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Citation for published version (APA):

Khoa, V. A., & Muntean, A. (2015). *Asymptotic analysis of a semi-linear elliptic system in perforated domains: well-posedness and correctors for the homogenization limit*. (CASA-report; Vol. 1535). Technische Universiteit Eindhoven.

Document status and date:

Published: 01/12/2015

Document Version:

Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

Please check the document version of this publication:

- A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
- The final author version and the galley proof are versions of the publication after peer review.
- The final published version features the final layout of the paper including the volume, issue and page numbers.

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EINDHOVEN UNIVERSITY OF TECHNOLOGY
Department of Mathematics and Computer Science

CASA-Report 15-35
December 2015

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by

V.A. Khoa, A. Muntean



Centre for Analysis, Scientific computing and Applications
Department of Mathematics and Computer Science
Eindhoven University of Technology
P.O. Box 513
5600 MB Eindhoven, The Netherlands
ISSN: 0926-4507

Asymptotic analysis of a semi-linear elliptic system in perforated domains: well-posedness and correctors for the homogenization limit

Vo Anh Khoa^{a,*}, Adrian Muntean^b

^a*Mathematics and Computer Science Division, Gran Sasso Science Institute, L'Aquila, Italy*

^b*Department of Mathematics and Computer Science, Karlstad University, Sweden*

Abstract

In this study, we prove results on the weak solvability and homogenization of a microscopic semi-linear elliptic system posed in perforated media. The model presented here explores the interplay between stationary diffusion and both surface and volume chemical reactions in porous media. Our interest lies in deriving homogenization limits (upscaling) for alike systems and particularly in justifying rigorously the obtained averaged descriptions. Essentially, we prove the well-posedness of the microscopic problem ensuring also the positivity and boundedness of the involved concentrations and then use the structure of the two scale expansions to derive corrector estimates delimitating this way the convergence rate of the asymptotic approximates to the macroscopic limit concentrations. Our techniques include Moser-like iteration techniques, a variational formulation, two-scale asymptotic expansions as well as energy-like estimates.

Keywords: Corrector estimates, Homogenization, Elliptic systems, Perforated domains

2010 MSC: 35B27, 35C20, 76M30, 35B09

1. Introduction

We study the semi-linear elliptic boundary-value problem of the form

$$(P^\varepsilon) : \begin{cases} \mathcal{A}^\varepsilon u_i^\varepsilon \equiv \nabla \cdot (-d_i^\varepsilon \nabla u_i^\varepsilon) = R_i(u^\varepsilon), & \text{in } \Omega^\varepsilon \subset \mathbb{R}^d, \\ d_i^\varepsilon \nabla u_i^\varepsilon \cdot \mathbf{n} = \varepsilon (a_i^\varepsilon u_i^\varepsilon - b_i^\varepsilon F_i(u_i^\varepsilon)), & \text{on } \Gamma^\varepsilon, \\ u_i^\varepsilon = 0, & \text{on } \Gamma^{ext}, \end{cases} \quad (1.1)$$

for $i \in \{1, \dots, N\}$ ($N \geq 2, d \in \{2, 3\}$). Following [1], this system models the diffusion in a porous medium as well as the aggregation, dissociation and surface deposition of N interacting populations of colloidal particles indexed by u_i^ε . As short-hand notation, $u^\varepsilon := (u_1^\varepsilon, \dots, u_N^\varepsilon)$ points out the vector of these concentrations. Such scenarios arise in drug-delivery mechanisms in human bodies and often includes cross- and thermo-diffusion which are triggers of our motivation (compare [2] for the Sorret and Dufour effects and [3, 4] for related cross-diffusion and chemotaxis-like systems).

The model (1.1) involves a number of parameters: d_i^ε represents molecular diffusion coefficients, R_i represents the volume reaction rate, $a_i^\varepsilon, b_i^\varepsilon$ are the so-called deposition coefficients, while F_i indicates a surface chemical reaction for the immobile species. We refer to (1.1) as problem (P^ε) .

The main purpose of this paper is to obtain corrector estimates that delimitate the error made when homogenizing (averaging, upscaling, coarse graining...) the problem (P^ε) , i.e. we want to estimate the speed of convergence as $\varepsilon \rightarrow 0$ of suitable norms of differences in micro-macro concentrations and micro-macro concentration gradients. This way we

*Corresponding author

Email addresses: khoa.vo@gssi.infn.it, vakhoha.hcmus@gmail.com (Vo Anh Khoa), adrian.muntean@kau.se (Adrian Muntean)

justify rigorously the upscaled models derived in [1] and prepare the playground to obtain corrector estimates for the thermo-diffusion scenario discussed in [5]. From the corrector estimates perspective, the major mathematical difficulty we meet here is the presence of the nonlinear surface reaction term. To quantify its contribution to the corrector terms we use an energy-like approach very much inspired by [6]. The main result of the paper is Theorem 10 where we state the corrector estimate. It is worth noting that this work goes along the line open by our works [7] (correctors via periodic unfolding) and [8] (correctors by special test functions adapted to the local periodicity of the microstructures). An alternative strategy to derive correctors for our scenario could in principle exclusively rely on periodic unfolding, refolding and defect operators approach if the boundary conditions along the microstructure would be of homogeneous Neumann type; compare [9] and [10].

The corrector estimates obtained with this framework can be further used to design convergent multiscale finite element methods for the studied PDE system (see e.g. [11] for the basic idea of the MsFEM approach and [12] for an application to perforated media).

The paper is organized as follows: In Section 2 we start off with a set of technical preliminaries focusing especially on the working assumptions on the data and the description of the microstructure of the porous medium. The weak solvability of the microscopic model is established in Section 3. The homogenization method is applied in Section 4 to the problem (P^ε) . This is the place where we derive the corrector estimates and establish herewith the convergence rate of the homogenization process. A brief discussion (compare Section 5) closes the paper.

2. Preliminaries

2.1. Description of the geometry

The geometry of our porous medium is sketched in Figure 2.1 (left), together with the choice of perforation (referred here to also as "microstructure") cf. Figure 2.1 (right). We refer the reader to [13] for a concise mathematical representation of the perforated geometry. In the same spirit, take Ω be a bounded open domain in \mathbb{R}^d with a piecewise smooth boundary $\Gamma = \partial\Omega$. Let Y be the unit representative cell, i.e.

$$Y := \left\{ \sum_{i=1}^d \lambda_i \vec{e}_i : 0 < \lambda_i < 1 \right\},$$

where we denote by \vec{e}_i by i th unit vector in \mathbb{R}^d .

Take Y_0 the open subset of Y with a piecewise smooth boundary ∂Y_0 in such a way that $\overline{Y_0} \subset \overline{Y}$. In the porous media terminology, Y is the unit cell made of two parts: the gas phase (pore space) $Y \setminus \overline{Y_0}$ and the solid phase Y_0 .

Let $Z \subset \mathbb{R}^d$ be a hypercube. Then for $X \subset Z$ we denote by X^k the shifted subset

$$X^k := X + \sum_{i=1}^d k_i \vec{e}_i,$$

where $k = (k_1, \dots, k_d) \in \mathbb{Z}^d$ is a vector of indices.

Setting $Y_1 = Y \setminus \overline{Y_0}$, we now define the pore skeleton by

$$\Omega_0^\varepsilon := \bigcup_{k \in \mathbb{Z}^d} \{ \varepsilon Y_0^k : Y_0^k \subset \Omega \},$$

where ε is observed as a given scale factor or homogenization parameter.

It thus comes out that the total pore space is

$$\Omega^\varepsilon := \Omega \setminus \overline{\Omega_0^\varepsilon},$$

for εY_0^k the ε -homotetic set of Y_0^k , while the total pore surface of the skeleton is denoted by

$$\Gamma^\varepsilon := \partial\Omega_0^\varepsilon = \bigcup_{k \in \mathbb{Z}^d} \{\varepsilon\Gamma^k : \Gamma^k \subset \Omega\}.$$

The exterior boundary of Ω^ε is certainly a hypersurface in \mathbb{R}^d , denoted by $\Gamma^{ext} = \partial\Omega^\varepsilon \setminus \Gamma^\varepsilon$, where it has a nonzero $(d-1)$ -dimensional measure, satisfies $\Gamma^{ext} \cap \Gamma^\varepsilon = \emptyset$ and coincides with Γ . Moreover, n denotes the unit normal vector to Γ^ε .

40 Finally, our perforated domain Ω^ε is assumed to be connected through the gas phase. Notice here that Γ^{ext} is smooth.

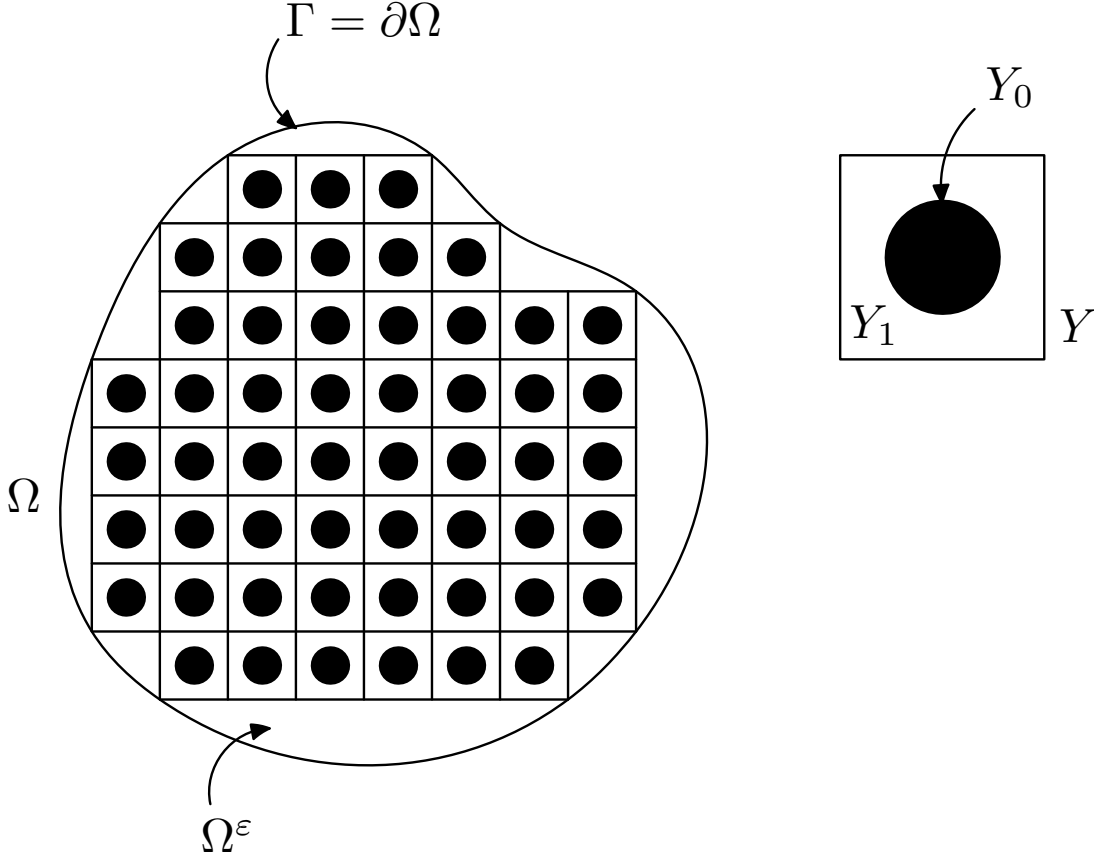


Figure 2.1: Admissible two-dimensional perforated domain (left) and basic geometry of the microstructure (right).

N.B. This paper aims at understanding the problem in two or three space dimensions. However, all our results hold also for $d \geq 3$. Throughout this paper, C denotes a generic constant which can change from line to line. If not otherwise stated, the constant C is independent of the choice of ε .

2.2. Notation. Assumptions on the data

We denote by $x \in \Omega^\varepsilon$ the macroscopic variable and by $y = x/\varepsilon$ the microscopic variable representing fast variations at the microscopic geometry. With this convention in view, we write

$$d_i^\varepsilon(x) = d_i\left(\frac{x}{\varepsilon}\right) = d_i(y).$$

45 A similar meaning is given to all involved "oscillating" data, e.g. to $a_i^\varepsilon(x)$, $b_i^\varepsilon(x)$.

We now make the following set of assumptions:

(A₁) the diffusion coefficient $d_i^\varepsilon \in L^\infty(\mathbb{R}^d)$ is Y -periodic, and it exists a positive constant α_i such that

$$d_i(y) \xi_i \xi_j \geq \alpha_i |\xi|^2 \quad \text{for any } \xi \in \mathbb{R}^d.$$

(A₂) the deposition coefficients $a_i^\varepsilon, b_i^\varepsilon \in L^\infty(\Gamma^\varepsilon)$ are positive and Y -periodic.

