Monotone contractions of the boundary of the disc

MONOTONE CONTRACTIONS OF THE BOUNDARY OF THE DISC

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ABSTRACT. In this paper, we study contractions of the boundary of a Riemannian 2-disc where the maximal length of the intermediate curves is minimized. We prove that with an arbitrarily small overhead ε in the lengths of the intermediate curves, there exists such an optimal contraction that is monotone, i.e., where the intermediate curves are simple closed curves which are pairwise disjoint. This proves a conjecture of Chambers and Rotman.

1. INTRODUCTION

This paper deals with monotone homotopies, which we first define. Throughout the article, for two disjoint non-contractible homotopic simple closed curves α and β, we denote by $A(\alpha, \beta)$ the annulus that they bound. For a contractible simple closed curve α, we denote by $D(\alpha)$ the disc that it bounds — since the surfaces we deal with in this paper always have at least one boundary, this disc is uniquely defined.

Definition 1.1. *Let (M, g) be a Riemannian annulus with boundaries γ_0 and γ_1, and let $H : \mathbb{S}^1 \times [0, 1] \to M$ be a homotopy between γ_0 and γ_1, i.e., a smooth map such that $H(t, 0) = \gamma_0$ and $H(t, 1) = \gamma_1$. We will say that H is* monotone *if every intermediate curve $\gamma_\tau := H(t, \tau)$ is a simple closed curve parameterized by t for each $\tau \in [0, 1]$ and if the closed 2-annuli $A(\gamma_\tau, \gamma_1) \subseteq M$ satisfy the inclusion $A(\gamma_{\tau_2}, \gamma_1) \subseteq A(\gamma_{\tau_1}, \gamma_1)$ for every $\tau_1 < \tau_2$.*

A monotone contraction *of a Riemannian 2-disc is a monotone homotopy from its boundary to a constant curve.*

We prove the following theorem, which was a conjecture by Chambers and Rotman [6, Conjecture 0.2].

Theorem 1.2. *Suppose that (D, g) is a Riemannian disc, and suppose that there is a contraction of ∂D through curves of length less than L. Then, for any $\varepsilon > 0$, there is a monotone contraction of ∂D through curves of length less than $L + \varepsilon$.*

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This theorem has numerous applications in Riemannian geometry, for which we refer to Chambers and Rotman [6, Section 0.1].

1.1. **Applications to applied topology.** From the computational topology literature, much recent work has focused on computing a "best" homotopy between two curves as a means of measuring similarity of the curves or determining optimal morphs between them [3, 4, 7]. The main goal in this setting is to determine the computational complexity of such a problem in the most common settings, generally where the two curves are in the plane (possibly with obstacles) or on a meshed surface, as it typically returned by surface reconstruction algorithms.

The type of optimality we study in this work has been investigated in a combinatorial setting, where it was called the "height" of the homotopy [2, 7], and in the graph theoretic setting, where it was called a "b-northward migration" [1]. However, the exact complexity of this problem remains open, and both papers include a conjecture that the best such morphings will proceed monotonically. The monotonicity result we present in this paper is a key ingredient in showing that this problem lies in the complexity class $\mathcal{NP}$.

2. PRELIMINARIES

Throughout the article, a closed curve γ in a Riemannian annulus A is called a *minimizing geodesic* if it is *essential* (i.e., homotopic to one of the boundaries), and its length is minimal among the essential curves.

Definition 2.1. *A* zigzag *Z is a collection of homotopies $H_1, \dots, H_n$ with the following properties:*

(1) *H_i alternates between outward and inwards monotone homotopies, i.e., each of the H_i is a monotone homotopy, but for any $i \in \{1, \dots, n-1\}$, the concatenation of H_i and H_{i+1} is not.*
(2) $H_i(1) = H_{i+1}(0)$

We define $\gamma_0 = H_1(0)$ and $\gamma_i = H_i(1)$ for $1 \leq i \leq n$.

Each H_i goes from γ_{i-1} to γ_i. We define the degree *of Z, $deg(Z)$, to be n.*

We define the area *of Z, $area(Z)$, to be $\sum_{i=1}^n area(A(\gamma_{i-1}, \gamma_i))$.*

We define the length *of Z, $length(Z)$, to be $length(\gamma_0) + \sum_{i=1}^n length(\gamma_i)$.*

We will also need the following definitions and a theorem from the article of Chambers and Rotman [6].

