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The data singular and the data isotropic loci for affine cones

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ABSTRACT

The generic number of critical points of the Euclidean distance function from a data point to a variety is called the Euclidean distance degree (or ED degree). The two special loci of the data points where the number of critical points is smaller than the ED degree are called the Euclidean distance *data singular locus* and the Euclidean distance *data isotropic locus*. In this article, we present connections between these two special loci of an affine cone and its dual cone.

1. Introduction

Models in science are often expressed as real solution sets of systems of polynomial equations, namely real algebraic varieties. One of the most fundamental optimization problems that can be formulated on such sets is the following: Given a real algebraic variety and given a general data point of the ambient space, minimize the Euclidean distance from the given data point to the variety.

To solve this problem algebraically, we examine the critical points of the squared Euclidean distance function. The number of such critical points is an important complexity parameter for both numerical and exact algorithms, [5, 13] for finding the optimal solution to the distance minimization problem and is called the *Euclidean Distance Degree (or ED degree).* This optimization problem arises in a wide range of applications, such as low-rank approximations (Example 8), control theory (Example 10), formation control (Example 13), algebraic statistics (Example 14), and multiview geometry (Example 15).

For a general data point u, the number of complex critical points is constant, while the number of real critical points is typically not constant for all general u. For example, if one of the critical points has a multiplicity, then the number of real critical points typically changes. This locus is called the *ED-discriminant* (or classically *focal locus*) [2, 3, 6, 9, 14].

In this article, we want to discuss the locus (different from the ED discriminant) of exceptional data points u for which the number of complex critical points is smaller than the ED degree. We consider three cases in which we can have a different number of critical points than expected. The first reason is because a critical point may wander off into the singular locus of the variety. The study of this special locus was proposed by Bernd Sturmfels, first examples were developed [4] and it was named *ED data singular locus.* In a similar fashion, the second case is when a critical point becomes isotropic with respect to the Euclidean inner product (i.e., it has norm zero); this locus will be called *ED data isotropic locus.* In these two cases, the number of critical points is smaller than the ED degree. Finally, a data point can have infinitely many critical points, but this phenomenon is apparently recorded by the ED discriminant, so we do not deal with it in this article. A classical example would be that there are infinitely many critical rank 2 approximations of a matrix with two identical singular values.

In this article, we aim to describe the data singular and the data isotropic loci of affine cones.

2. The special Loci of data points

To find the critical points algebraically, we consider X to be a variety in $\mathbb{C}^n$ and we examine all complex critical points of the complexified distance, induced by the standard symmetric bilinear form,

$$d_u(x) = (u - x | u - x) = \sum_{i=1}^{n} (u_i - x_i)^2,$$

with $x \in X^{\text{reg}}$, where X^{reg} denotes the locus of *regular points* of X, so we only allow those critical points that are nonsingular. Since the ED degree is additive over the components of a variety from now on, we assume that $X \subseteq \mathbb{C}^n$ is an irreducible algebraic variety of codimension c with defining radical ideal I. If $x \in X^{\text{reg}}$ is a critical point of d_u, then the following holds: $u - x \perp T_x X$. This latter condition can be formulated as $x \in X^{\text{reg}}$ is a critical point of d_u if and only if all the $(c+1) \times (c+1)$ minors of the matrix:

$$\begin{pmatrix} u - x \\ \text{Jac}_x(I) \end{pmatrix}$$

vanish, where $\text{Jac}_x(I)$ is the Jacobian of I at the point x.

We define the *ED-correspondence* to be the closure of the set of all pairs (u, x), such that $x \in X^{\text{reg}}$ is critical to d_u, and we denote it by $\mathcal{E}_X \subseteq \mathbb{C}^n_u \times \mathbb{C}^n_x$. In other words, $\mathcal{E}_X$ is the closure of:

$$\left\{ (u, x) \big| u \in \mathbb{C}^n, \ x \in X^{\text{reg}}, \ \text{rank} \begin{pmatrix} u - x \\ \text{Jac}_x(I) \end{pmatrix} \leq c \right\}.$$

We have two natural projection maps $\pi_1 : \mathcal{E}_X \to \mathbb{C}^n_u$ sending (u, x) to u and $\pi_2 : \mathcal{E}_X \to \mathbb{C}^n_x$ sending (u, x) to x. Let $\text{Sing}X$ denotes the *singular locus* of X, that is, the set of all points of $x \in X$ such that all the $c \times c$ minors of $\text{Jac}_x(I)$ vanish.

