Multibody dynamics notation

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1 Introduction

This document introduces a notation to be used for describing kinematics and dynamics quantities, as well as their coordinate transformations, of a mechanical system composed by rigid bodies. The notation strives to be compact, not ambiguous, and in harmony with Lie Group formalism.

The notation borrows from the well known notation introduced in [7], which is also used, with slight modifications, in [15]. The notation used in [7], is, unfortunately, not fully in accordance with Lie group formalism used in, e.g., [13, 14, 10], that is, however, less compact than [7], leading to long expressions when several rigid bodys are present. This report is an attempt to get the best from these two worlds, also focusing on an unambiguous notation to describe the Jacobians relating the generalized coordinates to Cartesian linear and angular velocities.

2 A quick overview on the developed notation

Quick reference list for the symbols used in this document. Precise definition is given in the text below.

A,B	coordinate frames
p	an arbitrary point
o_B	origin of B
$[A]$	orientation frame associated to A
$B[A]$	frame with origin o_B and orientation $[A]$
${}^A p$	coordinates of p w.r.t. to A
${}^A o_B$	coordinates of o_B w.r.t. to A
${}^A H_B$	homogeneous transformation from B to A
${}^A X_B$	velocity transformation from B to A
${}^C \mathrm{v}_{A,B}$	twist expressing the velocity of B wrt to A written in C
${}^C \mathrm{v}^{\wedge}_{A,B}$	4×4 matrix representation of ${}^C \mathrm{v}_{A,B}$
${}^C \mathrm{v}_{A,B} \times$	6×6 matrix representation of the twist cross product
${}^C \mathrm{v}_{A,B} \bar{\times}^*$	6×6 matrix representation of the dual cross product
${}_B \mathrm{f}$	coordinates of the wrench f w.r.t. B
${}_A X^B$	wrench transformation from B to A
$\left\langle {}_B \mathrm{f}, {}^B \mathrm{v}_{A,B} \right\rangle$	pairing between wrench and velocity
${}_B(\mathbb{M}_L)_B$	6×6 inertia tensor of link (=rigid body) L expressed with respect to frame B
${}^C J_{A,B}$	Jacobian relating the velocity of B with respect to A expressed in C
${}^C J_{A,B/F}$	Jacobian relating the velocity of B with respect to A expressed in C, where the floating base velocity is expressed in F

3 Math preliminaries

3.1 Notation

The following notation is used throughout the document.

- The set of real numbers is denoted by $\mathbb{R}$. Let u and v be two n-dimensional column vectors of real numbers, i.e. $u, v \in \mathbb{R}^n$, then their inner product is denoted as $u^T v$, with "T" the transpose operator.

- The identity matrix of dimension n is denoted $I_n \in \mathbb{R}^{n \times n}$; the zero column vector of dimension n is denoted $0_n \in \mathbb{R}^n$; the zero matrix of dimension $n \times m$ is denoted $0_{n \times m} \in \mathbb{R}^{n \times m}$.

- The set SO(3) is the set of $\mathbb{R}^{3 \times 3}$ orthogonal matrices with determinant equal to one, namely

$$\mathrm{SO}(3) := \{\, R \in \mathbb{R}^{3 \times 3} \mid R^T R = I_3,\ \det(R) = 1 \,\}. \tag{1}$$

  When endowed with matrix multiplication, SO(3) becomes a Lie group, the *Special Orthogonal* group of dimension three.

- The set $\mathfrak{so}(3)$, read *little* $\mathfrak{so}(3)$, is the set of 3×3 skew-symmetric matrices,

$$\mathfrak{so}(3) := \{\, S \in \mathbb{R}^{3 \times 3} \mid S^T = -S \,\}. \tag{2}$$

  When endowed with the matrix commutator as operation, the set becomes a Lie algebra.

- The set SE(3) is defined as

$$\mathrm{SE}(3) := \left\{ \begin{bmatrix} R & p \\ 0_{1 \times 3} & 1 \end{bmatrix} \in \mathbb{R}^{4 \times 4} \mid R \in \mathrm{SO}(3), p \in \mathbb{R}^3 \right\}. \tag{3}$$

  When endowed with matrix multiplication, it becames the *Special Euclidean* group of dimension three, a Lie group that can be used to represent rigid transformations and their composition in the 3D space.

- The set $\mathfrak{se}(3)$ is defined as

$$\mathfrak{se}(3) := \left\{ \begin{bmatrix} \Omega & v \\ 0_{1 \times 3} & 0 \end{bmatrix} \in \mathbb{R}^{4 \times 4} \mid \Omega \in \mathfrak{so}(3), v \in \mathbb{R}^3 \right\}. \tag{4}$$

  When endowed with the matrix commutator as operation, $\mathfrak{se}(3)$ becomes the Lie algebra of the Lie group SE(3).