Definition 2.2. *(*[6, Definition 0.6]*) Let $\alpha : [0,1] \longrightarrow M$ and $\beta : [0,1] \longrightarrow M$ be two simple closed curves in a Riemannian manifold M. If every two*

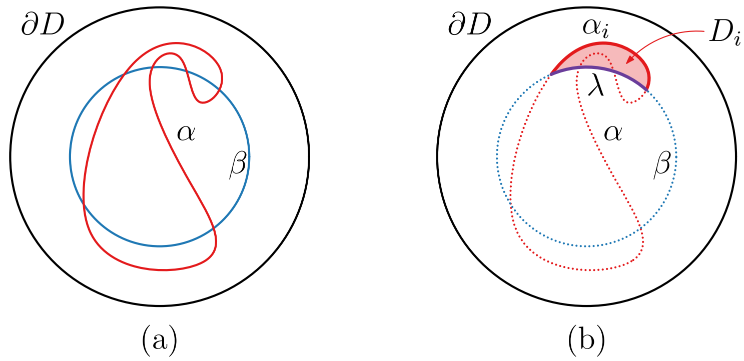


FIGURE 1. Meandering curves

points of intersection between α and β are consecutive on α if and only if they are consecutive on β, then α and β are said to satisfy the simple intersection property.

When α, β defined in 2.2 do not satisfy the simple intersection property, we will say that they are *meandering* with respect to each other.

Definition 2.3. *Let α, β be two simple closed curves in a closed topological 2-disk D. Let $\alpha_i = \alpha|_{[t_i,t_{i+1}]}$ be an arc of α, such that the interior of the arc does not intersect β, while its endpoints $\alpha(t_i), \alpha(t_{i+1}) \in \beta$. Then these points subdivide β into two arcs. Let λ be an arc that together with α_i bounds a disk in the closed annulus $A(\partial D, \beta(t))$ between ∂D and β. Then we will call λ a corresponding arc. We will refer to the disk D_i with the boundary $\alpha_i \cup \beta$ as a corresponding disk, (see fig. 1 (b). The disk that corresponds to arc α_i is shaded).*

Definition 2.4. *Let α, β be two simple closed curves in a closed topological 2-disk D. Suppose α is meandering with respect to β. We will call an arc α_i of α that intersects β only at its endpoints, maximal, if it is adjacent to the outer face in the planar graph obtained by superimposing α and β.*

Proposition 2.5. *Suppose that there is a contraction of ∂D through curves of length less than L, then for any $\varepsilon > 0$ there exists a zigzag of degree n such that $\gamma_0 = \partial D$ and γ_n is a constant curve, and such that all curves of all homotopies have length less than $L + \varepsilon$.*

Proof. First, by a result of Chambers and Liokumovitch [5, Theorem 1.1], we know that for any $\varepsilon > 0$, there exists a contraction of ∂D through *simple* closed curves of length less than $L + \varepsilon$.

We say that a corresponding disk between two arcs α and α' is δ-thin if there is a reparameterization of α such that $\alpha(t)$ and $\alpha'(t)$ are at distance at most δ for any $t \in [0, 1]$. Similarly, an annulus $A(\alpha, \beta)$ is δ-thin if there is a reparameterization of α such that $\alpha(t)$ and $\beta(t)$ are at distance at most δ. Now, we consider a discretized version of the contraction H, i.e., we consider an increasing sequence of n times $t_i \in [0,1]$ so that

- $t_0 = 0$,
- $t_n = 1$, and
- for $0 \leq i \leq n-1$, if $H(t_i)$ and $H(t_{i+1})$ intersect, they have the simple intersection property and the corresponding disks are δ-thin, for δ to be precised later. If they do not intersect, the annulus $A(H(t_i), H(t_{i+1}))$ is δ-thin.

