Finding all convex tangrams

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1 General Introduction

Some time ago I was strolling in a second-hand book shop. There I found a bilingual (German / Dutch) book "Tangram, / Das alte chinesische Formenspiel / Het oude Chinese vormenspel " by J. Elffers [2] on tangram puzzles with over 1600 examples and their solutions.

Let us start with explaining the idea of a tangram puzzle. A player is given

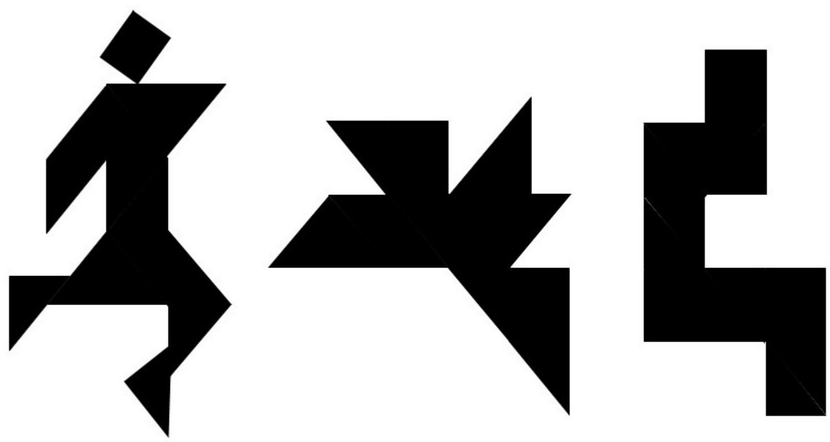

Figure 1: A few typical tangram figures.

a picture of a more or less abstract figurative or geometrical object. In Fig. 1 a few examples are given. Furthermore, the player has a set of 7 flat geometrical pieces, consisting of 5 rectangular isosceles triangles, one single square and one single parallelogram, see Fig. 2 The areas of these pieces are closely related. Indeed, let us identify the triangles by "large", "medium" and "small", conform their area. Then the ratios of the areas are given by "small" : "medium" = 1 : 2 and "medium" : "large" = 1 : 2. Moreover, the area of the square as well as the parallelogram is twice the area of a "small" triangle. Consequently, the 7 pieces can be arranged to form a single larger square, as shown in Fig. 3-Left. Notice that this square can be divided into 16 identical triangles of the type "small", see Fig. 3-Right.

When looking at the seven tangram pieces the question might arise how many different figures can be formed using these pieces. Clearly, the answer is "infi-

Figure 2: The 7 individual tangram pieces.

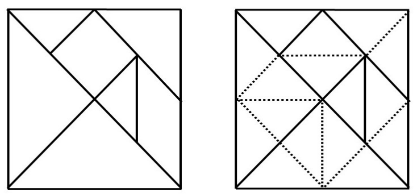

Figure 3: Left: The 7 tangram pieces forming a full square.
Right: The tangram pieces subdivided into 16 identical triangles.

nite". Indeed, even for one single figure (see for example Fig. 1-1) we can have an infinite number of variations, by rotating its head a bit. Thus, the general question is not quite interesting from a mathematical point of view. However, when we restrict ourselves considerably by just asking how many *convex* figures can be found, then we do have a challenging question for mathematicians. This question has already been addressed in 1976 in the book [2] by J. Elffers where a (rather) short mathematical treatment was given, with reference to the scientific paper "A Theorem on the Tangram" [1] by two Chinese mathematicians Fu Traing Wang and Chuan-Chih Hsiung in 1946. In that paper they presented an ingenious translation of this geometrical problem into an algebraic one. In this way they could prove that precisely 13 (essentially) different convex tangrams can be formed.
Unfortunately, due to the limited amount of allowed pages for publication a number of crucial details and proofs were not included in that paper. The main aim of this report is to repeat the results from [1] as well as to provide these missing details in a rather extensive way.
Finally, in the last section we will pay some attention to the problem of finding all convex figures formed by the pieces of the Japanese tangram, as discussed in the paper [3] by E. Fox-Epstein and R. Uehara.

2 Mathematical problem description

In this report we will closely follow the introduction to the problem formulation as given in paper [1].
We start with restating their main theorem.

Main Theorem :

By means of the tangram exactly 13 convex polygons can be formed.

It is easily seen from Fig. 3-Right that the tangram can be divided into 16 identical isosceles rectangular triangles. These triangles will be called *basic* triangles.
The two short sides of a basic triangle will be called the *rational* sides, and the long side will be denoted as *irrational* side. It is shown in paper [1] that when 16 basic triangles are arranged to a convex polygon, then the sides of the polygon consist of basic triangle sides of the same type (rational of irrational).
The sides of the polygon with rational triangle sides will be called *rational*, and analogously *irrational*.

Lemma 1. If sixteen equal isosceles right triangles are combined into a convex polygon, then a rational side of one triangle does not lie along an irrational side of another.

Proof : First of all, let us suppose that of the sixteen given triangles two, denoted by ABC and $A'B'C'$, are arranged so that their rational side $A'C'$ of the triangle $A'B'C'$ lies along the rational side AB of the triangle ABC. Since the given triangles are combined into a convex polygon, we may, without loss of generality, further suppose that the vertex A' coincides with the vertex A. See Fig. 4. In this case, at least another pair of the given triangles, denoted by DEF and $D'E'F'$, is such that one rational side $D'E'$ of the triangle $D'E'F'$ lies along the irrational side DF of the triangle DEF, and $D = B$, $D' = C'$, $E' = F$. If we fill the angle CDE or $B'D'F'$ with one or more of the given triangles, the case where one rational side of one triangle lies along the irrational side of another triangle will occur again. Repeatedly applying the above discussion, it may be easily seen that the polygon formed by the given triangles can not be convex. This contradicts the hypothesis, and establishes the lemma.

The following lemma follows immediately from Lemma 1 :

Lemma 2. If sixteen equal isosceles right triangles are combined into a convex polygon, then the sides of the polygon are formed by sides of the same kind (rational or irrational) of the triangles. Moreover, if a side of the polygon which is formed by the rational or the irrational sides of the triangles is said to be rational or

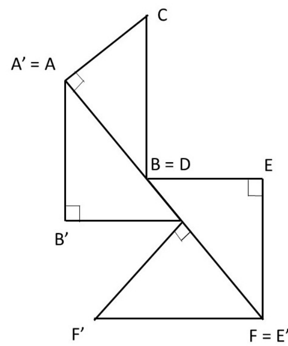

Figure 4: Covering the Tangram by basic triangles.

an irrational side (respectively) of the polygon, then the rational and the irrational sides of the polygon alternate, except in the case that an angle of the polygon is a right angle and then the two adjacent sides are both rational or both irrational.

Lemma 3. If sixteen equal isosceles right triangles are combined into a convex polygon, then the number of the sides of the polygon does not exceed eight.

Proof : Since the sum of all angles of a convex of n sides is equal to $(n-2)\pi$ (see Note below), and the maximal value of the angles formed by the given triangles is $\frac{3}{4}\pi$, we have $(n-2)\pi \leq \frac{3}{4}\pi n$. Hence, $n \leq 8$.

Note : Consider Fig. 5. Then we have $\sum_{i=1}^{N} \alpha_i = 2\pi$ and $\sum_{i=1}^{N}(\alpha_i + \beta_i + \gamma_i) = n\pi$. Thus, $\sum_{i=1}^{N}(\beta_i + \gamma_i) = (n-2)\pi$.

Since the angles of the convex polygon formed by the given triangles are $\frac{3}{4}\pi$, $\frac{1}{2}\pi$ or $\frac{1}{4}\pi$, by means of Lemma 2 and Lemma 3 we easily $^{*)}$ obtain Lemma 4.

Lemma 4. If sixteen equal isosceles right triangles are combined into a convex polygon, then this polygon can be inscribed in a rectangle with all the rational or the irrational sides of the polygon as the sides of the rectangle.

$^{*)}$ **Remark** :

For convenience, we hereafter will include more arguments for easier understanding the correctness of Lemma 4.

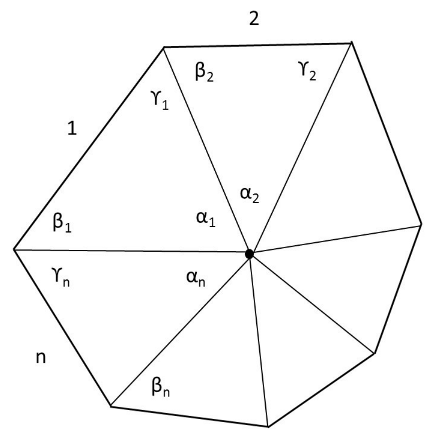

Figure 5: A convex polygon with n sides.

2.1 Additional arguments for proving Lemma 4

As before, consider a convex polygon consisting of 16 basic triangles, having n angles. We will call this shortly by *our polygon*.
Suppose our polygon has p acute, q rectangular and r obtuse angles. Thus,

$$n = p + q + r. \tag{1}$$

As shown in the Note above we know that in a convex polygon with n sides the sum of all its angles is equal to $(n-2)\pi$. So, for our polygon we have

$$p\frac{\pi}{4} + q\frac{\pi}{2} + r\frac{3\pi}{4} = (n-2)\pi. \tag{2}$$

Combining (1) and (2) gives

$$2p + q = 8 - n \tag{3}$$

Since $p \geq 0$ and $q \geq 0$ we see that $n \leq 8$. Moreover, $n \geq 3$ since we are dealing with a polygon. Combining $3 \leq n \leq 8$ and (3) we have the following possible combinations for p, q, r and n as shown in Table 1.

In the Figures 6 - 8 we show all typical shapes of our polygon for the cases indicated in Table 1, of course leaving out those shapes that can be obtained by applying obvious rotations and/or by mirroring. Notice that the *sizes* of the polygon sides are not (yet) known. In these Figures we also have included the rectangle in which the polygon can be inscribed such that all rational sides of the polygon lie along its sides.