So for a given data point u, the cardinality of the fiber of π_1 over u, $\pi_1^{-1}(u)$, measures the number of critical points.

We want to discuss the locus of exceptional data points u at which the number of complex critical points is different from the ED degree. As mentioned in the introduction, we consider three cases in which we can have different number of critical points than expected. The first one is because a critical point may wander off into $\text{Sing}X$ because of the closure appearing in the definition of $\mathcal{E}_X$. This locus is called the *ED data singular locus*.

2.1. Data singular locus

We use the precise definition of the ED data singular locus [4], that is, the Zariski closure of the set:

$$\pi_1(\mathcal{E}_X \cap \pi_2^{-1}(\text{Sing}X)).$$

We denote the ED data singular locus of an algebraic variety X by $\text{DS}(X)$ (abbreviating "data singular" locus) and we aim to describe the data singular locus of affine cones. We define X^* the *dual variety to X* to be the Zariski closure of the set:

$$\left\{ y \in \mathbb{C}^n \mid \exists x \in X^{\text{reg}} : y \perp T_x X \right\}.$$

More precisely, we view X^* as subset of $\mathbb{C}^n$ through the standard symmetric bilinear form on $\mathbb{C}^n$. Our main result in this section is the following theorem.

Theorem 1. *Let $X \subseteq \mathbb{C}^n$ be an irreducible affine cone that is not a linear space. Then the following two inclusions hold:*

$$X^* \subseteq_{(1)} \text{DS}(X) \subseteq_{(2)} X^* + \text{Sing}X.$$

Proof. First, we prove inclusion (1) for a dense subset of X^*. For this, take $u \in X^*$, such that there exists a regular point $x_r \in X^{\text{reg}}$, such that $u \perp T_{x_r} X$, that is, all the $(c+1) \times (c+1)$ minors of $\begin{pmatrix} u \\ \text{Jac}_{x_r}(I) \end{pmatrix}$ vanish,

where c is the codimension of X and $\text{Jac}_{x_r}(I)$ is the Jacobian of the (radical) ideal I of X at the point x_r. We denote an arbitrary $(c+1) \times (c+1)$ minor of this matrix by $\left(\begin{smallmatrix} u \\ \text{Jac}_{x_r}(I) \end{smallmatrix} \right)_{(c+1)}$.

We claim that $(u + \lambda x_r, \lambda x_r) \in \mathcal{E}_X$ for all real $\lambda \geq 0$. We have that if $f \in I$ is homogeneous of degree d, then $\nabla f(\lambda x) = \lambda^d \nabla f(x)$. So if x_r is a regular point, then λx_r is also regular, for any $\lambda > 0$. Moreover, we get that for any $(c+1) \times (c+1)$ minor:

$$\begin{pmatrix} (u + \lambda x_r) - \lambda x_r \\ \text{Jac}_{\lambda x_r}(I) \end{pmatrix}_{(c+1)} = \begin{pmatrix} u \\ \text{Jac}_{\lambda x_r}(I) \end{pmatrix}_{(c+1)} = \lambda^N \begin{pmatrix} u \\ \text{Jac}_{x_r}(I) \end{pmatrix}_{(c+1)} = 0,$$

where N is the sum of degrees of the defining polynomials of I, which appear in the particular $(c+1) \times (c+1)$ minor.