The existence of this sequence follows directly by compactness. Now, if $H(t_i)$ and $H(t_{i-1})$ intersect, for each $0 < i < n$, we define an auxiliary curve $H(t_i)^f$ from $H(t_i)$: $H(t_i)^f$ is obtained from $H(t_i)$ by considering all of the arcs of $H(t_i)$ in $D(H(t_{i-1}))$ and replacing the other ones by the arcs they correspond to in $H(t_{i-1})$. Then we claim that there are monotone homotopies between $H(t_i)$ and $H(t_i)^f$, and between $H(t_i)^f$ and $H(t_{i+1})$ such that the intermediate curves have length less than $L+\varepsilon$. Indeed, one can go from one to the other using monotone homotopies that interpolate within the corresponding disks, and if δ is chosen small enough, this interpolation can be done without incurring an overhead of more than ε on the lengths of the curves. If $H(t_i)$ and $H(t_{i-1})$ do not intersect, for δ small enough, the δ-thin assumption implies that there exists a monotone homotopy between $H(t_{i-1})$ and $H(t_i)$, where the intermediate closed curves have length less than $L+\varepsilon$.

Gluing together all of these monotone homotopies, we obtain a zigzag with curves of length at most $L+\varepsilon$, which concludes the proof.

□

One of our main technical tools is the following theorem.

Theorem 2.6. *Let H be a monotone homotopy between simple closed curves γ_0 and γ_1 such that the length of the intermediate curves does not exceed L, and let γ be another simple closed curve in $A(\gamma_0, \gamma_1)$ such that γ is a minimizing geodesic in $A(\gamma, \gamma_1)$. Then for any $\varepsilon > 0$, there exists a monotone homotopy between γ_0 and γ where the lengths of the intermediate curves do not exceed $L+\varepsilon$.*

Although being not explicitly stated in Chambers and Rotman [6], this theorem is implicit in the proof of their Theorem 0.7. More precisely, their proof is divided in two steps, and this is the result obtained by Step 1.

We will also need the following lemma.

Lemma 2.7. *Let $\alpha : [0,1] \longrightarrow D$, $\beta : [0,1] \longrightarrow D$ be two simple closed curves in a closed topological disk D. Let $A(\partial D, \alpha)$, $A(\partial D, \beta)$ be two (closed) annuli between, ∂D and α, and ∂D and β respectively, (see fig. 2*

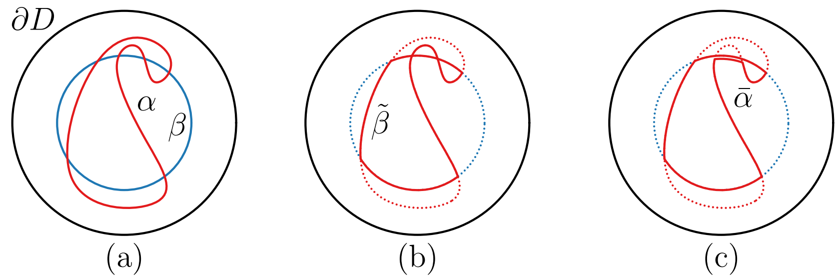


FIGURE 2. Obtaining $\bar{\alpha}$ from β

(a)). Let α be a shortest closed curve homotopic to ∂D in $A(\partial D, \alpha)$, while β is a shortest closed curve in $A(\partial D, \beta)$ homotopic to ∂D. We will refer to shortest curves in annuli, homotopic to the boundary curves as geodesics. One can construct a new closed curve $\bar{\alpha}$ by by replacing all of the arcs of α that lie in $A(\partial D, \beta)$ by the corresponding arcs of β, (see fig. 2 (c)). Then $length(\bar{\alpha}) \leq \min\{length(\alpha), length(\beta)\}$.

Proof. It is easy to see that $\bar{\alpha}$ is not longer than α. Indeed, let e be an arc of α that lies in $A(\partial D, \beta)$. Let f be the arc of β that corresponds to e. Then the length of f is less than or equal to the length of e. If it was not the case, we could have replaced f by e, obtaining a closed curve in $A(\partial D, \beta)$ that is homotopic to ∂D of length shorter than that of β, which would contradict β being a geodesic. We will now show that $\bar{\alpha}$ is also shorter than β.

In order to see it, note that one can obtain $\bar{\alpha}$ from β via the following two-step procedure.

Step 1. In a first step, we consider the maximal arcs of β with respect to α, and replace those by their corresponding arcs in α. This yields a curve $\tilde{\beta}$, which we claim is not longer than than β. Indeed, if f is a maximal arc of β, and its corresponding arc is e, then e cannot be longer than f, otherwise α would not be a minimizing geodesic in $A(\partial D, \alpha)$.