Table 1: All possible combinations (n, p, q, r).

id	n	$p \geq 0$	$q \geq 0$	$r \geq 0$	$2p + q = 8 - n$
A	8	0	0	8	0
B	7	0	1	6	1
C	6	1	0	5	2
D	6	0	2	4	2
E	5	1	1	3	3
F	5	0	3	2	3
G	4	2	0	2	4
H	4	1	2	1	4
I	4	0	4	0	4
J	3	2	1	0	5

2.2 Proof of the Main Theorem

Below we give the proof based on that given in [1]. In fact, we follow the main reasoning as in [1], but we included several comments and Figures for more clarification.

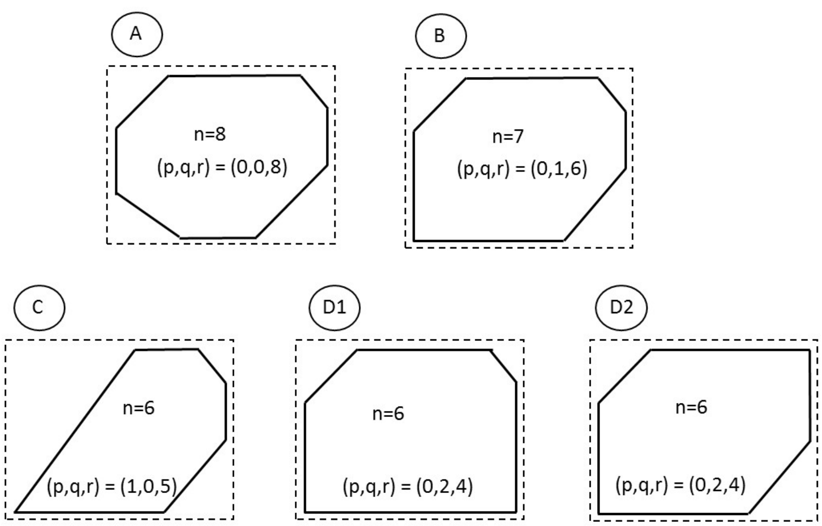

Figure 6: The typical shapes in case $n = 8$, $n = 7$ and $n = 6$.

We have to find all possible convex polygons formed by sixteen equal isosceles right triangles. First, since $n \leq 8$, we may assume that this convex polygon is an octagon (whether or not degenerated), denoted by $ABCDEFGH$. From Lemma 2 and Lemma 4 (and recalling Table 1 and the Figures 6 - 8) we may further assume that this polygon is inscribed in a rectangle $PQRS$ and that all the rational sides BC, DE, FG, HA of the polygon lie along the sides PQ, QR, RS, SP of the rectangle, respectively. See Fig. 9-Left.
Let a, b, c, d be the number of irrational sides of the given triangles on the (irrational) sides AB, CD, EF, GH. Then there are a rational sides of the given triangles along both PB and AP. See Fig. 9-Right. Similarly for b, c and d.
Since BC and FG were assumed to be a rational side of the polygon, we can conclude that PQ and RS consist of x rational sides of the given triangles. Similarly, QR and SP are composed of y rational sides.
Now we can derive a set of equations and constraints from which all convex polygons with the given basic triangles can be determined. Again consider Fig. 9-Left. We start by noting that the area of the rectangle $PQRS$ equals xy. Without loss of generality we may assume that the area of a basic triangle equals 1. Then the total area of the indicated corners of $PQRS$ equals $(a^2+b^2+c^2+d^2)/2$. Clearly, the area of the polygon itself consists of 16 basic triangles, so its area is 8.
Combining this, we find

$$2xy - 16 = a^2 + b^2 + c^2 + d^2 \tag{4}$$

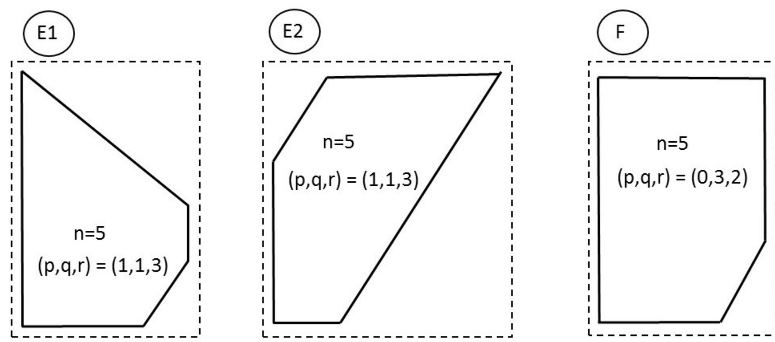

Figure 7: The typical shapes in case $n = 5$.

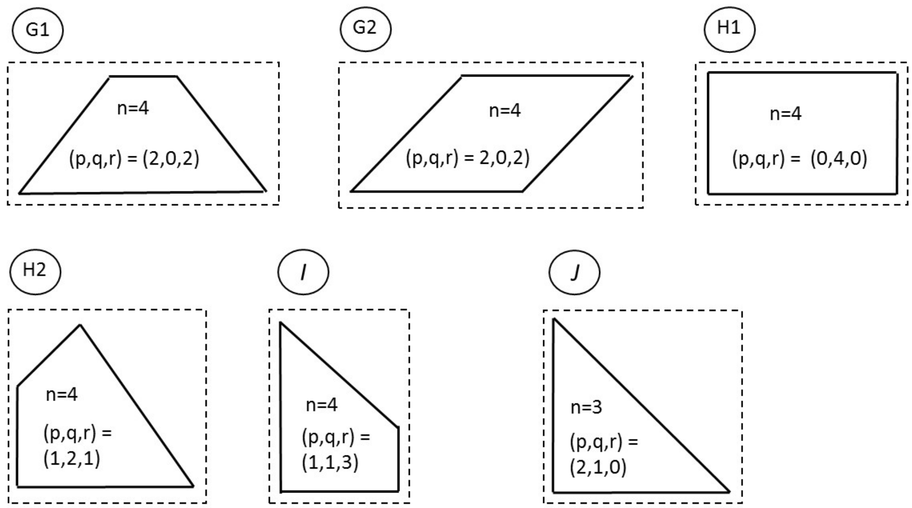

Figure 8: The typical shapes in case $n = 4$ and $n = 3$.

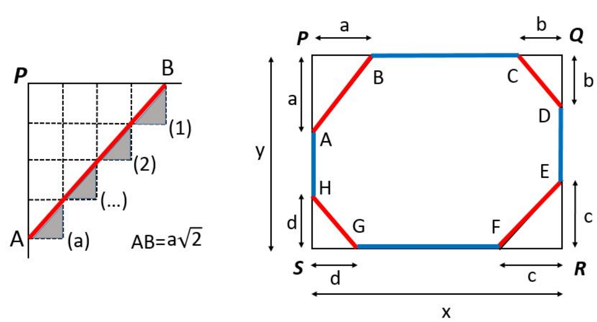

Figure 9: Left: Corner with an integer number (= a) of triangles.
Right: The convex polygon inscribed in a rectangle.
The irrational and rational sides are in red and blue, respectively

with the constraints

$$\begin{aligned} & a+b \leq x, \quad c+d \leq x, \\ & a+d \leq y, \quad b+c \leq y, \\ & a \geq 0,\ b \geq 0,\ c \geq 0,\ d \geq 0, \\ & a,\ b,\ c,\ d,\ x \text{ and } y \text{ are integers} \geq 0. \end{aligned} \tag{5}$$

2.3 Problem reformulation

Clearly, our original problem can be reduced to finding the integer solutions of equation (4) and inequalities (5) and keeping those solutions with the basic triangles that also can be formed with the Tangram pieces.
Without loss of generality we may make the following

$$\text{Assumption}: \quad y \geq x. \tag{6}$$

For each pair (x,y) we have to find all quadruples (a,b,c,d) satisfying both equation (4) and the constraints (5). Such a quadruple (a,b,c,d) will be called *feasible*.
Suppose (a,b,c,d) is feasible.
Then, by interchanging a and d (denoted as $a \Leftrightarrow d$), and b and c (denoted as $b \Leftrightarrow c$) we see that (d,c,b,a) is also feasible.
We call (a,b,c,d) and (d,c,b,a) equivalent, denoted as $(a,b,c,d) \equiv (d,c,b,a)$.
Similarly, (c,d,a,b) is also feasible and $(a,b,c,d) \equiv (c,d,a,b)$.
Moreover, in case $x = y$ we also have that (a,d,c,b) is equivalent to (a,b,c,d).
Indeed, consider (5) and interchange b and d. Then (5) becomes

$$a+d \leq x, \;\; c+b \leq x \;\; ; \;\; a+b \leq y, \;\; d+c \leq y. \tag{7}$$

Since $x = y$ we can interchange x and y in (7) giving again (5).

We will only include the alphabetically largest quadruple in the list of all solutions.
For reference convenience, we summarize this

- (a,b,c,d) is feasible for (x,y) if (a,b,c,d) satisfies (4) and (5);
- $(a,b,c,d) \equiv (d,c,b,a)$, by $a \Leftrightarrow d$ and $b \Leftrightarrow c$; if $y \geq x$
- $(a,b,c,d) \equiv (c,d,a,b)$, by $a \Leftrightarrow c$ and $b \Leftrightarrow d$; if $y \geq x$
- $(a,b,c,d) \equiv (a,d,c,b)$, by $b \Leftrightarrow d$; if $y = x$ (8)
- only the alphabetically largest one of (a,b,c,d), (d,c,b,a), (c,d,a,b) and (a,d,c,b), if $(x = y)$ will be included in the list of solutions.

