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A MULTIPOINT FLUX APPROXIMATION FINITE VOLUME SCHEME FOR TWO PHASE POROUS MEDIA FLOW WITH DYNAMIC CAPILLARITY

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Abstract. We study a two phase porous media flow model where dynamic effects are included in the capillary pressure. We present a finite volume method for the simulation of the solution. The method is based on a multi-point flux approximation. An energy estimate is derived for the numerical solution, and compactness arguments provide convergence to the weak solution as the mesh size and the time step tend to zero. Finally, we present some numerical results to confirm the theoretically proved convergence.

Key words. Dynamic capillary pressure, two-phase flow, finite volume scheme, O-method

AMS subject classifications.

1. Introduction. In this paper, we define and analyze a finite volume method for a two phase flow model in a porous medium:

$$\partial_t u - \nabla \cdot (k_o(u)\nabla \bar{p}) = 0, \tag{1.1}$$

$$\partial_t (1-u) - \nabla \cdot (k_w(u)\nabla p) = 0, \tag{1.2}$$

$$\bar{p} - p = p_c(u) + \tau \partial_t u, \tag{1.3}$$

which are defined in $Q := \Omega \times (0,T]$, where Ω is a bounded subset of $\mathbb{R}^2$, T is a given maximal time. The unknowns $u, \bar{p}$ and p are the non-wetting phase saturation, the non-wetting phase and wetting phase pressures. The equations (1.1), (1.2) are obtained by combing the mass balance and Darcy's law ([5, 30, 42]). The permeabilities $k_o(\cdot), k_w(\cdot)$ for non-wetting phase and wetting phase are given monotone functions. The gravity is neglected in the model. In order to close the above system, we prescribe the initial and boundary conditions

$$u(0,\cdot) = u^0, \qquad \text{in} \quad \Omega, \tag{1.4}$$

$$\bar{p} = p = 0, \qquad \text{at} \quad \partial\Omega \quad \text{for} \quad t > 0, \tag{1.5}$$

where u^0 is a given function, which will be specified later.

Equation (1.3) expresses the phase pressure difference $\bar{p} - p$, as a function of u and $\partial_t u$. In classical models (see [5, 30, 35]), one assumes

$$\bar{p} - p = p_c(u),$$

where p_c, the capillary pressure is a monotone function of saturation u. This however, holds only if measurements are obtained under equilibrium condition. Experiments (see [6, 16]) have invalidated this assumption, whenever flow is more rapid. One possible extension is (1.3) as proposed in [29], where τ is a positive damping factor ($\tau > 0$).

Standard models, obtained for $\tau = 0$ have been intensively investigated in the reservoir simulation. In this sense, existence and uniqueness of the weak solutions are proved in [35], but assuming

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that initial and boundary conditions bounded away from 0. This has been extended to the case of arbitrarily chosen saturation for initial and boundary conditions. The existence can be found in [3, 12] and the uniqueness of weak solution was proved in [12]. For numerical schemes, we refer to ([4, 8, 10, 17, 20, 22, 43, 45, 46]), where finite element method, mixed finite element method, discontinuous Galerkin method are analyzed, or linear iterative schemes are investigated. In particular, for finite volume schemes, we refer to [11, 24, 39]. Whenever $\tau > 0$, (1.1) - (1.3) becomes a so-called non-equilibrium model. In this case, the existence and uniqueness of a weak solution is obtained in [13, 26, 27, 40], but in a simplified context when the total flow is assumed to be known. This allows reducing one equation in (1.1) - (1.3). In this case, but in the heterogeneous case, if no entry pressure presents, numerical schemes are discussed in [31]. Also, variational inequality approaches have been considered in [32] for situations including an entry pressure. Further, we refer to [18] which gives the coupling conditions analysis. In [44], they consider numerical algorithms for unsaturated flow in highly heterogeneous media for this model. For the full model, the existence and uniqueness of the weak solutions are proved in [34, 14], but assuming that the equations are non-degenerate (i.e. all non-linearities are bounded away from 0 or $+\infty$). The authors in [33] present discontinuous Galerkin scheme for this case. The authors have given the numerical investigations in [28] in heterogeneous case. In the degenerate case, we refer to [15], which gives the existence of weak solutions for the model in an equivalent form.

In this paper, we show that the approximate solution of the system (1.1) - (1.3) obtained by a multi-point flux approximation finite volume scheme converges to its weak solution. The rest of the paper is organized as follows. In Section 2, we give the assumptions on the data and the define the weak solution. We introduce the finite volume scheme in Section 3, and show the existence of the numerical solution. In Section 4, we prove the convergence of the scheme by compactness arguments. In the last section, we present some numerical results that confirm the theoretically obtained convergence.

2. The weak solution. To investigate the system (1.1) - (1.5), we make the following assumptions

- **(A1)** Ω is an open, bounded and connected polygonal domain in $\mathbb{R}^2$ with Lipschitz continuous boundary $\partial\Omega$. $\bar{\Omega}$ denotes the closure of Ω.
- **(A2)** The functions k_o and k_w: $\mathbb{R} \to \mathbb{R}$ are C^1, and there exists $\delta > 0$ such that $\delta \leq k_o(u), k_w(u) \leq 1$ for all $u \in \mathbb{R}$. We assume k_o to be an increasing function with $k_o(u) = \delta$ for $u \leq 0$ and $k_o(u) = 1$ for $u \geq 1$. Also k_w will be considered to be a decreasing function with $k_w(u) = 1$ for $u \leq 0$ and $k_w(u) = \delta$ for $u \geq 1$.
- **(A3)** $p_c : \mathbb{R} \to \mathbb{R}$ is an increasing function of u, $p_c \in C^1$, $p_c(0) = 0$ and there exist $m_p, M_p > 0$ such that $m_p \leq p_c^{'}(u) \leq M_p < \infty$.
- **(A4)** $\tau > 0$ is a positive constant.
- **(A5)** The initial condition u^0 is in $C^0 \cap W_0^{1,2}(\Omega)$.

Remark 1. *It is not necessary to take $p_c(0) = 0$. We just expect to obtain a consistent boundary condition for $u|_{\partial\Omega} = 0$. If $p_c(0) \neq 0$, one can impose $\bar{p}|_{\partial\Omega} = p_c(0)$ or define a 'new non-wetting phase pressure' $\bar{p} := p - p_c(0) + p_c(u) + \partial_t u$ to make sure that $u = 0$ at the boundary (see [25]). Furthermore, the proofs here can be extended easily to other types of boundary conditions like non-homogeneous Dirichlet or Neumann.*

Remark 2. *The choice of $u^0 \in C(\bar{\Omega})$ is for the ease of presentation, since the proposed discretization of the gradients involves continuous functions. While these are available pointwise approximations due to the spaces where these are sought, taking $u^0 \in W^{1,2}(\Omega)\backslash\Omega$ would not be sufficient to define its discrete gradient. This is however, needed in the proof, but not for the* *scheme itself. If u^0 is not continuous, then one may take into convolution with a grid-size dependent mollifier.*

Furthermore, we define P_c as

$$P_c(u) = \int_0^u p_c(s)ds. \tag{2.1}$$

Clearly, by **(A3)**, P_c is convex and for all $u \in \mathbb{R}$

$$P_c(u) \geq 0, \quad \text{with} \quad P_c(0) = 0. \tag{2.2}$$

Also one has

$$p_c(a)(a-b) \leq P_c(a) - P_c(b) \quad \text{for all } a, b \in \mathbb{R}. \tag{2.3}$$

In the following, we define the solution for the system (1.1) - (1.5):

Definition 2.1. *$(u, \bar{p}, p)$ is a weak solution of the model (1.1) - (1.5) if $u \in W^{1,2}(0,T; L^2(\Omega))$, $\bar{p}, p \in L^2(0,T; W_0^{1,2}(\Omega))$, and for any $\phi, \psi \in L^2(0,T; W_0^{1,2}(\Omega))$, $\lambda \in L^2(0,T; L^2(\Omega))$ there hold*

$$(\partial_t u, \phi) + (k_o \nabla \bar{p}, \nabla \phi) = 0, \tag{2.4}$$

$$-(\partial_t u, \psi) + (k_w \nabla p, \nabla \psi) = 0, \tag{2.5}$$

$$(\bar{p} - p, \lambda) = (p_c(u), \lambda) + \tau(\partial_t u, \lambda). \tag{2.6}$$

As mentioned before, existence and uniqueness results can be found in [14, 15, 34]. Note that by **(A3)** we obtain $u \in W^{1,2}(0,T; W_0^{1,2}(\Omega))$ (see [25]).

3. The finite volume scheme. In this section, we introduce a finite volume scheme for the system (1.1) - (1.5), then give the a priori estimates.

3.1. Meshes and notations. To introduce the finite volume scheme to the system, we consider an admissible mesh (see [23] pp. 38).

Definition 3.1. *Let Ω be an open bounded polygonal subset of $\mathbb{R}^2$. An admissible finite volume mesh of Ω, denoted by $\mathcal{T}$ is a family of triangular disjoint subsets of Ω such that two triangles may either be disjoint, or share a node, or a full edge. The set of all edges including the boundary ones is denoted by $\mathcal{E}$. The geometric centers of the triangles form the set $\mathcal{P}$. In other words:*

- *The closure of the union of all the triangles is $\bar{\Omega}$;*
- *For any $K \in \mathcal{T}$, there exists a subset $\mathcal{E}_K$ of $\mathcal{E}$ such that $\partial K = \bar{K} \setminus K = \cup_{\sigma \in \mathcal{E}_K} \bar{\sigma}$. Furthermore, $\mathcal{E} = \cup_{K \in \mathcal{T}} \mathcal{E}_K$.*
- *For any $(K, L) \in \mathcal{T}^2$ with $K \neq L$, either the 1-dimension Lebesgue measure of $\bar{K} \cap \bar{L}$ is 0 or $\bar{K} \cap \bar{L} = \sum \bar{\sigma}$ for some $\sigma \in \mathcal{E}$. There exists a subset $\mathcal{E}_K$ of $\mathcal{E}$ such that $\partial K = \bar{K} \setminus K = \cup_{\sigma \in \mathcal{E}_K} \bar{\sigma}$.*
- *The family $\mathcal{P} = \{x_K\}_{K \in \mathcal{T}}$ is such that $x_K \in K$ (for all $K \in \mathcal{T}$) and it is the geometric center of the volume K.*

Further, we assume:

- **(A6)** The angles θ of any triangle $K \in \mathcal{T}$ satisfy $\arccos(\frac{2\sqrt{m_p M_p}}{m_p + M_p}) \leq \theta \leq \pi - \arccos(\frac{2\sqrt{m_p M_p}}{m_p + M_p})$.