Step 2. Next, let us consider the curve $\tilde{\beta}$. We will replace the arcs of $\tilde{\beta}$ that lie in $A(\partial D, \beta)$ with the corresponding arcs of β, (see fig. 2 (c)). Once again, the length of the curve cannot increase during this process, as it would contradict the fact that β is a minimizing geodesic in $A(\partial D, \beta)$.

It now remains to show that the curve that we obtain after performing steps 1 and 2 above is the same as $\bar{\alpha}$. To see that, let us first order the maximal arcs of α with respect to the parametrization of α, and denote the resulting sequence of maximal arcs by $a_1, \ldots, a_k$, and their corresponding arcs by $b_1, \ldots b_k$. Fig. 3 depicts the curve α that is meandering with respect to β with the ordered maximal arcs, a_1, a_2, a_3, a_4. Let us denote the arc of β that connects the endpoint of $a_i, i \in 1, ..., k$ with the starting point of a_{i+1}

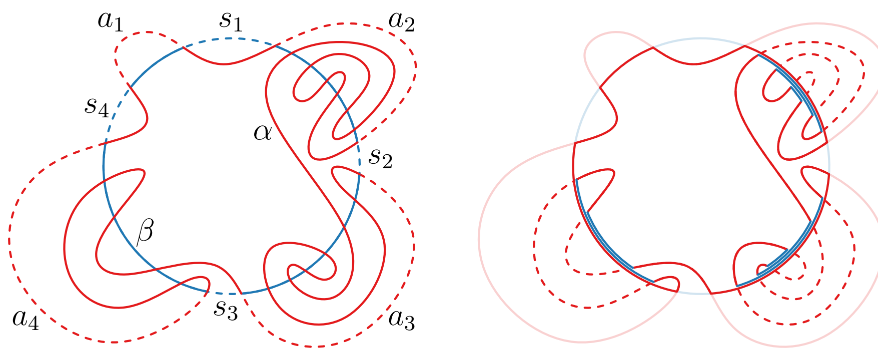


FIGURE 3.

by s_i (where we use the convention that that $a_{k+1} = a_1$, (see fig. 3). Observe that the s_i are exactly the maximal arcs of β with respect to α , and denote by $t_1, \ldots, d_k$ their corresponding arcs. Now, recall that step 1 replaces all of the maximal arcs $s_i, i = 1, \ldots, k$ with their corresponding arcs t_i. We claim that this yields the same curve as replacing all of the maximal arcs a_i of α with the corresponding arcs b_i. Indeed, α is the concatenation of the arcs $a_1, t_1, \ldots a_k, t_k$, while β is the concatenation of the arcs $s_1, b_1, \ldots s_k, b_k$. Both procedures yield the concatenation of the arcs $a_1, b_1, \ldots, a_k, b_k$, i.e., the same curve.

Then, step 2 switches the remaining arcs of α that are in $A(\partial D, \beta)$ to the corresponding arcs of β. Therefore we obtain $\bar{\alpha}$.

□

The proof of Theorem 1.2 relies on the following two propositions allowing us to modify small portions of zigzags. The first one follows rather directly from Theorem 2.6, but the second one requires more work.

Proposition 2.8. *Suppose that Z is a degree 2 zigzag where the intermediate curves have length at most L. If γ_1 is not a minimizing geodesic in $A(\gamma_1, \gamma_2)$, and if a minimizing geodesic γ in this annulus also lies in the interior of $A(\gamma_0, \gamma_1)$ and is essential in it, then, for any $\varepsilon > 0$, there is a zigzag Z' where the intermediate curves have length at most $L + \varepsilon$ and such that*

(1) *$deg(Z') = 2$, and Z' has the same order as Z.*
(2) *$length(Z') < length(Z)$.*
(3) *$\gamma'_0 = \gamma_0$, and $\gamma'_2 = \gamma_2$.*

Suppose that Z is a degree 2 zigzag where the intermediate curves have length at most L, and that γ_0 is a minimizing geodesic in $A(\gamma_1, \gamma_2)$, or that γ_2 is a minimizing geodesic in $A(\gamma_0, \gamma_1)$. Then for any $\varepsilon > 0$, there is a *degree 1 zigzag Z' of length less than that of Z, where the intermediate curves have length at most $L + \varepsilon$ and such that $\gamma_0' = \gamma_0$, and $\gamma_1' = \gamma_2$.*

Proof. The first part of the proposition follows from two applications of Theorem 2.6. We first apply it to the homotopy H_0 and the curve γ, and then to the reversal of the homotopy H_1 and the curve γ. This yields two new homotopies H_0' and H_1', going respectively from γ_0 to γ and from γ to γ_2 and their concatenation satisfies the needed properties.