So $(u + \lambda x_r, \lambda x_r) \in \mathcal{E}_X$ for all real $\lambda > 0$. But then taking the limit when λ goes to zero, we get that

$$(u, 0) \in \mathcal{E}_X \cap \pi_2^{-1}(\text{Sing}X),$$

since $\mathcal{E}_X \cap \pi_2^{-1}(\text{Sing}X)$ is Zariski closed (hence closed wrt. Euclidean topology as well) and since $0 \in \text{Sing}X$. Indeed, for every $x \in X$, the line $\{\lambda \cdot x\}$ is in the tangent space to 0, so $T_0 X$ is equal to the linear span of X, which has a greater dimension than X if and only if X is not a linear space, and hence $0 \in \text{Sing}X$. So then $u = \pi_1((u, 0)) \in \text{DS}(X)$.

For the proof of (2), take an element $(u, x_0) \in \mathcal{E}_X \cap \pi_2^{-1}(\text{Sing}X)$. Then, this point can be approximated by a sequence in the part of $\mathcal{E}_X$ over X^{reg}. That is, there exists a sequence $\delta_i \to 0$ in $\mathbb{C}^n$ and $x_i \to x_0$ with all the $x_i \in X^{\text{reg}}$, such that

$$(u + \delta_i, x_i) \in \mathcal{E}_X.$$

By the ED Duality Theorem for affine cones [3, Theorem 5.2], we get that $(u + \delta_i) - x_i \in X^*$, for all i. Now taking the limit, when i goes to infinity, we get that $u - x_0 \in X^*$, since X^* is closed (hence closed wrt. Euclidean topology as well). Finally, this means that $u \in x_0 + X^* \subseteq \text{Sing}X + X^*$. $\qquad \square$

Note that the condition in the theorem that X is not a linear space is necessary to prove the theorem. Otherwise, if X is a linear subspace of $\mathbb{C}^n$, then it has a nonempty dual (its orthogonal complement with respect to the inner product), but its singular locus is empty, hence its data singular locus is empty as well.

2.2. Data isotropic locus

A second possibility for a data point u to have smaller number of critical points than expected is by letting one of the critical points become isotropic. Let us denote by $Q = \{x \in \mathbb{C}^n : \sum_{i=1}^{n} x_i^2 = 0\}$ the isotropic quadric with respect to the standard symmetric bilinear form. Draisma et al. define the ED degree of a projective variety in $\mathbb{P}^{n-1}$ to be the ED degree of the corresponding affine cone in $\mathbb{C}^n$ [3]. Moreover, given a data point u, the critical points to these two objects are in a one-to-one correspondence, given that none of the critical points lies in the isotropic quadric [3, Lemma 2.8]. In particular, the role of Q shows that the computation of ED degree is a metric problem. This is the reason that even though in the definition of the affine $\mathcal{E}_X$ we keep the isotropic critical points, when we pass to projective varieties we will exclude the isotropic points. This way the data isotropic locus represents the locus of data points which have a different number of critical points when X is considered as an affine cone compared to when X is considered as a projective variety. More precisely, we define the *ED data isotropic locus* to be the Zariski closure of the set:

$$\pi_1(\mathcal{E}_X \cap \pi_2^{-1}(Q \cap X)).$$

We denote the ED data isotropic locus of an algebraic variety X by $\text{DI}(X)$ (abbreviating "data isotropic" locus). We have the following theorem for the ED data isotropic locus of affine cones.

Theorem 2. *Let $X \subseteq \mathbb{C}^n$ be an irreducible affine cone. Then the following two inclusions hold:*

$$X^* \subseteq_{(1)} \text{DI}(X) \subseteq_{(2)} X^* + (Q \cap X),$$

where X^* denotes the dual variety to X.

Proof. The proof follows the lines of the proof of Theorem 1, keeping in mind that $0 \in X$ is always an isotropic point. □

In the following two sections, we will give examples to show that both inclusions appearing in Theorems 1 and 2 can be strict and/or equalities.

3. Examples of the ED data singular locus

In this section, we present several useful examples concerning the ED data singular locus of an affine cone. Before we get to the examples, we present how one can computationally determine the objects we are working with. We illustrate the main algorithms with code in Macaulay2 [7]. For an affine cone $X \subseteq \mathbb{C}^n$, of codimension c with defining radical ideal I, one can determine its dual X^* using the following code [12, Algorithm 5.1].