Remark 3. *If $p_c(\cdot)$ is a linear function with respect to u, the assumption **(A6)** can be relaxed to $0 < \theta < \pi$, which is practically fulfilled by any triangular mesh.*

Throughout this paper, the following notations are used: the mesh size is defined by $\text{size}(\mathcal{T}) = \sup\{\text{diam}(K),\ K \in \mathcal{T}\}$. The sets of interior and boundary edges are denoted by $\mathcal{E}_{\text{int}}$, resp. $\mathcal{E}_{\text{ext}}$:

$\mathcal{E}_{\text{int}} = \{\sigma \in \mathcal{E}; \sigma \not\subset \partial\Omega\}$, and $\mathcal{E}_{\text{ext}} = \{\sigma \in \mathcal{E}; \sigma \subset \partial\Omega\}$. For a triangle, we denote by $\mathrm{m}(K)$ its measure. We introduce some notations for the triangle $K \in \mathcal{T}$ (see Figure 1). P_i, P_j, P_k denote the vertices of the triangle K, x_K is the geometric center of K, $P_{i,j}, P_{j,k}, P_{k,i}$ are the midpoints of the segments P_iP_j, P_jP_k, P_kP_i. $P_{i/2,j}$ is the point on P_iP_j which satisfies $\mathrm{m}(P_iP_{i/2,j})/\mathrm{m}(P_iP_j) = 1/3$, similar to $P_{i/2,k}$. We let $\mathcal{P}_v$ stand for the set of all vertices P_i, $\mathcal{P}_M$ for all edges midpoint, $\mathcal{P}_T$ for all points $P_{i/2,j}$ introduced above. We use K_r $(r = i, j, k)$ to denote the quadrilateral determined by P_r, x_K and the midpoints $P_{r,\cdot}$, $P_{\cdot,r}$ of the edges. Let $\sigma^1_{K_i}$ denote the segment $P_iP_{i,j}$, $\sigma^2_{K_i}$ denote the segment $P_iP_{k,i}$. Then we denote $e_{\sigma^1_{K_i}} = \overrightarrow{x_KP_{i/2,j}}$, $e_{\sigma^2_{K_i}} = \overrightarrow{x_KP_{i/2,k}}$ as the vectors. Let $\mathbf{n}_{\sigma^1_{K_i}}$ and $\mathbf{n}_{\sigma^2_{K_i}}$ be the normal vectors to P_iP_j and P_kP_i outward to K_i. Finally, we define two vectors $\mu_{\sigma^1_{K_i}}, \mu_{\sigma^2_{K_i}}$ as follows (see [37, 41]):

$$\begin{cases} \mu_{\sigma^1_{K_i}} \cdot e_{\sigma^1_{K_i}} = 1, \\ \mu_{\sigma^1_{K_i}} \cdot e_{\sigma^2_{K_i}} = 0, \\ \mu_{\sigma^2_{K_i}} \cdot e_{\sigma^1_{K_i}} = 0, \\ \mu_{\sigma^2_{K_i}} \cdot e_{\sigma^2_{K_i}} = 1. \end{cases} \tag{3.1}$$

Observe that $\mu_{\sigma^1_{K_i}}$ and $\mathbf{n}_{\sigma^1_{K_i}}$, respectively $\mu_{\sigma^2_{K_i}}$ and $\mathbf{n}_{\sigma^2_{K_i}}$ are parallel.

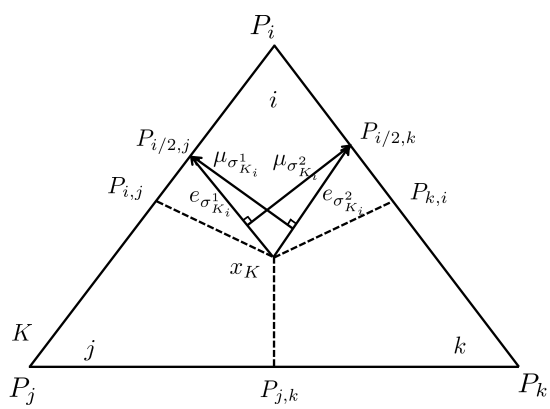

FIG. 1. *A triangular finite volume and the associated nodes, edges and vectors*

3.2. The scheme. To define the scheme, some notations are needed.

DEFINITION 3.2. *Let Ω be an open bounded polygonal subset of $\mathbb{R}^2$, $\mathcal{T}$ be an admissible mesh in Section 3.1. $h = \frac{T}{N}$ denotes the time step for any $N \in \mathbb{N}$ and t_n denotes the time at $t = nh$ for $n \in \{0, ..., N\}$. Let $X(\mathcal{T}, h)$ be the set of functions that are piecewise constant in both time and space, i.e. v from $\Omega \times (0, Nh)$ to $\mathbb{R}$ such that there exists a family of real values $\{v^n_K, K \in \mathcal{T}, n \in \{0, ..., N\}\}$, with $v(x, t) = v^n_K$ for a.e. $x \in K$, $K \in \mathcal{T}$ and for a.e. $t \in (nh, (n+1)h]$, $n \in \{0, ..., N-1\}$.*

Further, for considering discrete gradients, additional values at edges σ will be needed. In [1], Aavatsmark has defined four freedoms in each triangle: one in the center and three at the midpoint of each edge. In [38], the authors define six freedoms at each edge but not in the center of triangle. Here, we define seven freedoms in each triangle. To give the full discretization for the system (1.1) -

(1.5), we use $\{u_K^n, K\in\mathcal{T}, n\in\{0,...,N\}\}$ to denote the discrete approximation of u, the value u_K^n is the approximation of $u(x_K,nh)$, and the same for p_K^n, $\bar p_K^n$. For given $K\in\mathcal{T}$ and with P_r being one of its nodes, $r\in\{i,j,k\}$ a counterclockwise ordering, $u^n_{\sigma^1_{K_r}}$, $\bar p^n_{\sigma^1_{K_r}}$, $p^n_{\sigma^1_{K_r}}$ denote the approximations of $u(x_{P_{r/2,r+1}},t^n)$, $\bar p(x_{P_{r/2,r+1}},t^n)$, $p(x_{P_{r/2,r+1}},t^n)$ and it is similar for $u^n_{\sigma^2_{K_r}}$, $\bar p^n_{\sigma^2_{K_r}}$, $p^n_{\sigma^2_{K_r}}$.

Observe that, due to (3.1), given a vector $\mathbf{v}\in\mathbb{R}^2$ one has

$$\mathbf{v} = (\mathbf{v}\cdot e_{\sigma^1_{K_r}})\ \mu_{\sigma^1_{K_r}} + (\mathbf{v}\cdot e_{\sigma^2_{K_r}})\ \mu_{\sigma^2_{K_r}}. \tag{3.2}$$

This inspires the definition of discrete gradient: for $K\in\mathcal{T}$ and $r\in\{i,j,k\}$, let the values v_K, $v_{\sigma^1_{K_r}}$, $v_{\sigma^2_{K_r}}$ be given, the discrete gradient in the quadrilateral K_r is

$$\nabla_{K_r} v_K := (v_{\sigma^1_{K_r}} - v_K)\cdot \mu_{\sigma^1_{K_r}} + (v_{\sigma^2_{K_r}} - v_K)\cdot \mu_{\sigma^2_{K_r}}. \tag{3.3}$$

Then for any $n=0,1,...,N-1$, we define the scheme as follows

$$\begin{aligned} &\mathrm{m}(K)\frac{u_K^{n+1}-u_K^n}{h} \\ =&k_o(u_K^{n+1})\sum_{r=i,j,k}\Bigg(\mathrm{m}(\sigma^1_{K_r})\Big((\bar p^{n+1}_{\sigma^1_{K_r}}-\bar p^{n+1}_K)\ \mu_{\sigma^1_{K_r}} + (\bar p^{n+1}_{\sigma^2_{K_r}}-\bar p^{n+1}_K)\ \mu_{\sigma^2_{K_r}}\Big)\cdot\mathbf{n}_{\sigma^1_{K_r}} \\ &\qquad + \mathrm{m}(\sigma^2_{K_r})\Big((\bar p^{n+1}_{\sigma^1_{K_r}}-\bar p^{n+1}_K)\ \mu_{\sigma^1_{K_r}} + (\bar p^{n+1}_{\sigma^2_{K_r}}-\bar p^{n+1}_K)\ \mu_{\sigma^2_{K_r}}\Big)\cdot\mathbf{n}_{\sigma^2_{K_r}}\Bigg), \end{aligned} \tag{3.4}$$

$$\begin{aligned} &-\mathrm{m}(K)\frac{u_K^{n+1}-u_K^n}{h} \\ =&k_w(u_K^{n+1})\sum_{r=i,j,k}\Bigg(\mathrm{m}(\sigma^1_{K_r})\Big((p^{n+1}_{\sigma^1_{K_r}}-p^{n+1}_K)\ \mu_{\sigma^1_{K_r}} + (p^{n+1}_{\sigma^2_{K_r}}-p^{n+1}_K)\ \mu_{\sigma^2_{K_r}}\Big)\cdot\mathbf{n}_{\sigma^1_{K_r}} \\ &\qquad + \mathrm{m}(\sigma^2_{K_r})\Big((p^{n+1}_{\sigma^1_{K_r}}-p^{n+1}_K)\ \mu_{\sigma^1_{K_r}} + (p^{n+1}_{\sigma^2_{K_r}}-p^{n+1}_K)\ \mu_{\sigma^2_{K_r}}\Big)\cdot\mathbf{n}_{\sigma^2_{K_r}}\Bigg), \end{aligned} \tag{3.5}$$

$$\bar p_K^{n+1} - p_K^{n+1} = p_c(u_K^{n+1}) + \tau\frac{u_K^{n+1}-u_K^n}{h}, \tag{3.6}$$

for all $K\in\mathcal{T}$. Similarly, at each edge $\sigma\in\mathcal{E}_{\mathrm{int}}$, we impose

$$\bar p^{n+1}_{\sigma^{I_d}_{K_r}} - p^{n+1}_{\sigma^{I_d}_{K_r}} = p_c(u^{n+1}_{\sigma^{I_d}_{K_r}}) + \tau\frac{u^{n+1}_{\sigma^{I_d}_{K_r}} - u^n_{\sigma^{I_d}_{K_r}}}{h}, \quad (I_d = 1,2) \tag{3.7}$$

and the flux continuity of each phase

$$\begin{aligned} &k_o(u_K^{n+1})\Big((\bar p^{n+1}_{\sigma^1_{K_r}}-\bar p^{n+1}_K)\ \mu_{\sigma^1_{K_r}} + (\bar p^{n+1}_{\sigma^2_{K_r}}-\bar p^{n+1}_K)\ \mu_{\sigma^2_{K_r}}\Big)\cdot\mathbf{n}_{K|L} \\ +&k_o(u_L^{n+1})\Big((\bar p^{n+1}_{\sigma^1_{L_r}}-\bar p^{n+1}_L)\ \mu_{\sigma^1_{L_r}} + (\bar p^{n+1}_{\sigma^2_{L_r}}-\bar p^{n+1}_L)\ \mu_{\sigma^2_{L_r}}\Big)\cdot\mathbf{n}_{L|K} = 0, \end{aligned} \tag{3.8}$$

$$k_w(u_K^{n+1})\Big((p_{\sigma^1_{K_r}}^{n+1}-p_K^{n+1})\ \mu_{\sigma^1_{K_r}}+(p_{\sigma^2_{K_r}}^{n+1}-p_K^{n+1})\ \mu_{\sigma^2_{K_r}}\Big)\cdot\mathbf{n}_{K|L}$$

$$+k_w(u_L^{n+1})\Big((p_{\sigma^1_{L_r}}^{n+1}-p_L^{n+1})\ \mu_{\sigma^1_{L_r}}+(p_{\sigma^2_{L_r}}^{n+1}-p_L^{n+1})\ \mu_{\sigma^2_{L_r}}\Big)\cdot\mathbf{n}_{L|K}=0. \tag{3.9}$$

Here L is the neighboring element of K sharing the edge σ, and $\mathbf{n}_{L|K}$ is the unit normal vector from L into K. Whenever $\sigma\in\mathcal{E}_{\text{ext}}$, the values $\bar p_{\sigma^1_{K_r}},\bar p_{\sigma^2_{K_r}},p_{\sigma^1_{K_r}},p_{\sigma^2_{K_r}}$ are set to 0. One takes $u^{n+1}_{\sigma^{I_d}_{K_r}}=0$ $(I_d=1,2)$ for any $K\in\mathcal{T}$ and $r=i,j,k$ such that $\sigma^{I_d}_{K_r}\in\mathcal{E}_{\text{ext}}$. Also, flux continuity holds each half edge σ and for the discrete gradient is used.

Initially, we take $u_K^0=u^0(x_K)$ for any $K\in\mathcal{T}$. This makes sense since $u^0\in C(\bar\Omega)$. If $u^0\notin C(\bar\Omega)$, then one takes as explanation in Remark 2, $u^0_{\mathcal{T}}=\eta_{\mathcal{T}}*u^0$, where η is the standard mollifier ([21]). Clearly, since $u^0\in W_0^{1,2}(\Omega)$, one has $\|u^0_{\mathcal{T}}-u^0\|_{W_0^{1,2}(\Omega)}\to 0$ as size$(\mathcal{T})\to 0$.

3.3. A priori estimates and existence of the fully discrete solution. In this section, we discuss the fully discrete solution to (3.4) - (3.9). We first provide some elementary results that will be used later.