For the second part of the proposition, let us first deal with the first case where γ_0 is a minimizing geodesic in $A(\gamma_1, \gamma_2)$. Then one application of Theorem 2.6 to the homotopy H_1 and γ_0 yields the homotopy from γ_0 to γ_2. The other case is obtained by applying the theorem to H_0 and γ_2 instead. □

Proposition 2.9. *Suppose that Z is a zigzag of degree 3 where the intermediate curves have length at most L and such that*
(1) γ_1 is a minimizing geodesic in $A(\gamma_1, \gamma_2)$ but not a constant curve, and
(2) One of the following two conditions if fulfilled:

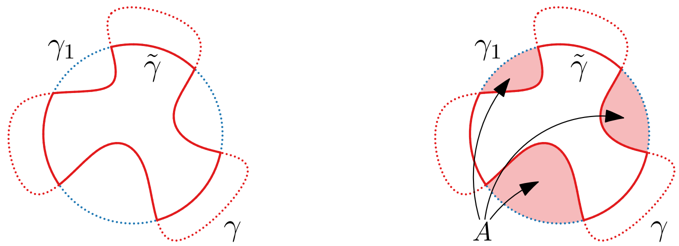
Case a. *There is a minimizing geodesic $\gamma \in A(\gamma_2, \gamma_3)$ which is not fully contained in the interior of $A(\gamma_1, \gamma_2)$.*

Case b. *There is a minimizing geodesic $\gamma \in A(\gamma_2, \gamma_3)$ which is fully contained in the interior of $A(\gamma_1, \gamma_2)$ but not essential in $A(\gamma_1, \gamma_2)$.*

Then for any $\varepsilon > 0$, there is a zigzag Z' of degree 3 where the intermediate curves have length at most $L + \varepsilon$ and such that

(1) $length(Z') \leq length(Z)$.
(2) $area(Z') > area(Z)$.
(3) $\gamma_0' = \gamma_0$, *and* $\gamma_3' = \gamma_3$.

Proof. First, note that one can always modify the homotopy H_3 to obtain a new homotopy H_3', such that the maximal length of curves of H_3' is not larger than the maximal length of curves in H_3, the area spanned by H_3' is the same as the area spanned by H_3, and γ is one of the curves of H_3'. One achieves this by applying Theorem 2.6 twice to the homotopy H_3 and γ: let D_γ be a closed disk that has γ as its boundary. Applying the theorem in this particular case amounts to first replacing all of the curves, or the segments of the curves of the homotopy H_3 that lie in D_γ by the segments of γ, which results in the new monotone homotopy S between the original curve of H_3 and γ, in which the lengths of curves in the homotopy is not increased in comparison with those H_3. Likewise, we can next replace all of the curves, or the segments of the curves of H_3 that lie outside of D_γ resulting in a new monotone homotopy P between γ and the final curve of H_3. After concatenating S and P we obtain the desired homotopy H_3'.



(a) (b)

FIGURE 4. $\tilde{\gamma}$

Thus, without loss of generality assume that $\gamma = (H_3)_t$ for some $t \in [0,1]$. Let S be the restriction of H_3 to the interval $[0,t]$.

Case a. In that case, the disk bounded by γ intersects the disk bounded by γ_1. Then the new zigzag Z' is obtained in the following manner: the first forward step is obtained by replacing the forward monotone homotopy H_1 between the curves γ_0 and γ_1 by the new forward monotone homotopy H_1' between γ_0 and a new curve $\tilde{\gamma}$ (see fig. 4 (a)). This homotopy is constructed by "gluing" the two forward monotone homotopies H_1 between γ_0 and γ_1 and S between γ_2 and γ. The construction of H_1' is completely analogous to the proof of Theorem 0.7 in [6] and will be summarized at the end of the proof. The new homotopy H_1' will be a concatenation of two homotopies: an old homotopy H_1 and a new homotopy $\tilde{H}$ obtained from S by replacing all of the arcs that lie outside the closed disk bounded by γ_1 by their corresponding arcs in γ_1, and when a whole intermediate closed curve of S lies outside of the closed disk bounded by γ_1, we replace it by γ_1.