- Given the vector $w = (x; y; z) \in \mathbb{R}^3$, we define $w^\wedge$ (read w *hat*) as the 3×3 *skew-symmetric* matrix

$$w^\wedge = \begin{bmatrix} x \\ y \\ z \end{bmatrix}^\wedge := \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix} \in \mathfrak{so}(3). \tag{5}$$

Given the *skew-symmetric matrix* $W = w^\wedge$, we define $W^\vee \in \mathbb{R}^3$ (read W *vee*) as

$$W^\vee = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix}^\vee := \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3. \tag{6}$$

Clearly, the vee operator is the inverse of the hat operator.

- Given a vector $\mathrm{v} = (v; \omega) \in \mathbb{R}^6$, v and $\omega \in \mathbb{R}^3$, we define

$$\mathrm{v}^\wedge = \begin{bmatrix} v \\ \omega \end{bmatrix}^\wedge := \begin{bmatrix} \omega^\wedge & v \\ 0_{1\times 3} & 0 \end{bmatrix} \in \mathfrak{se}(3). \tag{7}$$

- Similarly to what done for vectors in $\mathbb{R}^3$ few lines above, we define the vee operator as the inverse of the hat operator such that

$$\begin{bmatrix} \omega^\wedge & v \\ 0_{1\times 3} & 0 \end{bmatrix}^\vee := \begin{bmatrix} v \\ \omega \end{bmatrix} = \mathrm{v} \in \mathbb{R}^6. \tag{8}$$

- Given two normed vector spaces E and F and a function between them $f : E \mapsto F$ we define as the derivative of f in $x_0 \in E$ as the linear function $Df(x_0) : E \mapsto F$ such that:

$$\lim_{x \to x_0} \frac{||f(x) - f(x_0) - Df(x_0) \cdot (x - x_0)||}{||x - x_0||} = 0 \tag{9}$$

If $E = \mathbb{R}$ and $F = \mathbb{R}$ then $Df(x_0)$ is the linear function tangent to $f(x)$, that is usually indicated as:

$$Df(x_0) \cdot x = \left.\frac{df}{dx}\right|_{x=x_0} (x - x_0) + f(x_0).$$

If $E = \mathbb{R}^n$ and $F = \mathbb{R}^m$ then $Df(x_0)$ is:

$$Df(x_0) \cdot x = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}_{x=x_0} (x - x_0) + f(x_0).$$

Given a function f of p normed vector spaces $f : E_1 \times E_2 \dots E_p \mapsto F$ we define as $D_1 f$ the derivative of f with respect to E_1, with $D_2 f$ the derivative of f with respect to E_2 and so on so forth.

We refer to [17, Chapter 2] and to [1, Section 2.3] for further details on the derivative notation (D, D_1, D_2, ...) used in this document.

4 Points and coordinate frames

A *frame* is defined as the combination of a point (called *origin*) and an *orientation frame* in the 3D space [4, 18]. We typically employ a capital letter to indicate a frame. Given a frame A, we will indicate with o_A its origin and with $[A]$ its orientation frame. Formally, we write this as $A = (o_A, [A])$.

Frames can be time varying. They can be used, e.g., to describe the position and orientation in space of a rigid body as time evolves. They are also used to expressed a coordinate system for a wrench exchanged by two bodies or used to define a coordinate system to describe a robot task (like a frame attached to the center of mass and oriented as the inertial frame).

Newton's mechanics requires the definition of an *inertial* frame. In this document, we usually indicate this frame simply with A (the Absolute frame). As common practice, for robots operating near the Earth surface, we will assume the frame A to be fixed to the world's surface, disregarding non-inertial effects due to the Earth's motion.

4.1 Coordinate vector of a point

Given a point p, its coordinates with respect to a frame $A = (o_A, [A])$ are collected in the *coordinate vector* ${}^A p$. The coordinate vector ${}^A p$ represents the coordinates of the 3D geometric vector $\vec{r}_{o_A,p}$ connecting the origin of frame A with the point p, pointing towards p, expressed in the orientation frame $[A]$, that is

$$ {}^A p := \begin{bmatrix} \vec{r}_{o_A,p} \cdot \vec{x}_A \\ \vec{r}_{o_A,p} \cdot \vec{y}_A \\ \vec{r}_{o_A,p} \cdot \vec{z}_A \end{bmatrix} \in \mathbb{R}^3, \tag{10} $$

with $\cdot$ denoting the scalar product between two vectors and $\vec{x}_A$, $\vec{y}_A$, $\vec{z}_A$, the unit vectors defining the orientation frame $[A]$.