LEMMA 3.1. *Let $m\geq 1$ and $\boldsymbol{a}^j,\boldsymbol{b}^j\in\mathbb{R}^m$ be an m-dimensional real vectors, $j\in\{0,...,N\}$. We have the following identities:*

$$\sum_{j=1}^N\langle\boldsymbol{a}^j-\boldsymbol{a}^{j-1},\sum_{n=j}^N\boldsymbol{b}^n\rangle=\sum_{j=1}^N\langle\boldsymbol{a}^j,\boldsymbol{b}^j\rangle-\langle\boldsymbol{a}^0,\sum_{j=1}^N\boldsymbol{b}^j\rangle, \tag{3.10}$$

$$\sum_{j=1}^N\langle\boldsymbol{a}^j-\boldsymbol{a}^{j-1},\boldsymbol{a}^j\rangle=\frac12(|\boldsymbol{a}^N|^2-|\boldsymbol{a}^0|^2+\sum_{j=1}^N|\boldsymbol{a}^j-\boldsymbol{a}^{j-1}|^2), \tag{3.11}$$

$$\sum_{n=1}^N\langle\sum_{j=n}^N\boldsymbol{a}^j,\boldsymbol{a}^n\rangle=\frac12|\sum_{j=1}^N\boldsymbol{a}^j|^2+\frac12\sum_{j=1}^N|\boldsymbol{a}^j|^2, \tag{3.12}$$

where $<\cdot,\cdot>$ is the inner product in $\mathbb{R}^m$.

LEMMA 3.2. ***Discrete Gronwall inequality:*** *If $\{y_n\}$, $\{f_n\}$ and $\{g_n\}$ are nonnegative sequences and*

$$y_n\leq f_n+\sum_{0\leq k<n}g_ky_k,\quad \textit{for all } n\geq 0,$$

then

$$y_n\leq f_n+\sum_{0\leq k<n}f_kg_k\exp(\sum_{k<j<n}g_j),\quad \textit{for all } n\geq 0.$$

The existence of a solution to the discrete system (3.4) - (3.9) can be obtained by a Leray - Schauder argument, as done in [39]. One has

LEMMA 3.3. *Let $n\in\{0,1,...,N-1\}$, and assume u^n given. Then there exists a solution $(u_K^{n+1},u^{n+1}_{\sigma^1_{K_r}},u^{n+1}_{\sigma^2_{K_r}},\bar p_K^{n+1},\bar p^{n+1}_{\sigma^1_{K_r}},\bar p^{n+1}_{\sigma^2_{K_r}},p_K^{n+1},p^{n+1}_{\sigma^1_{K_r}},p^{n+1}_{\sigma^2_{K_r}})_{K\in\mathcal{T}}$ to the discrete system* (3.4) - (3.9).

Without entering into details, the proof requires a priori estimates, which are obtained below.

LEMMA 3.4. *A $C>0$ not depending on h or size($\mathcal{T}$) exists such that, for any $N^* \in \{0,...,N-1\}$ we have the following:*

$$\sum_{n=0}^{N^*} h \sum_{K\in\mathcal{T}} k_o(u_K^{n+1}) \sum_{r=i,j,k} m(K_r)\nabla_{K_r}\bar{p}_K^{n+1}\cdot\nabla_{K_r}\bar{p}_K^{n+1}$$
$$+\sum_{n=0}^{N^*} h \sum_{K\in\mathcal{T}} k_w(u_K^{n+1}) \sum_{r=i,j,k} m(K_r)\nabla_{K_r}p_K^{n+1}\cdot\nabla_{K_r}p_K^{n+1}$$
$$+\tau\sum_{n=0}^{N^*} h \sum_{K\in\mathcal{T}} m(K)\Big(\frac{u_K^{n+1}-u_K^n}{h}\Big)^2 + \sum_{K\in\mathcal{T}} m(K)P_c(u_K^{N^*+1}) \le C. \tag{3.13}$$

Proof. We start by proving the following:

$$\mu_{\sigma^1_{K_r}} = \frac{\mathrm{m}(\sigma^1_{K_r})\cdot\mathbf{n}_{\sigma^1_{K_r}}}{\mathrm{m}(K_r)}, \qquad \mu_{\sigma^2_{K_r}} = \frac{\mathrm{m}(\sigma^2_{K_r})\cdot\mathbf{n}_{\sigma^2_{K_r}}}{\mathrm{m}(K_r)}. \tag{3.14}$$

To see this, we refer to Figure 1 and take without losing of generality $r=i$. Note that $\mathrm{m}(P_i x_K P_{k,i}) = m(P_i x_K P_{i,j}) = \frac{1}{6}\mathrm{m}(P_i P_j P_k)$ since x_K is the geometric center and $P_{k,i}$, $P_{i,j}$ are midpoints. This gives $\mathrm{m}(K_i) = \frac{1}{3}\mathrm{m}(P_i P_j P_k)$. With θ_{I_d} being the angle spanned by $e_{\sigma^{I_d}_{K_i}}$ and $\mu_{\sigma^{I_d}_{K_i}}$, the matching height of x_K to $\sigma^{I_d}_{K_i}$ is $|e_{\sigma^{I_d}_{K_i}}|cos\theta_i = \frac{1}{|\mu_{\sigma^{I_d}_{K_i}}|}$, due to (3.1). Therefore, one has

$$\mathrm{m}(K_i) = 2\mathrm{m}(P_i x_K P_{i,j}) = \frac{1}{|\mu_{\sigma^{I_d}_{K_i}}|}\mathrm{m}(\sigma^{I_d}_{K_i}) \quad (I_d = 1,2),$$

so

$$|\mu_{\sigma^{I_d}_{K_i}}| = \frac{\mathrm{m}(\sigma^{I_d}_{K_i})}{\mathrm{m}(K_i)}.$$

This immediately implies (3.14) since $\mu_{\sigma^{I_d}_{K_i}}$ and $\mathbf{n}_{\sigma^{I_d}_{K_i}}$ are parallel and have the same sense.

Then multiplying (3.4) by $\bar{p}_K^{n+1}$, (3.8) by $\mathrm{m}(\sigma^1_{K_r})\bar{p}^{n+1}_{\sigma^1_{K_r}}$ and $\mathrm{m}(\sigma^2_{K_r})\bar{p}^{n+1}_{\sigma^2_{K_r}}$ respectively, adding the three equalities and summing the resulting over $K\in\mathcal{T}$, we find that

$$-\sum_{K\in\mathcal{T}} \mathrm{m}(K)(u_K^{n+1}-u_K^n)\bar{p}_K^{n+1} = h\sum_{K\in\mathcal{T}} k_o(u_K^{n+1}) \sum_{r=i,j,k} \mathrm{m}(K_r)\nabla_{K_r}\bar{p}_K^{n+1}\cdot\nabla_{K_r}\bar{p}_K^{n+1}. \tag{3.15}$$

Similarly, we also obtain

$$\sum_{K\in\mathcal{T}} \mathrm{m}(K)(u_K^{n+1}-u_K^n)p_K^{n+1} = h\sum_{K\in\mathcal{T}} k_w(u_K^{n+1}) \sum_{r=i,j,k} \mathrm{m}(K_r)\nabla_{K_r}p_K^{n+1}\cdot\nabla_{K_r}p_K^{n+1}. \tag{3.16}$$

Adding (3.15) and (3.16) gives

$$
\begin{aligned}
(3.17)\qquad & h\sum_{K\in\mathcal{T}} k_o(u_K^{n+1}) \sum_{r=i,j,k} \mathrm{m}(K_r)\nabla_{K_r}\bar{p}_K^{n+1}\cdot\nabla_{K_r}\bar{p}_K^{n+1} \\
& +h\sum_{K\in\mathcal{T}} k_w(u_K^{n+1}) \sum_{r=i,j,k} \mathrm{m}(K_r)\nabla_{K_r}p_K^{n+1}\cdot\nabla_{K_r}p_K^{n+1} \\
& +\sum_{K\in\mathcal{T}} \mathrm{m}(K)(u_K^{n+1}-u_K^n)(\bar{p}_K^{n+1}-p_K^{n+1}) = 0.
\end{aligned}
$$

Further, multiplying (3.6) by $\mathrm{m}(K)(u_K^{n+1}-u_K^n)$ and summing the resulting over $K\in\mathcal{T}$ leads to

$$
\begin{aligned}
(3.18)\qquad & \sum_{K\in\mathcal{T}} \mathrm{m}(K)(\bar{p}_K^{n+1}-p_K^{n+1})(u_K^{n+1}-u_K^n) \\
& =\sum_{K\in\mathcal{T}} \mathrm{m}(K)p_c(u_K^{n+1})(u_K^{n+1}-u_K^n)+\tau\sum_{K\in\mathcal{T}} \mathrm{m}(K)\frac{u_K^{n+1}-u_K^n}{h}(u_K^{n+1}-u_K^n).
\end{aligned}
$$

Using this into (3.17) gives

$$
\begin{aligned}
(3.19)\qquad & h^2\sum_{K\in\mathcal{T}} k_o(u_K^{n+1}) \sum_{r=i,j,k} \mathrm{m}(K_r)\nabla_{K_r}\bar{p}_K^{n+1}\cdot\nabla_{K_r}\bar{p}_K^{n+1} \\
& +h^2\sum_{K\in\mathcal{T}} k_w(u_K^{n+1}) \sum_{r=i,j,k} \mathrm{m}(K_r)\nabla_{K_r}p_K^{n+1}\cdot\nabla_{K_r}p_K^{n+1} \\
& +\tau\sum_{K\in\mathcal{T}} \mathrm{m}(K)(u_K^{n+1}-u_K^n)^2+h\sum_{K\in\mathcal{T}} \mathrm{m}(K)p_c(u_K^{n+1})(u_K^{n+1}-u_K^n) = 0.
\end{aligned}
$$

Recalling (2.3), one gets

$$
\begin{aligned}
(3.20)\qquad & h^2\sum_{K\in\mathcal{T}} k_o(u_K^{n+1}) \sum_{r=i,j,k} \mathrm{m}(K_r)\nabla_{K_r}\bar{p}_K^{n+1}\cdot\nabla_{K_r}\bar{p}_K^{n+1} \\
& +h^2\sum_{K\in\mathcal{T}} k_w(u_K^{n+1}) \sum_{r=i,j,k} \mathrm{m}(K_r)\nabla_{K_r}p_K^{n+1}\cdot\nabla_{K_r}p_K^{n+1} \\
& +\tau\sum_{K\in\mathcal{T}} \mathrm{m}(K)(u_K^{n+1}-u_K^n)^2+h\sum_{K\in\mathcal{T}} \mathrm{m}(K)P_c(u_K^{n+1}) \le h\sum_{K\in\mathcal{T}} \mathrm{m}(K)P_c(u_K^n).
\end{aligned}
$$

Summing the above equation from 0 to N^* for any $N^*\in\{0,...,N-1\}$ gives

$$
\begin{aligned}
(3.21)\qquad & \sum_{n=0}^{N^*} h\sum_{K\in\mathcal{T}} k_w(u_K^{n+1}) \sum_{r=i,j,k} \mathrm{m}(K_r)\nabla_{K_r}p_K^{n+1}\cdot\nabla_{K_r}p_K^{n+1} \\
& +\sum_{n=0}^{N^*} h\sum_{K\in\mathcal{T}} k_o(u_K^{n+1}) \sum_{r=i,j,k} \mathrm{m}(K_r)\nabla_{K_r}\bar{p}_K^{n+1}\cdot\nabla_{K_r}\bar{p}_K^{n+1} \\
& +\tau\sum_{n=0}^{N^*} h\sum_{K\in\mathcal{T}} \mathrm{m}(K)\Big(\frac{u_K^{n+1}-u_K^n}{h}\Big)^2+\sum_{K\in\mathcal{T}} \mathrm{m}(K)P_c(u_K^{N^*+1}) \le \sum_{K\in\mathcal{T}} \mathrm{m}(K)P_c(u_K^0).
\end{aligned}
$$

This proof is then concluded by using the continuity of $p_c(\cdot)$ and **(A5)**, yielding

$$\sum_{K\in\mathcal{T}} \mathrm{m}(K)P_c(u_K^0) \le C.$$

□

To obtain estimates in terms of the discrete gradients of the saturation, we first prove the result below:

PROPOSITION 3.1. *Given $\alpha,\beta \in [m_p, M_p]$ and two vectors $\boldsymbol{a},\boldsymbol{b}\in\mathbb{R}^d$ ($d\ge 2$) such that the angle in between is $\gamma\in[arccos(\frac{2\sqrt{m_pM_p}}{m_p+M_p}), \pi - arccos(\frac{2\sqrt{m_pM_p}}{m_p+M_p})]$, one has*

$$\alpha|\boldsymbol{a}|^2+\beta|\boldsymbol{b}|^2+(\alpha+\beta)|\boldsymbol{a}|\ |\boldsymbol{b}|cos\gamma \ge 0. \tag{3.22}$$

Proof. The case $\mathbf{b}=\mathbf{0}$ is trivial. If $\mathbf{b}\neq\mathbf{0}$, let $x=\frac{|\mathbf{a}|}{|\mathbf{b}|}$. Then, the proof reduces to showing that

$$\alpha x^2+(\alpha+\beta)cos\gamma\ x+\beta\ge 0,$$

for all $x\in\mathbb{R}$. Since $|cos\gamma|\le\frac{2\sqrt{m_pM_p}}{m_p+M_p}$, one has

$$\begin{aligned}\Delta &:= (\alpha+\beta)^2(cos\gamma)^2-4\alpha\beta\\ &\le(\alpha+\beta)^2\frac{4m_pM_p}{(m_p+M_p)^2}-4\alpha\beta\\ &=4\alpha^2\Big(\frac{m_pM_p}{(m_p+M_p)^2}(1+\frac{\beta}{\alpha})^2-\frac{\beta}{\alpha}\Big).\end{aligned}$$

Observing that $\frac{m_p}{M_p}\le\frac{\beta}{\alpha}\le\frac{M_p}{m_p}$, one immediately sees that $\Delta\le 0$, which concludes the proof. □

Now we can provide the a priori estimates

LEMMA 3.5. *If $h<\tau$, for any $N^*\in\{0,...,N-1\}$ it holds*

$$\sum_{K\in\mathcal{T}}\sum_{r=i,j,k} m(K_r)|\nabla_{K_r}u_K^{N^*+1}|^2\le C, \tag{3.23}$$

where C is independent of h, size($\mathcal{T}$), or N^*.

Proof. Subtracting (3.6) from (3.7) gives

$$\begin{aligned}&h(\bar{p}^{n+1}_{\sigma^{I_d}_{K_r}}-\bar{p}^{n+1}_K)-h(p^{n+1}_{\sigma^{I_d}_{K_r}}-p^{n+1}_K)\\ =&h(p_c(u^{n+1}_{\sigma^{I_d}_{K_r}})-p_c(u^{n+1}_K))+\tau(u^{n+1}_{\sigma^{I_d}_{K_r}}-u^{n+1}_K)-\tau(u^{n}_{\sigma^{I_d}_{K_r}}-u^{n}_K)\quad(I_d=1,2),\end{aligned} \tag{3.24}$$

Multiplying (3.24) by $\mathrm{m}(K_r)\mu_{\sigma^{I_d}_{K_r}}\nabla_{K_r}u^{n+1}_K$, adding the resulting for $I_d=1$ and 2 and summing over $r \in \{i, j, k\}$, $K \in \mathcal{T}$ and $n \in \{0, ..., N^*\}$ for any fixed $N^* < N$ gives

$$\begin{aligned}
&\sum_{n=0}^{N^*} h \sum_{K\in\mathcal{T}} \sum_{r=i,j,k} \mathrm{m}(K_r)(\nabla_{K_r}\bar{p}_K^{n+1} - \nabla_{K_r}p_K^{n+1}) \cdot \nabla_{K_r}u_K^{n+1} \\
=&\sum_{n=0}^{N^*} h \sum_{K\in\mathcal{T}} \sum_{r=i,j,k} \mathrm{m}(K_r)\nabla_{K_r}p_c(u_K^{n+1}) \cdot \nabla_{K_r}u_K^{n+1} \\
+&\sum_{n=0}^{N^*} \tau \sum_{K\in\mathcal{T}} \sum_{r=i,j,k} \mathrm{m}(K_r)(\nabla_{K_r}u_K^{n+1} - \nabla_{K_r}u_K^{n}) \cdot \nabla_{K_r}u_K^{n+1}.
\end{aligned}$$

Applying on the left the Cauchy-Schwarz inequality and using Lemma 3.1 for the last term on the right gives

$$\begin{aligned}
&\frac{\tau}{2}\sum_{K\in\mathcal{T}} \sum_{r=i,j,k} \mathrm{m}(K_r)|\nabla_{K_r}u_K^{N^*+1}|^2 + \frac{\tau}{2}\sum_{n=0}^{N^*}\sum_{K\in\mathcal{T}} \sum_{r=i,j,k} \mathrm{m}(K_r)|\nabla_{K_r}u_K^{n+1} - \nabla_{K_r}u_K^{n}|^2 \\
&+\sum_{n=0}^{N^*} h \sum_{K\in\mathcal{T}} \sum_{r=i,j,k} \mathrm{m}(K_r)\nabla_{K_r}p_c(u_K^{n+1}) \cdot \nabla_{K_r}u_K^{n+1} \\
&\leq \frac{\tau}{2}\sum_{K\in\mathcal{T}} \sum_{r=i,j,k} \mathrm{m}(K_r)|\nabla_{K_r}u_K^{0}|^2 + \frac{1}{2}\sum_{n=0}^{N^*} h \sum_{K\in\mathcal{T}} \sum_{r=i,j,k} \mathrm{m}(K_r)|\nabla_{K_r}\bar{p}_K^{n+1} - \nabla_{K_r}p_K^{n+1}|^2 \\
&\quad + \frac{1}{2}\sum_{n=0}^{N^*} h \sum_{K\in\mathcal{T}} \sum_{r=i,j,k} \mathrm{m}(K_r)|\nabla_{K_r}u_K^{n+1}|^2.
\end{aligned}$$

By Lemma 3.4, the second term on the right is bounded uniformly in h, size($\mathcal{T}$) and N^*. This gives:

$$\begin{aligned}
&\frac{\tau}{2}\sum_{K\in\mathcal{T}} \sum_{r=i,j,k} \mathrm{m}(K_r)|\nabla_{K_r}u_K^{N^*+1}|^2 + \frac{\tau}{2}\sum_{n=0}^{N^*}\sum_{K\in\mathcal{T}} \sum_{r=i,j,k} \mathrm{m}(K_r)|\nabla_{K_r}u_K^{n+1} - \nabla_{K_r}u_K^{n}|^2 \\
&+\sum_{n=0}^{N^*} h \sum_{K\in\mathcal{T}} \sum_{r=i,j,k} \mathrm{m}(K_r)\nabla_{K_r}p_c(u_K^{n+1}) \cdot \nabla_{K_r}u_K^{n+1} \\
\leq& C + \frac{1}{2}\sum_{n=0}^{N^*} h \sum_{K\in\mathcal{T}} \sum_{r=i,j,k} \mathrm{m}(K_r)|\nabla_{K_r}u_K^{n+1}|^2.
\end{aligned}$$

Furthermore, the third term on the left is positive. To see this, observe that for any $n \in \{0, ..., N^*\}$,

$K \in \mathcal{T}$, $r \in \{i,j,k\}$ there exist $\xi_1, \xi_2 \in \mathbb{R}$ such that

$$\begin{aligned}
&\sum_{n=0}^{N^*} h \sum_{K\in\mathcal{T}} \sum_{r=i,j,k} \mathrm{m}(K_r) \nabla_{K_r} p_c(u_K^{n+1}) \cdot \nabla_{K_r} u_K^{n+1} \\
&= \sum_{n=0}^{N^*} h \sum_{K\in\mathcal{T}} \sum_{r=i,j,k} \mathrm{m}(K_r) \Big(\big(p_c(u^{n+1}_{\sigma^1_{K_r}}) - p_c(u_K^{n+1})\big)\mu_{\sigma^1_{K_r}} + \big(p_c(u^{n+1}_{\sigma^2_{K_r}}) - p_c(u_K^{n+1})\big)\mu_{\sigma^2_{K_r}} \Big) \\
&\qquad \cdot \Big(\big(u^{n+1}_{\sigma^1_{K_r}} - u_K^{n+1}\big)\mu_{\sigma^1_{K_r}} + \big(u^{n+1}_{\sigma^2_{K_r}} - u_K^{n+1}\big)\mu_{\sigma^2_{K_r}} \Big) \\
&= \sum_{n=0}^{N^*} h \sum_{K\in\mathcal{T}} \sum_{r=i,j,k} \mathrm{m}(K_r) \Bigg(p_c'(\xi_1) \Big(\big(u^{n+1}_{\sigma^1_{K_r}} - u_K^{n+1}\big)|\mu_{\sigma^1_{K_r}}| \Big)^2 + p_c'(\xi_2) \Big(\big(u^{n+1}_{\sigma^2_{K_r}} - u_K^{n+1}\big)|\mu_{\sigma^2_{K_r}}| \Big)^2 \\
&\qquad + \big(p_c'(\xi_1) + p_c'(\xi_2)\big)\big(u^{n+1}_{\sigma^1_{K_r}} - u_K^{n+1}\big)|\mu_{\sigma^1_{K_r}}| \cdot \big(u^{n+1}_{\sigma^2_{K_r}} - u_K^{n+1}\big)|\mu_{\sigma^2_{K_r}}| \cdot \cos(\pi-\theta) \Bigg).
\end{aligned}$$

Note that to avoid an excess of notions, we omitted any additional indices for ξ_1, ξ_2, which actually depend on the particular n, K or r. Observing that γ the angle between $\mu_{\sigma^1_{K_r}}$ and $\mu_{\sigma^2_{K_r}}$, satisfies $\gamma = \pi - \theta$ and by **(A6)**, $|\cos\gamma| \le \frac{2\sqrt{m_p M_p}}{m_p + M_p}$, using **(A3)** and Proposition 3.1, one immediately gets that the above is positive. This gives

$$\frac{\tau - h}{2} \sum_{K\in\mathcal{T}} \sum_{r=i,j,k} \mathrm{m}(K_r) |\nabla_{K_r} u_K^{N^*+1}|^2 \le C + \frac{1}{2} \sum_{n=0}^{N^*-1} h \sum_{K\in\mathcal{T}} \sum_{r=i,j,k} \mathrm{m}(K_r) |\nabla_{K_r} u_K^{n+1}|^2,$$

and the conclusion is a direct consequence of Lemma 3.2. □

4. Convergence of the scheme.

4.1. Compactness results. To prove the convergence we recall Definition 3.2 and use the time-space discrete values to construct a sequence of triples defined in $\Omega \times (0,T]$:

$$v_{\mathcal{T},h}(x,t) = v_K^n \quad \text{for all} \quad x \in K \quad \text{and} \quad t \in (nh,(n+1)h], \quad n = 0,...,N-1, \tag{4.1}$$

and we define the discrete gradient in Ω as

$$\nabla_{\mathcal{T}} v_{\mathcal{T},h}(x,t) = \sum_{r=i,j,k} \nabla_{K_r} v_K^n \quad \text{for all} \quad x \in K \quad \text{and} \quad t \in (nh,(n+1)h], \quad n = 0,...,N-1. \tag{4.2}$$

Further, we will use the discrete version of the seminorm in the space $L^2(0,T;W^{1,2}(\Omega))$.