(A₃) the reaction rates $R_i : \Omega^\varepsilon \times [0, \infty)^N \rightarrow \mathbb{R}$ and $F_i : \Gamma^\varepsilon \times [0, \infty) \rightarrow \mathbb{R}$ are Carathéodory functions, i.e. they are, respectively, continuous in $[0, \infty)^N$ and $[0, \infty)$ with respect to x variable (in the “almost all” sense), and measurable in Ω^ε and Γ^ε with essential boundedness with respect to concentrations $u_i^\varepsilon \geq 0$.

(A₄) The chemical rate R_i and F_i are sublinear in the sense that for any $p = (p_1, \dots, p_N)$

$$R_i(p) \leq C \left(1 + \sum_{j=1, j \neq i}^N p_j \right) \quad \text{for } p \geq 0,$$

$$F_i(p_i) \leq C(1 + p_i) \quad \text{for } p_i \geq 0,$$

for any $p = (p_1, \dots, p_N)$.

Furthermore, assume that $R_i(p)/p_i$ is decreasing and $F_i(p_i)/p_i$ is increasing in p_i for any $p > 0$.

(A₅) For every $\varepsilon > 0$, there exist vectors (x -dependent) $r_0^\varepsilon, r_\infty^\varepsilon, f_0^\varepsilon, f_\infty^\varepsilon$ whose elements are

$$r_{0,i}^\varepsilon = \lim_{u_i^\varepsilon \rightarrow 0^+} \frac{R_i(u^\varepsilon)}{u_i^\varepsilon}, \quad r_{\infty,i}^\varepsilon = \lim_{u_i^\varepsilon \rightarrow \infty} \frac{R_i(u^\varepsilon)}{u_i^\varepsilon},$$

$$f_{0,i}^\varepsilon = \lim_{u_i^\varepsilon \rightarrow 0^+} \varepsilon \left(a_i^\varepsilon - b_i^\varepsilon \frac{F_i(u_i^\varepsilon)}{u_i^\varepsilon} \right), \quad f_{\infty,i}^\varepsilon = \lim_{u_i^\varepsilon \rightarrow \infty} \varepsilon \left(a_i^\varepsilon - b_i^\varepsilon \frac{F_i(u_i^\varepsilon)}{u_i^\varepsilon} \right).$$

(A₆) R_i and F_i satisfy the growth conditions:

$$|R_i(x, p)| \leq C \sum_{i=1}^N (1 + p_i)^2 \quad \text{for } p \geq 0, \quad (2.1)$$

$$|a_i^\varepsilon p_i - b_i^\varepsilon F_i(p_i)| \leq C(1 + p_i)^2 \quad \text{for } p_i \geq 0. \quad (2.2)$$

Let us define the function space

$$V^\varepsilon := \{v \in H^1(\Omega^\varepsilon) \mid v = 0 \text{ on } \Gamma^{\text{ext}}\},$$

which is a closed subspace of the Hilbert space $H^1(\Omega^\varepsilon)$, and thus endowed with the semi-norm

$$\|v\|_{V^\varepsilon} = \left(\sum_{i=1}^d \int_{\Omega^\varepsilon} \left| \frac{\partial v}{\partial x_i} \right|^2 dx \right)^{1/2} \quad \text{for all } v \in V^\varepsilon.$$

Obviously, this norm is equivalent to the usual H^1 -norm by the Poincaré inequality. Moreover, this equivalence is uniform in ε (cf. [6, Lemma 2.1]).

We introduce the Hilbert spaces

$$\mathcal{H}(\Omega^\varepsilon) = L^2(\Omega^\varepsilon) \times \dots \times L^2(\Omega^\varepsilon), \quad \mathcal{V}^\varepsilon = V^\varepsilon \times \dots \times V^\varepsilon,$$

with the inner products defined respectively by

$$\langle u, v \rangle_{\mathcal{H}(\Omega^\varepsilon)} := \sum_{i=1}^N \int_{\Omega^\varepsilon} u_i v_i dx, \quad u = (u_1, \dots, u_N), v = (v_1, \dots, v_N) \in \mathcal{H}(\Omega^\varepsilon),$$

$$\langle u, v \rangle_{\mathcal{V}^\varepsilon} := \sum_{i=1}^N \sum_{j=1}^n \int_{\Omega^\varepsilon} \frac{\partial u_i}{\partial x_j} \frac{\partial v_i}{\partial x_j} dx, \quad u = (u_1, \dots, u_N), v = (v_1, \dots, v_N) \in \mathcal{V}^\varepsilon.$$

Furthermore, the notation $\mathcal{H}(\Gamma^\varepsilon)$ indicates the corresponding product of $L^2(\Gamma^\varepsilon)$ spaces. For $q \in (2, \infty]$, the following spaces are also used

$$\mathcal{W}^q(\Omega^\varepsilon) = L^q(\Omega^\varepsilon) \times \dots \times L^q(\Omega^\varepsilon),$$

$$\mathcal{W}^q(\Gamma^\varepsilon) = L^q(\Gamma^\varepsilon) \times \dots \times L^q(\Gamma^\varepsilon).$$

Before studying the asymptotics behaviour as $\varepsilon \rightarrow 0$ (the homogenization limit), we must ensure the well-posedness of the microstructure model. In this section we focus only on the weak solvability of the problem, the stability with respect to the initial data and all parameter being straightforward to prove. We remark at this stage that the structure of the model equation has attracted much attention. For example, Amann used in [14] the method of sub- and super-
60 solutions to prove the existence of positive solutions when a Robin boundary condition is considered. Brezis and Oswald introduced in [15] an energy minimization approach to guarantee the existence, uniqueness and positivity results for the semi-linear elliptic problem with zero Dirichlet boundary conditions. Very recently, García-Melián et al. [16] extended the result in [15] (and also of other previous works including [17, 18]) to problems involving nonlinear boundary conditions of mixed type. For what we are concerned here, we will use Moser-like iterations technique (see the original works by Moser
65 [19, 20]) to prove L^∞ -bounds for all concentrations and then follow the strategy provided by Brezis and Oswald [15] to study the well-posedness of (P^ε) .

Definition 1. A function $u^\varepsilon \in \mathcal{V}^\varepsilon$ is a weak solution to (P^ε) provided that

$$\sum_{i=1}^N \int_{\Omega^\varepsilon} (d_i^\varepsilon \nabla u_i^\varepsilon \nabla \varphi_i - R_i(u^\varepsilon) \varphi_i) dx - \sum_{i=1}^N \varepsilon \int_{\Gamma^\varepsilon} (a_i^\varepsilon u_i^\varepsilon - b_i^\varepsilon F_i(u_i^\varepsilon)) \varphi_i dS_\varepsilon = 0 \quad \text{for all } \varphi \in \mathcal{V}^\varepsilon. \quad (3.1)$$

Definition 2. By means of the usual variational characterization, the principal eigenvalue of (P^ε) is defined by

$$\lambda_1(p^\varepsilon, q^\varepsilon) := \inf_{\substack{u^\varepsilon \in \mathcal{V}^\varepsilon, \\ \sum_{i=1}^N |u_i^\varepsilon|^2 \neq 0}} \frac{\sum_{i=1}^N \left(\alpha \int_{\Omega^\varepsilon} |\nabla_x u_i^\varepsilon|^2 dx - N \int_{\Omega^\varepsilon} p_i^\varepsilon |u_i^\varepsilon|^2 dx - N \int_{\Gamma^\varepsilon} q_i^\varepsilon |u_i^\varepsilon|^2 dS_\varepsilon \right)}{\sum_{i=1}^N \int_{\Omega^\varepsilon} |u_i^\varepsilon|^2 dx}, \quad (3.2)$$

where p_i^ε and q_i^ε are measurable such that either they are simultaneously bounded from above or from below (this leads to $\lambda_1 \in (-\infty, \infty]$ or $\lambda_1 \in [-\infty, \infty)$, correspondingly). Here, we denote $\alpha := \min \{\alpha_1, \dots, \alpha_N\}$.

Lemma 3. Assume (A_1) - (A_5) and replace (A_4) by (A_6) . Let $u^\varepsilon \in \mathcal{V}^\varepsilon \cap \mathcal{H}(\Gamma^\varepsilon)$ be a weak solution to (P^ε) , then $u^\varepsilon \in \mathcal{W}^\infty(\Omega^\varepsilon)$ and it exists an ε -independent constant $C > 0$ such that

$$\|u^\varepsilon\|_{\mathcal{W}^\infty(\Omega^\varepsilon)} \leq C \left(1 + \|u^\varepsilon\|_{\mathcal{H}(\Omega^\varepsilon)} + \|u^\varepsilon\|_{\mathcal{H}(\Gamma^\varepsilon)} \right).$$

Proof. Let $\beta \geq 1$ and $k_i > 1$ for all $i = \overline{1, N}$. We begin by introducing a vector φ^ε of test functions $\varphi_i^\varepsilon = \min \left\{ v_i^{\beta + \frac{1}{2}}, k_i^{\beta + \frac{1}{2}} \right\}$ -
70 1 where $v_i = u_i^\varepsilon + 1$ with u_i^ε as in (3.1). Thus, it is straightforward to show that $\varphi^\varepsilon \in \mathcal{V}^\varepsilon \cap \mathcal{H}(\Gamma^\varepsilon)$. We have

$$\begin{aligned} \alpha \left(\beta + \frac{1}{2} \right) \sum_{i=1}^N \int_{\{v_i < k_i\}} v_i^{\beta - \frac{1}{2}} |\nabla v_i|^2 &\leq \sum_{i=1}^N \int_{\Omega^\varepsilon} R_i(x, u^\varepsilon) \varphi_i^\varepsilon dx + \sum_{i=1}^N \int_{\Gamma^\varepsilon} F_i(x, u_i^\varepsilon) \varphi_i^\varepsilon dS_\varepsilon \\ &\leq C \sum_{i=1}^N \int_{\Omega^\varepsilon} |1 + u_i^\varepsilon|^2 v_i^{\beta + \frac{1}{2}} dx \\ &\quad + C \sum_{i=1}^N \int_{\Gamma^\varepsilon} |1 + u_i^\varepsilon|^2 v_i^{\beta + \frac{1}{2}} dS_\varepsilon \\ &\leq C \sum_{i=1}^N \left(\int_{\Omega^\varepsilon} v_i^{\beta + \frac{3}{2}} dx + \int_{\Gamma^\varepsilon} v_i^{\beta + \frac{3}{2}} dS_\varepsilon \right), \end{aligned} \quad (3.3)$$

where we have used (2.1) and (2.2).

Now, for every $i \in \{1, \dots, N\}$, if we assign $\psi_i = \min \left\{ v_i^{\frac{\beta+\frac{3}{2}}{2}}, k_i^{\frac{\beta+\frac{3}{2}}{2}} \right\}$, then one has

$$\left(\beta + \frac{1}{2} \right) v_i^{\beta-\frac{1}{2}} |\nabla v_i|^2 \chi_{\{v_i < k_i\}} = \frac{4 \left(\beta + \frac{1}{2} \right)}{\left(\beta + \frac{3}{2} \right)^2} |\nabla \psi_i|^2. \quad (3.4)$$

Since Ω^ε is a Lipschitz domain, then the trace embedding $H^1(\Omega^\varepsilon) \subset L^q(\partial\Omega^\varepsilon)$ holds for $1 \leq q \leq 2_{\partial\Omega^\varepsilon}^*$, where $2_{\partial\Omega^\varepsilon}^* = 2(d-1)/(d-2)$ if $d \geq 3$, and $2_{\partial\Omega^\varepsilon}^* = \infty$ if $d = 2$ (cf. [21]). Therefore, given $q \in (2, 2^*]$ we apply this embedding to (3.3) with the aid of (3.4) and then obtain

$$\frac{4\alpha \left(\beta + \frac{1}{2} \right)}{\left(\beta + \frac{3}{2} \right)^2} \sum_{i=1}^N \left[\left(\int_{\Gamma^\varepsilon} |\psi_i|^q dS_\varepsilon \right)^{\frac{2}{q}} - \int_{\Omega^\varepsilon} |\psi_i|^2 dx \right] \leq C \sum_{i=1}^N \left(\int_{\Omega^\varepsilon} v_i^{\beta+\frac{3}{2}} dx + \int_{\Gamma^\varepsilon} v_i^{\beta+\frac{3}{2}} dS_\varepsilon \right). \quad (3.5)$$

We see that $\psi_i^2 \leq v_i^{\beta+\frac{3}{2}}$ and also

$$\frac{1}{\beta + \frac{3}{2}} \leq \frac{4 \left(\beta + \frac{1}{2} \right)}{\left(\beta + \frac{3}{2} \right)^2} \leq 4$$

holds for all $\beta \geq 1$. As a result, (3.5) yields

$$\sum_{i=1}^N \left(\int_{\Gamma^\varepsilon} |\psi_i|^q dS_\varepsilon \right)^{\frac{2}{q}} \leq C\alpha^{-1} \left(\beta + \frac{3}{2} \right) \sum_{i=1}^N \left(\int_{\Omega^\varepsilon} v_i^{\beta+\frac{3}{2}} dx + \int_{\Gamma^\varepsilon} v_i^{\beta+\frac{3}{2}} dS_\varepsilon \right). \quad (3.6)$$