If all a,b,c,d are equal to zero then equation (4) is reduced to $xy = 8$, corresponding to the solutions $(x,y) = (1,8)$, $(2,4)$, $(4,2)$ and $(8,1)$. Since $y \geq x$ was assumed we have

$$(x,y) = (1,8), \;\; (2,4), \quad \text{if} \;\; (a,b,c,d) = (0,0,0,0). \tag{9}$$

Let us now assume that not all of the parameters a,b,c,d are equal to zero. Then it follows from (4) and the fact that x and y are integers that

$$xy > 8, \text{ and } x \geq 1, \; y \geq 1. \tag{10}$$

Since we assumed that $y \geq x$ (see (6)) we have $y^2 \geq xy \geq 8$. Thus,

$$y \geq 3. \tag{11}$$

3 Some considerations on ways to solve our problem

Before presenting the full approach as given in [1], we like to mention a few considerations that might be helpful for understanding why specific choices in [1] have been made.

First we notice that a trial and error approach to find all solutions to (4)-(5) will not work. Instead we should find one or more systematic ways to tackle the problem. Below we will give some ideas that have been used in the paper [1].

- Trying to exclude several parameter choices might be helpful, for example by assuming $y \geq x$.
- Derive necessary conditions for a particular parameter as done in the previous section ($y \geq 3$).
- split the original problem in a few number of smaller ones, which might be solved easy. For example, consider the two cases $(a,b,c,d) = (0,0,0,0)$ and $(a,b,c,d) \neq (0,0,0,0)$ separately, also done in the previous section. Or, try to solve the problem just for $x = 1$ or $x = y$.
- discard a (large) number of solutions by showing that equivalent solutions can be ignored.
- Try to use simple inequalities like $a^2+b^2+c^2+d^2 \geq 0, \;\; a^2+b^2 \leq (a+b)^2$ and similarly $c^2+d^2 \leq (c+d)^2$.

Last but not leat we want to mention the powerful technique of eliminating one parameter by rewriting (a part of) the original expression. For example, we can write $2xy-16 = 2x(y+1)-2x-16$ and try to replace the term $2x(y+1)$ by an inequality in x only.

In the next section we will give the details of this idea. It will turn out that by using this approach we can derive several different cases for solving the problem, as was done in the paper [1].

3.1 Using an effective elimination technique

For convenience, we recall the equation to be considered.

$$2xy-16=2x(y+1)-2x-16. \tag{12}$$

We assume that $y>x$.
Suppose $x\geq q$ with integer $q>0$, still to be fixed.
Since x and y are integers and $y>x$, we have $y\geq x+1$, and so $y+1\geq 2+x$.
If $x\geq q$ then $2x>2q-1$, so $2>\frac{2q-1}{x}$. Thus,

$$y+1\geq x+2>\frac{2q-1}{x}+x, \quad \text{for } x\geq q. \tag{13}$$

Hence,

$$x(y+1)>x^2+(2q-1). \tag{14}$$

Consequently,

$$\begin{aligned} 2xy-16 &= 2x(y+1)-2x-16>2x^2+2(2q-1)-2x-16 \\ &= 2x^2-2x+4q-18, \quad \text{for } x\geq q. \end{aligned} \tag{15}$$

Recalling (5) we have $c+d\leq x$. Then, using (4)

$$\begin{aligned} 2xy-16 &= a^2+b^2+c^2+d^2\leq a^2+b^2+(c+d)^2 \\ &\leq a^2+b^2+x^2. \end{aligned} \tag{16}$$

Combining (15) and (16) we find

$$a^2+b^2+x^2>2x^2-2x+4q-18. \tag{17}$$

Hence,

$$a^2+b^2>x^2-2x+4q-18=(x-1)^2+4q-19. \tag{18}$$

Clearly, inequality (18) is only helpful for further analysis if its right-hand side is nonnegative. So, we should require that $4q-19\geq 0$, i.e., $q\geq 5$.
Consequently, since we assumed $x\geq q$, we should require

$$\text{Requirement}:\ x\geq 5. \tag{19}$$

By choosing $q=5$ we can rewrite (18) as

$$a^2+b^2>(x-1)^2+1. \tag{20}$$

Notice that it follows from the above that one of the special cases for solving our problem is the case $y>x,\ y>5,\ x\geq 5$.
And then we might expect that another case is $y>x,\ y>5,\ 1\leq x\leq 5$. This is a (partial) clarification for the choices made in the paper [1], see also next section.

4 Investigation of the various cases

We will distinguish the following cases:

A: $y > x$

A1: $y > 5$

A1.1: $y > 5,\ x \geq 5$

A1.2: $y > 5,\ 1 < x < 5$

A1.3: $y > 5,\ x = 1$

A2: $5 \geq y > x$

B: $y = x$

Notice that the case $y < x$ need not to be considered due to the assumption $y \geq x$.

4.1 Case A1.1: $y > x,\ y > 5,\ x \geq 5$

In this case we can completely reuse the results as given by the inequalities (13) up to (20) in the previous section for $q = 5$. Thus, we have now

$$a^2 + b^2 > (x-1)^2 + 1, \text{ for } x \geq 5. \tag{21}$$

It follows from (21) that a and b are not both zero.
On the other hand, a and b are not both different from zero.
Indeed, suppose $a \geq 1,\ b \geq 1$.
Without loss of generality $a \geq b$, say $a = b + p,\ p \geq 0$. Then

$$\begin{aligned} ab - (a+b) + 1 &= b^2 + bp - (2b+p) + 1 \\ &= (b-1)^2 + p(b-1) \geq 0. \end{aligned} \tag{22}$$

Hence,

$$(a+b-1)^2 + 1 = a^2 + b^2 + 2\big(ab - (a+b) + 1\big) \geq a^2 + b^2. \tag{23}$$

Using (5) and the assumption $a \geq 1,\ b \geq 1$ we find $2 \leq a + b \leq x$.
So, $1 \leq a + b - 1 \leq x - 1$. Hence,

$$(a+b-1)^2 \leq (x-1)^2. \tag{24}$$

Combining (23) and (24) we then have

$$a^2+b^2 \leq (a+b-1)^2+1 \leq (x-1)^2+1. \tag{25}$$

Clearly, this contradicts (21). Thus,

$$\text{Conclusion : } a \text{ and } b \text{ are not both different from zero.} \tag{26}$$

Let, for instance, $b=0$, then $a \leq x$. If, further $a < x$, then $a \leq x-1$, which contradicts (21). So, $a=x$. Similarly, if $a=0$, then we find $b=x$.
Summarizing, we have

$$\begin{aligned}&\text{if } \ y>x, \ \ x \geq 5, \ \ y>5 \ \text{ then}\\ &\text{either } \big(a=0, \ \ b=x\big) \ \text{ or } \ \big(b=0, \ \ a=x\big).\end{aligned} \tag{27}$$

Similarly, we can show that

$$\begin{aligned}&\text{if } \ y>x, \ \ x \geq 5, \ \ y>5 \ \text{ then}\\ &\text{either } \big(c=0, \ \ d=x\big) \ \text{ or } \ \big(d=0, \ \ c=x\big).\end{aligned} \tag{28}$$

Combining (27) and (28) we find

$$\begin{aligned}&\text{in \ case } \ y>x, \ \ x \geq 5, \ \ y>5 \text{ then, \ with } y=x+p, \ p>0\\ &2x^2=a^2+b^2+c^2+d^2=2xy-16=2x^2+2px-16.\end{aligned} \tag{29}$$

It follows from (29) that

$$px=8 \ \text{ with integer } p>0 \ \text{ and } \ x \geq 5. \tag{30}$$

Clearly, (30) has only the solution (x,y) = (8,9), corresponding to $p=1$.

Now we will determine the feasible tuples (a,b,c,d) corresponding to $(x,y)=(8,9)$.
In this case $2xy-16=128=8^2+8^2+0^2+0^2$. Using (26) we have $(a,b)=(8,0)$ or $(0,8)$, and $(c,d)=(8,0)$ or $(0,8)$.
This gives $(a,b,c,d)=(8,0,8,0)$, $(8,0,0,8)$, $(0,8,8,0)$ or $(0,8,0,8)$.
Recalling Remark (8) we have $(0,8,8,0) \equiv (8,0,8,0)$ by $a \Leftrightarrow c$, $b \Leftrightarrow d$ and $(0,8,0,8) \equiv (8,0,8,0)$ by $a \Leftrightarrow d$, $b \Leftrightarrow c$. So, we find the candidates $(8,0,8,0)$ and $(8,0,0,8)$, with $(8,0,8,0)$ being feasible, but $(8,0,0,8)$ is not (since $a+d=16>9=y$).

4.2 Case A1.2: $y > x,\ y > 5,\ 1 < x < 5$

Recalling (4) and the restrictions (5) , we have

$$\begin{aligned} 2xy - 16 = a^2 + b^2 + c^2 + d^2 \le (a+b)^2 + (c+d)^2 \le 2x^2, \quad \text{i.e.,} \\ xy \le x^2 + 8. \end{aligned} \tag{31}$$

Combination of (31) and $1 < x < 5$ gives $x = 2,\ 3$ or $x = 4$. Hence, we have

$$\begin{aligned} x = 2:\ \ 2y \le 12,\ \text{ so }\ y \le 6; \\ x = 3:\ \ 3y \le 17,\ \text{ so }\ y \le 6; \\ x = 4:\ \ 4y \le 24,\ \text{ so }\ y \le 6. \end{aligned} \tag{32}$$

Combining (32) and $y > 5$ we find $(x,y) = (2,6),\ (3,6)$ or $(x,y) = (4,6)$.
Now we have to find all feasible (a,b,c,d) corresponding to these pairs (x,y).