Definition 4.1. **(Discrete seminorms)** *For $v \in X(\mathcal{T},h)$ enriched with values $\{(v^n_{\sigma^1_{K_r}}, v^n_{\sigma^2_{K_r}})|K \in \mathcal{T}, r = i,j,k\}$, define*

$$|v(\cdot,t)|_{1,\mathcal{T}} = \left(\sum_K \sum_{r=i,j,k} \mathrm{m}(K_r) \Big(|v^n_{\sigma^1_{K_r}} - v^n_K|^2 |\mu_{\sigma^1_{K_r}}|^2 + |v^n_{\sigma^2_{K_r}} - v^n_K|^2 |\mu_{\sigma^2_{K_r}}|^2 \Big) \right)^{1/2},$$

for all $t \in (nh,(n+1)h]$, $\quad n = 0, ..., N-1$, *and*

$$|v|_{1,\mathcal{T},h} = \left(\sum_{n=0}^{N} h \sum_{K} \sum_{r=i,j,k} \mathrm{m}(K_r) \Big(|v^n_{\sigma^1_{K_r}} - v^n_K|^2 |\mu_{\sigma^1_{K_r}}|^2 + |v^n_{\sigma^2_{K_r}} - v^n_K|^2 |\mu_{\sigma^2_{K_r}}|^2 \Big) \right)^{1/2}.$$

Note that $|\cdot|_{1,\mathcal{T}}$ and $|\cdot|_{1,\mathcal{T},h}$ are the discrete counterparts of the gradient norms for functions in $W^{1,2}(\Omega)$, respectively $L^2(0,T;W^{1,2}(\Omega))$. Clearly, these are seminorms in the corresponding spaces. Following the Lemma 3.4 and Lemma 3.5, we have

LEMMA 4.1. *Under assumption* **(A3)**, *if* $(u_{\mathcal{T},h}, \bar{p}_{\mathcal{T},h}, p_{\mathcal{T},h}) \in (X(\mathcal{T},h)))^3$ *solves the system* (3.4) - (3.9), *one has*

$$|p_{\mathcal{T},h}|^2_{1,\mathcal{T},h} + |\bar{p}_{\mathcal{T},h}|^2_{1,\mathcal{T},h} + |u_{\mathcal{T},h}|^2_{1,\mathcal{T},h} \le C, \quad \textit{and} \quad |u_{\mathcal{T},h}|^2_{1,\mathcal{T}} \le C \quad \textit{for all } t \in (0,T],$$

where C *does not depend on size*$(\mathcal{T})$ *or* h.

The following is a discrete counterpart of the Poincaré inequality. Before stating it, let $(v_K, v_{\sigma^1_{K_i}}, v_{\sigma^2_{K_i}}, v_{\sigma^1_{K_j}}, v_{\sigma^2_{K_j}}, v_{\sigma^1_{K_k}}, v_{\sigma^2_{K_k}})$ be given 7-tuples for any $K \in \mathcal{T}$ satisfying $v_{\sigma^1_{K_r}} = v_{\sigma^2_{K_r}} = 0$ whenever $\sigma^1_{K_r}, \sigma^2_{K_r} \in \mathcal{E}_{\text{ext}}$. Let $X^0_{\mathcal{T}}$ be the space of piecewise function $v : \Omega \to \mathbb{R}, v|_K = v_K$, endowed with the discrete gradient $\nabla_K v$ by using the additional values, and $\|\cdot\|_{L^2(\Omega)}$ we mean the L^2-norm of the piecewise constant v. Then we have

LEMMA 4.2. **(Discrete Poincaré inequality)** *A constant* $C > 0$ *depending on* Ω, *but not on size*$(\mathcal{T})$ *exists such that*

$$\|v(\cdot)\|^2_{L^2(\Omega)} \le C|v|^2_{1,\mathcal{T}}.$$

Proof. We essentially apply the technique in [23]. For $\sigma \in \mathcal{E}$, define χ_σ from $\mathbb{R}^2 \times \mathbb{R}^2$ to $\{0,1\}$ as

$$\chi_\sigma(x,y) := \begin{cases} 1, & [x,y] \cap \sigma \neq \emptyset, \\ 0, & [x,y] \cap \sigma = \emptyset. \end{cases} \tag{4.3}$$

Let $\mathbf{e}$ be a given vector and $x \in \Omega$. Let $\mathcal{D}_x$ be the semi-line having the origin x and continuing in the direction of $\mathbf{e}$. Let $y(x)$ be such that $y(x) \in \mathcal{D}_x \cap \partial\Omega$ and $[x, y(x)] \subset \bar{\Omega}$, where $[x,y(x)] = \{\beta x + (1-\beta) y(x), \beta \in [0,1]\}$ (i.e. $y(x)$ is the first point where $\mathcal{D}_x$ meets $\partial\Omega$). For $y(x)$ such that $\chi_\sigma(x, y(x)) = 1$ and if $\sigma \in \mathcal{E}_{\text{int}}$, let $K|L$ be the triangles adjacent to σ, where K is the one closest to x. Let v_K, v_L be the corresponding values. Also, let v_σ be the value from $\{v_{\sigma^{I_d}_{K_r}}, I = 1,2, r = i,j,k\}$ corresponding to the past of σ in K intersected by $\mathcal{D}_x$. For $\sigma \in \mathcal{E}_{\text{ext}}$, one takes $v_L = v_\sigma = 0$. Also if the intersection point happens to be vertex in $\mathcal{P}$ or edge midpoint in $\mathcal{P}_M$, then one takes arbitrary v_σ, v_K, v_L. The choice is not important as finally, we integrate for $x \in \Omega$.

Along $[x, y(x)]$, we define

$$D_\sigma v := \begin{cases} |v_\sigma - v_K| + |v_\sigma - v_L|, & \text{if } \sigma \in \mathcal{E}_{\text{int}} \text{ and } \chi_\sigma(x,y) = 1, \\ |v_\sigma - v_K|, & \text{if } \sigma \in \mathcal{E}_{\text{ext}} \text{ and } \chi_\sigma(x,y) = 1, \\ 0, & \text{if } \chi_\sigma(x,y) = 0. \end{cases} \tag{4.4}$$

Let now $K \in \mathcal{T}$ arbitrary. For a.e. $x \in K$, one has

$$|v_K| \le \sum_{\sigma \in \mathcal{E}} \chi_\sigma(x, y(x)) D_\sigma v. \tag{4.5}$$

Using the Cauchy - Schwarz inequality, this gives

$$|v_K|^2 \leq \sum_{\sigma\in\mathcal{E}} \chi_\sigma(x,y(x)) \frac{(D_\sigma v)^2}{d_\sigma c_\sigma} \sum_{\sigma\in\mathcal{E}} \chi_\sigma(x,x+\eta) d_\sigma c_\sigma, \tag{4.6}$$

for a.e. $x \in \mathbb{R}^2$, where $c_\sigma = |\mathbf{n}_\sigma \cdot \mathbf{e}|$, $\mathbf{n}_\sigma$ denotes a unit normal vector to σ, and

$$d_\sigma := \begin{cases} \frac{1}{|\mu_{\sigma_K}|} + \frac{1}{|\mu_{\sigma_L}|}, & \text{if } \sigma \in \mathcal{E}_{\text{int}}, \\ \frac{1}{|\mu_{\sigma_K}|}, & \text{if } \sigma \in \mathcal{E}_{\text{ext}}. \end{cases} \tag{4.7}$$

As in [23], we show that for a.e. $x \in \Omega$

$$\sum_{\sigma\in\mathcal{E}} \chi_\sigma(x,y(x)) d_\sigma c_\sigma \leq 2\text{diam}(\Omega). \tag{4.8}$$

Given now $\mathbf{e}$ and $x \in K$ for any $K \in \mathcal{T}$ and assuming that $\mathcal{D}_x$ does not go through any vertex. Assuming $\sigma_L \in \bar{L} \cap \partial\Omega$, $L \in \mathcal{T}$, let $x_{\sigma_x} \in \bar{\Omega}$ be the the point of the intersection $e_{\sigma^{I_d}_{L_r}}$ ($I_d = 1$ or 2, $r = i, j$ or k) to σ_L whenever $y(x) \in \sigma^{I_d}_{L_r}$. Since the control volumes are convex, there exists $x_C \in K$ and lies on the extension of $\sigma^{I_d}_{K_r}$ in opposite direction such that

$$\sum_{\sigma\in\mathcal{E}} \chi_\sigma(x,y(x)) d_\sigma c_\sigma \leq |(x_C - x_{\sigma_x}) \cdot \mathbf{e}| + \text{diam}(\Omega). \tag{4.9}$$

Further, using $x_C, x_{\sigma_x} \in \bar{\Omega}$ gives (4.8). Integrating (4.6) over Ω and using to (4.8) this gives

$$\begin{aligned} &\sum_{K\in\mathcal{T}} \int_K |v_K|^2 dx \\ \leq &2\text{diam}(\Omega) \Big(\sum_{\sigma\in\mathcal{E}_{\text{int}}} \int_\Omega \chi_\sigma(x,y(x)) dx \frac{(|v_\sigma - v_K| + |v_\sigma - v_L|)^2}{d_\sigma c_\sigma} + \sum_{\sigma\in\mathcal{E}_{\text{ext}}} \int_\Omega \chi_\sigma(x,y(x)) dx \frac{|v_\sigma - v_K|^2}{d_\sigma c_\sigma} \Big). \end{aligned}$$

Since $\int_\Omega \chi_\sigma(x,y(x))dx \leq \text{diam}(\Omega)\text{m}(\sigma)c_\sigma$, one has

$$\begin{aligned} &\sum_{K\in\mathcal{T}} \int_K |v_K|^2 dx \\ \leq &4(\text{diam}(\Omega))^2 \Big(\sum_{\sigma\in\mathcal{E}_{\text{int}}} \text{m}(\sigma)(|\mu_{\sigma_K}||v_\sigma - v_K|^2 + |\mu_{\sigma_L}||v_\sigma - v_L|^2) + \sum_{\sigma\in\mathcal{E}_{\text{ext}}} \text{m}(\sigma)(|\mu_{\sigma_K}||v_\sigma - v_K|^2 \Big). \end{aligned}$$

Recalling (3.14), Definition 4.1 and Lemma 4.1, we have

$$\|v\|^2_{L^2(\Omega)} \leq C|v|^2_{1,\mathcal{T}}.$$

□

With this lemma, one has uniformly with respect to $\mathcal{T}$ and h

$$\|u_{\mathcal{T},h}\|^2_{L^2(0,T;L^2(\Omega))} + \|\bar{p}_{\mathcal{T},h}\|^2_{L^2(0,T;L^2(\Omega))} + \|p_{\mathcal{T},h}\|^2_{L^2(0,T;L^2(\Omega))} \leq C.$$

Now we show the following lemma about space translations.

LEMMA 4.3. *Given the trianglarization $\mathcal{T}$ and $v \in X^0_{\mathcal{T}}$, let $\tilde{v}$ be the extension of v by 0 to the entire $\mathbb{R}$. Then for any $\eta \in \mathbb{R}^2$, one has*

$$\|\tilde{v}(\cdot+\eta) - \tilde{v}(\cdot)\|^2_{L^2(\mathbb{R}^2)} \leq 2|v|^2_{1,\mathcal{T}}|\eta|(|\eta| + C\text{size}(\mathcal{T})), \tag{4.10}$$

with $C > 0$ only depending on Ω and not on v, η or $\mathcal{T}$.