Our next aim is to show that if for some $s \geq 2$ we have $u^\varepsilon \in \mathcal{W}^s(\Omega^\varepsilon) \cap \mathcal{W}^s(\Gamma^\varepsilon)$, then $u^\varepsilon \in \mathcal{W}^{ks}(\Omega^\varepsilon) \cap \mathcal{W}^{ks}(\Gamma^\varepsilon)$ for $k > 1$ arbitrary at each ε -level. In fact, assume that $u^\varepsilon \in \mathcal{W}^{\beta+\frac{3}{2}}(\Omega^\varepsilon) \cap \mathcal{W}^{\beta+\frac{3}{2}}(\Gamma^\varepsilon)$ then letting $k \rightarrow \infty$ in (3.6) gives

$$\sum_{i=1}^N \left(\int_{\Gamma^\varepsilon} |v_i|^{\frac{q}{2}(\beta+\frac{3}{2})} dS_\varepsilon \right)^{\frac{2}{q}} \leq C \left(\beta + \frac{3}{2} \right) \sum_{i=1}^N \left(\int_{\Omega^\varepsilon} v_i^{\beta+\frac{3}{2}} dx + \int_{\Gamma^\varepsilon} v_i^{\beta+\frac{3}{2}} dS_\varepsilon \right). \quad (3.7)$$

One obtains in the same manner that by the embedding $H^1(\Omega^\varepsilon) \subset L^q(\Omega^\varepsilon)$ (this is valid for $1 \leq q \leq 2_{\Omega^\varepsilon}^*$ where $2_{\Omega^\varepsilon}^* = 2d/(d-2)$ if $d \geq 3$, and $2_{\Omega^\varepsilon}^* = \infty$ if $d = 2$; thus q given before is definitely valid), we are led to the following estimate

$$\sum_{i=1}^N \left(\int_{\Omega^\varepsilon} |v_i|^{\frac{q}{2}(\beta+\frac{3}{2})} dx \right)^{\frac{2}{q}} \leq C \left(\beta + \frac{3}{2} \right) \sum_{i=1}^N \left(\int_{\Omega^\varepsilon} v_i^{\beta+\frac{3}{2}} dx + \int_{\Gamma^\varepsilon} v_i^{\beta+\frac{3}{2}} dS_\varepsilon \right). \quad (3.8)$$

Combining (3.7), (3.8) and the Minkowski inequality enables us to get

$$\left(\int_{\Omega^\varepsilon} |v_i|^{\frac{q}{2}(\beta+\frac{3}{2})} dx + \int_{\Gamma^\varepsilon} |v_i|^{\frac{q}{2}(\beta+\frac{3}{2})} dS_\varepsilon \right)^{\frac{2}{q}} \leq C \left(\beta + \frac{3}{2} \right) \sum_{i=1}^N \left(\int_{\Omega^\varepsilon} v_i^{\beta+\frac{3}{2}} dx + \int_{\Gamma^\varepsilon} v_i^{\beta+\frac{3}{2}} dS_\varepsilon \right),$$

for all $i \in \{1, \dots, N\}$, which easily leads to, by raising to the power $1/(\beta + \frac{3}{2})$, the fact that $u_i^\varepsilon \in L^{\frac{q}{2}(\beta+\frac{3}{2})}(\Omega^\varepsilon) \cap L^{\frac{q}{2}(\beta+\frac{3}{2})}(\Gamma^\varepsilon)$ for all $i \in \{1, \dots, N\}$; and hence $u^\varepsilon \in \mathcal{W}^{\frac{q}{2}(\beta+\frac{3}{2})}(\Omega^\varepsilon) \cap \mathcal{W}^{\frac{q}{2}(\beta+\frac{3}{2})}(\Gamma^\varepsilon)$.

The constant k is indicated by $q/2 > 1$. Thus, if we choose q and β such that

$$\beta + \frac{3}{2} = 2 \left(\frac{q}{2} \right)^n \quad \text{for } n = 0, 1, 2, \dots,$$

and iterating the above estimate, we obtain, by induction, that

$$\|v\|_{2(\frac{q}{2})^n} \leq \prod_{j=0}^n \left(2 \left(\frac{q}{2} \right)^j C \right)^{\frac{1}{2} \left(\frac{2}{q} \right)^j} \|v\|_2, \quad (3.9)$$

where we have denoted by

$$\|v\|_r := \sum_{i=1}^N \left(\int_{\Omega^\varepsilon} |v_i|^r dx + \int_{\Gamma^\varepsilon} |v_i|^r dS_\varepsilon \right)^{\frac{1}{r}}.$$

It is interesting to point out that since the series $\sum_{n=0}^{\infty} \left(\frac{2}{q}\right)^n$ and $\sum_{n=0}^{\infty} n \left(\frac{2}{q}\right)^n$ are convergent for $q > 2$, we have

$$\prod_{j=0}^n \left(2 \left(\frac{q}{2}\right)^j C\right)^{\frac{1}{2} \left(\frac{2}{q}\right)^j} < \sqrt{(2C)^{\sum_{n=0}^{\infty} \left(\frac{2}{q}\right)^n} q^{\sum_{n=0}^{\infty} n \left(\frac{2}{q}\right)^n}} = C.$$

Therefore, the constant in the right-hand side of (3.9) is indeed independent of n , and by passing $n \rightarrow \infty$ in (3.9), i.e. in the inequality,

$$\|v\|_{\mathcal{W}^{2(\frac{q}{2})^n}(\Omega^\varepsilon)} \leq C \left(\|v\|_{\mathcal{H}(\Omega^\varepsilon)} + \|v\|_{\mathcal{H}(\Gamma^\varepsilon)} \right),$$

we finally obtain

$$\|v\|_{\mathcal{W}^\infty(\Omega^\varepsilon)} \leq C \left(\|v\|_{\mathcal{H}(\Omega^\varepsilon)} + \|v\|_{\mathcal{H}(\Gamma^\varepsilon)} \right).$$

Consequently, recalling $v_i = u_i^\varepsilon + 1$, we have:

$$\|u^\varepsilon\|_{\mathcal{W}^\infty(\Omega^\varepsilon)} \leq C \left(1 + \|u^\varepsilon\|_{\mathcal{H}(\Omega^\varepsilon)} + \|u^\varepsilon\|_{\mathcal{H}(\Gamma^\varepsilon)} \right).$$

This step completes the proof of the lemma. \square

Remark 4. Using the trace inequality (cf. [6, Lemma 2.31]) and the norm equivalence between V^ε and $H^1(\Omega^\varepsilon)$, if $u^\varepsilon \in \mathcal{V}^\varepsilon$ then the result in Lemma 3 reads

$$\begin{aligned} \|u^\varepsilon\|_{\mathcal{W}^\infty(\Omega^\varepsilon)} &\leq C \left(1 + \varepsilon^{-1/2} \|u^\varepsilon\|_{\mathcal{H}(\Omega^\varepsilon)} + \|u^\varepsilon\|_{\mathcal{V}^\varepsilon} \right) \\ &\leq C \left(1 + \varepsilon^{-1/2} \|u^\varepsilon\|_{\mathcal{V}^\varepsilon} \right). \end{aligned}$$

Lemma 5. Assume (A_1) - (A_5) and that $\lambda_1(r_\infty^\varepsilon, f_\infty^\varepsilon) > 0$ and $\lambda_1(r_0^\varepsilon, f_0^\varepsilon) < 0$ hold. We define the following functional

$$J[u^\varepsilon] := \frac{1}{2} \sum_{i=1}^N \int_{\Omega^\varepsilon} d_i^\varepsilon |\nabla u_i^\varepsilon|^2 dx - \sum_{i=1}^N \int_{\Omega^\varepsilon} \mathcal{R}_i(x, u^\varepsilon) dx - \sum_{i=1}^N \int_{\Gamma^\varepsilon} \mathcal{F}_i(x, u_i^\varepsilon) dS_\varepsilon,$$

where

$$\begin{aligned} \mathcal{R}_i(x, u^\varepsilon) &:= \int_0^{u_i^\varepsilon} R_i(x, u_1^\varepsilon, \dots, s_i, \dots, u_N^\varepsilon) ds_i, \\ \mathcal{F}_i(x, u_i^\varepsilon) &:= \int_0^{u_i^\varepsilon} (a_i^\varepsilon s - b_i^\varepsilon F_i(s)) ds, \end{aligned}$$

and the nonlinear terms are extended to be $R_i(x, 0)$ and $F_i(x, 0)$ for $u_i^\varepsilon \leq 0$. Then J is coercive on \mathcal{V}^ε and lower semi-continuous for \mathcal{V}^ε . Moreover, there exists $\phi \in \mathcal{V}^\varepsilon$ such that $J[\phi] < 0$.

Proof. Step 1: (Coerciveness)

Suppose, by contradiction, that it exists a sequence $\{u^{\varepsilon, m}\} \subset \mathcal{V}^\varepsilon$ such that $\|u^{\varepsilon, m}\|_{\mathcal{V}^\varepsilon} \rightarrow \infty$ while $J[u^{\varepsilon, m}] \leq C$. Setting

$$s_{i,m} = \left(\int_{\Gamma^\varepsilon} |u_i^{\varepsilon, m}|^2 dS_\varepsilon \right)^{1/2}, \quad t_{i,m} = \left(\int_{\Omega^\varepsilon} |u_i^{\varepsilon, m}|^2 dx \right)^{1/2}, \quad (3.10)$$

we say that $\sum_{i=1}^N t_{i,m}^2 \rightarrow \infty$ up to a subsequence as $m \rightarrow \infty$. Indeed, the assumption $J[u^{\varepsilon, m}] \leq C$ yields that

$$\frac{1}{2} \sum_{i=1}^N \int_{\Omega^\varepsilon} d_i^\varepsilon |\nabla u_i^{\varepsilon, m}|^2 dx \leq \sum_{i=1}^N \int_{\Omega^\varepsilon} \mathcal{R}_i(x, u^{\varepsilon, m}) dx + \sum_{i=1}^N \int_{\Gamma^\varepsilon} \mathcal{F}_i(x, u_i^{\varepsilon, m}) dS_\varepsilon + C, \quad (3.11)$$

which, in combination with (3.10) and (A_4) , leads to

$$\frac{1}{2} \sum_{i=1}^N \int_{\Omega^\varepsilon} d_i^\varepsilon |\nabla u_i^{\varepsilon, m}|^2 dx \leq C(N) \left(1 + \sum_{i=1}^N t_{i,m}^2 + \sum_{i=1}^N s_{i,m}^2 \right). \quad (3.12)$$

Here, if $\sum_{i=1}^N t_{i,m}^2$ is convergent, then $\sum_{i=1}^N s_{i,m}^2$ cannot be bounded. While putting

$$v_{i,m} = u_i^{\varepsilon,m} / \sum_{i=1}^N s_{i,m},$$

it enables us to derive that

$$\sum_{i=1}^N \int_{\Omega^\varepsilon} |\nabla v_{i,m}|^2 dx = \frac{\sum_{i=1}^N \int_{\Omega^\varepsilon} |\nabla u_i^{\varepsilon,m}|^2 dx}{\left(\sum_{i=1}^N s_{i,m} \right)^2} \leq \frac{\sum_{i=1}^N \int_{\Omega^\varepsilon} |\nabla u_i^{\varepsilon,m}|^2 dx}{\sum_{i=1}^N s_{i,m}^2}. \quad (3.13)$$

80 If we assign $\alpha := \min \{ \alpha_1, \dots, \alpha_N \} > 0$, then it follows from (3.12) and (3.13) that

$$\begin{aligned} \frac{\alpha}{2} \sum_{i=1}^N \int_{\Omega^\varepsilon} |\nabla v_{i,m}|^2 dx &\leq \frac{\sum_{i=1}^N \int_{\Omega^\varepsilon} d_i^\varepsilon |\nabla u_i^{\varepsilon,m}|^2 dx}{2 \sum_{i=1}^N s_{i,m}^2} \\ &\leq C(N) \left(1 + \frac{\sum_{i=1}^N t_{i,m}^2}{N} + \frac{1}{N \sum_{i=1}^N s_{i,m}^2} \right) \leq C(N). \end{aligned}$$

Now, we claim that there exists $v_i \in V^\varepsilon$ such that $v_{i,m} \rightharpoonup v_i$ weakly in V^ε , and then strongly in $L^2(\Omega^\varepsilon)$ and in $L^2(\Gamma^\varepsilon)$. However, it implies here a contradiction. It is because we have $v_i \equiv 0$ in Ω^ε for all $i = \overline{1, N}$ while

$$\sum_{i=1}^N \int_{\Gamma^\varepsilon} |v_i|^2 dS_\varepsilon = \left(\sum_{i=1}^N s_i \right)^{-2} \sum_{i=1}^N \int_{\Gamma^\varepsilon} |u_i^\varepsilon|^2 dS_\varepsilon \geq N^{-1} > 0.$$