Example 3 (Computing the dual variety). We present the algorithm for the real affine cone $X \subseteq \mathbb{C}^3$ defined by the homogeneous equation $f = x_1^3 + x_2^2 x_3$.

```
n=3;
kk=QQ[x_1..x_n,y_1..y_n];
f=x_1^3+x_2^2*x_3;
I=ideal(f);
c=codim I;
Y=matrix{{y_1..y_n}};
Jac= jacobian gens I;
S=submatrix(Jac,{0..n-1},{0..numgens(I)-1});
Jbar=S|transpose(Y);
EX = I + minors(c+1,Jbar);
SingX=I+minors(c,Jac);
EXreg=saturate(EX,SingX);
IDual=eliminate(toList(x_1..x_n),EXreg)
```

The output reveals that X^* is the zero locus of the polynomial $f^* = 4x_1^3 - 27x_2^2 x_3$.

Following the definition of the data singular locus, the next example contains an algorithm for calculating the ideal of it.

Example 4 (Computing the data singular locus). We present the algorithm for the real affine cone $X \subseteq \mathbb{C}^3$ defined by the homogeneous equation $f = x_1^3 + x_2^2 x_3$.

```
n=3;
kk=QQ[x_1..x_n,y_1..y_n];
f=x_1^3+x_2^2*x_3;
I=ideal(f);
c=codim I;
Y=matrix{{x_1..x_n}}-matrix{{y_1..y_n}};
Jac= jacobian gens I;
S=submatrix(Jac,{0..n-1},{0..numgens(I)-1});
Jbar=S|transpose(Y);
EX = I + minors(c+1,Jbar);
SingX=I+minors(c,Jac);
EXreg=saturate(EX,SingX);
DSX=radical eliminate(toList(x_1..x_n),EXreg+SingX)
```

From the output, we see that the data singular locus is the zero set of the polynomial $x_1(4x_1^3 - 27x_2^2 x_3)$.

Now, we arrived at the point to present a sequence of interesting varieties and the corresponding duals and data singular loci. The first example is the one we used for presenting the algorithms previously. In this example, both inclusions (1) and (2) are strict, as it will be seen.

Example 5 (Cuspidal cubic cone). Let $X \subseteq \mathbb{C}^3$ be the real variety defined by the homogeneous equation $f = x_1^3 + x_2^2 x_3$. Since it is an affine cone, it has a dual X^*, which is defined by the dual equation $f^* = 4x_1^3 - 27x_2^2 x_3$. For the data singular locus, we get that $DS(X)$ is the zero locus of the polynomial $x_1(4x_1^3 - 27x_2^2 x_3)$. So we can see that X^* is even a component of $DS(X)$. Moreover, $X^* + \mathrm{Sing}X$ is something much larger and not equal to $DS(X)$. For example, the point:

$$(3, 2, 1) + (0, 0, 1) \in X^* + \mathrm{Sing}X,$$

but is not on $DS(X)$. Figure 1 shows X in blue and X^* in green and $DS(X)$ is the union of the green-colored X^* and the additional surface in red.

The next example shows that both inclusions (1) and (2) can in fact be equalities. More generally, we have the following corollary to Theorem 1.

Corollary 6. *Let $X \subseteq \mathbb{C}^n$ be an affine cone, with $\mathrm{Sing}X = \{0\}$, then $DS(X) = X^*$. Moreover, if X is a general hypersurface of degree d, then*

$$\deg(DS(X)) = d(d-1)^{n-1}.$$

Proof. The first part follows directly from the claim of Theorem 1. The "moreover" part is classical [12, Exercise 5.14]. □