Proof. For $\sigma \in \mathcal{E}$, using χ_σ as defined in (4.3), and for $\eta \in \mathbb{R}^2$, one has

$$|\tilde{v}(x+\eta) - \tilde{v}(x)| \leq \sum_{\sigma\in\mathcal{E}} \chi_\sigma(x, x+\eta) D_\sigma v, \text{ for a.e. } x \in \Omega,$$

where K, L are the volumes adjacent to σ. Following again [23], but defining d_σ as in (4.7), one obtains

$$\begin{aligned} &|\tilde{v}(x+\eta,t) - \tilde{v}(x,t)|^2 \\ \leq&\Big(\sum_{\sigma\in\mathcal{E}_{\text{int}}} \chi_\sigma(x,x+\eta)\frac{(|v_\sigma - v_K| + |v_\sigma - v_L|)^2}{d_\sigma c_\sigma} \\ &+ \sum_{\sigma\in\mathcal{E}_{\text{ext}}} \chi_\sigma(x,x+\eta)\frac{|v_\sigma - v_K|^2}{d_\sigma c_\sigma}\Big) \cdot \sum_{\sigma\in\mathcal{E}} \chi_\sigma(x,x+\eta) d_\sigma c_\sigma, \end{aligned} \tag{4.11}$$

for a. e. $x \in \mathbb{R}^2$. Here $c_\sigma = |\mathbf{n}_\sigma \cdot \frac{\eta}{|\eta|}|$, and $\mathbf{n}_\sigma$ denotes a unit normal vector to σ. First, by [23] there exists $C > 0$, only depending on Ω such that

$$\sum_{\sigma\in\mathcal{E}} \chi_\sigma(x,x+\eta) d_\sigma c_\sigma \leq |\eta| + C\text{size}(\mathcal{T}), \text{ for a.e. x} \in \mathbb{R}^2. \tag{4.12}$$

Further, observe that for all $\sigma \in \mathcal{E}$,

$$\int_{\mathbb{R}^2} \chi_\sigma(x,x+\eta)dx \leq \text{m}(\sigma) c_\sigma |\eta|.$$

Therefore, integrating (4.11) over $\mathbb{R}^2$ and using (3.14) one gets

$$\begin{aligned} &\|\tilde{v}(\cdot+\eta,\cdot) - \tilde{v}(\cdot,\cdot)\|^2_{L^2(\mathbb{R}^2)} \\ \leq&\int_{\mathbb{R}^2} \sum_{\sigma\in\mathcal{E}_{\text{int}}} \chi_\sigma(x,x+\eta)\frac{(|v_\sigma - v_K| + |v_\sigma - v_L|)^2}{d_\sigma c_\sigma} \sum_{\sigma\in\mathcal{E}} \chi_\sigma(x,x+\eta) d_\sigma c_\sigma dx \\ &+ \int_{\mathbb{R}^2} \sum_{\sigma\in\mathcal{E}_{\text{ext}}} \chi_\sigma(x,x+\eta)\frac{|v_\sigma - v_K|^2}{d_\sigma c_\sigma} \sum_{\sigma\in\mathcal{E}} \chi_\sigma(x,x+\eta) d_\sigma c_\sigma dx \\ \leq&\Big(\sum_{\sigma\in\mathcal{E}_{\text{int}}} \frac{\text{m}(\sigma)}{d_\sigma}(|v_\sigma - v_K| + |v_\sigma - v_L|)^2 + \sum_{\sigma\in\mathcal{E}_{\text{ext}}} \frac{\text{m}(\sigma)}{d_\sigma}|v_\sigma - v_K|^2\Big)|\eta|(|\eta| + C\text{size}(\mathcal{T})) \\ \leq&2\Big(\sum_{\sigma\in\mathcal{E}_{\text{int}}} \text{m}(\sigma)\frac{|\mu_{\sigma_K}||\mu_{\sigma_L}|}{|\mu_{\sigma_K}| + |\mu_{\sigma_L}|}(|v_\sigma - v_K|^2 + |v_\sigma - v_L|^2) + \sum_{\sigma\in\mathcal{E}_{\text{ext}}} \text{m}(\sigma)|\mu_{\sigma_K}||v_\sigma - v_K|^2\Big)|\eta|(|\eta| + C\text{size}(\mathcal{T})) \\ \leq&2\Big(\sum_{\sigma\in\mathcal{E}_{\text{int}}} \text{m}(\sigma)(|\mu_{\sigma_K}||v_\sigma - v_K|^2 + |\mu_{\sigma_L}||v_\sigma - v_L|^2) + \sum_{\sigma\in\mathcal{E}_{\text{ext}}} \text{m}(\sigma)|\mu_{\sigma_K}||v_\sigma - v_K|^2\Big)|\eta|(|\eta| + C\text{size}(\mathcal{T})) \\ =&2|v|^2_{1,\mathcal{T}}|\eta|(|\eta| + C\text{size}(\mathcal{T})), \end{aligned}$$

which completes the proof. □

The result in Lemma 4.3 extends straightforwardly to the case where v is time dependent as well, namely if v is piecewise constant in the space-time volumes as in the case of $X(\mathcal{T}, h)$ elements. Clearly, when estimating the $L^2(0, T; L^2(\mathbb{R}^2))$ norm, in this case the norm $|v|^2_{1,\mathcal{T},h}$ will appear on the right. We continue with the estimates for the time translations:

LEMMA 4.4. *Let* $\{u_K^{n+1}, n = 0, ..., N-1\}$ *be the* u *components of the solution of* (3.4) - (3.9) *and* $u_{\mathcal{T},h}$ *the extension to* $\Omega \times (0, T]$ *defined in* (4.1). *A* $C > 0$ *exists such that for any* $\xi \in (0, T)$

$$\|u_{\mathcal{T},h}(\cdot, \cdot + \xi) - u_{\mathcal{T},h}(\cdot, \cdot)\|^2_{L^2(\Omega\times(0,T-\xi))} \leq C,$$

where $C > 0$ *only depending on* Ω *and not on* ξ, h *or* $\mathcal{T}$.

Proof. Letting

$$B(t) = \int_\Omega \Big(u_{\mathcal{T},h}(x, t+\xi) - u_{\mathcal{T},h}(x, t)\Big)^2 dx,$$

for $t \in (0, T-\xi)$, one has obviously

$$\int_{\Omega\times(0,T-\xi)} \Big(u_{\mathcal{T},h}(x, t+\xi) - u_{\mathcal{T},h}(x, t)\Big)^2 dxdt = \int_0^{T-\xi} B(t)dt.$$

With $n_0(t), n_1(t) \in \{0, ..., N-1\}$ such that $n_0(t)h \leq t \leq (n_0(t)+1)h$ and $n_1(t)h \leq t+\xi \leq (n_1(t)+1)h$, B rewrites

$$B(t) = \sum_{K\in\mathcal{T}} \mathrm{m}(K)\Big(u_K^{n_1(t)} - u_K^{n_0(t)}\Big)^2 = \sum_{K\in\mathcal{T}} \mathrm{m}(K)\Big(\sum_{n=n_0(t)}^{n_1(t)-1} u_K^{n+1} - u_K^n\Big)^2. \tag{4.13}$$

Using the Cauchy-Schwarz Inequality gives

$$B(t) \leq N \sum_{K\in\mathcal{T}} \mathrm{m}(K) \sum_{n=n_0(t)}^{n_1(t)-1} \big(u_K^{n+1} - u_K^n\big)^2. \tag{4.14}$$

Defining $\chi(n; t, t+\xi)$ as

$$\chi(n; t, t+\xi) = \begin{cases} 1, & \text{if} \quad nk \in (t, t+\xi], \\ 0, & \text{if} \quad nk \notin (t, t+\xi], \end{cases}$$

(4.14) becomes

$$B(t) \leq N \sum_{n=0}^{N-1} \chi_n(t, t+\xi) \sum_{K\in\mathcal{T}} \mathrm{m}(K)\Big(u_K^{n+1} - u_K^n\Big)^2.$$

Since $0 \leq \int_0^{T-\xi} \chi_n(t, t+\xi)dt \leq \xi$, we have

$$\int_0^{T-\xi} B(t)dt \leq N\xi \sum_{n=0}^{N-1} \sum_{K\in\mathcal{T}} \mathrm{m}(K)\Big(u_K^{n+1} - u_K^n\Big)^2.$$

Following (3.20) and according to **(A5)**, one has

$$\tau\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}}\mathrm{m}(K)\Big(u_K^{n+1}-u_K^n\Big)^2\le h\sum_{K\in\mathcal{T}}\mathrm{m}(K)P_c(u_K^0),$$

which gives

$$\int_0^{T-\xi}B(t)dt\le CT\xi,$$

and the proof is concluded. □

4.2. Convergence results. In this section, we show the convergence of the finite volume scheme. Following the a priori estimates obtained above, one has

THEOREM 4.1. *There exists a sequence $(\mathcal{T}_m,h_m)$ such that size$(\mathcal{T}_m)\to 0$, $h_m\to 0$ as $m\to\infty$, and the triple $(u_{\mathcal{T}_m,h_m},\bar{p}_{\mathcal{T}_m,h},p_{\mathcal{T}_m,h_m})$ converges weakly in $L^2(Q)$ to the solution $(u,\bar{p},p)$ in the sense of Definition 2.1. Moreover, $u_{\mathcal{T}_m,h_m}$ converges strongly to u in $L^2(0,T;L^2(\Omega))$.*

Proof. Lemma 4.1 gives that $(u_{\mathcal{T},h},\bar{p}_{\mathcal{T},h},p_{\mathcal{T},h})$ is bounded uniformly in $L^2(0,T;L^2(\Omega))$. This gives immediately the existence of a sequence $(\mathcal{T}_m,h_m)$ and of a triple $(u_{\mathcal{T}_m,h_m},\bar{p}_{\mathcal{T}_m,h},p_{\mathcal{T}_m,h_m})$ such that it converges weakly to a triplet $(u,\bar{p},p)$ in $L^2(Q)$. Then, by Lemma 4.3, Lemma 4.4 and Theorem 3.11 in [23], we obtain $u\in W^{1,2}(0,T;W_0^{1,2}(\Omega))$, $\bar{p},p\in L^2(0,T;W_0^{1,2}(\Omega))$. Furthermore, Lemma 4.3 and Lemma 4.4 and the Kolmogorov-M. Riesz-Frécht Theorem (Theorem 4.26 in [7]) also give the strong convergence: $u_{\mathcal{T}_m,h_m}\to u$ as $m\to\infty$. In the following, we show $(u,\bar{p},p)$ is the weak solution of Problem P. To do so, we let $\phi,\psi\in C^2(\bar{\Omega}\times[0,T])$ such that $\phi=\psi=0$ on $\partial\Omega\times[0,T]$, $\phi(\cdot,T)=\psi(\cdot,T)=0$. For λ, we make the assumption as $\lambda\in C^1(\bar{\Omega}\times[0,T])$, $\lambda(\cdot,T)=0$, which means that pointwise values make sense. Multiplying (3.6) by $h_m\lambda_{\mathcal{T}_m,h_m}(x_K,(n+1)h_m)\mathrm{m}(K)$, summing the resulting for $n\in\{0,...,N-1\}$ and $K\in\mathcal{T}$ gives

$$\begin{aligned}&\sum_{n=0}^{N-1}h_m\sum_{K\in\mathcal{T}_m}\mathrm{m}(K)(\bar{p}_K^{n+1}-p_K^{n+1})\lambda(x_K,(n+1)h_m)\\&=\sum_{n=0}^{N-1}h_m\sum_{K\in\mathcal{T}_m}\mathrm{m}(K)p_c(u_K^{n+1})\lambda(x_K,(n+1)h_m)\\&\quad+\tau\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}}\mathrm{m}(K)(u_K^{n+1}-u_K^n)\lambda(x_K,(n+1)h_m).\end{aligned}\tag{4.15}$$

We denote the last term of (4.15) by T_1 and rewrite it as

$$\begin{aligned}T_1=&\tau\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_m}\mathrm{m}(K)(u_K^{n+1}-u_K^n)\lambda(x_K,(n+1)h_m)\\=&\tau\sum_{n=1}^{N-1}\sum_{K\in\mathcal{T}_m}\mathrm{m}(K)u_K^n(\lambda(x_K,nh_m)-\lambda(x_K,(n+1)h_m))\\&+\tau\sum_{K\in\mathcal{T}_m}\mathrm{m}(K)\Big(u_K^N\lambda(x_K,T)-u_K^0\lambda(x_K,h_m)\Big).\end{aligned}$$

First we have $\lambda(x_K, T) = 0$. Then according to the property of the initial condition, one has

$$\sum_{K\in\mathcal{T}_m} \mathrm{m}(K) u_K^0 \lambda(x_K, h_m) \longrightarrow \int_\Omega u^0(x)\lambda(x,0)dx \quad \text{as} \quad m \longrightarrow \infty.$$

Further, since $\lambda \in C^1(\bar{\Omega}\times[0,T])$, $\lambda(\cdot, T) = 0$, one has

$$\tau \sum_{n=0}^{N-1} \sum_{K\in\mathcal{T}_m} \mathrm{m}(K) u_K^{n+1}(\lambda(x_K,(n-1)h_m) - \lambda(x_K, nh_m)) \to \tau \int_0^T \int_\Omega u(x,t)\partial_t \lambda dx dt \text{ as } m \to \infty.$$

Similarly, since $\bar{p}_{\mathcal{T}_m,h_m} - p_{\mathcal{T}_m,h_m}$ converges weakly to $\bar{p} - p$, one has

$$\sum_{n=0}^{N-1} \sum_{K\in\mathcal{T}_m} (\bar{p}_K^{n+1} - p_K^{n+1})\lambda(x_K, nh) \longrightarrow \int_0^T \int_\Omega (\bar{p} - p)\lambda dt dx \quad \text{as} \quad m \longrightarrow \infty.$$

From the above, one gets that $(u, p, \bar{p})$ satisfies (2.6).