Case A1.2.1 : $(\mathbf{x},\mathbf{y}) = (\mathbf{2},\mathbf{6})$
In this case $2xy - 16 = 8 = 2^2 + 2^2 + 0^2 + 0^2$. Without loss of generality we may assume $a \ge b$. Then we find the combinations (in alphabetically decreasing order) $(a,b,c,d) = (2,2,0,0),\ (2,0,2,0),\ (2,0,0,2)$ or $(0,0,2,2)$.
Clearly, $(2,2,0,0)$ is not feasible since $a+b = 4 > 2 = x$. Recalling Remark (8) we have $(0,0,2,2) \equiv (2,2,0,0)$ by $a \Leftrightarrow d,\ b \Leftrightarrow c$. So, $(0,0,2,2)$ is also not feasible. The remaining candidates are $(a,b,c,d) = (2,0,2,0)$ and $(2,0,0,2)$. It is easily checked that they are feasible. See also Table 2.

Case A1.2.2 : $(\mathbf{x},\mathbf{y}) = (\mathbf{3},\mathbf{6})$
In this case $2xy - 16 = 20 = 4^2 + 2^2 + 0^2 + 0^2$. Without loss of generality we may assume $a \ge b$. Then we find the combinations (in alphabetically decreasing order) $(a,b,c,d) = (4,2,0,0),\ (4,0,2,0),\ (4,0,0,2),\ (2,0,4,0), (2,0,0,4), (0,0,4,2)$ or $(0,0,2,4)$.
Clearly, $(4,2,0,0), (4,0,2,0)$ and $(4,0,0,2)$ are not feasible since in these cases $a + b > x$. Furthermore, $(2,0,4,0) \equiv (4,0,2,0)$ and $(0,0,4,2) \equiv (4,2,0,0)$, both by $a \Leftrightarrow c,\ b \Leftrightarrow d$. Finally, $(2,0,0,4) \equiv (4,0,2,0)$ and $(0,0,2,4) \equiv (4,2,0,0)$, both by $a \Leftrightarrow d,\ b \Leftrightarrow c$.
So we can conclude that in this case none of the candidates (a,b,c,d) are feasible.

Case A1.2.3 : $(\mathbf{x},\mathbf{y}) = (\mathbf{4},\mathbf{6})$
In this case $2xy - 16 = 32 = 4^2 + 4^2 + 0^2 + 0^2$. Similar to case A1.2.1, we find the tuples $(a,b,c,d) = (4,4,0,0),\ (4,0,4,0),\ (4,0,0,4)$ or $(0,0,4,4)$ with both $(4,4,0,0)$ and $(0,0,4,4)$ being not feasible. The remaining candidates are $(4,0,4,0)$ and $(4,0,0,4)$. It is easily seen that $(4,0,4,0)$ is feasible, but $(4,0,0,4)$ is not since

$a+d=8>6=y$.

Summary of the cases in A1.1 and A1.2

In Table 2 we have collected all candidates (a,b,c,d) found above (leaving out the equivalent ones), listed in alphabetically increasing order. In case (a,b,c,d) is not feasible, the reason of violation is indicated by †.

Table 2: All candidates (a,b,c,d) in case A1.1 $y>x,\ y>5,\ x\geq 5$ and case A1.2 $y>x,\ y>5,\ 1<x<5$.
If (a,bc,d) is not feasible then the reason is indicated by †.

x	y	a	b	c	d	$2xy-16$	$a+b$	$c+d$	$a+d$	$b+c$	$feasible$
8	9	8	0	0	8	128	8	8	†16	0	† $(a+d>y)$
8	9	8	0	8	0	128	8	8	8	8	$\surd$
2	6	2	0	0	2	8	2	2	4	0	$\surd$
2	6	2	0	2	0	8	2	2	2	2	$\surd$
3	6	4	2	0	0	20	†6	0	4	2	† $(a+b>x)$
3	6	4	0	2	0	20	†4	2	4	2	† $(a+b>x)$
3	6	4	0	0	2	20	†4	2	6	0	† $(a+b>x)$
4	6	4	0	0	4	32	4	0	†8	0	† $(a+d>y)$
4	6	4	0	4	0	32	4	4	4	4	$\surd$

4.3 Case A1.3: $y>x,\ \ y>5,\ \ x=1$

In this case we have (using the constraints (5)) $a+b\leq 1,\ c+d\leq 1$. Thus, (a,b) is either $(0,1)$ or $(1,0)$. The same holds for (c,d). So, at most two parameters in (a,b,c,d) are equal to 1. However, we only can have the following situations:

(i) 4 parameters are equal to 0, i.e., $a=b=c=d=0$,

(ii) 3 parameters are equal to 0, and one is equal to 1,

(iii) 2 parameters are equal to 0, the other two are equal to 1.

Case (i): Clearly, in this case we have $(x,y)=(1,8)$.

Case (ii): Suppose $a=b=c=0,\ d=1$.
Then $2xy-16=2y-16=d^2=1$, so $y=17/2$, being not feasible.
Similarly, the cases $a=b=d=0,\ c=1$, $a=c=d=0,\ b=1$ and $b=c=d=0,\ a=1$ do not give a feasible (a,b,c,d).

Case (iii): In this case we have $2xy-16=2y-16=2$. Hence, $(x,y)=(1,9)$ with 4 candidates for (a,b,c,d), being $(0,1,0,1),(0,1,1,0),(1,0,0,1)$ and $(1,0,1,0)$. Similar as before, we see that $(0,1,0,1)\equiv(1,0,1,0)$ by $a\Leftrightarrow d,\ b\Leftrightarrow c$, while $(0,1,1,0)\equiv(1,0,0,1)$ by $a\Leftrightarrow c,\ b\Leftrightarrow d$.

Summarizing, we have

Table 3: All feasible combinations (x,y) in case $y>x,\ y>5,\ x=1$.

x	y	a	b	c	d	$2xy-16$	$a+b$	$c+d$	$a+d$	$b+c$	$feasible$
1	8	0	0	0	0	0	0	0	0	0	$\surd$
1	9	1	0	0	1	2	1	1	2	0	$\surd$
1	9	1	0	1	0	2	1	1	1	1	$\surd$

4.4 Case A2: $5\geq y>x$

In this case we can test for every set of integer values of x,y,a,b,c,d directly from the equation (4) and the inequalities (5) and so obtain the required solutions.
Below we will give more details of this approach.
We consider two cases: (i) $(a,b,c,d)=(0,0,0,0)$ and (ii) $(a,b,c,d)\neq(0,0,0,0)$.

4.4.1 Case A2.1: $(a,b,c,d)=(0,0,0,0)$

In this case it follows from (4) that $xy=8$. Recalling (9) the solutions are $(x,y)=(1,8)$ and $(2,4)$. Since $y\leq 5$ the only solution is $(x,y,a,b,c,d)=(2,4,0,0,0,0)$.

4.4.2 CaseA2.2: $(a,b,c,d)\neq(0,0,0,0)$

Clearly, we now have $0<2xy-16<2y^2-16$. So, $y\geq 3$. Furthermore,
$0<2xy-16<10x-16$. So, $x\geq 2$. If $(x,y)=(2,3)$ then $2xy-16<0$ and no solution exists. If $(x,y)=(2,4)$ then $2xy-16=0$ which is not possible since $(a,b,c,d)\neq(0,0,0,0)$. So, we still need to investigate the

$$\text{Candidates}:\ (x,y)=(2,5),\ (3,4),\ (3,5)\text{ and }(4,5). \tag{33}$$

In Table 4 we list these candidate pairs (x,y), the value $2xy-16$ and all corresponding sums of squares $a^2+b^2+c^2+d^2$.

For these pairs (x,y) we have to find all feasible quadruples (a,b,c,d).
Without loss of generality we may assume that $a \geq b$. Notice that then no additional assumptions for c and d can be made.

Table 4: All (x,y) with $2xy-16 \geq 0$ in Cases (i) and (ii).

Case	x	y	$2xy-16$	$a^2+b^2+c^2+d^2$
A2.2.1	2	5	4	$2^2+0^2+0^2+0^2$ or
				$1^2+1^2+1^2+1^2$
A2.2.2	3	4	8	$2^2+2^2+0^2+0^2$
A2.2.3	3	5	14	$3^2+2^2+1^2+0^2$
A2.2.4	4	5	34	$5^2+3^2+0^2+0^2$ or
				$4^2+4^2+1^2+1^2$

Case A2.2.1: In case $(x,y)=(2,5)$ we have $2xy-16=4$.
It is easily seen that we have either $4=2^2+0^2+0^2+0^2$ or
$4=1^2+1^2+1^2+1^2$.
This gives (with $a \geq b$): $(a,b,c,d)=(2,0,0,0)$, $(0,0,2,0)$, $(0,0,0,2)$ or $(1,1,1,1)$.
We have $(0,0,2,0) \equiv (2,0,0,0)$ by $a \Leftrightarrow c$, $b \Leftrightarrow d$.
Also $(0,0,0,2) \equiv (2,0,0,0)$ by $a \Leftrightarrow d$, $b \Leftrightarrow c$.
So, if $(x,y)=(2,5)$ then $(a,b,c,d)=(2,0,0,0)$ or $(1,1,1,1)$, ignoring the equivalent quadruples.

Case A2.2.2: In case $(x,y)=(3,4)$ we have $2xy-16=8$.
Clearly, $8=2^2+2^2+0^2+0^2$. This gives $(a,b,c,d)=(2,2,0,0),(2,0,2,0),(2,0,0,2)$ or $(0,0,2,2)$. Notice that $(2,2,0,0)$ is not feasible, since $a+b=4>x$. Furthermore, $(0,0,2,2)$ is equivalent to $(2,2,0,0)$, so this quadruple is also not feasible.
So, if $(x,y)=(3,4)$ then $(a,b,c,d)=(2,0,2,0)$ or $(2,0,0,2)$.