Let us now assume that $\sum_{i=1}^N t_{i,m}^2$ is divergent. By putting

$$w_{i,m} = u_i^{\varepsilon,m} / \sum_{i=1}^N t_{i,m},$$

we have, in the same manner, that

$$\frac{\alpha}{2} \sum_{i=1}^N \int_{\Omega^\varepsilon} |\nabla w_{i,m}|^2 dx \leq C(N) \left(1 + \frac{1}{N \sum_{i=1}^N t_{i,m}^2} + \frac{\sum_{i=1}^N s_{i,m}^2}{N \sum_{i=1}^N t_{i,m}^2} \right).$$

From (3.10), we know that

$$\sum_{i=1}^N \int_{\Omega^\varepsilon} |w_{i,m}|^2 dx = \left(\sum_{i=1}^N t_{i,m} \right)^{-2} \sum_{i=1}^N \int_{\Omega^\varepsilon} |u_i^{\varepsilon,m}|^2 dx \leq 1, \quad (3.14)$$

and

$$\sum_{i=1}^N \int_{\Gamma^\varepsilon} |w_{i,m}|^2 dS_\varepsilon \geq N^{-1} \left(\sum_{i=1}^N t_{i,m}^2 \right)^{-1} \sum_{i=1}^N \int_{\Gamma^\varepsilon} |u_i^{\varepsilon,m}|^2 dS_\varepsilon \geq \frac{\sum_{i=1}^N s_{i,m}^2}{N \sum_{i=1}^N t_{i,m}^2}. \quad (3.15)$$

Combining the trace inequality (cf. [6, Lemma 2.31]) with (3.14) and (3.15), we obtain

$$\begin{aligned}
\frac{\sum_{i=1}^N s_{i,m}^2}{\sum_{i=1}^N t_{i,m}^2} &\leq N \sum_{i=1}^N \int_{\Gamma^\varepsilon} |w_{i,m}|^2 dS_\varepsilon \\
&\leq CN \left(2 \sum_{i=1}^N \left(\int_{\Omega^\varepsilon} |w_{i,m}|^2 dx \right)^{1/2} \left(\int_{\Omega^\varepsilon} |\nabla w_{i,m}|^2 dx \right)^{1/2} + \varepsilon^{-1} \sum_{i=1}^N \int_{\Omega^\varepsilon} |w_{i,m}|^2 dx \right) \\
&\leq CN \left(2 \sum_{i=1}^N \left(\int_{\Omega^\varepsilon} |\nabla w_{i,m}|^2 dx \right)^{1/2} + \varepsilon^{-1} \right).
\end{aligned}$$

It yields that

$$\sum_{i=1}^N \int_{\Omega^\varepsilon} |\nabla w_{i,m}|^2 dx \leq \frac{2C(N)}{\alpha} \left[\sum_{i=1}^N \left(\int_{\Omega^\varepsilon} |\nabla w_{i,m}|^2 dx \right)^{1/2} + C(\varepsilon) + \left(\sum_{i=1}^N t_{i,m}^2 \right)^{-1} \right],$$

which finally leads to

$$\left| \left(\int_{\Omega^\varepsilon} |\nabla w_{i,m}|^2 dx \right)^{1/2} - \frac{C(N)}{\alpha} \right| \leq C(N, \varepsilon) \left(1 + \left(\sum_{i=1}^N t_{i,m}^2 \right)^{-1} \right)^{1/2} \quad \text{for all } i = \overline{1, N}. \quad (3.16)$$

Therefore, $\int_{\Omega^\varepsilon} |\nabla w_{i,m}|^2 dx$ is bounded by the inequality (3.16). So, up to a subsequence, $w_{i,m} \rightharpoonup w_i$ weakly in V^ε , and then strongly in $L^2(\Omega^\varepsilon)$ and $L^2(\Gamma^\varepsilon)$. In addition, it can be proved that $\sum_{i=1}^N \int_{\Omega^\varepsilon} |w_i|^2 dx \geq N^{-1} > 0$, and from (3.11), it gives us that

$$\frac{\alpha}{2} \sum_{i=1}^N \int_{\Omega^\varepsilon} |\nabla w_{i,m}|^2 dx \leq \frac{C}{\sum_{i=1}^N t_{i,m}^2} + \sum_{i=1}^N \int_{\Omega^\varepsilon} \frac{\mathcal{R}_i(x, u_i^{\varepsilon, m})}{\sum_{i=1}^N t_{i,m}^2} dx + \sum_{i=1}^N \int_{\Gamma^\varepsilon} \frac{\mathcal{F}_i(x, u_i^{\varepsilon, m})}{\sum_{i=1}^N t_{i,m}^2} dS_\varepsilon. \quad (3.17)$$

We now consider the second integral on the right-hand side of the above inequality, then the third one is totally similar. Using the fact that $w_{i,m} \rightarrow w_i$ strongly in $L^2(\Omega^\varepsilon)$ and the assumptions (A₄)-(A₅) in combination with the Fatou lemma, we get

$$\limsup_{m \rightarrow \infty} \sum_{i=1}^N \int_{\Omega^\varepsilon} \frac{\mathcal{R}_i(x, u_i^{\varepsilon, m})}{\sum_{i=1}^N t_{i,m}^2} dx \leq \frac{N}{2} \sum_{i=1}^N \int_{\Omega^\varepsilon \cap \{w > 0\}} r_{\infty, i}^\varepsilon |w_{i,m}|^2 dx,$$

where we have also applied the following inequalities

$$\begin{aligned}
N^{-1} \left(\sum_{i=1}^N t_{i,m}^2 \right)^{-1} &\leq \left(\sum_{i=1}^N t_{i,m} \right)^{-2} \leq |w_{i,m}|^2 |u_i^{\varepsilon, m}|^{-2}, \\
\limsup_{u_i^\varepsilon \rightarrow \infty} \frac{\mathcal{R}_i(x, u_i^\varepsilon)}{|u_i^\varepsilon|^2} &\leq \frac{1}{2} r_{\infty, i}^\varepsilon(x) \quad \text{for a.e. } x \in \Omega^\varepsilon.
\end{aligned}$$

Thus, passing to the limit in (3.17) we are led to

$$\frac{\alpha}{2} \sum_{i=1}^N \int_{\Omega^\varepsilon} |\nabla w_i|^2 dx \leq \frac{N}{2} \left(\sum_{i=1}^N \int_{\Omega^\varepsilon \cap \{w > 0\}} r_{\infty, i}^\varepsilon |w_i|^2 dx + \sum_{i=1}^N \int_{\Gamma^\varepsilon \cap \{w > 0\}} f_{\infty, i}^\varepsilon |w_i|^2 dS_\varepsilon \right).$$

Recall that $\lambda_1(r_\infty^\varepsilon, f_\infty^\varepsilon) > 0$, it then gives us that $w_i^+ \equiv 0$ for all $i = \overline{1, N}$. As a consequence, $w_i \equiv 0$ while it contradicts the above result $\sum_{i=1}^N \int_{\Omega^\varepsilon} |w_i|^2 dx \geq N^{-1}$.

Hence, J is coercive.

Step 2: (Lower semi-continuity)

It can be proved as in [15, 16] that: if $u^{\varepsilon, m} \rightharpoonup u^\varepsilon$ in \mathcal{V}^ε , then we obtain

$$\begin{aligned} \limsup_{m \rightarrow \infty} \int_{\Omega^\varepsilon} \mathcal{R}_i(x, u^{\varepsilon, m}) dx &\leq \int_{\Omega^\varepsilon} \mathcal{R}_i(x, u^\varepsilon) dx, \\ \limsup_{m \rightarrow \infty} \int_{\Gamma^\varepsilon} \mathcal{F}_i(x, u_i^{\varepsilon, m}) dS_\varepsilon &\leq \int_{\Gamma^\varepsilon} \mathcal{F}_i(x, u_i^\varepsilon) dS_\varepsilon, \end{aligned}$$

by using the growth assumptions (A₄) in combination with the Fatou lemma. Thus, J is lower semi-continuous.

This result tells us that J achieves the global minimum at a function $u^\varepsilon \in \mathcal{V}^\varepsilon$. If we replace u^ε by $(u^\varepsilon)^+$, u^ε can be supposed to be non-negative. Moreover, the last step shows that u^ε is non-trivial.

Step 3: (Non-triviality of the minimisers)

What we need to prove now is that there exists $\phi \in \mathcal{V}^\varepsilon$ such that $J[\phi] < 0$. In fact, given $\psi \in \mathcal{V}^\varepsilon \cap \mathcal{W}^\varepsilon$ satisfying $\|\psi\|_{\mathcal{W}^\varepsilon} = 1$ and

$$\alpha \sum_{i=1}^N \int_{\Omega^\varepsilon} |\nabla \psi_i|^2 dx < N \sum_{i=1}^N \left(\int_{\Omega^\varepsilon} r_{0,i}^\varepsilon |\psi_i|^2 dx + \int_{\Gamma^\varepsilon} f_{0,i}^\varepsilon |\psi_i|^2 dS_\varepsilon \right).$$

In fact, here we assume that ψ is non-negative. By the assumptions (A₄)-(A₅), we have

$$\liminf_{\delta \rightarrow 0^+} \frac{\mathcal{R}_i(x, \delta \psi)}{\delta^2} \geq \frac{1}{2} r_{0,i}^\varepsilon(x) |\psi|^2 \geq \frac{1}{2} r_{0,i}^\varepsilon(x) |\psi_i|^2 \quad \text{for a.e. } x \in \Omega^\varepsilon,$$

and

$$\liminf_{\delta \rightarrow 0^+} \frac{\mathcal{F}_i(x, \delta \psi_i)}{\delta^2} \geq \frac{1}{2} f_{0,i}^\varepsilon(x) |\psi_i|^2 \quad \text{for a.e. } x \in \Gamma^\varepsilon.$$

This coupling with the Fatou lemma enable us to obtain the following

$$\begin{aligned} &\sum_{i=1}^N \left(\liminf_{\delta \rightarrow 0^+} \int_{\Omega^\varepsilon} \frac{\mathcal{R}_i(x, \delta \psi)}{\delta^2} dx + \liminf_{\delta \rightarrow 0^+} \frac{\mathcal{F}_i(x, \delta \psi_i)}{\delta^2} \right) \\ &\geq \frac{1}{2} \sum_{i=1}^N \left(\int_{\Omega^\varepsilon} r_{0,i}^\varepsilon |\psi_i|^2 dx + \int_{\Gamma^\varepsilon} f_{0,i}^\varepsilon |\psi_i|^2 dS_\varepsilon \right), \end{aligned}$$

which leads to

$$\limsup_{\delta \rightarrow 0^+} \frac{J[\delta \psi]}{\delta^2} < 0.$$

Hence, to complete the proof, we need to choose $\phi = \delta \psi$. □

Theorem 6. Assume (A₁)-(A₅) and $\lambda_1(r_\infty^\varepsilon, f_\infty^\varepsilon) > 0$, $\lambda_1(r_0^\varepsilon, f_0^\varepsilon) < 0$ hold. Then (P^ε) admits at least a non-negative weak solution $u^\varepsilon \in \mathcal{V}^\varepsilon \cap \mathcal{W}^\infty(\Omega^\varepsilon)$.

Proof. We begin the proof by introducing the approximate system

$$(P^{k,\varepsilon}) : \begin{cases} \nabla \cdot (-d_i^\varepsilon \nabla u_i^\varepsilon) = R_i^k(u^\varepsilon), & \text{in } \Omega^\varepsilon \subset \mathbb{R}^d, \\ d_i^\varepsilon \nabla u_i^\varepsilon \cdot \mathbf{n} = G_i^k(u_i^\varepsilon), & \text{on } \Gamma^\varepsilon, \\ u_i^\varepsilon = 0, & \text{on } \Gamma^{\text{ext}}, \end{cases}$$

in which we have defined that for each integer $k > 0$ the truncated reaction rates

$$R_i^k(u^\varepsilon) := \begin{cases} \max\{-ku_i^\varepsilon, R_i(u^\varepsilon)\}, & \text{if } u_i^\varepsilon \geq 0, \\ R_i(0), & \text{if } u_i^\varepsilon < 0, \end{cases}$$

and

$$G_i^k(u_i^\varepsilon) := \begin{cases} \varepsilon \max \{-k u_i^\varepsilon, a_i^\varepsilon u_i^\varepsilon - b_i^\varepsilon F_i(u_i^\varepsilon)\}, & \text{if } u_i^\varepsilon \geq 0, \\ -\varepsilon b_i^\varepsilon F_i(0), & \text{if } u_i^\varepsilon < 0. \end{cases}$$

It is easy to check that our truncated functions R_i^k and G_i^k fulfill both (A₄) and (A₆). In addition, if we set elements $R_{0,i}^k, R_{\infty,i}^k, G_{0,i}^k, G_{\infty,i}^k$ as functions in (A₅) by R_i^k and G_i^k , one may prove that

$$r_{0,i}^\varepsilon \leq R_{0,i}^k, r_{\infty,i}^\varepsilon \leq R_{\infty,i}^k, f_{0,i}^\varepsilon \leq G_{0,i}^k, f_{\infty,i}^\varepsilon \leq G_{\infty,i}^k \quad \text{for all } i \in \{1, \dots, N\},$$

and $\lambda_1(R_0^k, G_0^k) < 0$ and $\lambda_1(R_\infty^k, G_\infty^k) > 0$ for k large (see, e.g. [16]).