4.2 Change of orientation frame

Given two frames A and B, we will employ the notation

$$ {}^A R_B \in \mathrm{SO}(3) \tag{11} $$

to denote the coordinate transformation from frame B to frame A. The coordinate transformation ${}^A R_B$ only depends on the relative orientation between the orientation frames $[A]$ and $[B]$, irrespectively of the position of the origins o_A and o_B.

4.3 Homogeneous transformation

To describe the position and orientation of a frame B with respect to another frame A, we employ the 4×4 homogeneous matrix

$$
{}^{A}H_{B} := \begin{bmatrix} {}^{A}R_{B} & {}^{A}o_{B} \\ 0_{1\times 3} & 1 \end{bmatrix}. \tag{12}
$$

Given a point p, the homogeneous transformation matrix ${}^{A}H_{B}$ can be also used to map the coordinate vector ${}^{A}p$ to ${}^{B}p$ as follows. Let ${}^{A}\bar{p}$ and ${}^{B}\bar{p}$ denote the *homogenous representation* of ${}^{A}p$ and ${}^{B}p$, respectively. That is, let ${}^{A}\bar{p} := ({}^{A}p; 1) \in \mathbb{R}^{4}$ and likewise for ${}^{B}\bar{p}$ (note that ; indicates row concatenation). Then

$$
{}^{A}\bar{p} = {}^{A}H_{B}{}^{B}\bar{p}, \tag{13}
$$

which is the matrix form of ${}^{A}p = {}^{A}R_{B}{}^{B}p + {}^{A}o_{B}$. We refer to [13, Chapter 2] for further details on homogeneous rapresentation of rigid transformations.

5 Velocity vectors (twists)

In the following, given a point p and a frame A, we define

$$
{}^{A}\dot{p} := \frac{d}{dt}\left({}^{A}p\right). \tag{14}
$$

In particular, when p is the origin of a frame, e.g., $p = o_{B}$, we have

$$
{}^{A}\dot{o}_{B} = \frac{d}{dt}\left({}^{A}o_{B}\right).
$$

It is important to note that, by itself, expressions like $\dot{o}_{B}$ or $\dot{p}$ have *no* meaning. Similarly to (14), we also define

$$
{}^{A}\dot{R}_{B} := \frac{d}{dt}\left({}^{A}R_{B}\right) \tag{15}
$$

and

$$
{}^{A}\dot{H}_{B} := \frac{d}{dt}\left({}^{A}H_{B}\right) = \begin{bmatrix} {}^{A}\dot{R}_{B} & {}^{A}\dot{o}_{B} \\ 0_{1\times 3} & 0 \end{bmatrix}. \tag{16}
$$

The relative velocity between a frame B with respect to a frame A can be represented by the time derivative of the homogenous matrix ${}^{A}H_{B} \in \mathrm{SE}(3)$. A more compact representation of ${}^{A}\dot{H}_{B}$ can be obtained multiplying it by the inverse of ${}^{A}H_{B}$ on the left or on the right. In both cases, the result is an element of $\mathfrak{se}(3)$ that will be called a *twist*. Premultipliying on the left, one obtains

$$
\begin{aligned}
{}^{A}H_{B}^{-1}{}^{A}\dot{H}_{B} &= \begin{bmatrix} {}^{A}R_{B}^{T} & -{}^{A}R_{B}^{T}{}^{A}o_{B} \\ 0_{1\times 3} & 1 \end{bmatrix} \begin{bmatrix} {}^{A}\dot{R}_{B} & {}^{A}\dot{o}_{B} \\ 0_{1\times 3} & 0 \end{bmatrix} \\
&= \begin{bmatrix} {}^{A}R_{B}^{T}{}^{A}\dot{R}_{B} & {}^{A}R_{B}^{T}{}^{A}\dot{o}_{B} \\ 0_{1\times 3} & 0 \end{bmatrix}.
\end{aligned} \tag{17}
$$

Note that ${}^A R_B^T {}^A \dot{R}_B$ appearing on the right hand side of (17) is skew symmetric. Define ${}^B v_{A,B}$ and ${}^B \omega_{A,B} \in \mathbb{R}^3$ so that

$$
\begin{aligned}
{}^B v_{A,B} &:= {}^A R_B^T {}^A \dot{o}_B, &(18)\\
{}^B \omega_{A,B}^{\wedge} &:= {}^A R_B^T {}^A \dot{R}_B. &(19)
\end{aligned}
$$