Example 7 (Cone over ellipse). Let $X \subseteq \mathbb{C}^3$ the cone over an ellipse, defined by the homogeneous equation $f = x_1^2 + 4x_2^2 - 9x_3^2$. The singular locus $\mathrm{Sing}X$ only contains 0, so as a consequence of Theorem 1,
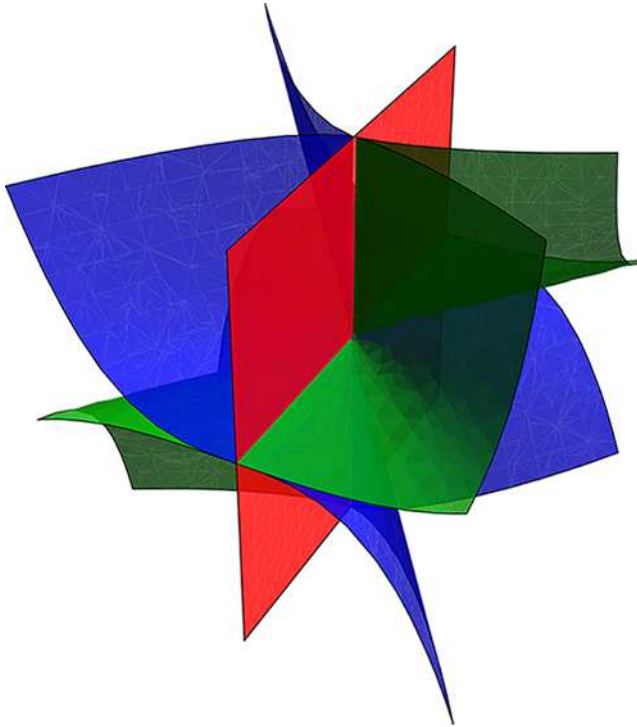

Figure 1. $V(x_1^3 + x_2^2 x_3)$ together with its dual and its data singular locus.
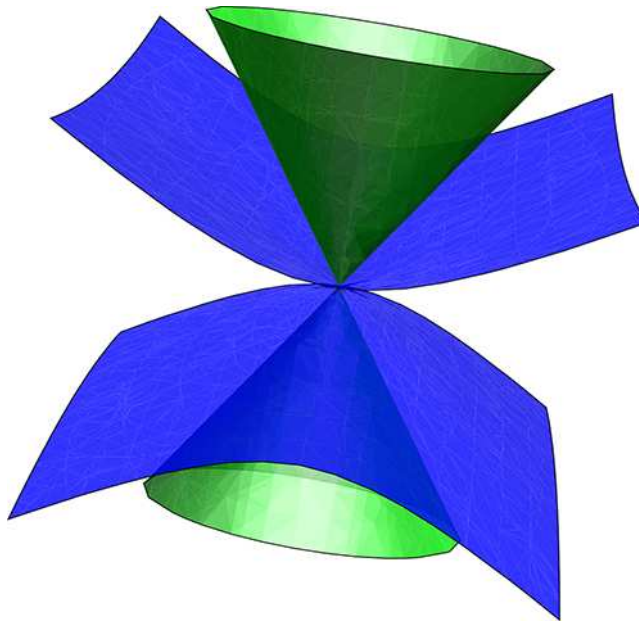
Figure 2. $V(x_1^2 + 4x_2^2 - 9x_3^2)$ together with its dual.

we have that $DS(X)$ equals the dual variety X^*, defined by the dual equation $f^* = x_1^2 + x_2^2/4 - x_3^2/9$. Figure 2 shows X in blue and X^* in green.

The next example concerned the well-known and much used *determinantal varieties*. We will see that for this variety, inclusion (1) is strict and inclusion (2) is an equality.

Example 8 (Determinantal varieties). Denote by $M_{n\times m}^{\leq r}$ the variety of $n \times m$ matrices (suppose $n \leq m$) of rank at most r. It is classical that the singular locus is the variety $M_{n\times m}^{\leq r-1}$. We have that the dual variety is exactly $M_{n\times m}^{\leq n-r}$ [6, Chapter 1, Proposition 4.11]. So applying Theorem 1, we get that

$$M_{n\times m}^{\leq n-r} \subseteq DS(M_{n\times m}^{\leq r}) \subseteq M_{n\times m}^{\leq n-r} + M_{n\times m}^{\leq r-1} = M_{n\times m}^{\leq n-1}.$$

So for rank-one matrices ($r = 1$), we get that $DS(M_{n\times m}^{\leq 1}) = M_{n\times m}^{\leq n-1}$, which is not a surprise based on Corollary 6, since $M_{n\times m}^{\leq 1}$ is smooth, except at 0. But something more is true for general r. We claim that the upper bound for the inclusions is always attained. For this, we have the following proposition.