Thanks to Lemma 5, the problem $(P^{k,\varepsilon})$ admits a global non-trivial and non-negative minimizer, denoted by $u^{k,\varepsilon}$, which belongs to \mathcal{V}^ε and it is associated with the following functional

$$J^k[u^\varepsilon] := \frac{1}{2} \sum_{i=1}^N \int_{\Omega^\varepsilon} d_i^\varepsilon |\nabla u_i^\varepsilon| dx - \sum_{i=1}^N \int_{\Omega^\varepsilon} \mathcal{R}_i^k(x, u^\varepsilon) dx - \sum_{i=1}^N \int_{\Gamma^\varepsilon} \mathcal{F}_i^k(x, u_i^\varepsilon) dS_\varepsilon.$$

Furthermore, $u^{k,\varepsilon}$ defines a weak solution to the problem $(P^{k,\varepsilon})$ for every k and thus, $u^{k,\varepsilon} \in \mathcal{W}^\infty(\Omega^\varepsilon)$ by Lemma 3.

Now, we assign a vector v^ε whose elements are defined by $v_i^\varepsilon := \min\{u_i^\varepsilon, u_i^{k,\varepsilon}\}$ where $u \in \mathcal{V}^\varepsilon$ is the global minimizer constructed from the functional J . We shall prove that $J[v^\varepsilon] \leq J[u^\varepsilon]$. Note that when doing so, $v^\varepsilon \in \mathcal{W}^\infty(\Omega^\varepsilon)$ and then define a weak solution $u \in \mathcal{V}^\varepsilon \cap \mathcal{W}^\infty(\Omega^\varepsilon)$ to (P^ε) .

In fact, one has

$$J^k[u^{k,\varepsilon}] \leq J[\phi] \quad \text{for all } \phi \in \mathcal{V}^\varepsilon.$$

Then by choosing ϕ such that $\phi_i := \max\{u_i^\varepsilon, u_i^{k,\varepsilon}\}$ we have

$$\begin{aligned} & \sum_{i=1}^N \int_{\{u_i^{k,\varepsilon} < u_i^\varepsilon\} \cap \Omega^\varepsilon} \left(\frac{1}{2} d_i^\varepsilon |\nabla u_i^{k,\varepsilon}|^2 - \mathcal{R}_i^k(x, u^{k,\varepsilon}) \right) dx - \sum_{i=1}^N \int_{\{u_i^{k,\varepsilon} < u_i^\varepsilon\} \cap \Gamma^\varepsilon} \mathcal{F}_i^k(x, u_i^{k,\varepsilon}) dS_\varepsilon \\ & \leq \sum_{i=1}^N \int_{\{u_i^{k,\varepsilon} < u_i^\varepsilon\} \cap \Omega^\varepsilon} \left(\frac{1}{2} d_i^\varepsilon |\nabla u_i^\varepsilon|^2 - \mathcal{R}_i^k(x, u^\varepsilon) \right) dx - \sum_{i=1}^N \int_{\{u_i^{k,\varepsilon} < u_i^\varepsilon\} \cap \Gamma^\varepsilon} \mathcal{F}_i^k(x, u_i^\varepsilon) dS_\varepsilon. \end{aligned} \quad (3.18)$$

In addition, by the choice of J (see in Lemma 5) we deduce that

$$\begin{aligned} J[v^\varepsilon] - J[u^\varepsilon] &= \sum_{i=1}^N \int_{\{u_i^{k,\varepsilon} < u_i^\varepsilon\} \cap \Omega^\varepsilon} \frac{1}{2} d_i^\varepsilon \left(|\nabla u_i^{k,\varepsilon}|^2 - |\nabla u_i^\varepsilon|^2 \right) dx \\ & \quad - \sum_{i=1}^N \int_{\{u_i^{k,\varepsilon} < u_i^\varepsilon\} \cap \Omega^\varepsilon} \left(\mathcal{R}_i^k(x, u^{k,\varepsilon}) - \mathcal{R}_i^k(x, u^\varepsilon) \right) dx \\ & \quad - \sum_{i=1}^N \int_{\{u_i^{k,\varepsilon} < u_i^\varepsilon\} \cap \Gamma^\varepsilon} \left(\mathcal{F}_i^k(x, u_i^{k,\varepsilon}) - \mathcal{F}_i^k(x, u_i^\varepsilon) \right) dS_\varepsilon. \end{aligned} \quad (3.19)$$

On the other hand, (3.18) yields

$$\sum_{i=1}^N \int_{\{u_i^{k,\varepsilon} < u_i^\varepsilon\} \cap \Omega^\varepsilon} \left[\mathcal{R}_i^k(x, u^{k,\varepsilon}) - \mathcal{R}_i^k(x, u^\varepsilon) - (\mathcal{R}_i^k(x, u^{k,\varepsilon}) - \mathcal{R}_i^k(x, u^\varepsilon)) \right] \leq 0, \quad (3.20)$$

and

$$\sum_{i=1}^N \int_{\{u_i^{k,\varepsilon} < u_i^\varepsilon\} \cap \Gamma^\varepsilon} \left[\mathcal{F}_i^k(x, u^{k,\varepsilon}) - \mathcal{F}_i^k(x, u^\varepsilon) - (\mathcal{F}_i^k(x, u^{k,\varepsilon}) - \mathcal{F}_i^k(x, u^\varepsilon)) \right] \leq 0. \quad (3.21)$$

Hence, combining (3.18)-(3.21) we complete the proof of the lemma. This tells us that under assumptions (A₁)-(A₅)

the problem (P^ε) admits a non-negative, non-trivial and bounded weak vector of solutions u^ε at each ε -level. \square

Remark 7. If $R_i(u^\varepsilon) \geq -Mu_i^\varepsilon$ in Ω^ε (or for each subdomain of Ω^ε if rigorously stated) for some ε -dependent constant $M > 0$ and all $i \in \{1, \dots, N\}$, then (P^ε) has at least a positive, non-trivial and bounded weak solution u^ε by the Hopf strong maximum principle. Furthermore, one may prove in the same vein in [16, Lemma 13] that the solution is unique by using vectors of test functions $\varphi_\delta^\varepsilon$ and ψ_δ^ε whose elements are given by

$$\varphi_{\delta,i}^\varepsilon = \frac{(u_i^\varepsilon + \delta)^2 - (v_i^\varepsilon + \delta)^2}{u_i^\varepsilon + \delta}, \quad \psi_{\delta,i}^\varepsilon = \frac{(u_i^\varepsilon + \delta)^2 - (v_i^\varepsilon + \delta)^2}{v_i^\varepsilon + \delta},$$

where u_i^ε and v_i^ε are two solutions of (P^ε) at each layer $i \in \{1, \dots, N\}$, which are expected to equal to each other.

Remark 8. In the case of zero Neumann boundary condition on Γ^ε , if the nonlinearity R_i is globally Lipschitz with the Lipschitz constant, denoted by L_i , independent of the scale ε for any $i \in \{1, \dots, N\}$, then we may use an iterative scheme to deal with the existence and uniqueness of solutions to our problem. In fact, for $n \in \mathbb{N}$ such an iterative scheme is given by

$$(P_n^\varepsilon) : \begin{cases} \nabla \cdot (-d_i^\varepsilon \nabla u_i^{\varepsilon,n+1}) = R_i(u^{\varepsilon,n}), & \text{in } \Omega^\varepsilon, \\ d_i^\varepsilon \nabla u_i^{\varepsilon,n+1} \cdot \mathbf{n} = 0, & \text{on } \Gamma^\varepsilon, \\ u_i^{\varepsilon,n+1} = 0, & \text{on } \Gamma^{ext}, \end{cases} \quad (3.22)$$

where the starting point is $u^{\varepsilon,0} = 0$.

This global Lipschitz assumption is an alternative to (A_4) for R_i and it is termed as (A'_4) .

Theorem 9. *Assume (A_1) and (A_3) hold (without F_i) and suppose that the nonlinearity R_i satisfy (A'_4) replaced by (A_4) . Then, the problem (P^ε) with zero Neumann boundary condition on Γ^ε has a unique solution in \mathcal{V}^ε if the constant $\alpha^{-1} \max_{1 \leq i \leq N} \{L_i\} N$ is small enough.*

Proof. It is worth noting that the problem (3.22) admits a unique solution in \mathcal{V}^ε for any n . Then, the functional $w_i^{\varepsilon,n} = u_i^{\varepsilon,n+1} - u_i^{\varepsilon,n} \in V^\varepsilon$ satisfies the following problem:

$$\begin{cases} \nabla \cdot (-d_i^\varepsilon \nabla w_i^{\varepsilon,n}) = R_i(u^{\varepsilon,n}) - R_i(u^{\varepsilon,n-1}), & \text{in } \Omega^\varepsilon, \\ d_i^\varepsilon \nabla w_i^{\varepsilon,n} \cdot \mathbf{n} = 0, & \text{on } \Gamma^\varepsilon, \\ w_i^{\varepsilon,n} = 0. & \text{on } \Gamma^{ext}, \end{cases}$$

Using the test function $\psi_i \in V^\varepsilon$ we arrive at

$$\langle d_i^\varepsilon w_i^{\varepsilon,n}, \psi_i \rangle_{V^\varepsilon} = \langle R_i(u^{\varepsilon,n}) - R_i(u^{\varepsilon,n-1}), \psi_i \rangle_{L^2(\Omega^\varepsilon)}$$

We may consider an estimate for the above expression:

$$\alpha \sum_{i=1}^N |\langle w_i^{\varepsilon,n}, \psi_i \rangle_{V^\varepsilon}| \leq \sum_{i=1}^N L_i N \left| \langle w_i^{\varepsilon,n-1}, \psi_i \rangle_{L^2(\Omega^\varepsilon)} \right| \quad (3.23)$$

Thanks to Hölder's and Poincaré inequalities, we have

$$\sum_{i=1}^N |\langle w_i^{\varepsilon,n}, \psi_i \rangle_{V^\varepsilon}| \leq C_p \alpha^{-1} \max_{1 \leq i \leq N} \{L_i\} N \|w^{\varepsilon,n-1}\|_{\mathcal{V}^\varepsilon} \|\psi\|_{\mathcal{V}^\varepsilon},$$

where $C_p > 0$ is the Poincaré constant independent of the choice of ε , but the dimension d of the media (see, e.g. [6, Lemma 2.1] and [22]).

At this point, if the constant $\alpha^{-1} \max_{1 \leq i \leq N} \{L_i\} N$ is small enough such that $\kappa_p := C_p \alpha^{-1} \max_{1 \leq i \leq N} \{L_i\} N < 1$, then choosing $\psi_i = w_i^{\varepsilon, n}$ for $i \in \{1, \dots, N\}$ we obtain that

$$\|w^{\varepsilon, n}\|_{\mathcal{V}^\varepsilon} \leq \kappa_p \|w^{\varepsilon, n-1}\|_{\mathcal{V}^\varepsilon}.$$

Consequently, for some $k \in \mathbb{N}$ we get

$$\begin{aligned} \|u^{\varepsilon, n+k} - u^{\varepsilon, n}\|_{\mathcal{V}^\varepsilon} &\leq \|u^{\varepsilon, n+k} - u^{\varepsilon, n+k-1}\|_{\mathcal{V}^\varepsilon} + \dots + \|u^{\varepsilon, n+1} - u^{\varepsilon, n}\|_{\mathcal{V}^\varepsilon} \\ &\leq \kappa_p^{n+k-1} \|u^{\varepsilon, 1} - u^{\varepsilon, 0}\|_{\mathcal{V}^\varepsilon} + \dots + \kappa_p^n \|u^{\varepsilon, 1} - u^{\varepsilon, 0}\|_{\mathcal{V}^\varepsilon} \\ &\leq \kappa_p^n (\kappa_p^{k-1} + \kappa_p^{k-2} + \dots + 1) \|u^{\varepsilon, 1}\|_{\mathcal{V}^\varepsilon} \\ &\leq \frac{\kappa_p^n (1 - \kappa_p^k)}{1 - \kappa_p} \|u^{\varepsilon, 1}\|_{\mathcal{V}^\varepsilon}. \end{aligned} \tag{3.24}$$

Therefore, $\{u^{\varepsilon, n}\}$ is a Cauchy sequence in \mathcal{V}^ε , and then there exists uniquely $u^\varepsilon \in \mathcal{V}^\varepsilon$ such that $u^{\varepsilon, n} \rightarrow u^\varepsilon$ strongly in \mathcal{V}^ε as $n \rightarrow \infty$. Remarkably, this convergence combining with the Lipschitz property of R_i leads to the fact that $R_i(u^{\varepsilon, n}) \rightarrow R_i(u^\varepsilon)$ strongly in \mathcal{V}^ε as $n \rightarrow \infty$. As a result, the function u^ε is the solution of the problem (P^ε) when passing to the limit in n .