In particular, the new curve $\tilde{\gamma}$ is constructed by replacing the segments of γ that lie in $A(\gamma_1, \gamma_2)$ by the corresponding segments of γ_1. Note that if there are none, $\tilde{\gamma} = \gamma$.

Recall that, during $\tilde{H}$, when we replace the arcs of γ by the corresponding arcs of γ_1

the length of the resulting curve does not increase.

Note that any closed curve α that lies in the disk bounded by γ_2 and intersects $A(\gamma_1, \gamma_2)$ can be modified by γ_1 in such a way to obtain $\tilde{\alpha}$, with the length not larger than the length of α. Moreover, the resulting curve $\tilde{\gamma}$ will lie in the closed disk bounded by γ_1. Note also that (1) the area spanned by performing the first step of the new zigzag will be larger than the area spanned by the first step of the original zigzag Z, (see fig. 4 (b)), (2) By Lemma 2.7 the length of $\tilde{\gamma}$ will be at most the length of γ_1. To apply this lemma we let $\partial D = \gamma_0$, $\alpha = \gamma$ and $\beta = \gamma_1$.

The second, backward step of the zigzag will be obtained by running $\tilde{H}$ back, and then following it by the monotone homotopy between γ_1 and γ_2.

Finally, the third step of the new zigzag will be the same as the third step of the old zigzag. Thus, H_1' will be the forward homotopy between $\gamma_0' = \gamma_0$ and $\gamma_1' = \tilde{\gamma}$, H_2' will be the backward homotopy between γ_1' and $\gamma_2' = \gamma_2$, and $H_3' = H_3$.

Case b. In that case, the disk bounded by γ does not intersect the disk bounded by γ_1. Thus, by monotonicity, the homotopy S "sweeps" $D(\gamma_1)$ completely. Denote by p one of the last points of $D(\gamma_1)$ swept by S, i.e., a point on γ_1 such that $p \in D(S(t))$ for some t but $D(S(t') \cap D(\gamma_1)) = \emptyset$ for any $t' > t$. We will construct a new homotopy H_1' between γ_0 and the constant curve p, and a new homotopy H_2' between p and γ_2. Let us denote by S' the restriction of S to the interval $[0, t]$. The two homotopies H_1' and H_2' are built almost identically to the ones in Case 1: H_1' results from the gluing of the homotopy H_1, and a new homotopy $\tilde{H}$ obtained from S' by replacing all the arcs that lie outside of $D(\gamma_1)$ by the corresponding arcs of γ_1 (and as before, whole intermediate closed curves lying of S' lying in $A(\gamma_1, \gamma_2)$ are replaced by γ_1). Once again, details of this construction are deferred to the end of the proof. Notice that by definition of p, the curve $\tilde{\gamma}$ we obtain at the end of $\tilde{\gamma}$ is contained in the boundary of $D(\gamma_1)$, and is homotopic to p within this boundary. Since this homotopy can be performed without increasing the lengths of the curve, we can concatenate $\tilde{H}$ with it to obtain a homotopy to the constant curve p. The second, backward step of the zigzag will be obtained by running $\tilde{H}$ back, and then following it by the monotone homotopy between γ_1 and γ_2. Finally, the third step of the new zigzag will be the same as the third step of the old zigzag. Thus, H_1' will be the forward homotopy between $\gamma_0' = \gamma_0$ and $\gamma_1' = \tilde{\gamma}$, H_2' will be the backward homotopy between γ_1' and $\gamma_2' = \gamma_2$, and $H_3' = H_3$.

Note that since γ_1 is a minimizing geodesic in $A(\gamma_1, \gamma_2)$, the lengths of the curves in H_1' and H_2' do not exceed L, and the area spanned by the new zigzag is larger than that spanned by H_1. Since p is a constant curve, its length has zero, and thus the length of the new zigzag is not larger than that of the original one.