Proposition 9. *The ED data singular locus of the determinantal variety $M_{n\times m}^{\leq r}$ is equal to $M_{n\times m}^{\leq n-1}$, for all $1 \leq r \leq n - 1$.*

Proof. An $n \times m$ matrix U lies in the interior of $DS(M_{n\times m}^{\leq r})$ if and only if it has a singular critical point. All the critical points of U look like

$$T_1 \cdot \text{Diag}(0, 0, \ldots, \sigma_{i_1}, 0, \ldots, 0, \sigma_{i_r}, 0, \ldots, 0) \cdot T_2,$$

where the singular value decomposition of U is equal to $U = T_1 \cdot \text{Diag}(\sigma_1, \ldots, \sigma_n) \cdot T_2$, with $\sigma_1 \geq \cdots \geq \sigma_n$ singular values and T_1, T_2 orthogonal matrices of size $n \times n$ and $m \times m$ [3, Example 2.3]. Such a critical point is singular if and only if it has rank at most $r - 1$, which can only happen if one of the singular values $\sigma_{i_1}, \ldots, \sigma_{i_r}$ is zero. So there exists a singular critical point to U if and only if there is a zero singular value of U, which can only happen if U has a rank defect. Hence all the $(n-1) \times (n-1)$ minors are zero, that is, $U \in M_{n\times m}^{\leq n-1}$. Now since $M_{n\times m}^{\leq n-1}$ is Zariski closed, we have the desired equality. $\square$

The next example shows that X^* is a subvariety of DS(X) but not necessarily a component of it.

Example 10 (Hurwitz determinant). In control theory, to check whether a given polynomial is stable, one builds up the so-called Hurwitz matrix H_n and checks if every leading principal minor of H_n is positive. Take $n = 4$, then the 4th Hurwitz matrix looks like:

$$H_4 = \begin{pmatrix} x_2 & x_4 & 0 & 0 \\ x_1 & x_3 & x_5 & 0 \\ 0 & x_2 & x_4 & 0 \\ 0 & x_1 & x_3 & x_5 \end{pmatrix}.$$

The ratio $\Gamma_4 = \det(H_4)/x_5$ is a homogeneous polynomial and is called the *Hurwitz determinant* for $n = 4$ [3, Example 3.5].

Let $X \subseteq \mathbb{C}^5$ be the affine cone defined by Γ_4. Then its dual variety has one irreducible component given by:

$$X^* = V(-x_3x_4 + x_2x_5, -x_3^2 + x_1x_5, -x_2x_3 + x_1x_4),$$

while its data singular locus DS(X) has two irreducible components and it is defined by:

$$V((x_1x_2^2 + x_2x_3x_4 + x_4^2x_5)(x_2^4x_3 - x_1x_2^3x_4 - 2x_1x_2x_4^3 - x_3x_4^4 + 2x_2^3x_4x_5 + x_2x_4^3x_5)).$$

Clearly, X^* is not a component of DS(X). Moreover, DS(X) is not equal to $X^* + $SingX, since SingX $= V(x_2, x_4)$ and the point:

$$(2, 1, 1, 0, 1) = (1, 1, 0, 0, 0) + (1, 0, 1, 0, 1)$$

lies on $X^* + $SingX but it is not on DS(X).

We have thus seen examples of varieties with: both inclusions in Theorem 1 being strict, both inclusions in Theorem 1 being equalities and the second inclusion being an equality, while the first one is strict. It is natural to ask if there are examples where the first inclusion is an equality, while the second one is strict. The author could not find such an example, so the following question arises.