Case A2.2.3: In case $(x,y)=(3,5)$ we have $2xy-16=14$.
Now we can write 14 as a sum of 4 squared integers (apart from ordering) only as $14=3^2+2^2+1^2+0^2$. This gives the following quadruples (assuming $a \geq b$):

- $(3,2,1,0)$, $(3,2,0,1)$: being not feasible since $a+b>x$;
- $(3,1,2,0)$, $(3,1,0,2)$: being not feasible since $a+b>x$;
- $(3,0,2,1)$, $(3,0,1,2)$: being feasible ;
- $(2,1,3,0) \equiv (3,0,2,1)$, by $a \Leftrightarrow c$ and $b \Leftrightarrow d$; so, being feasible
- $(2,1,0,3) \equiv (3,0,1,2)$, by $a \Leftrightarrow d$ and $b \Leftrightarrow c$; so, being feasible

- $(2,0,3,1) \equiv (3,1,2,0)$: so, being not feasible;
- $(2,0,1,3) \equiv (3,1,0,2)$: so, being not feasible ;
- $(1,0,3,2) \equiv (3,2,1,0)$: so, being not feasible ;
- $(1,0,2,3) \equiv (3,2,0,1)$: so, being not feasible ;

So, if $(x,y) = (3,5)$ then $(a,b,c,d,) = (3,0,2,1)$ or $(3,0,1,2)$.

Case A2.2.4: In case $(x,y) = (4,5)$ we have $2xy - 16 = 34$ with $34 = 5^2 + 3^2 + 0^2 + 0^2$ or $34 = 4^2 + 4^2 + 1^2 + 1^2$.
This gives (with $a \geq b$) the following quadruples:

- $(3,0,5,0)$, $(3,0,0,5)$, $(0,0,5,3)$, $(0,0,3,5)$: all not feasible since $c+d > x$;
- $(5,3,0,0)$: being not feasible since $a+b > x$;
- $(5,0,3,0) \equiv (3,0,5,0)$, by $a \Leftrightarrow c$ and $b \Leftrightarrow d$; so, being not feasible;
- $(5,0,0,3)$: being not feasible since $a+d > y$;
- $(4,4,1,1)$, $(4,1,4,1)$, $(4,1,1,4)$: all being not feasible since $a+b > x$;

Thus, no feasible (a,b,c,d) exists in this case.

Summarizing the Cases A2.1 and A2.2.1-A2.2.4, we have the following solutions (the equivalent ones being left out) as given in Table 5.

Table 5: All feasible combinations (x,y) in case $x < y \leq 5$.

x	y	a	b	c	d	$2xy-16$	$a+b$	$c+d$	$a+d$	$b+c$	$feasible$
2	4	0	0	0	0	0	0	0	0	0	$\surd$
2	5	1	1	1	1	4	2	2	2	2	$\surd$
2	5	2	0	0	0	4	2	0	2	0	$\surd$
3	4	2	0	0	2	8	2	2	4	0	$\surd$
3	4	2	0	2	0	8	2	2	2	2	$\surd$
3	5	3	0	1	2	14	3	3	5	1	$\surd$
3	5	3	0	2	1	14	3	3	4	2	$\surd$

4.5 Case B: $y = x$

For case B we start with proving that $3 \leq x \leq 5$.

4.5.1 Part I : Proof of $3 \le x \le 5$ in case $y = x$

First notice that if $a = b = c = d = 0$, then $x^2 = 8$, so there is no integer solution x. Now assume $(a,b,c,d) \neq (0,0,0,0)$. Recalling (4) we have $0 < a^2+b^2+c^2+d^2 = 2xy-16 = 2x^2-16$, so $x^2 > 8$. Thus, we have

$$\text{Necessary condition :} \quad x \ge 3. \tag{34}$$

Secondly, if $a,b,c,d < x$ then it can be shown (see Part II below) that

$$a^2+b^2 \le (x-1)^2+1 \ \text{ and } \ c^2+d^2 \le (x-1)^2+1. \tag{35}$$

From (4) and (35) we obtain

$$2x^2-16 \le 2(x-1)^2+2, \tag{36}$$

which gives $x \le 5$. So, recalling (34) we have the following

$$\text{Restriction :} \quad 3 \le x \le 5. \tag{37}$$

Thirdly, we consider the case where one of a,b,c,d is equal to x.
If, for instance, $a = x$, then $b = 0$ and $d = 0$, since $a+b \le x$ and $a+d \le x$, see (5). Also $c+d \le y$, so $c \le x$.
Then (4) becomes $2x^2-16 = x^2+c^2$, so $x^2 = c^2+16$. Thus, $x \ge 4$.
Clearly, $c^2 = x^2-16$. So, $c < x$, i.e., $c \le x-1$. Then $x^2 = c^2+16 \le (x-1)^2+16$. Thus, $x \le 8$. Now it can easily be seen from $4 \le x \le 8$ that $x^2 = c^2+16$ only has the solutions $(x,c) = (4,0)$ or $(x,c) = (5,3)$.
Finally, we can show (see Proof $\mathscr{P}$ below) that when $a = b = 0$ or $c = d = 0$, then $x < 4$; and when $a = b = c = 0$, then $x = d = 4$.

We conclude from above that in all cases $3 \le x \le 5$. The proof is completed.

Proof $\mathscr{P}$
(i): Suppose $a = b = 0$. Since $(a,b,c,d) \neq (0,0,0,0)$ at least one of c, d is not equal to zero. So, $c+d > 0$ and Thus,

$$2x^2-16 = c^2+d^2 = (c+d)^2-2cd < (c+d)^2 \le x^2. \tag{38}$$

Hence, $x < 4$.
(ii): Similarly, the case $c = d = 0$ also gives $x < 4$.
(iii): When $a = b = c = 0$ then $0 < d = c+d \le x$. So, $0 < 2x^2-16 = d^2 \le x^2$. Hence, $8 \le x^2 \le 16$, i.e., $x = 3$ or $x = 4$. If $x = 3$ then $d^2 = 2$, being not feasible. If $x = 4$ then $d^2 = 16$, Hence, $x = d = 4$.
End of Proof $\mathscr{P}$

4.5.2 Part II : Finding all feasible pairs (a,b) and (c,d) in case $y=x$

For convenience, let us recall the notion of a *feasible* quadruple (a,b,c,d), see (8): (a,b,c,d) is called *feasible* if $a^2+b^2+c^2+d^2=2xy-16$ and $a+b\leq x,\ \ c+d\leq x$ and $a+d\leq y,\ \ b+c\leq y$, for a given pair (x,y).
In addition to this, we will introduce the notion of a *feasible pair*:
A pair (a,b) is a *feasible pair* to (x,y) if

$$a^2+b^2\leq 2xy-16 \ \text{ and } \ a+b\leq x. \tag{39}$$

Notice that if (a,b,c,d) is a feasible quadruple to (x,y), then clearly (a,b) as well as (c,d) are a feasible pair to (x,y).

In Table 6 below we show all feasible pairs (a,b) for $3\leq x\leq 5$.
Without loss of generality we (again) assume $a\geq b$.
For completeness, the values for $(x-1)^2+1$ are also shown.
Notice that in this table in all cases (except $x=a=4$ and $x=a=5$) we have $a^2+b^2\leq(x-1)^2+1$. So, this proves the first part of (35). By renaming a into c into c and b into d in Table 6 we also have $c^2+d^2\leq(x-1)^2+1$, proving the second part of (35).
Moreover, Table 6 gives also all feasible pairs (c,d) with $c\geq d$.

4.5.3 Part III : Finding all feasible quadruples (a,b,c,d) in case $y=x$

We will now determine all feasible quadruples (a,b,c,d) satisfying the equality $a^2+b^2+c^2+d^2=2x^2-16$ and the constraints (5) for $3\leq x\leq 5$.
As indicated above, all feasible pairs (a,b) with $a\geq b$ as well as (c,d) with $c\geq d$ can be found using Table 6.
So, we only need to combine each feasible pair (a,b) with all feasible pairs (c,d) to find a (possibly) feasible combination (a,b,c,d) for $x=3,\ 4,\ 5$.
However, we still may assume that $a\geq b$, but we may not assume anymore that either $c\geq d$ or $d\geq c$. Thus, when combining a pair (a,b) with (c,d) we also have to consider pairs (c,d) with $d\geq c$. Clearly, these can also be found using Table 6.
Notice that in Table 6 we have $2x^2-16=2,\ 16$ *or* 34. It might be helpful for finding feasible quadruples to notice that
$2=1^2+1^2+0^2+0^2,\ \ 16=4^2+0^2+0^2+0^2$ or $16=2^2+2^2+0^2+0^2$, and
$34=5^2+3^2+0^2+0^2$ or $34=4^2+4^2+1^2+1^2$.
Now consider Table 6 for both pairs (a,b) and (c,d). By taking all possible combinations a^2+b^2 and c^2+d^2, computing their sum and comparing this with $2x^2-16$, it can easily be checked that we only have the following candidates for (a,b,c,d) satisfying equation (4), but not necessarily satisfying all constraints (5). This gives

Table 6: All feasible pairs (a,b) in case $x=y$, with $a \geq b$

x	a	b	$a+b$	a^2+b^2	$2x^2-16$	$(x-1)^2+1$
3	0	0	0	0	2	5
3	1	0	1	1	2	5
3	1	1	2	2	2	5
4	0	0	0	0	16	10
4	1	0	1	1	16	10
4	1	1	2	4	16	10
4	2	0	2	4	16	10
4	2	1	3	5	16	10
4	2	2	4	8	16	10
4	3	0	3	9	16	10
4	3	1	4	10	16	10
4	4	0	4	16	16	10
5	0	0	0	0	34	17
5	1	0	1	1	34	17
5	1	1	2	4	34	17
5	2	0	2	4	34	17
5	2	1	3	5	34	17
5	2	2	4	8	34	17
5	3	0	3	9	34	17
5	3	1	4	10	34	17
5	3	2	5	13	34	17
5	4	0	4	16	34	17
5	4	1	5	17	34	17
5	5	0	5	25	34	17

Table 7 below. Moreover, for each (a,b,c,d) it is indicated (by $\surd$) if it is feasible or not (by † and the reason of violation).