Furthermore, given $\varphi \in (C_0^\infty(\Omega \times [0,T)))^2$, for any $K \in \mathcal{T}$ and $n \in \{0, ..., N\}$ set

$$\varphi_K^n = \varphi(x_K, t^n), \quad \mathrm{div}_K \varphi_K^n = \frac{1}{\mathrm{m}(K)} \sum_{L\in\mathcal{N}_K} \sum_{I_d=1}^{2} \mathrm{m}(\sigma_{K|L}^{I_d})\ \varphi_K^n \cdot \mathbf{n}_{K|L}. \tag{4.16}$$

Letting $\chi_{K\times(t^n,t^{n+1}]}$ be characteristic function of $K \times (t^n, t^{n+1}]$, we use (4.16) to define $\varphi_{\mathcal{T},h}$: $\Omega \times (0,T)$ as

$$\varphi_{\mathcal{T},h} = \sum_{n=0}^{N-1} \sum_{K\in\mathcal{T}} \varphi_K^{n+1} \chi_{K\times(t^n,t^{n+1}]}. \tag{4.17}$$

Further, its discrete divergence is:

$$\mathrm{div}_{\mathcal{T}} \varphi_{\mathcal{T},h} = \sum_{n=0}^{N-1} \sum_{K\in\mathcal{T}} \left(\mathrm{div}_K \varphi_K^{n+1}\right) \chi_{K\times(t^n,t^{n+1}]}. \tag{4.18}$$

With the definitions (4.16), (4.17) and (4.18), we have $\varphi_{\mathcal{T}_m,h_m} \to \varphi$ and $\mathrm{div}_{\mathcal{T}_m} \varphi_{\mathcal{T}_m} \to \mathrm{div}\varphi$ uniformly as $m \to \infty$. By the compactness results and Lemma 3.5 there exists a ζ such that $\int_0^T \nabla_{\mathcal{T}} u_{\mathcal{T}_m} \mathrm{div}\varphi \to \int_0^T \zeta \mathrm{div}\varphi$ as $m \to \infty$.

Now we identify the discrete gradient limit ζ with ∇u:

$$\begin{aligned}
&\int_0^T \int_\Omega u_{\mathcal{T}_m,h_m} \mathrm{div}_{\mathcal{T}} \varphi_{\mathcal{T},h} \\
&= \sum_{n=0}^{N-1} h_m \sum_{K\in\mathcal{T}} \mathrm{m}(K) u_K^{n+1} (\mathrm{div}_K \varphi_K^{n+1}) \\
&= \sum_{n=0}^{N-1} h_m \sum_{K\in\mathcal{T}} u_K^{n+1} \sum_{L\in\mathcal{N}_K} \sum_{I_d=1}^{2} \mathrm{m}(\sigma_{K|L}^{I_d})\ \varphi_K^{n+1} \cdot \mathbf{n}_{K|L} \\
&= -\sum_{n=0}^{N-1} h_m \sum_{K|L\in\mathcal{E}_{\mathrm{int}}} (u_L^{n+1} - u_K^{n+1}) \sum_{I_d=1}^{2} \mathrm{m}(\sigma_{K|L}^{I_d})\ \varphi_K^{n+1} \cdot \mathbf{n}_{K|L} \\
&= -\sum_{n=0}^{N-1} h_m \sum_{K|L\in\mathcal{E}_{\mathrm{int}}} \sum_{I_d=1}^{2} \mathrm{m}(\sigma_{K|L}^{I_d})\ \varphi_K^{n+1} \cdot \mathbf{n}_{K|L} \Big((u_L^{n+1} - u_{\sigma_{K|L}^{I_d}}^{n+1}) + (u_{\sigma_{K|L}^{I_d}}^{n+1} - u_K^{n+1}) \Big) \\
&= -\sum_{n=0}^{N-1} h_m \sum_{K|L\in\mathcal{E}_{\mathrm{int}}} \sum_{I_d=1}^{2} \Big(\frac{\mathrm{m}(\sigma_{L|K}^{I_d})}{|\mu_{L|K}^{I_d}|} (u_{\sigma_{L|K}^{I_d}}^{n+1} - u_L^{n+1}) |\mu_{L|K}^{I_d}|\ \mathbf{n}_{L|K} \\
&\qquad\qquad + \frac{\mathrm{m}(\sigma_{K|L}^{I_d})}{|\mu_{K|L}^{I_d}|} (u_{\sigma_{K|L}^{I_d}}^{n+1} - u_K^{n+1}) |\mu_{K|L}^{I_d}|\ \mathbf{n}_{K|L} \Big) \cdot \varphi_K^{n+1} \\
&= -\sum_{n=0}^{N-1} h_m \sum_{K|L\in\mathcal{E}_{\mathrm{int}}} \sum_{I_d=1}^{2} \Big(\frac{\mathrm{m}(L)}{3} (u_{\sigma_{L|K}^{I_d}}^{n+1} - v_L^{n+1})\ \mu_{L|K}^{I_d} + \frac{\mathrm{m}(K)}{3} (u_{\sigma_{K|L}^{I_d}}^{n+1} - u_K^{n+1})\ \mu_{K|L}^{I_d} \Big) \cdot \varphi_K^{n+1} \\
&= -\int_0^T \int_\Omega \nabla_{\mathcal{T}_m} u_{\mathcal{T}_m,h_m} \cdot \varphi_K \to -\int_0^T \int_\Omega \xi \cdot \varphi.
\end{aligned}$$

Therefore $\nabla u = \xi$ in the sense of distributions, and in particular, $u \in L^2(0,T;W^{1,2}(\Omega))$. Similarly, we can also obtain $\nabla_{\mathcal{T}_m} \bar{p}_{\mathcal{T}_m,h_m} \to \nabla \bar{p}$, $\nabla_{\mathcal{T}_m} p_{\mathcal{T}_m,h_m} \to \nabla p$ weakly in $L^2(0,T;L^2(\Omega))$ as $m \to \infty$.

Now we concentrate on $A = \int_0^T \int_\Omega k_o(u) \nabla \bar{p} \cdot \nabla \phi dx dt$. To do so, we define the discretization and approximation of ϕ denoted by Φ and $\phi_{\mathcal{T},h}$:

$$\begin{cases} \Phi_K^{n+1} = \phi(x_K, (n+1)h), \quad K \in \mathcal{T}, \quad n \in \{0,...N-1\}, \\ \Phi_\sigma^{n+1} = \phi(x_\sigma, (n+1)h), \quad \sigma \in \mathcal{E}, \quad n \in \{0,...N-1\}, \\ \phi_{\mathcal{T},h} = \Phi_K^{n+1}, \quad x \in K, \quad t \in (nh,(n+1)h) \quad \text{for all } n = \{1,...,N-1\}. \end{cases} \tag{4.19}$$

Then multiplying (3.4) by $h_m \phi_K^{n+1} := h_m \phi(x_K, (n+1)h)$ and (3.8) by $h_m \phi_{\sigma_{K_r}^{I_d}}^{n+1} := h_m \phi(x_{\sigma_{K_r}^{I_d}}, (n+$

$1)h_m)$ and summing over $K \in \mathcal{T}_m$ and $n \in \{0,...,N-1\}$, with $r = P_i, P_j, P_k$ and $I_d = 1,2$, one has

$$A_{\mathcal{T}_m,h_m} = \sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_m}\sum_{r=i,j,k}\int_{t^n}^{t^{n+1}}\int_{K_r} k_o(u_K^{n+1})\Big((\bar{p}^{n+1}_{\sigma^1_{K_r}} - \bar{p}^{n+1}_K)\cdot\mu_{\sigma^1_{K_r}} + (\bar{p}^{n+1}_{\sigma^2_{K_r}} - \bar{p}^{n+1}_K)\cdot\mu_{\sigma^2_{K_r}}\Big)\cdot$$
$$\Big((\phi^{n+1}_{\sigma^1_{K_r}} - \phi^{n+1}_K)\cdot\mu_{\sigma^1_{K_r}} + (\phi^{n+1}_{\sigma^2_{K_r}} - \phi^{n+1}_K)\cdot\mu_{\sigma^2_{K_r}}\Big)dxdt.$$
$$= \int_0^T\int_\Omega k_o(u_{\mathcal{T}_m,h_m})\nabla_{\mathcal{T}_m}\bar{p}_{\mathcal{T}_m,h_m}\nabla_{\mathcal{T}_m}\phi_{\mathcal{T}_m,h_m}dxdt$$

Then we have

$$\begin{aligned} T_2 =& A_{\mathcal{T}_m,h_m} - A \\ =& \int_0^T\int_\Omega k_o(u_{\mathcal{T}_m,h_m})\nabla_{\mathcal{T}_m}\bar{p}_{\mathcal{T}_m,h_m}\cdot\nabla_{\mathcal{T}_m}\phi_{\mathcal{T}_m,h_m}dxdt - \int_0^T\int_\Omega k_o(u)\nabla\bar{p}\cdot\nabla\phi dxdt \\ =& \left.\int_0^T\int_\Omega k_o(u_{\mathcal{T}_m,h_m})\nabla_{\mathcal{T}_m}\bar{p}_{\mathcal{T}_m,h_m}\nabla_{\mathcal{T}_m}\phi_{\mathcal{T}_m,h}dxdt - \int_0^T\int_\Omega k_o(u)\nabla_{\mathcal{T}}\bar{p}_{\mathcal{T}_m,h_m}\cdot\nabla_{\mathcal{T}_m}\phi_{\mathcal{T}_m,h_m}dxdt\right\} T_{21} \\ &+ \left.\int_0^T\int_\Omega k_o(u)\nabla_{\mathcal{T}_m}\bar{p}_{\mathcal{T}_m,h_m}\cdot\nabla_{\mathcal{T}_m}\phi_{\mathcal{T}_m,h_m}dxdt - \int_0^T\int_\Omega k_o(u)\nabla\bar{p}\cdot\nabla\phi dxdt\right\} T_{22} \end{aligned}$$

By the assumption **(A2)**, the compactness of $\nabla_{\mathcal{T}}\bar{p}_{\mathcal{T},h}$, the regularity of ϕ and $u_{\mathcal{T}_m,h_m} \to u$ as $m \to \infty$, we easily obtain

$$T_{21} \rightharpoonup 0 \quad \text{as } m \to \infty. \tag{4.20}$$

Furthermore, for T_{22} since we have $\nabla_{\mathcal{T}_m}\bar{p}_{\mathcal{T}_m,h_m} \to \nabla\bar{p}$, it is also easily obtained that

$$T_{22} \rightharpoonup 0 \quad \text{as } m \to \infty, \tag{4.21}$$

which together with (4.20) implies $A_{\mathcal{T}_m,h_m}$ converges weakly to A as $m \to \infty$. In the same way, one gets the convergence for $\int_0^T\int_\Omega k_w(u)\nabla p\cdot\nabla\phi dxdt$, which concludes the proof. □

5. Numerical results. We consider a test problem similar to (1.1) - (1.3) but with constant diffusion coefficients and linear relationship for $p_c(u)$. Without losing the generality, we set the diffusion coefficients to be 1 and $p_c(u) = u$. Specifically, for $\Omega = (0,1)\times(0,1)$, we consider the following problem

$$\partial_t u - \Delta\bar{p} = 0, \tag{5.1}$$
$$\partial_t(1-u) - \Delta p = 0, \tag{5.2}$$
$$\bar{p} - p = u + \tau\partial_t u. \tag{5.3}$$

To close the system, we prescribe the boundary conditions:

$$\bar{p} = p = 0 \quad \text{at } \partial\Omega,$$

the initial condition

$$u(x,y,0) = \sin(2\pi x)\cdot\sin(3\pi y).$$

In this case, an explicit solution can be found:

$$u(x,y,t) = \exp(\frac{-13\pi^2 t}{2+13\tau\pi^2}) \cdot \sin(2\pi x) \cdot \sin(3\pi y),$$

$$\bar{p}(x,y,t) = \exp(\frac{-13\pi^2 t}{2+13\tau\pi^2}) \cdot \sin(2\pi x) \cdot \sin(3\pi y) \cdot (\frac{1}{2} - \frac{13\tau\pi^2}{2(2+13\tau\pi^2)}), \quad \text{and}$$

$$p(x,y,t) = \exp(\frac{-13\pi^2 t}{2+13\tau\pi^2}) \cdot \sin(2\pi x) \cdot \sin(3\pi y) \cdot (-\frac{1}{2} + \frac{13\tau\pi^2}{2(2+13\tau\pi^2)}).$$

The convergence results obtained here are based on compactness arguments, without having rigorous error estimates. Nevertheless, for this specific example, since an explicit solution is known, we estimate the order of the scheme as follows.