Problem 11. *Find an affine cone X, such that $X^* = $ DS $\subset X^* + $ Sing(X) or prove that there is no such X.*

4. Examples of the ED data isotropic locus

In this section, we present several application-oriented examples concerning the ED data isotropic locus of an affine cone. We begin with presenting how can one computationally determine the data isotropic locus of a variety.

Example 12 (Computing the data isotropic locus). We present the algorithm for the affine cone defined by $f = x_1x_6 - x_2x_5 + x_3x_4$, representing the Grassmannian of planes in 4-space.

```
n=6;
kk=QQ[x_1..x_n,y_1..y_n];
f=x_1*x_6-x_2*x_5+x_3*x_4;
I=ideal(f);
c=codim I;
Y=matrix{{x_1..x_n}}-matrix{{y_1..y_n}};
Jac= jacobian gens I;
S=submatrix(Jac,{0..n-1},{0..numgens(I)-1});
Jbar=S|transpose(Y);
EX = I + minors(c+1,Jbar);
SingX=I+minors(c,Jac);
q=sum for i from 1 to n list x_i^2;
Q=ideal(q);
EXreg=saturate(EX,SingX);
DIX=radical eliminate(toList(x_1..x_n),EXreg+Q)
```

From the output, we learn that DI(X) is the zero locus of the polynomial $x_1x_6 - x_2x_5 + x_3x_4$, so we get that the data isotropic locus is equal to the dual variety which in this case equals the variety.

The next example shows that the data isotropic locus can be equal to the dual and strictly contained in $X^* + (X \cap Q)$.

Example 13 (Cayley–Menger variety). Let X denote the variety in $\mathbb{C}^3$ with parametric representation:

$$\begin{cases} x_1 = (z_1 - z_2)^2, \\ x_2 = (z_1 - z_3)^2, \\ x_3 = (z_2 - z_3)^2. \end{cases}$$

Based on [1] and on [3, Example 3.7], the points in X record the squared distances among three interacting agents with coordinates z_1, z_2, and z_3 on the line $\mathbb{R}$. The prime ideal of X is given by the determinant of the *Cayley–Menger matrix*

$$\begin{pmatrix} 2x_2 & x_2 + x_3 - x_1 \\ x_2 + x_3 - x_1 & 2x_3 \end{pmatrix}$$

So X is defined by the irreducible polynomial:

$$f = x_1^2 - 2x_1x_2 + x_2^2 - 2x_1x_3 - 2x_2x_3 + x_3^2.$$

After running the computations, one can see that the data isotropic locus equals the dual variety, which is defined by $f^* = x_1x_2 + x_1x_3 + x_2x_3$ (Figure 3). And it does not equal $X^* + (Q \cap X)$, for example, because the point $(1, 0, 0) + (0, 1, i) \in X^* + (Q \cap X)$, but it does not lie on DI(X).

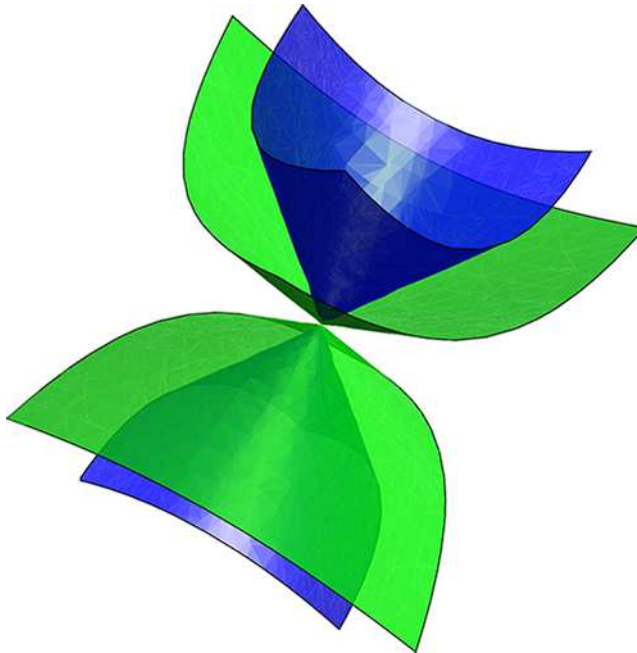The next example shows that both inclusions from Theorem 2 can be strict.