In addition, when $k \rightarrow \infty$, it follows from (3.24) that

$$\|u^{\varepsilon, n} - u^\varepsilon\|_{\mathcal{V}^\varepsilon} \leq \frac{\kappa_p^n}{1 - \kappa_p} \|u^{\varepsilon, 1}\|_{\mathcal{V}^\varepsilon},$$

which implies the convergence rate of the linearization and guarantees the stability of the problem (P^ε) . \square

4. Homogenization asymptotics. Corrector estimates

4.1. Two-scale asymptotic expansions

For every $i \in \{1, \dots, N\}$, we introduce the following M th-order expansion ($M \geq 2$):

$$u_i^\varepsilon(x) = \sum_{m=0}^M \varepsilon^m u_{i,m}(x, y) + \mathcal{O}(\varepsilon^{M+1}), \quad x \in \Omega^\varepsilon, \tag{4.1}$$

where $u_{i,m}(x, \cdot)$ is Y -periodic for $0 \leq m \leq M$.

It follows from (4.1) that

$$\begin{aligned} \nabla u_i^\varepsilon &= (\nabla_x + \varepsilon^{-1} \nabla_y) \left(\sum_{m=0}^M \varepsilon^m u_{i,m} + \mathcal{O}(\varepsilon^{M+1}) \right) \\ &= \varepsilon^{-1} \nabla_y u_{i,0} + \sum_{m=0}^{M-1} \varepsilon^m (\nabla_x u_{i,m} + \nabla_y u_{i,m+1}) + \mathcal{O}(\varepsilon^M). \end{aligned} \tag{4.2}$$

Using the relation of the operator \mathcal{A}^ε and (4.2), we compute that

$$\begin{aligned} \mathcal{A}_\varepsilon u_i^\varepsilon &= (\nabla_x + \varepsilon^{-1} \nabla_y) \cdot \left(-d_i(y) \left[\varepsilon^{-1} \nabla_y u_{i,0} + \sum_{m=0}^{M-1} \varepsilon^m (\nabla_x u_{i,m} + \nabla_y u_{i,m+1}) \right] \right) \\ &\quad + \mathcal{O}(\varepsilon^{M-1}), \end{aligned}$$

then after collecting those having the same powers of ε , we obtain

$$\begin{aligned} \mathcal{A}_\varepsilon u_i^\varepsilon &= \varepsilon^{-2} \nabla_y \cdot (-d_i(y) \nabla_y u_{i,0}) \\ &\quad + \varepsilon^{-1} [\nabla_x \cdot (-d_i(y) \nabla_y u_{i,0}) + \nabla_y \cdot (-d_i(y) (\nabla_x u_{i,0} + \nabla_y u_{i,1}))] \\ &\quad + \sum_{m=0}^{M-2} \varepsilon^m [\nabla_x \cdot (-d_i(y) (\nabla_x u_{i,m} + \nabla_y u_{i,m+1})) \\ &\quad + \nabla_y \cdot (-d_i(y) (\nabla_x u_{i,m+1} + \nabla_y u_{i,m+2}))] + \mathcal{O}(\varepsilon^{M-1}). \end{aligned} \tag{4.3}$$

In the same vein, we take into consideration the boundary condition at Γ^ε as follows:

$$\begin{aligned} -d_i^\varepsilon \nabla u_i^\varepsilon \cdot \mathbf{n} &:= -d_i(y) \left(\varepsilon^{-1} \nabla_y u_{i,0} + \sum_{m=0}^{M-1} \varepsilon^m (\nabla_x u_{i,m} + \nabla_y u_{i,m+1}) \right) \cdot \mathbf{n} \\ &= \varepsilon b_i(y) F_i \left(\sum_{m=0}^{M-1} \varepsilon^m u_{i,m} \right) - a_i(y) \sum_{m=0}^{M-1} \varepsilon^{m+1} u_{i,m} + \mathcal{O}(\varepsilon^M). \end{aligned} \quad (4.4)$$

It is worth noting that in order to investigate the convergence analysis, we give assumptions that allow to pull the ε -dependent quantities out of the nonlinearities R_i and F_i :

$$R_i \left(\sum_{m=0}^M \varepsilon^m u_{1,m}, \dots, \sum_{m=0}^M \varepsilon^m u_{N,m} \right) = \sum_{m=0}^M \varepsilon^m \bar{R}_i(u_{1,m}, \dots, u_{N,m}) + \mathcal{O}(\varepsilon^{M+1}), \quad (4.5)$$

$$F_i \left(\sum_{m=0}^M \varepsilon^m u_{i,m} \right) = \sum_{m=0}^M \varepsilon^m \bar{F}_i(u_{i,m}) + \mathcal{O}(\varepsilon^{M+1}), \quad (4.6)$$

in which \bar{R}_i and \bar{F}_i are global Lipschitz functions corresponding to the Lipschitz constant L_i and K_i , respectively, for $i \in \{1, \dots, N\}$.

From now on, collecting the coefficients of the same powers of ε in (4.3) and (4.4) in combination with using (4.5) and (4.6), we are led to the following systems of elliptic problems, which we refer to the auxiliary problems:

$$\begin{cases} \mathcal{A}_0 u_{i,0} = 0, & \text{in } Y_1, \\ -d_i(y) \nabla_y u_{i,0} \cdot \mathbf{n} = 0, & \text{on } \partial Y_0, \\ u_{i,0} \text{ is } Y\text{-periodic in } y, \end{cases} \quad (4.7)$$

$$\begin{cases} \mathcal{A}_0 u_{i,1} = -\mathcal{A}_1 u_{i,0}, & \text{in } Y_1, \\ -d_i(y) (\nabla_x u_{i,0} + \nabla_y u_{i,1}) \cdot \mathbf{n} = 0, & \text{on } \partial Y_0, \\ u_{i,1} \text{ is } Y\text{-periodic in } y, \end{cases} \quad (4.8)$$

$$\begin{cases} \mathcal{A}_0 u_{i,m+2} = \bar{R}_i(u_m) - \mathcal{A}_1 u_{i,m+1} - \mathcal{A}_2 u_{i,m}, & \text{in } Y_1, \\ -d_i(y) (\nabla_x u_{i,m+1} + \nabla_y u_{i,m+2}) \cdot \mathbf{n} = b_i(y) \bar{F}_i(u_{i,m}) - a_i(y) u_{i,m}, & \text{on } \partial Y_0, \\ u_{i,m+2} \text{ is } Y\text{-periodic in } y, \end{cases} \quad (4.9)$$

for $0 \leq m \leq M-2$.

125 Here, the notation u_m is ascribed to the vector containing elements $u_{i,m}$ for all $i \in \{1, \dots, N\}$, and we have denoted by

$$\begin{aligned} \mathcal{A}_0 &:= \nabla_y \cdot (-d_i(y) \nabla_y), \\ \mathcal{A}_1 &:= \nabla_x \cdot (-d_i(y) \nabla_y) + \nabla_y \cdot (-d_i(y) \nabla_x), \\ \mathcal{A}_2 &:= \nabla_x \cdot (-d_i(y) \nabla_x). \end{aligned}$$

For the first auxiliary problem (4.7), it is trivial to prove that the solution to (4.7) is independent of y , and hence we obtain

$$u_{i,0}(x, y) = \tilde{u}_{i,0}(x). \quad (4.10)$$

For the second auxiliary problem (4.8), we recall the result in [23, Lemma 2.1] to ensure the existence and uniqueness of periodic solutions to the elliptic problem, which is called the solvability condition. In this case, this condition satisfies

itself because we easily get from the PDE in (4.8) that

$$-\int_{\partial Y_1} d_i(y) \nabla_y u_{i,1} \cdot n dS_y = \int_{\partial Y_0} d_i(y) \nabla_x \tilde{u}_{i,0} \cdot n dS_y,$$

by Gauß's theorem. Thus, it claims the existence of a unique weak solution to (4.8).

Moreover, this solution is sought by using separation of variables:

$$u_{i,1}(x, y) = -\chi_i(y) \cdot \nabla_x \tilde{u}_{i,0}(x) + \mathcal{C}_i(x). \quad (4.11)$$

Substituting (4.11) into (4.8), we obtain the i th cell problem:

$$\begin{cases} \mathcal{A}_0 \chi_i = \nabla_y d_i(y), & \text{in } Y_1, \\ -d_i(y) \nabla_y \chi_i \cdot n = d_i(y) \cdot n, & \text{on } \partial Y_0, \\ \chi_i \text{ is } Y\text{-periodic in } y, \end{cases} \quad (4.12)$$

in which the field $\chi_i(y)$ is called cell function. Additionally, by the definition of the mean value, we have

$$\mathcal{M}_Y(\chi_i) := \frac{1}{|Y|} \int_{Y_1} \chi_i dy = 0. \quad (4.13)$$

As a consequence, it can be proved that χ_i belongs to the space $H_{\#}^1(Y_1)/\mathbb{R}$ and satisfies (4.13).

Now, it only remains to consider the third auxiliary problem (4.9). Assume that we have in mind the functions u_m and u_{m+1} , then to find u_{m+2} let us remark that the right-hand side of the PDE in (4.9) can be rewritten as

$$\begin{aligned} \bar{R}_i(u_m) - \mathcal{A}_1 u_{i,m+1} - \mathcal{A}_2 u_{i,m} &= \bar{R}_i(u_m) + \nabla_y (d_i(y) \nabla_x u_{i,m+1}) \\ &\quad + \nabla_x (d_i(y) (\nabla_x u_{i,m} + \nabla_y u_{i,m+1})). \end{aligned} \quad (4.14)$$

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We define the operator $\mathcal{L}_i(\psi)$ for $i \in \{1, \dots, N\}$ by multiplying (4.14) by a test function $\psi \in C_{\#}^{\infty}(Y_1)$, as follows:

$$\begin{aligned} \mathcal{L}_i(\psi) &= \int_{Y_1} \bar{R}_i(u_m) \psi dy + \int_{Y_1} \nabla_y (d_i(y) \nabla_x u_{i,m+1}) \psi dy \\ &\quad + \int_{Y_1} \nabla_x (d_i(y) (\nabla_x u_{i,m} + \nabla_y u_{i,m+1})) \psi dy \\ &= \int_{Y_1} \bar{R}_i(u_m) \psi dy - \int_{Y_1} d_i(y) \nabla_x u_{i,m+1} \nabla_y \psi dy \\ &\quad + \int_{Y_1} \nabla_x (d_i(y) (\nabla_x u_{i,m} + \nabla_y u_{i,m+1})) \psi dy. \end{aligned}$$

To apply the Lax-Milgram type lemma provided by [6, Lemma 2.2], we need $\mathcal{L}_i(\psi_1) = \mathcal{L}_i(\psi_2)$ for $\psi_1, \psi_2 \in H_{\#}^1(Y_1)/\mathbb{R}$ with $\psi_1 \simeq \psi_2$, or it is equivalent to

$$\int_{Y_1} \bar{R}_i(u_m) (\psi_1 - \psi_2) dy + \int_{Y_1} \nabla_x (d_i(y) (\nabla_x u_{i,m} + \nabla_y u_{i,m+1})) (\psi_1 - \psi_2) dy = 0. \quad (4.15)$$

Note that $\psi_1 - \psi_2$ is independent of y . Hence, (4.15) becomes

$$\int_{Y_1} \nabla_x (-d_i(y) (\nabla_x u_{i,m} + \nabla_y u_{i,m+1})) dy = \int_{Y_1} \bar{R}_i(u_m) dy. \quad (4.16)$$

For simplicity, we first take $m = 0$. Remind from (4.10) and (4.11) that $u_{i,0}$ and $u_{i,1}$ are known, while the term $R_i(u_0)$ depends on x only, then one has

$$\int_{Y_1} \nabla_x (-d_i(y) (-\nabla_y \chi_i \nabla_x \tilde{u}_{i,0} + \nabla_x \tilde{u}_{i,0})) dy = |Y_1| \bar{R}_i(u_0),$$

or equivalently,

$$\int_{Y_1} \nabla_x (-d_i(y) (-\nabla_y \chi_i + \mathbb{I}) \nabla_x \tilde{u}_{i,0}) dy = |Y_1| \bar{R}_i(u_0).$$

Thus, if we set the homogenized (or effective) coefficient

$$q_i = \frac{1}{|Y|} \int_{Y_1} d_i(y) (-\nabla_y \chi_i + \mathbb{I}) dy,$$

the $\tilde{u}_{i,0}$ must satisfy (in the ‘‘almost all’’ sense)

$$-\nabla_x (q_i \nabla_x \tilde{u}_{i,0}) = |Y|^{-1} |Y_1| \bar{R}_i(u_0), \quad \text{in } \Omega. \quad (4.17)$$

On the other hand, it is associated with $\tilde{u}_{i,0} = 0$ at Γ^{ext} and still satisfies the ellipticity condition.