4.5.4 Part IV: Identifying all equivalent feasible (a,b,c,d) in case $y=x$

Clearly, all feasible quadruples (a,b,c,d) have been identified in Table 7. However, a few of them are equivalent. Recalling (8), we have $(d,c,b,a) \equiv (a,b,c,d)$, by $a \Leftrightarrow d$, and $b \Leftrightarrow c$, and also $(c,d,a,b) \equiv (a,b,c,d)$, by $a \Leftrightarrow c$, and $b \Leftrightarrow d$. Moreover, also $(a,d,c,b) \equiv (a,b,c,d)$ by $b \Leftrightarrow d$, since $y=x$.

Table 7: All possible (a,b,c,d) with (a,b) and (c,d) from Table 6, with $x = y$, $a \geq b$ and satisfying (4). If (a,b,c,d) is not feasible then its reason is marked by †.

x	a	b	c	d	$2x^2-16$	$a+b$	$c+d$	$a+d$	$b+c$	feasible
3	0	0	1	1	2	0	2	1	1	$\surd$
3	1	0	0	1	2	1	1	2	0	$\surd$
3	1	0	1	0	2	1	1	1	1	$\surd$
3	1	1	0	0	2	2	0	1	1	$\surd$
4	0	0	0	4	16	0	4	4	0	$\surd$
4	0	0	4	0	16	0	4	0	4	$\surd$
4	2	2	2	2	16	4	4	4	4	$\surd$
4	4	0	0	0	16	4	0	4	0	$\surd$
5	0	0	3	5	34	0	†8	5	3	† $c+d>x$
5	0	0	5	3	34	0	†8	3	5	† $c+d>x$
5	1	1	4	4	34	2	†8	5	5	† $c+d>x$
5	3	0	0	5	34	3	5	†8	0	† $a+d>x$
5	3	0	5	0	34	3	5	3	5	$\surd$
5	4	1	1	4	34	5	5	†8	2	† $a+d>x$
5	4	1	4	1	34	5	5	5	5	$\surd$
5	5	0	0	3	34	5	3	†8	0	† $a+d>x$
5	5	0	3	0	34	5	3	5	3	$\surd$
5	5	3	0	0	34	†8	0	5	3	† $a+b>x$

The following quadruples in Table 7 are equivalent:

- $x=3:$ $(0,0,1,1) \equiv (1,1,0,0)$ by $a \Leftrightarrow c$, $b \Leftrightarrow d$;
- $x=3:$ $(1,0,0,1) \equiv (1,1,0,0)$ by just $b \Leftrightarrow d$;
- $x=4:$ $(0,0,0,4) \equiv (4,0,0,0)$ by $a \Leftrightarrow d$, $b \Leftrightarrow c$;
- $x=4:$ $(0,0,4,0) \equiv (4,0,0,0)$ by $a \Leftrightarrow c$, $b \Leftrightarrow d$;
- $x=5:$ $(3,0,5,0) \equiv (5,0,3,0)$ by $a \Leftrightarrow c$, $b \Leftrightarrow d$;

From each set of equivalent (feasible) quadruples we will only keep the alphabetically largest one, see Table 8.

Table 8: All feasible (a,b,c,d) in case $x=y$, with equivalent tuples removed.

x	a	b	c	d	$2x^2-16$	$a+b$	$c+d$	$a+d$	$b+c$	feasible
3	1	1	0	0	2	2	0	1	1	$\surd$
3	1	0	1	0	2	1	1	1	1	$\surd$
4	4	0	0	0	16	4	0	4	0	$\surd$
4	2	2	2	2	16	4	4	4	4	$\surd$
5	4	1	4	1	34	5	5	5	5	$\surd$
5	5	0	3	0	34	5	3	5	3	$\surd$

4.6 Summary: All feasible (a,b,c,d) as found in all cases.

All feasible solutions collected from the Tables 2, 3, 5, and 8 are given in the Table 9. The solutions in this table are ordered conform those given in [1], i.e., in (almost) all cases first by increasing y, and next in decreasing alphabetical order of the quadruple (a,b,c,d) if more solutions are available for a fixed pair (x,y).

Remark 1
It can easily be seen (see section 6) that the solutions in the rows marked by * in Table 9 can not be formed by the set of tangram pieces.

Remark 2
When carefully comparing both parts in Table 9, we see that they are identical, except their last line with $(x,y,a,b,c,d) = (3,3,1,1,0,0)$ and $(3,3,1,0,0,1)$ at left and right, respectively.
However, recalling Remark (8) we have $(1,1,0,0) \equiv (1,0,0,1)$.
So, we can conclude that both tables with their solutions are fully equivalent.

Table 9: Left: All calculated feasible quadruples (a,b,c,d).
Right: All feasible quadruples (a,b,c,d) as copied from [1].
Solutions in the rows marked by * can not be formed by the Tangram pieces.

x	y	a	b	c	d	Case
*1	8	0	0	0	0	
*1	9	1	0	1	0	$y > 5,\ y > x$
*1	9	1	0	0	1	$x = 1$
*8	9	8	0	8	0	
*4	6	4	0	4	0	$y > 5,\ y > x$
2	6	2	0	2	0	$x > 1$
2	6	2	0	0	2	
*5	5	4	1	4	1	$y = x$
*5	5	5	0	3	0	
3	5	3	0	1	2	
3	5	3	0	2	1	$5 \geq y > x$
2	5	1	1	1	1	
2	5	2	0	0	0	
4	4	2	2	2	2	$y = x$
4	4	4	0	0	0	
3	4	2	0	2	0	
3	4	2	0	0	2	$5 \geq y > x$
2	4	0	0	0	0	
3	3	1	0	1	0	$y = x$
3	3	1	1	0	0	

x	y	a	b	c	d
*1	8	0	0	0	0
*1	9	1	0	1	0
*1	9	1	0	0	1
*8	9	8	0	8	0
*4	6	4	0	4	0
2	6	2	0	2	0
2	6	2	0	0	2
*5	5	4	1	4	1
*5	5	5	0	3	0
3	5	3	0	1	2
3	5	3	0	2	1
2	5	1	1	1	1
2	5	2	0	0	0
4	4	2	2	2	2
4	4	4	0	0	0
3	4	2	0	2	0
3	4	2	0	0	2
2	4	0	0	0	0
3	3	1	0	1	0
3	3	1	0	0	1

5 The convex polygons composed of the Tangram pieces

Using Table 9 we can show the 13 convex polygons composed of the 7 Tangram pieces. See Fig. 10. The shapes types mentioned in Figs. 6 - 8 are also indicated. Notice that the types A, B, C or $E1$ are absent.

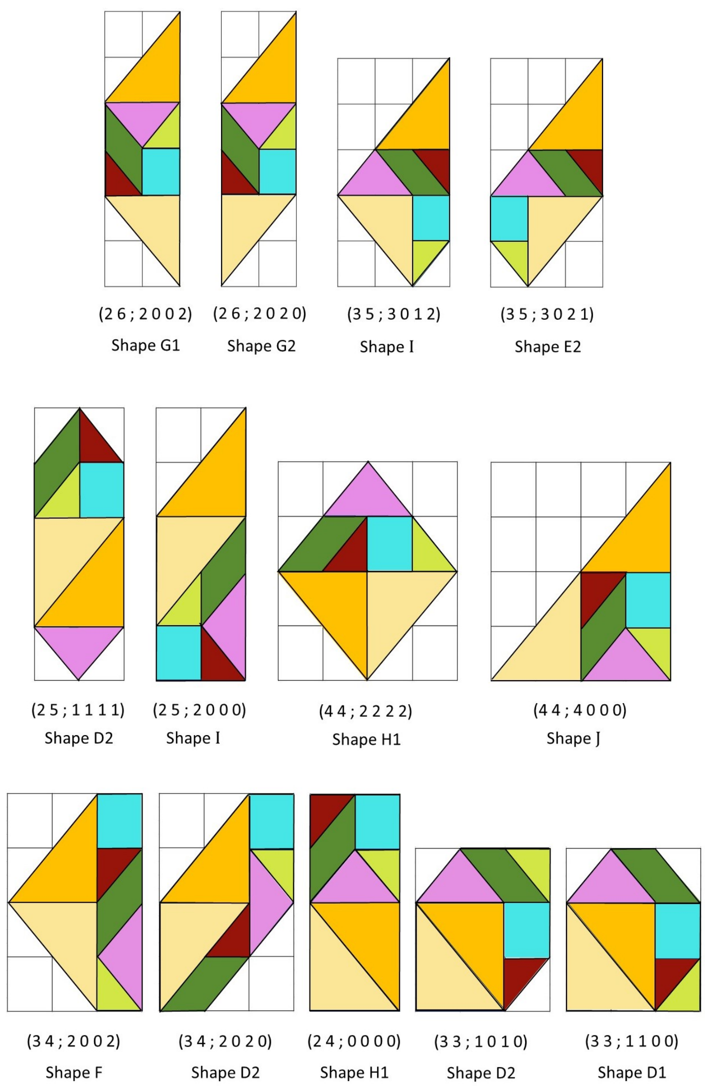

Figure 10: The 13 convex polygons constructed with the Tangram pieces.

6 The infeasible polygons for the Tangram pieces

In the figure below we show the 7 convex polygons that can not be made with the Tangram pieces.