Figure 3. Cayley–Menger variety (in blue) together with its dual (in green).

Example 14 (Cayley's cubic). Let X be defined by $f = x_1^3 - x_1 x_2^2 - x_1 x_3^2 + 2x_2 x_3 x_4 - x_1 x_4^2$, the 3×3 symmetric determinant in $\mathbb{C}^4$. This hypersurface is sometimes called the *Cayley's cubic surface* and receives much attention in the study of elliptopes and exponential varieties in algebraic statistics ([12, Example 5.44], [11, Example 1.1], [10]). Its dual variety is the *quartic Steiner surface* defined by $f^* = x_2^2 x_3^2 - 2x_1 x_2 x_3 x_4 + x_2^2 x_4^2 + x_3^2 x_4^2$. After running the computations, one finds that the data isotropic locus is the union:

$$\mathrm{DI}(X) = V(x_1^{18} + 4x_1^{16} x_2^2 + 6x_1^{14} x_2^4 - \cdots + 729 x_3^4 x_4^{14}) \cup X^*.$$

So it is clearly not equal to the dual variety. And it is not equal to $X^* + (Q \cap X)$ either, because, for example, the point:

$$(1, 1, 0, 0) + (0, 0, 1, i) \in X^* + (Q \cap X)$$

but it is not in $\mathrm{DI}(X)$

Our next example shows that the second inclusion in Theorem 2 can be equality and moreover, it can give the whole space.

Example 15 (Special essential variety). Essential matrices play an important role in *multiview geometry* [8]. The connections between the ED degree theory and multiview geometry were investigated [3, Example 3.3]. The set of essential matrices is called the *essential variety* and is defined as follows:

$$\mathcal{E} = \{X \in M_{3\times3} | \det X = 0, \ 2XX^T X - \mathrm{trace}(XX^T)X = 0\}.$$

It is a codimension 3 variety of degree 10. We are interested in the data isotropic locus of this variety, but for computational reasons, we will take a linear section of it and we will only consider the symmetric, constant diagonal essential matrices, which we will call the special essential variety and will denote by $\mathcal{SE}$. More precisely, we define $\mathcal{SE}$ to be:

$$\left\{ X = \begin{pmatrix} x_1 & x_2 & x_3 \\ x_2 & x_1 & x_4 \\ x_3 & x_4 & x_1 \end{pmatrix} \middle| \det X = 0, \ 2XX^T X - \mathrm{trace}(XX^T)X = 0 \right\}.$$

Since this variety is not irreducible, we will perform our computations, componentwise. When running the computations, one will find that the data isotropic locus is the whole space. Indeed, one can observe that $\mathcal{SE}$ is inside the isotropic quadric Q, so every critical point is isotropic. We have that

$$\mathrm{DI}(X) = X^* + (X \cap Q) = X^* + X = \mathbb{C}^4.$$

Moreover, $\mathrm{DI}(X)$ is not equal to the dual variety, since X^* is a proper variety defined by $f^* = (x_2^2 + x_4^2)(x_2^2 + x_3^2)(x_3^2 + x_4^2)$. Moreover, clearly, the dual is not a component of $\mathrm{DI}(X)$.

In the last example, the reader can see that both inclusions from Theorem 2 can be equalities.

Example 16 (Line through the origin). In what follows let X be the line through the origin in $\mathbb{C}^3$ defined by the vanishing of the polynomials $x_1 + 2x_2 + 3x_3$ and $4x_1 + 5x_2 + 6x_3$. Then, we get that X intersects the quadric Q only in the point 0, so by Theorem 2, we immediately get that $X^* = \mathrm{DI}(X) = X^* + \{0\}$, and the dual is the orthogonal complement of X, so it is defined by $x_1 - 2x_2 + x_3$.

Acknowledgments

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