Therefore, the proof of the lemma will follow if we can show that there exists a monotone homotopy H_1' between the curves γ_1 and $\tilde{\gamma}$. To keep the proof simple, we focus on Case a, Case b being completely analogous. The existence of such homotopy follows from the construction given in the proof of Theorem 0.7 in [6].

H_1' will be a concatenation of two homotopies: H_1 and a monotone homotopy G between γ_1 and $\tilde{\gamma}$ obtained from H_3 by replacing the segments of curves of the homotopy S that lie outside the closed disk bounded by γ_1 by segments of γ_1 that are not longer than the corresponding segments that they are replacing via the procedure described in the previous paragraph.

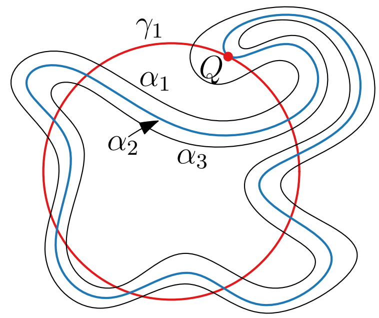


FIGURE 5.

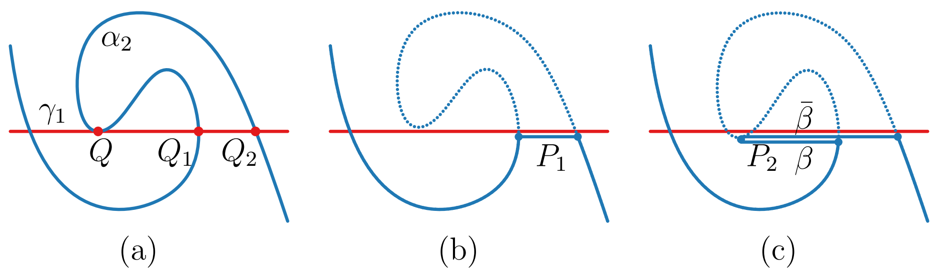


FIGURE 6.

The main difficulty lies in implementing this procedure continuously with respect to the curves in the homotopy. In fact, stated as it is the procedure can result in discontinuities, which appear when the replacement algorithm is not unique. Let α_s denote the curves of homotopy S. If for some $s \in [0, 1]$ the intersection between α_s and γ_1 is not transversal, the procedure can be discontinuous at α_s. Fig. 5 depicts such a situation. Here α_2 touches γ_1 at point Q. There are two ways to exchange the segments of γ_2 in the neighborhood of Q, (see fig. 6 (a) that depicts this situation locally). One way is to replace the segment of α_2 that connects the points Q_1 and Q_2 that lies outside of the open disk bounded by γ_1 by path P_1, (see fig. 6 (b)). Let us call this replacement the type 1 replacement. Another way is depicted in fig. 6 (c). Here we replace the segment of α_2 that connects Q_1 and Q_2 by P_2. P_2 is a path that consists of two paths: the first one replaces the segment of α_2 that connects Q and Q_2, while the second one, β, replaces the segment of α_2 that connects Q_1 and Q. Let us call this replacement the type 2 replacement. However, while there are two ways of replacing this segment of α_2, our procedure gives one canonical way to replace the relevant part of α_1, a curve that is close to α_2 and is outside of the disk bounded by α_2 (fig. 5). If we want the procedure to result in a homotopy, that forces us to choose the type 2 replacement on α_2. On the other hand, there is also, one type of replacement that can be performed on α_3, the curve that is close to α_2 and lies inside the disk that is bounded by α_2. Again, if we want that our procedure to result in a homotopy, it forces us to choose the type 1 replacement for α_2. Hence, we have a discontinuity at α_2. To avoid this discontinuity, note that $P_2 = \beta * \bar{\beta} * P_1$, (see fig. 6 (c)). Here $\bar{\beta}$ denotes path β traversed in the opposite direction. Therefore, P_1 and P_2 can be connected by the obvious length non-increasing path homotopy, which amounts to contracting $\beta * \bar{\beta}$ to Q_1. This path homotopy extends to the homotopy between the two curves derived from α_2. Allowing both the type 1 and type 2 replacements for the segment of α_2 and including the homotopy between the two different resulting curves solves the discontinuity problem.

