the scalar curvature of the $G(\mathbf{R})$–equivariant (Kähler-Einstein) metric and let λ be as before (cf. (8)). Set $\gamma(D) := R/n\lambda$. This is a positive integer and*

$$\mathsf{H}^{0,q}_{(2)}(X, T_X) = 0, \quad \text{for all } q \text{ for which } q < \gamma(D) - 1.$$

If no irreducible factor of D has dimension 1, one has $\gamma(D) \geq 3$. In particular, the resulting pairs $(\overline{X}, \partial X)$ are infinitesimally rigid.

Proof. Since X admits a Kähler-Einstein metic h_X, by Lemma 4.1 its fundamental $(1,1)$-form has Poincaré growth near the boundary. So the first assumption of 2.1 is fulfilled. By example 4.2.3 the second condition is also fulfilled.

In order to apply Prop. 3.1, one observes that the Kähler manifold X is homogeneous and that therefore $\lambda = \lambda_x$, $x \in X$, a constant. Since the scalar curvature of D is known to be negative, this proves the result, except that $\gamma(D)$ is an integer ≥ 2. The calculation of $\gamma(D)$ is local and has been done in [Bo, Cal-V] and it implies that it is an integer ≥ 2. Also, it is shown there that $\gamma(D) \geq 3$ whenever D has no irreducible factor of dimension 1. For details, see [Cal-V, Sect. 3] and [Bo, Sect. 2]. See also Remark 4.4 below.

I apply this to infinitesimal deformations of $(\overline{X}, \partial X)$ as follows. As is well known, these correspond bijectively to elements of $H^1(\overline{X}, T_{\overline{X}}(-\log \partial X))$. See e.g. [Sern, Prop. 3.4.17].

Now assume that $\alpha \in \mathcal{A}^{0,1}_{\overline{X}}(T_{\overline{X}}(-\log \partial X))$ represents a given cohomology class $[\alpha] \in H^1(\overline{X}, T_{\overline{X}}(-\log \partial X))$. By Prop. 2.4, the class $\beta = \alpha|X$ is an L^2-harmonic form and it suffices to show that $\beta = 0$ which follows from the vanishing of $\mathsf{H}^{0,1}_{(2)}(X, T_X)$. □

Remark 4.4. For irreducible D there is a table for the values of $\gamma(D)$ in [Cal-V] and [Bo]. I copy their result:

type	$I_{p,q}$	$II_m, m \geq 2$	$III_m, m \geq 1$	$IV_m, m \geq 3$	V	VI
$\gamma(D)$	$p+q$	$2(m-1)$	$m+1$	m	12	18
$\dim_{\mathbf{C}} D$	pq	$\frac{1}{2}m(m-1)$	$\frac{1}{2}m(m+1)$	m	16	27

If $D = D_1 \times \cdots \times D_N$ is the decomposition into irreducible factors, one has $\gamma(D) = \min_j \gamma(D_j)$. One sees from this that $\gamma(D) \geq 2$ with equality precisely when D contains a factor of type $I_{1,1} \simeq II_2 \simeq III_1$. One also sees that the best vanishing result is for the unit ball $I_{p,1}$ where all groups vanish.

Corollary 4.5. *Under the assumptions of Theorem 4.3, the pair $(\overline{X}, \partial X)$ has a unique model over a number field.*

Proof. This follows using spreads. For details see [Pe, Sh].

Remark. The above theorem is false for Shimura curves (one dimensional locally homogeneous algebraic manifolds). However, the corollary is true since all Shimura curves have models over $\overline{\mathbf{Q}}$. A proof which is a variant of the above method was given in [F] which motivated in fact this note.

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