After constructing a mesh and taking uniform time step, we refine it uniformly three times by having the mesh size and time step. We start with a uniform mesh as shown in Figure 2 (a). For each of the discretization parameters, we compute the L^2 and $W^{1,2}$ errors at $t = 1/16$:

$$(5.4)\quad E^u_{\mathcal{T},h} = \Big(\int_\Omega (u(x,t) - u_{\mathcal{T},h}(x,t))^2 dx\Big)^{1/2}, \quad E^p_{\mathcal{T},h} = \Big(\int_\Omega (\nabla\bar{p}(x,t) - \nabla_{\mathcal{T}}\bar{p}_{\mathcal{T},h}(x,t))^2 dx\Big)^{1/2}.$$

The results in Table 1 refer to $E^u_{\mathcal{T},h}$, $E^p_{\mathcal{T},h}$, which are representative for the scheme. All other errors have similar behavior. We estimate the order by computing

$$(5.5)\qquad \alpha = \log_2(\frac{E^u_{\mathcal{T},h}}{E^u_{\mathcal{T}/2,h/2}}), \qquad \beta = \log_2(\frac{E^p_{\mathcal{T},h}}{E^p_{\mathcal{T}/2,h/2}}).$$

Based on this , the scheme is first order convergence in both L^2 and $W^{1,2}$. Observe that the order in the approximation of the gradient is the same as the L^2-order, this being a consequence of the multipoint flux approximation. One of the advantage of the proposed scheme is that, theoretically, it is robust with respect to the meshing. Since p_c is linear, no restriction applies for the meshing. To evaluate the behavior of the scheme for non-uniform meshes, we use as starting point the non-uniform mesh in Figure 2 (b). Note that this mesh is built without any connection with the solution, such at the changes in or magnitude of the gradient. The results presented in Table 2 show practically no change in the order of the scheme.

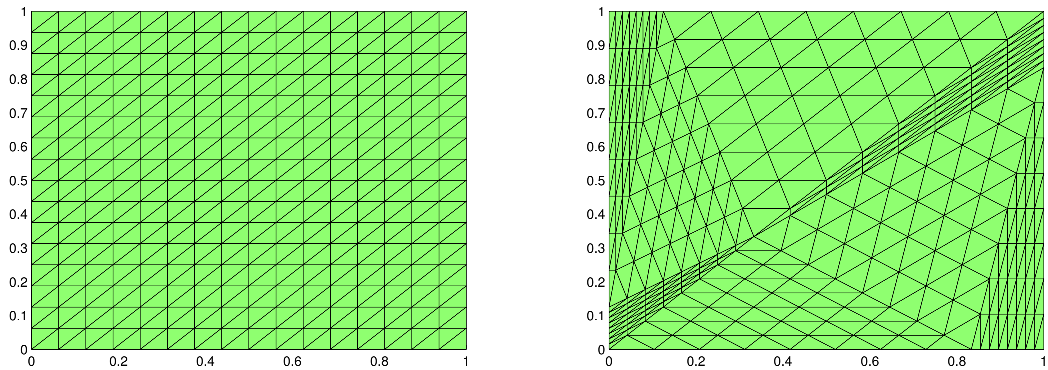

Fig. 2. *The uniform mesh (a) and nonuniform mesh (b).*

No. of cells	$E^u_{\mathcal{T},h}$	α	$E^p_{\mathcal{T},h}$	β
$4^3 \times 8$	8.6546×10^{-4}	–	5.7211×10^{-3}	–
$4^4 \times 8$	4.3880×10^{-4}	0.9799	2.8433×10^{-3}	1.0087
$4^5 \times 8$	2.2097×10^{-4}	0.9897	1.4191×10^{-3}	1.0026
$4^6 \times 8$	1.1088×10^{-4}	0.9949	7.0912×10^{-4}	1.0009

Table 1
Convergence results for uniform mesh, $\tau = 1$.

No. of cells	$E^u_{\mathcal{T},h}$	α	$E^p_{\mathcal{T},h}$	β
$4^3 \times 8$	8.7661×10^{-4}	–	7.8924×10^{-3}	–
$4^4 \times 8$	4.4138×10^{-4}	0.9899	3.8945×10^{-3}	1.0190
$4^5 \times 8$	2.2160×10^{-4}	0.9941	1.9390×10^{-3}	1.0061
$4^6 \times 8$	1.1104×10^{-4}	0.9969	9.6800×10^{-4}	1.0022

Table 2
Convergence results for nonuniform mesh, $\tau = 1$.

The two results up to now were obtained for the case of an isotropic diffusion operator. However, the multipoint flux approximation considered here applies to anisotropic cases too. To see this, we consider the following problem:

$$\partial_t u - \nabla \cdot (K \nabla \bar{p}) = 0, \tag{5.6}$$

$$\partial_t (1-u) - \nabla \cdot (K \nabla p) = 0, \tag{5.7}$$

$$\bar{p} - p = u + \tau \partial_t u. \tag{5.8}$$

With $k_1 = 1$, $k_2 = 5$, K is defined as

$$K = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}. \tag{5.9}$$

The boundary conditions remain unchanged:

$$\bar{p} = p = 0 \quad \text{at } \partial\Omega,$$

as the initial condition

$$u(x, y, 0) = \sin(2\pi x) \cdot \sin(3\pi y).$$

Again, an explicit solution can be found:

$$u(x, y, t) = \exp(\frac{-49\pi^2 t}{2 + 49\tau\pi^2}) \cdot \sin(2\pi x) \cdot \sin(3\pi y),$$

$$\bar{p}(x, y, t) = \exp(\frac{-49\pi^2 t}{2 + 49\tau\pi^2}) \cdot \sin(2\pi x) \cdot \sin(3\pi y) \cdot (\frac{1}{2} - \frac{49\tau\pi^2}{2(2 + 49\tau\pi^2)}), \quad \text{and}$$

$$p(x, y, t) = \exp(\frac{-49\pi^2 t}{2 + 49\tau\pi^2}) \cdot \sin(2\pi x) \cdot \sin(3\pi y) \cdot (-\frac{1}{2} + \frac{49\tau\pi^2}{2(2 + 49\tau\pi^2)}).$$

For the numerical tests we carried out the same steps as before: two meshes (uniform and non-uniform) are refined successively three times. We compute the same errors, and observe that even in the anisotropic case, the scheme still remains first order convergence for both meshes. The results are given in Tables 3 and 4.

No. of cells	$E^u_{\mathcal{T},h}$	α	$E^p_{\mathcal{T},h}$	β
$4^3 \times 8$	8.7790×10^{-4}	–	1.8338×10^{-3}	–
$4^4 \times 8$	4.4681×10^{-4}	0.9777	9.0883×10^{-4}	1.0127
$4^5 \times 8$	2.2544×10^{-4}	0.9869	4.5329×10^{-4}	1.0035
$4^6 \times 8$	1.1324×10^{-4}	0.9934	2.2647×10^{-4}	1.0011

TABLE 3
Convergence results for uniform mesh, $\tau = 1$ and in the anisotropic case.

No. of cells	$E^u_{\mathcal{T},h}$	α	$E^p_{\mathcal{T},h}$	β
$4^3 \times 8$	8.8311×10^{-4}	–	3.1188×10^{-3}	–
$4^4 \times 8$	4.4791×10^{-4}	0.9794	1.4995×10^{-3}	1.0565
$4^5 \times 8$	2.2569×10^{-4}	0.9889	7.3982×10^{-4}	1.0192
$4^6 \times 8$	1.1330×10^{-4}	0.9942	3.6810×10^{-4}	1.0071

TABLE 4
Convergence results for nonuniform mesh, $\tau = 1$ and in the anisotropic case.

One of the known features of the model (1.1) - (1.3) is that their solution does not satisfy a maximal principle. Instead, effects like saturation overshoot can be observed both experimentally [6, 16] and analytically [19]. To investigate this aspect, we present some numerical experiments carried out with the relative permeability functions as

$$k_w(s) = s^{1.5}, \qquad k_o = (1 - s)^{1.5},$$

where $s := 1 - u$ denotes the water saturation. Further, the capillary pressure function is

$$p_c(s) = 1 - s.$$

We take the domain $(x, y) \in \Omega = (-5, 10) \times (0, 10)$. The initial condition is (see Figure 3)

$$s^0 = (s_r - s_l)/(1 + \exp(-4x) + s_l), \quad \text{for any } y \in (0, 10),$$

where $s_l, s_r \in [0, 1]$ are two constant values, $s_l = 0.9$, $s_r = 0.1$.

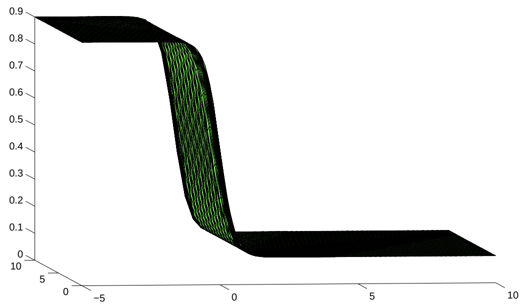

FIG. 3. *The initial saturation.*

At the lateral boundary, we assume 0 flux in the y direction for each phase:

$$-k_o(s)\bar{p}_y = -k_w(s)p_y = 0, \quad \text{along } y = 0 \text{ and } y = 10.$$

At the inflow and outflow boundary, we assume a given, constant total flux in the x direction:

$$-k_o(s)\bar{p}_x - k_w(s)p_x = 1, \quad \text{along } x = -5 \text{ and } x = 10.$$

Further, we assume that

$$s(-5, y) = s_l, \quad s(10, y) = s_r \quad \text{for any } y \in (0, 10),$$

and compute the pressures accordingly.

The numerical approximation of the saturation is displayed in Figure 4 for 2 times. Observe the occurrence of an overshoot.

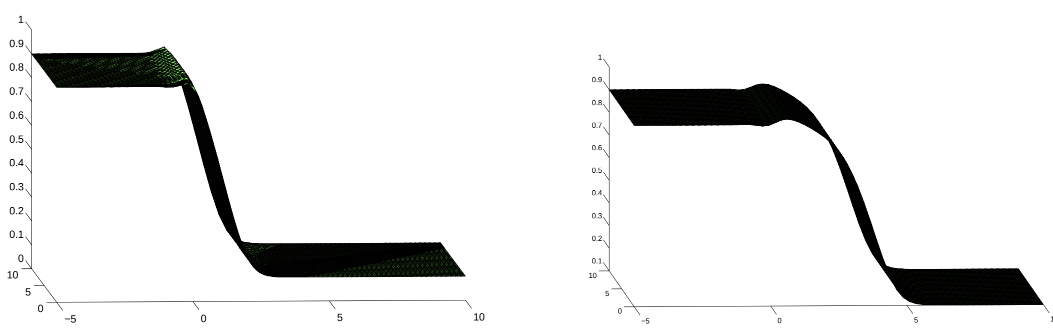

FIG. 4. *The saturation at $t = 1$ (a) and $t = 3$ (b) with $\tau = 1$.*

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