Let us now determine $u_{i,2}$. At first, the PDE in (4.9) (for $m = 0$) is given by

$$\mathcal{A}_0 u_{i,2} = \bar{R}_i(u_0) - d_i(y) \nabla_y \chi_i \nabla_x^2 \tilde{u}_{i,0} - \nabla_y (d_i(y) \chi_i) \nabla_x^2 \tilde{u}_{i,0} + d_i(y) \nabla_x^2 \tilde{u}_{i,0}, \quad \text{in } Y_1. \quad (4.18)$$

Next, the boundary condition reads

$$-d_i(y) \nabla_y u_{i,2} \cdot \mathbf{n} = b_i(y) \bar{F}_i(u_{i,0}) - a_i(y) u_{i,0} - d_i(y) \chi_i \nabla_x^2 \tilde{u}_{i,0} \cdot \mathbf{n}, \quad \text{on } \partial Y_0.$$

Note that (4.18) can be rewritten as

$$\mathcal{A}_0 u_{i,2} - \nabla_y (d_i(y) \chi_i \nabla_x^2 \tilde{u}_{i,0}) = \bar{R}_i(u_0) - d_i(y) (\nabla_y \chi_i - \mathbb{I}) \nabla_x^2 \tilde{u}_{i,0}.$$

Using the relation (4.17), we have

$$\mathcal{A}_0 u_{i,2} + \mathcal{A}_0 (\chi_i \nabla_x^2 \tilde{u}_{i,0}) = -|Y_1|^{-1} |Y| \nabla_x (q_i \nabla_x \tilde{u}_{i,0}) - d_i(y) (\nabla_y \chi_i - \mathbb{I}) \nabla_x^2 \tilde{u}_{i,0}. \quad (4.19)$$

Therefore, (4.19) allows us to look for $u_{i,2}$ of the form

$$u_{i,2}(x, y) = \theta_i(y) \nabla_x^2 \tilde{u}_{i,0}, \quad (4.20)$$

in which such a function θ_i is the solution of the following problem

$$\begin{cases} \mathcal{A}_0 (\nabla_y \theta_i - \chi_i) = -|Y_1|^{-1} |Y| q_i - d_i(y) (\nabla_y \chi_i - \mathbb{I}), & \text{in } Y_1, \\ -d_i(y) (\nabla_y \theta_i - \chi_i) \cdot \mathbf{n} = b_i(y) \bar{F}_i(u_{i,0}) - a_i(y) u_{i,0}, & \text{on } \partial Y_0, \\ \theta_i \text{ is } Y\text{-periodic in } y. \end{cases} \quad (4.21)$$

In conclusion, we have derived an expansion of $u_i^\varepsilon(x)$ up to the second-order corrector. In particular, we deduced that

$$u_i^\varepsilon(x) = \tilde{u}_{i,0}(x) - \varepsilon \chi_i \left(\frac{x}{\varepsilon} \right) \cdot \nabla_x \tilde{u}_{i,0}(x) + \varepsilon^2 \theta_i \left(\frac{x}{\varepsilon} \right) \nabla_x^2 \tilde{u}_{i,0}(x) + \mathcal{O}(\varepsilon^3), \quad x \in \Omega^\varepsilon, \quad (4.22)$$

where $\tilde{u}_{i,0}$ can be solved by the microscopic problem (4.7), χ_i satisfies the cell problem (4.12), and θ_i satisfies the cell problem (4.21). Moreover, the homogenized equation is defined in (4.17).

For the time being, it remains to derive the macroscopic equation from the PDE for $u_{i,2}$ in (4.9) for $m = 0$. When doing so, the following solvability condition has to be fulfilled:

$$\int_{Y_1} (\bar{R}_i(u_0) - \mathcal{A}_1 u_{i,1} - \mathcal{A}_2 \tilde{u}_{i,0}) dy = \int_{\partial Y_0} (b_i(y) \bar{F}_i(\tilde{u}_{i,0}) - a_i(y) \tilde{u}_{i,0} + d_i(y) \nabla_x u_{i,1} \cdot \mathbf{n}) dS_y. \quad (4.23)$$

The left-hand side of (4.23) can be rewritten as

$$\int_{Y_1} \bar{R}_i(u_0) dy + \int_{Y_1} \nabla_y (d_i(y) \nabla_x u_{i,1}) dy + \int_{Y_1} \nabla_x (d_i(y) (\nabla_x \tilde{u}_{i,0} + \nabla_y u_{i,1})) dy. \quad (4.24)$$

Let us consider the last two integrals in (4.24). In fact, we have

$$\int_{Y_1} \nabla_x (d_i(y) \nabla_x \tilde{u}_{i,0}) dy = \nabla_x \cdot \left[\left(\int_{Y_1} d_i(y) dy \right) \nabla_x \tilde{u}_{i,0} \right] = \left(\int_{Y_1} d_i(y) dy \right) : \nabla_x \nabla_x \tilde{u}_{i,0}, \quad (4.25)$$

where we have used the inner product (or exactly, double dot product) between two matrices

$$A : B := \text{tr}(A^T B) = \sum_{ij} a_{ij} b_{ij}.$$

In addition, by periodicity and Gauß's theorem we obtain

$$\int_{Y_1} \nabla_y (d_i(y) \nabla_x u_{i,1}) dy = \int_{\partial Y_0} d_i(y) \nabla_x u_{i,1} \cdot \text{nd} S_y. \quad (4.26)$$

Next, employing the double dot product again, we get

$$\int_{Y_1} \nabla_x (d_i(y) \nabla_y u_{i,1}) dy = - \int_{Y_1} (d_i(y) \nabla_y \chi_i) dy : \nabla_x \nabla_x \tilde{u}_{i,0}. \quad (4.27)$$

Combining (4.23), (4.25)-(4.27) yields the macroscopic equation:

$$\left(\int_{Y_1} (d_i(y) - d_i(y) \nabla_y \chi_i) dy \right) : \nabla_x \nabla_x \tilde{u}_{i,0} = \langle b_i \rangle \bar{F}_i(\tilde{u}_{i,0}) - \langle a_i \rangle \tilde{u}_{i,0} - |Y_1| \bar{R}_i(u_0),$$

where we have denoted by

$$\begin{aligned} \langle a_i \rangle &:= \int_{\partial Y_0} a_i(y) dy, \\ \langle b_i \rangle &:= \int_{\partial Y_0} b_i(y) dy. \end{aligned}$$

135 Furthermore, this equation is associated with the boundary condition $\tilde{u}_{i,0} = 0$ at Γ^{ext} .

4.2. Corrector estimates. Justification of the asymptotics

From the point of view of applications, upper bound estimates on convergence rates over the space \mathcal{V}^ε in terms of quantitative analysis tells how fast one can approximate both u^ε , the solution of systems (P^ε) , and ∇u^ε by the asymptotic expansion (4.22). On the other hand, it also gives rise to the question that: how much information on data will we need
140 via such averaging techniques?

We introduce the well-known cut-off function $m^\varepsilon \in C_c^\infty(\Omega)$ such that $\varepsilon |\nabla m^\varepsilon| \leq C$ and

$$m^\varepsilon(x) := \begin{cases} 1, & \text{if } \text{dist}(x, \Gamma) \leq \varepsilon, \\ 0, & \text{if } \text{dist}(x, \Gamma) \geq 2\varepsilon. \end{cases}$$

For $i \in \{1, \dots, N\}$, we define the function Ψ_i^ε by

$$\Psi_i^\varepsilon := \varphi_i^\varepsilon + (1 - m^\varepsilon) (\varepsilon u_{i,1} + \varepsilon^2 u_{i,2}),$$

where we have denoted by

$$\varphi_i^\varepsilon := u_i^\varepsilon - (u_{i,0} + \varepsilon u_{i,1} + \varepsilon^2 u_{i,2}).$$

Due to the auxiliary problems (4.7)-(4.9), we have

$$\mathcal{A}^\varepsilon \varphi_i^\varepsilon = R_i(u^\varepsilon) - \bar{R}_i(u_0) - \varepsilon(\mathcal{A}_2 u_{i,1} + \mathcal{A}_1 u_{i,2}) - \varepsilon^2 \mathcal{A}_2 u_{i,2}, \quad x \in \Omega^\varepsilon, \quad (4.28)$$

while on the boundary Γ^ε , the function φ_i^ε satisfies

$$-d_i^\varepsilon \nabla_x \varphi_i^\varepsilon \cdot \mathbf{n} = \varepsilon^2 d_i^\varepsilon \nabla_x u_{i,2} \cdot \mathbf{n} + \varepsilon [a_i^\varepsilon (u_{i,0} - u_i^\varepsilon) + b_i^\varepsilon (F_i(u_i^\varepsilon) - \bar{F}_i(u_{i,0}))]. \quad (4.29)$$

Rewriting the above information, the function φ_i^ε satisfies the following system:

$$\begin{cases} \mathcal{A}_\varepsilon \varphi_i^\varepsilon = \bar{R}_i(u^\varepsilon) - \bar{R}_i(u_0) + \varepsilon g_i^\varepsilon, & \text{in } \Omega^\varepsilon, \\ -d_i^\varepsilon \nabla_x \varphi_i^\varepsilon \cdot \mathbf{n} = \varepsilon^2 h_i^\varepsilon \cdot \mathbf{n} + \varepsilon l_i^\varepsilon, & \text{on } \Gamma^\varepsilon, \end{cases} \quad (4.30)$$

where we have denoted by

$$\begin{aligned} g_i^\varepsilon &:= -d_i \left(\frac{x}{\varepsilon} \right) \chi_i \left(\frac{x}{\varepsilon} \right) \nabla_x^3 \tilde{u}_{i,0} + d_i \left(\frac{x}{\varepsilon} \right) \theta_i \left(\frac{x}{\varepsilon} \right) \nabla_x^3 \tilde{u}_{i,0} \\ &\quad + \nabla_y \left(d_i \left(\frac{x}{\varepsilon} \right) \theta_i \left(\frac{x}{\varepsilon} \right) \right) \nabla_x^3 \tilde{u}_{i,0} + \varepsilon d_i \left(\frac{x}{\varepsilon} \right) \theta_i \left(\frac{x}{\varepsilon} \right) \nabla_x^4 \tilde{u}_{i,0}, \\ h_i^\varepsilon &:= d_i \left(\frac{x}{\varepsilon} \right) \theta_i \left(\frac{x}{\varepsilon} \right) \nabla_x^3 \tilde{u}_{i,0}, \\ l_i^\varepsilon &:= a_i \left(\frac{x}{\varepsilon} \right) (\tilde{u}_{i,0} - u_i^\varepsilon) + b_i \left(\frac{x}{\varepsilon} \right) (F_i(u_i^\varepsilon) - \bar{F}_i(\tilde{u}_{i,0})). \end{aligned}$$

Now, multiplying the PDE in (4.30) by $\varphi_i \in V_\varepsilon$ for $i \in \{1, \dots, N\}$ and integrating by parts, we get that

$$\begin{aligned} \langle d_i^\varepsilon \varphi_i^\varepsilon, \varphi_i \rangle_{V^\varepsilon} &= \langle \bar{R}_i(u^\varepsilon) - \bar{R}_i(u_0), \varphi_i \rangle_{L^2(\Omega^\varepsilon)} + \varepsilon \langle g_i^\varepsilon, \varphi_i \rangle_{L^2(\Omega^\varepsilon)} \\ &\quad - \varepsilon \langle l_i^\varepsilon, \varphi_i \rangle_{L^2(\Gamma^\varepsilon)} - \varepsilon^2 \int_{\Gamma^\varepsilon} h_i^\varepsilon \cdot \mathbf{n} \varphi_i dS_\varepsilon. \end{aligned} \quad (4.31)$$

To guarantee all the derivatives appearing in g_i^ε (up to higher order correctors), $\tilde{u}_{i,0}$, which is the solution to (4.17), needs to be smooth enough, says $L^\infty(\Omega^\varepsilon)$ (cf. [24]), and the cell functions χ_i and θ_i to (4.12) and (4.21), respectively, belong at least to $H_{\#}^1(Y_1)$ as derived above. Consequently, it allows us to estimate g_i^ε by an ε -independent constant, i.e.

$$\|g_i^\varepsilon\|_{L^2(\Omega^\varepsilon)} \leq C \quad \text{for all } i \in \{1, \dots, N\}. \quad (4.32)$$