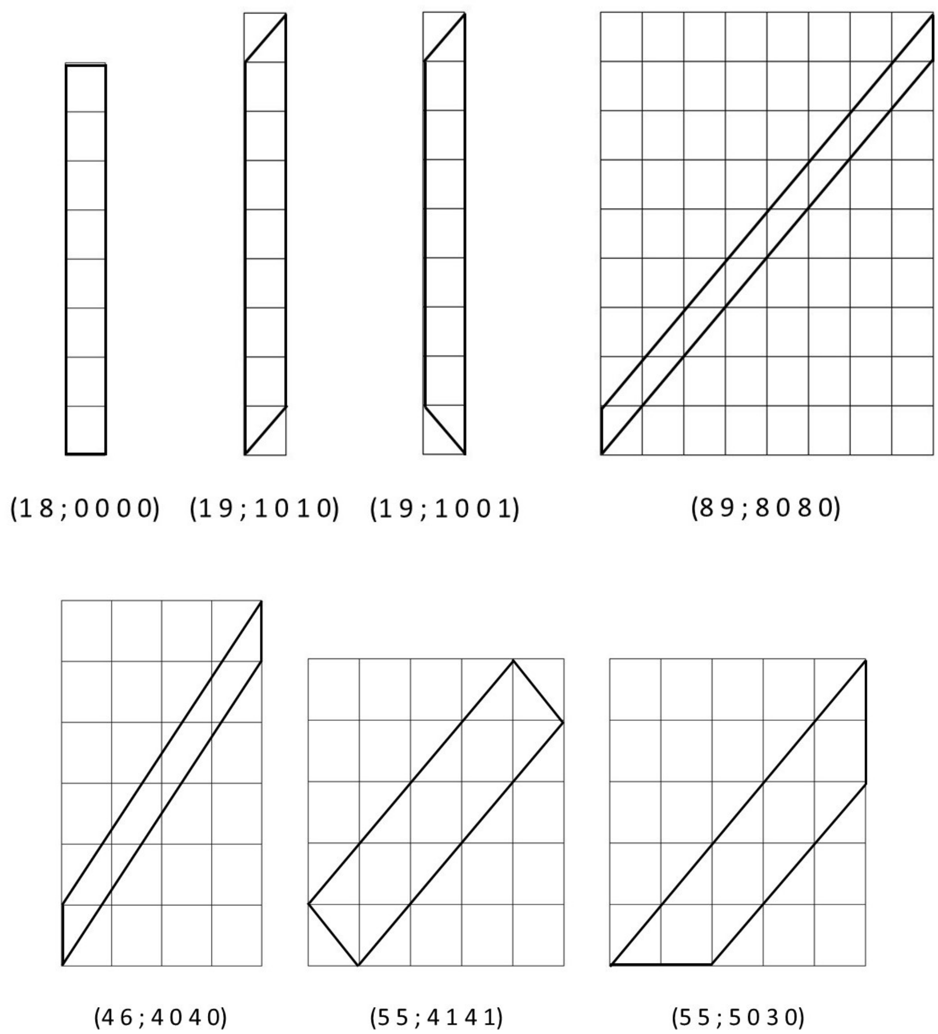

Figure 11: The 7 convex polygons that can not be made with the Tangram pieces.

7 Convex polygons formed with the Japanese Tangram

7.1 Introduction

In the previous sections we discussed the problem how many convex polygons can be formed by the seven pieces of the Chinese tangram. However, there is a similar but maybe less famous set of seven pieces called “Sei Shōganon Chie no Ita”, originated from Japan. Just like the pieces of the Chinese tangram, the pieces of Sei Shōganon Chie no Ita can form a full square, as shown in Fig. 12.

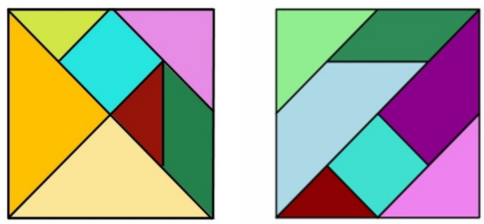

Figure 12: Left: the tangram in full square. Right: Sei Shōganon Chie no Ita in full sqare.

As described in [3], “Sei Shōganon” was a female courtier and famous novelist in the former Japanese empire about a millennium ago. “Chie no Ita” means “wisdom plates”, referring to the physical puzzle. It is said that the puzzle is named after Sei Shōganon’s wisdom. The Sei Shōganon Chie no Ita first appeared in literature in 1742, see [4].
For brevity reasons the “Sei Shōganon Chie no Ita” pieces will be called the Japanese tangram pieces.

7.2 The convex polygons using the Japanese Tangram

As shown above we know that exactly form thirteen convex polygons can be found using the seven pieces of the Chinese tangram. This was demonstrated by first noting that the tangram pieces could be subdivided to form a set of sixteen identical isosceles triangles (the so-called basic triangles) and next by showing that these triangles can form exactly twenty convex polygons. See Fig. 13-Left. Finally, by discarding those polygons that apparently can not be formed by the original Chinese tangram pieces we found thirteen forms, see Fig. 13-Right.
Clearly, we can apply the above idea of subdivision into basic triangles to the Japanese tangram pieces as well and we again can conclude that potentially twenty

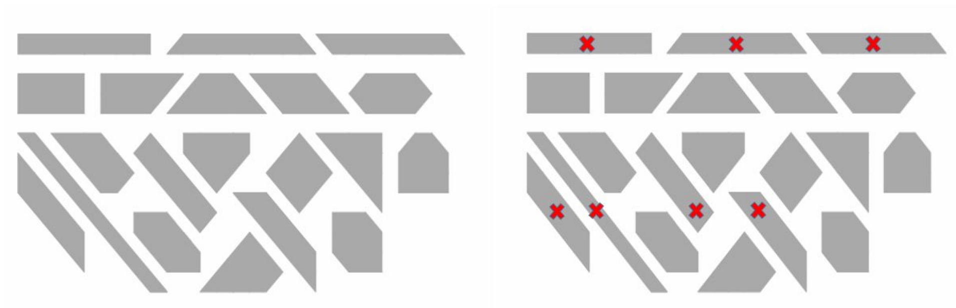

Figure 13: Left: All 20 potential convex polygons. Right: The 13 convex polygons (not marked by a red cross) that can be formed by the Chinese tangram.

convex polygons can be formed with the Japanese tangram pieces.
In fact, it was shown in [3] that exactly sixteen convex polygons exist that can be composed with the Japanese tangram pieces. See Fig. 14.

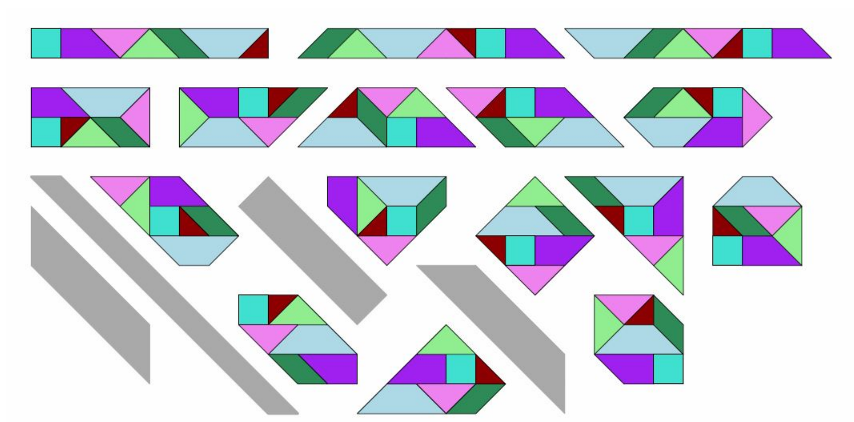

Figure 14: The 16 convex polygons made by the Japanese tangram pieces.

7.3 The optimal seven pieces puzzle

It is clear from the previous section that the Japanese tangram is more expressive than the Chinese tangram. However, The Japanese tangram is not the optimal set of seven pieces for forming as many convex polygons as possible. Indeed, in [3] the authors presented a set of seven pieces that can form nineteen distinct convex polygons. This set consists of 3 parallelograms, 3 triangles and one trapezium. See Fig. 15. Moreover, it was proven by Theorem 3 in [3] that no set of seven pieces can form 20 distinct convex polygons.

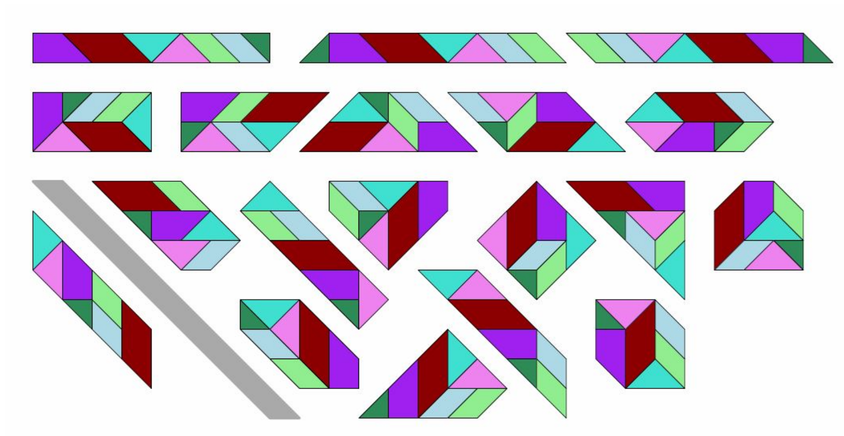

Figure 15: Seven pieces forming nineteen convex polygons.

Finally, we would like to mention Theorem 3 in [3] that deals with the question how many pieces (built up with basic triangles) are needed to form all 20 convex polygons.

Theorem 3 in [**3**] :

Ten or fewer pieces formed from sixteen identical isosceles right triangles cannot form 20 *convex polygons*.

However, *eleven pieces can*.

The last statement in Theorem 3 is visualized in Fig. 16. Notice that the eleven pieces consist of 5 identical parallelograms and 6 basic triangles. However, the basic triangles are not individually shown in Fig. 16.