Note that this new zigzag Z' still satisfies the first property of the hypothesis of the proposition.

□

3. Proof of Theorem 1.2

The proof of Theorem 1.2 uses a variational method. We will find a zigzag which starts at the boundary of the Riemannian disc, ends at a constant curve, and traverses curves of length less than L which optimizes several quantities.

Definition 3.1. *Let $\mathcal{Z}_L$ denote the set of all zigzags that start at ∂D, end at a constant curves, and pass through curves of length less than L.*

Let

$$\mathfrak{L} = \inf_{Z \in \mathcal{Z}_L} length(Z),$$

and let

$$\mathcal{Z}_{L,Length} = \{Z \in \mathcal{Z}_L : length(Z) = \mathfrak{L}\}.$$

Let

$$\mathfrak{A} = \sup_{Z \in \mathcal{Z}_{L,Length}} Area(Z),$$

and let

$$\mathcal{Z}_{L,Length,Area} = \{Z \in \mathcal{Z}_{L,Length} : Area(Z) = \mathfrak{A}\}.$$

Proposition 3.2. *Suppose that there exists a contraction of ∂D through curves of length less than L. Then for any $\varepsilon > 0$, the set $\mathcal{Z}_{L+\varepsilon,Length,Area}$ is not empty.*

The proof follows directly from Proposition 2.5

We now have all the tools to prove our main theorem.

Proof of Theorem 1.2. Let $Z \in \mathcal{Z}_{L+\varepsilon,length,Area}$; by Proposition 3.2, there exists such a Z. We will show that Z has degree 1, and so Z consists of a single monotone homotopy which starts at ∂D, and ends at a constant curve. This completes the proof.

Suppose that the degree of Z is greater than 1. If γ_1 is a constant curve, then the first homotopy in Z satisfies the conclusions of the theorem. If not, then the degree of Z must be at least 3, as the zigzag must end at a constant curve.

Suppose that $n = deg(Z)$ is greater than or equal to 3. We will show that for every i between 1 and $n-2$, γ_i is a minimizing geodesic in $A(\gamma_i, \gamma_{i+1})$, and every minimizing geodesic in $A(\gamma_{i+1}, \gamma_{i+2})$ lies in $A(\gamma_i, \gamma_{i+1})$ and is essential in it. Note that my minimality of Z, none of the curves γ_i outside of the γ_0 and the last one are constant.

We will prove this by induction, and by using Proposition 2.9 and Proposition 2.8. We begin by proving this for $i = 1$. Since $\gamma_0 = \partial D$, $A(\gamma_1, \gamma_2) \subset A(\gamma_0, \gamma_1)$. As a result of this fact, due to Proposition 2.8, γ_1 must be a minimizing geodesic in $A(\gamma_1, \gamma_2)$. Furthermore, since Z minimizes area in $\mathcal{Z}_{L+\varepsilon,length}$, and due to Proposition 2.9, every minimizing geodesic in $A(\gamma_2, \gamma_3)$ is contained in $A(\gamma_1, \gamma_2)$ and is essential in it.

The proof of the inductive step works in the same way. Suppose that the result holds for $i < n-2$. By Proposition 2.8, and since every minimizing geodesic in $A(\gamma_{i+1}, \gamma_{i+2})$ lies in $A(\gamma_i, \gamma_{i+1})$ and is essential in it, γ_{i+1} is a minimizing geodesic in $A(\gamma_{i+1}, \gamma_{i+2})$. As a result of this fact and Proposition 2.9, every minimizing geodesic in $A(\gamma_{i+2}, \gamma_{i+3})$ lies in $A(\gamma_{i+1}, \gamma_{i+2})$ and is essential in it. This completes the proof of the induction.

In particular, every minimizing geodesic in $A(\gamma_{n-1}, \gamma_n)$ lies in $A(\gamma_{n-2}, \gamma_{n-1})$. Since γ_n is a constant curve, it is a minimizing geodesic, this implies that γ_n lies in $A(\gamma_{n-1}, \gamma_{n-2})$. In particular, the second part of Proposition implies that there is a zigzag Z' which lies in $\mathcal{Z}_L$, and of smaller length than Z. This contradicts the definition of Z, completing the proof. □

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