Furthermore, it is easily to estimate the integral including h_i^ε in (4.31) by the following (see, e.g. [6]):

$$\int_{\Gamma^\varepsilon} h_i^\varepsilon \cdot \mathbf{n} dS_\varepsilon \approx C \varepsilon^{-1},$$

which leads to

$$\|h_i^\varepsilon \cdot \mathbf{n}\|_{L^2(\Gamma^\varepsilon)} \leq C \varepsilon^{-1/2}. \quad (4.33)$$

Now, it remains to estimate the third integral in (4.31). Thanks to (A₂) and (4.6), we may have

$$\left| \langle l_i^\varepsilon, \varphi_i \rangle_{L^2(\Gamma^\varepsilon)} \right| \leq C (1 + \bar{K}_i) \|u_i^\varepsilon - \tilde{u}_{i,0}\|_{L^2(\Gamma^\varepsilon)} \|\varphi_i\|_{L^2(\Gamma^\varepsilon)}. \quad (4.34)$$

In the same vein, we get:

$$\left| \langle R_i(u^\varepsilon) - \bar{R}_i(u_0), \varphi_i \rangle_{L^2(\Omega^\varepsilon)} \right| \leq C \bar{L}_i \|u^\varepsilon - \tilde{u}_0\|_{V^\varepsilon} \|\varphi_i\|_{L^2(\Omega^\varepsilon)}. \quad (4.35)$$

Combining (4.31)-(4.35) with (A₁) and putting $\bar{L} := \max\{\bar{L}_1, \dots, \bar{L}_N\}$ and $\bar{K} := 1 + \max\{\bar{K}_1, \dots, \bar{K}_N\}$, we are led to the estimate:

$$\begin{aligned} \alpha \sum_{i=1}^N |\langle \varphi_i^\varepsilon, \varphi_i \rangle_{V^\varepsilon}| &\leq C (\bar{L} \|u^\varepsilon - \tilde{u}_0\|_{V^\varepsilon} + \varepsilon) \|\varphi\|_{V^\varepsilon} + C \left(\bar{K} \|u^\varepsilon - \tilde{u}_0\|_{\mathcal{H}(\Gamma^\varepsilon)} + \varepsilon^{3/2} \right) \|\varphi\|_{\mathcal{H}(\Gamma^\varepsilon)} \\ &\leq C \left(\varepsilon + \varepsilon^{1/2} \right) \|\varphi\|_{V^\varepsilon} \leq C \varepsilon^{1/2} \|\varphi\|_{V^\varepsilon}, \end{aligned} \quad (4.36)$$

145 where we have made use of the trace inequality $\|\varphi\|_{\mathcal{H}(\Gamma^\varepsilon)} \leq C\varepsilon^{-1/2} \|\varphi\|_{\mathcal{V}^\varepsilon}$ and the Poincaré inequality $\|\varphi\|_{\mathcal{H}(\Omega^\varepsilon)} \leq C \|\varphi\|_{\mathcal{V}^\varepsilon}$.

Recall that our aim is to estimate $\|\Psi^\varepsilon\|_{\mathcal{V}^\varepsilon}$, it remains to control the term $\langle (1 - m^\varepsilon) (\varepsilon u_{i,1} + \varepsilon^2 u_{i,2}), \varphi_i \rangle_{\mathcal{V}^\varepsilon}$ for $\varphi_i \in V^\varepsilon$.

In fact, one easily has that

$$\begin{aligned} \sum_{i=1}^N \left| \langle (1 - m^\varepsilon) (\varepsilon u_{i,1} + \varepsilon^2 u_{i,2}), \varphi_i \rangle_{\mathcal{V}^\varepsilon} \right| &\leq C\varepsilon \|\nabla (1 - m^\varepsilon)\|_{\mathcal{H}(\Omega^\varepsilon)} \|\varphi\|_{\mathcal{V}^\varepsilon} \\ &\quad + C \|(1 - m^\varepsilon) \nabla (\varepsilon u_1 + \varepsilon^2 u_2)\|_{\mathcal{H}(\Omega^\varepsilon)} \|\varphi\|_{\mathcal{V}^\varepsilon} \\ &\leq C \left(\varepsilon^{1/2} + \varepsilon^{3/2} \right) \|\varphi\|_{\mathcal{V}^\varepsilon} \leq C\varepsilon^{1/2} \|\varphi\|_{\mathcal{V}^\varepsilon}, \end{aligned} \quad (4.37)$$

where we have used

$$\|\nabla (1 - m^\varepsilon)\|_{\mathcal{H}(\Omega^\varepsilon)}^2 \leq N \left(\int_{\Omega^\varepsilon \cap \{x | \text{dist}(x, \Gamma) \leq 2\varepsilon\}} |\nabla m^\varepsilon|^2 dx \right) \leq C\varepsilon^{-1},$$

$$\begin{aligned} \|(1 - m^\varepsilon) \nabla (\varepsilon u_1 + \varepsilon^2 u_2)\|_{\mathcal{H}(\Omega^\varepsilon)}^2 &\leq N\varepsilon^2 |\Omega^\varepsilon| \int_{\Omega^\varepsilon \cap \{x | \text{dist}(x, \Gamma) \leq 2\varepsilon\}} |\nabla m^\varepsilon|^2 dx \\ &\leq C\varepsilon^3. \end{aligned}$$

Hence, by using the triangle inequality in (4.36) and (4.37) yields that

$$\sum_{i=1}^N |\langle \Psi_i^\varepsilon, \varphi_i \rangle_{\mathcal{V}^\varepsilon}| \leq C\varepsilon^{1/2} \|\varphi\|_{\mathcal{V}^\varepsilon},$$

which finally leads to

$$\|\Psi^\varepsilon\|_{\mathcal{V}^\varepsilon} \leq C\varepsilon^{1/2},$$

by choosing $\varphi = \Psi^\varepsilon$.

Summarizing, we can now state of the following theorem.

Theorem 10. *Let u^ε be the solution of the elliptic system (P^ε) with assumptions (A_1) – (A_3) and (4.5)–(4.6) up to $M = 2$. Suppose that the unique pair $(u_0, u_m) \in \mathcal{W}^\infty(\Omega^\varepsilon) \times \mathcal{W}^\infty(\Omega^\varepsilon; H_{\#}^1(Y_1)/\mathbb{R})$ for $m \in \{1, 2\}$. The following corrector with second order for the homogenization limit holds:*

$$\|u^\varepsilon - u_0 - m^\varepsilon (\varepsilon u_1 + \varepsilon^2 u_2)\|_{\mathcal{V}^\varepsilon} \leq C\varepsilon^{1/2},$$

150 where u_0, u_1 and u_2 are vectors whose elements are defined by (4.10), (4.11) and (4.20), respectively.

5. Discussion

In real-world applications, the nonlinear reaction term R_i is often locally Lipschitz. However, relying on Lemma 3 the L^∞ -type estimate of the positive solution makes the nonlinearity globally Lipschitz. For example, we choose $N = 2$ and only consider the $R_1(u_1, u_2) = u_1 u_2 - u_1^2$. We have

$$|R_1(u_1, u_2) - R_1(v_1, v_2)| \leq \max\{\|u_2\|_{L^\infty}, \|u_1\|_{L^\infty} + \|v_1\|_{L^\infty}\} (|u_1 - v_1| + |u_2 - v_2|).$$

In addition, for $M = 1$ we compute that

$$R_1(u_{1,0} + \varepsilon u_{1,1}, u_{2,0} + \varepsilon u_{2,1}) = u_{1,0} u_{2,0} + \varepsilon (u_{1,1} u_{2,0} + u_{1,0} u_{2,1} - 2u_{1,0} u_{1,1}) + \mathcal{O}(\varepsilon^2). \quad (5.1)$$

Consequently, it follows from (5.1) that

$$R_1 \left(\sum_{m \in \{0,1\}} \varepsilon^m u_{1,m}, \sum_{m \in \{0,1\}} \varepsilon^m u_{2,m} \right) = \sum_{m \in \{0,1\}} \varepsilon^m [(1-m) u_{1,0} u_{2,0} + \\ + m (u_{1,1} u_{2,0} + u_{1,0} u_{2,1} - 2u_{1,0} u_{1,1})] + \mathcal{O}(\varepsilon^2).$$

which implies $\bar{R}_1 := (1-m) u_{1,0} u_{2,0} + m (u_{1,1} u_{2,0} + u_{1,0} u_{2,1} - 2u_{1,0} u_{1,1})$.

If $u_{i,m}, v_{i,m} \in L^\infty(\Omega^\varepsilon)$ for all i, m we thus arrive at

$$|\bar{R}_1(u_{1,0}, u_{1,1}, u_{2,0}, u_{2,1}) - \bar{R}_1(v_{1,0}, v_{1,1}, v_{2,0}, v_{2,1})| \leq L_1 \sum_{m \in \{0,1\}, i \in \{1,2\}} |u_{i,m} - v_{i,m}|,$$

where $L_1 = 4 \max \left\{ \|u_{2,0}\|_{L^\infty(\Omega^\varepsilon)}, \|v_{1,0}\|_{L^\infty(\Omega^\varepsilon)}, \|v_{1,1}\|_{L^\infty(\Omega^\varepsilon)}, \|v_{2,1}\|_{L^\infty(\Omega^\varepsilon)}, \|u_{1,0}\|_{L^\infty(\Omega^\varepsilon)}, 1 \right\}$.

155 A similar discussion for the nonlinear surface rates F_i . In particular, note that that if L^∞ bounds are available (up to the boundary) then also the exponential function $F(u) = e^u$ can be treated conveniently.

We may repeat the homogenization procedure by the auxiliary problems (4.7)-(4.9) to obtain not only the general expansion of the concentrations and corresponding problems, but also the higher order of corrector estimate due to the \tilde{u}_0 -based construction of u_m . Taking the M -level expansion (4.1) into consideration, the general corrector can be found
160 easily. Indeed, by induction we have from (4.28) that for $x \in \Omega^\varepsilon$

$$\begin{aligned} \mathcal{A}^\varepsilon \varphi_i^\varepsilon &= \mathcal{A}^\varepsilon u_i^\varepsilon - \varepsilon^{-2} \mathcal{A}_0 u_{i,0} - \varepsilon^{-1} (\mathcal{A}_0 u_{i,1} + \mathcal{A}_1 u_{i,0}) \\ &\quad - \sum_{m=0}^{M-2} \varepsilon^m (\mathcal{A}_0 u_{i,m+2} + \mathcal{A}_1 u_{i,m+1} + \mathcal{A}_2 u_{i,m}) \\ &\quad - \varepsilon^{M-1} (\mathcal{A}_1 u_{i,M} + \mathcal{A}_2 u_{i,M-1}) - \varepsilon^M \mathcal{A}_2 u_{i,M} \\ &= R_i(u^\varepsilon) - \sum_{m=0}^{M-2} \varepsilon^m \bar{R}_i(u_m) - \varepsilon^{M-1} (\mathcal{A}_1 u_{i,M} + \mathcal{A}_2 u_{i,M-1}) - \varepsilon^M \mathcal{A}_2 u_{i,M}, \end{aligned}$$

while (4.29) becomes

$$-d_i^\varepsilon \nabla_x \varphi_i^\varepsilon \cdot \mathbf{n} = \varepsilon^M d_i^\varepsilon \nabla_x u_{i,M} + \varepsilon \left[a_i^\varepsilon \left(\sum_{m=0}^{M-2} \varepsilon^m u_{i,m} - u_i^\varepsilon \right) + b_i^\varepsilon \left(F(u_i^\varepsilon) - \sum_{m=0}^{M-2} \varepsilon^m \bar{F}(u_{i,m}) \right) \right].$$

Thanks to the assumptions (4.5) and (4.6), we are totally in a position to prove the generalization of Theorem 10. Since we just need to follow the above procedure, we shall give the following theorem while skipping the proof.

Theorem 11. *Let u^ε be the solution of the elliptic system (P^ε) with assumptions $(A_1) - (A_3)$ and (4.5)-(4.6) up to M -level of expansion. Suppose that the unique pair $(u_0, u_m) \in \mathcal{W}^\infty(\Omega^\varepsilon) \times \mathcal{W}^\infty(\Omega^\varepsilon; H_\#^1(Y_1)/\mathbb{R})$ for all $0 \leq m \leq M$. The following correctors for the homogenization limit hold:*

$$\left\| u^\varepsilon - \sum_{m=0}^M \varepsilon^m u_m \right\|_{\mathcal{V}^\varepsilon} \leq C \left(\varepsilon^{M-1} + \varepsilon^{M-1/2} \right),$$

$$\left\| u^\varepsilon - u_0 - \varepsilon \sum_{m=1}^M \varepsilon^m u_m \right\|_{\mathcal{V}^\varepsilon} \leq C \sum_{m=1}^M \varepsilon^{m-1/2}.$$

Acknowledgment

AM thanks NWO MPE "Theoretical estimates of heat losses in geothermal wells" (grant nr. 657.014.004) for funding.
165 VAK gratefully acknowledges the hospitality of Department of Mathematics and Computer Science, Eindhoven University of Technology in The Netherlands.

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