7.4 Proof of Theorem 3

We can reformulate Theorem 3 with the following three statements:

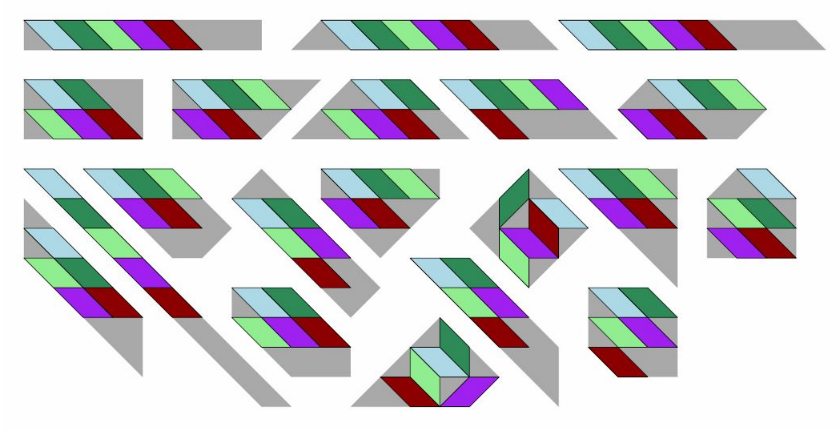

Figure 16: Eleven pieces forming all twenty convex polygons.
The six individual isosceles right triangles are not shown.

Reformulated Theorem 3 :

- Statement 1: Covering 20 convex polygons by any set of 7 pieces is not possible.
- Statement 2: Covering 20 convex polygons by any set of 8, 9 or 10 pieces is not possible.
- Statement 3: Covering 20 convex polygons by a set of 11 pieces is possible.

Now we will prove these statements separately.

7.4.1 Proof of Statement 1

It is clear from Fig. 17 that any set of 7 pieces covering shape (a) must have a piece that consists of at least 3 basic triangles, but such a piece does not fit in shape (b). Thus, we can conclude that no set of 7 pieces can form 20 distinc convex polygons. So, we need a set of at least 8 pieces for covering all 20 possible convex polygons. (Recall that the set of 16 identical basic triangles can cover all these 20 polygons).

End of Proof of Statement 1.

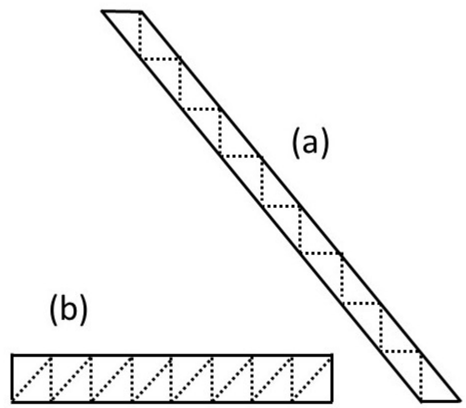

Figure 17: Any set of 7 pieces covering shape (a) must have a piece that consists of at least 3 triangles, which does not fit in shape (b)

.

7.4.2 Proof of Statement 2

Before proving this statement we will first introduce a few notations.
We will denote a basic triangle by B and a parallelogram consisting of 2 basic triangles by P. We will write "B's" as abbrevation for a set of at least two basic triangles. Similarly, for "P's". We will denote a trapezoidal piece consisting of one P and one B by T. Finally, when writing N pieces below, we assume $8 \leq N \leq 10$.
The proof of Statement 2 consists of several steps.

Step 1:
Consider shape (a) in Fig. 17 being a 1 x $8\sqrt{2}$ parallelogram.
We formulate the following

Assumption $\mathscr{A}$: shape (a) in Fig. 17 can be covered by N pieces.

Assuming $\mathscr{A}$ we we need at least 6 P's. (otherwise, if 5 P's were sufficient, then 10 B's of shape (a) can be covered. For covering the remaining part of shape (a) we need 6 B's more, so then we need 11 pieces in total which contradicts the assumption $N \leq 10$).

Conclusion of Step 1:
Assuming $\mathscr{A}$ we we need at least 6 P's.

Step 2.
Consequences of Conclusion Step 1:
Shape (a) can be covered (among others) by 8P, 7P + 2B or 6P + 4 B. See Fig. 18-Left. Clearly, there are no other sets consisting of only P's or a combination of P's and B's. (Of course, the arrangement of the pieces shown is not unique).
Notice that shape (a) in Fig. 17 can also be covered by N pieces being a combination of B's, P's and larger pieces containing at least 3 B's.
For example, shape (a) = T + 6P + B = 2T + 4P + 2B. See Fig. 18-Right.
However, any larger piece that might be used in covering shape (a) contains at least one P and therefore it does not fit in shape (b).

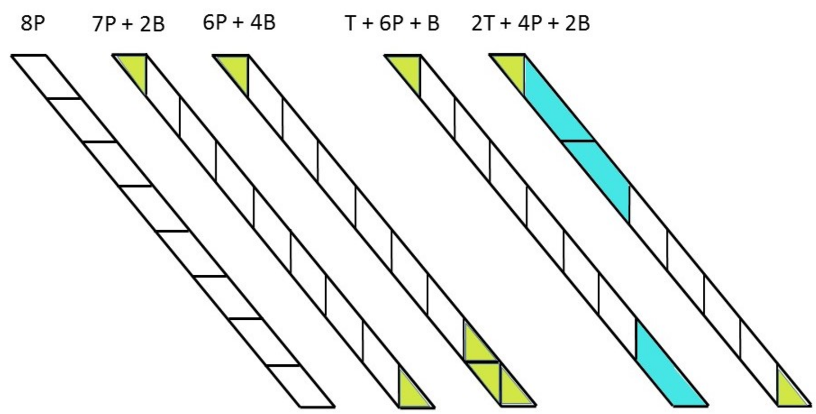

Figure 18: A few examples of pieces covering shape (a).
Left: Three coverings with only P's and B's.
Right: Two coverings where also at least one T is used.

Conclusion of Step 2:
Assuming $\mathscr{A}$, a set of N pieces that covers both shapes (a) and (b) can consist of only 8 P's, 7 P's + 2 B's or 6 P's + 4 B's.

Step 3.
It is easily seen from Fig. 17 (b) that 8 P's can not cover shape (b), but 7 P's + 2 B's or 6 P's + 4 B's can (see Fig. 19).

Conclusion of Step 3:
Assuming $\mathscr{A}$, a set of N pieces that covers both shapes (a) and (b) consists of either

7 P's + 2 B's or 6 P's + 4 B's.

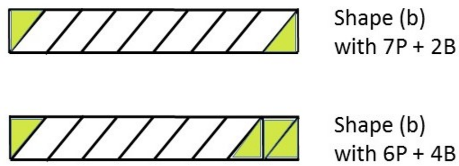

Figure 19: Covering shape (b) by 7P + 2B and 6P + 4B

Consequences of Conclusion Step 3:
If a set of N pieces would exist to cover *all* 20 polygons, then this set contains either 7 P's + 2 B's or 6 P's + 4 B's.

We will discuss these two cases in the Steps 4 and 5.

Step 4.
Consider the set consisting of 6 P's + 4 B's.
We will show that one of the 20 convex polygons can not be formed with these pieces.
To this end, consider the $2\sqrt{2}$-sidelength square $\mathscr{R}$ as shown in Fig. 20.

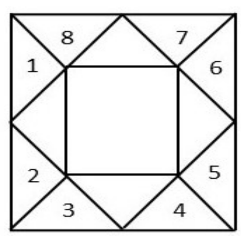

Figure 20: The $2\sqrt{2}$-sidelength square $\mathscr{R}$ with 8 triangles in the corners.

Square $\mathscr{R}$ has 8 triangles in the corners. Since in this case 4 B's are available for covering square $\mathscr{R}$, at most 4 triangles in the corners can be covered by these B's. Thus, at least 4 triangles in the corners have to be covered by P's.

It is easily seen from Fig. 21 that the 12 triangles along the boundary of square $\mathscr{R}$

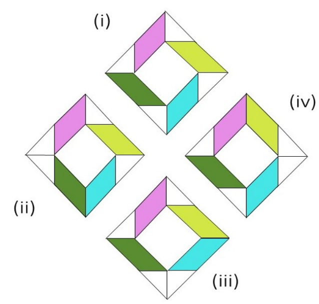

Figure 21: Six parallelograms do not fit in square $\mathscr{R}$.

can only be covered by 4 sets of pairs, each pair having one P and one B. Such a pair is denoted by PB or BP. Without loss of generality we can assume that the most upper piece on the left boundary of square $\mathscr{R}$ is a P.
Along the whole boundary we can have the following 4 different sequences formed by these pairs (by cyclic interchanging P and B in a pair).

(i) PB, PB, PB, PB
(ii) PB, BP, PB, PB
(iii) PB, PB, BP, PB
(iv) PB, PB, PB, BP

These 4 sequences are visualized in the 4 squares in Fig. 21.
Since we have 6 P's in total, the remaining 2 P's must be used for covering the smaller square in the centre of $\mathscr{R}$, but this is not possible.

Conclusion of Step 4:
The $2\sqrt{2}$-sidelength square can not be covered by 6 P's + 2 B's.

Step 5.
Suppose that the complete set of 20 polygons can be formed by 7 P's + 2 B's. Then the complete set can also be formed by 6 P's + 4 B's, by splitting one P into 2 B's. Using this fact and the conclusion from Step 4 we can conclude as follows.

Conclusion of Step 5:
The $2\sqrt{2}$-sidelength square can not be covered by 7 P's + 2 B's.

Final Conclusion of Steps 1-5:
The complete set of 20 polygons can not be formed by any set of 8, 9 or 10 pieces.

End of Proof of Statement 2.

7.4.3 Proof of Statement 3

As was shown earlier in Fig.16, five 1 x $\sqrt{2}$ parallelograms and 6 single triangles can realize all 20 convex polygons.

End of Proof of Statement 3.

Final Conclusion

By proving the Statements 1, 2 and 3 we have proven the reformulated Theorem 3, and so also Theorem 3. □

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