The *left trivialized* velocity of frame B with respect to frame A is

$$
{}^B \mathrm{v}_{A,B} := \begin{bmatrix} {}^B v_{A,B} \\ {}^B \omega_{A,B} \end{bmatrix} \in \mathbb{R}^6. \tag{20}
$$

By construction,

$$
{}^B \mathrm{v}_{A,B}^{\wedge} = {}^A H_B^{-1} {}^A \dot{H}_B. \tag{21}
$$

Note the slight abuse of notation in using the hat operator $\wedge$ in (19) and (21) that maps a vector into its corresponding matrix representation (respectively, from $\mathbb{R}^3$ to $\mathbb{R}^{3\times3}$ using (5) in (19) and from $\mathbb{R}^6$ to $\mathbb{R}^{4\times4}$ using (7) in (21)). Specularly to what done in (17), right multiplying ${}^A \dot{H}_B$ by the inverse of ${}^A H_B$ leads to

$$
\begin{aligned}
{}^A \dot{H}_B {}^A H_B^{-1} &= \begin{bmatrix} {}^A \dot{R}_B & {}^A \dot{o}_B \\ 0_{1\times3} & 0 \end{bmatrix} \begin{bmatrix} {}^A R_B^T & -{}^A R_B^T {}^A o_B \\ 0_{1\times3} & 1 \end{bmatrix} \\
&= \begin{bmatrix} {}^A \dot{R}_B {}^A R_B^T & {}^A \dot{o}_B - {}^A \dot{R}_B {}^A R_B^T {}^A o_B \\ 0_{1\times3} & 0 \end{bmatrix}. &(22)
\end{aligned}
$$

Define ${}^A v_{A,B}$ and ${}^A \omega_{A,B} \in \mathbb{R}^3$ as

$$
\begin{aligned}
{}^A v_{A,B} &:= {}^A \dot{o}_B - {}^A \dot{R}_B {}^A R_B^T {}^A o_B &(23)\\
{}^A \omega_{A,B}^{\wedge} &:= {}^A \dot{R}_B {}^A R_B^T. &(24)
\end{aligned}
$$

The *right trivialized* velocity of B with respect to A is then defined as

$$
{}^A \mathrm{v}_{A,B} := \begin{bmatrix} {}^A v_{A,B} \\ {}^A \omega_{A,B} \end{bmatrix} \in \mathbb{R}^6. \tag{25}
$$

By construction,

$$
{}^A \mathrm{v}_{A,B}^{\wedge} = {}^A \dot{H}_B {}^A H_B^{-1}. \tag{26}
$$

5.1 Expressing a twist with respect to an arbitrary frame

Straightforward algebraic calculations allow to show that the right and left trivialized velocities ${}^A \mathrm{v}_{A,B}$ and ${}^B \mathrm{v}_{A,B}$ are related via a linear trasformation. Following the notation introduced in [7], we denote this transformation ${}^A X_B$ and define it as

$$
{}^A X_B := \begin{bmatrix} {}^A R_B & {}^A o_B^{\wedge} {}^A R_B \\ 0_{3\times3} & {}^A R_B \end{bmatrix} \in \mathbb{R}^{6\times6}. \tag{27}
$$

We have therefore

$$^{A}\mathrm{v}_{A,B} = {}^{A}X_B \, {}^{B}\mathrm{v}_{A,B}. \tag{28}$$

The inverse transformation is simply given by ${}^{B}X_A = {}^{A}X_B^{-1}$, an exercise we leave to the reader (recall that ${}^{A}o_B = -{}^{A}R_B \, {}^{B}o_A$).

Technical note. To draw a connection with Lie group theory, indicating with $g = g_{A,B} := {}^{A}H_B \in \mathrm{SE}(3)$ an arbitrary element of the Special Euclidean group (i.e., a rigid transformation), ${}^{A}X_B$ is nothing else than Ad_g. Given $g \in \mathrm{SE}(3)$ and $\xi \in \mathfrak{se}(3)$,

$$\mathrm{Ad}_g \, \xi := g \xi g^{-1} \in \mathfrak{se}(3). \tag{29}$$

The operator $\mathrm{Ad} : \mathrm{SE}(3) \times \mathfrak{se}(3) \to \mathfrak{se}(3)$ is called the *adjoint action of the group SE(3) to its algebra* $\mathfrak{se}(3)$. Taking $g = {}^{A}H_B$ and $\xi = {}^{B}\mathrm{v}^{\wedge}_{A,B}$, one sees immediately that $g\xi g^{-1}$ appearing in the right hand side of (29) equals

$${}^{A}H_B \, {}^{B}\mathrm{v}^{\wedge}_{A,B} \, {}^{A}H_B^{-1}, \tag{30}$$

which, recalling the definition of ${}^{B}\mathrm{v}_{A,B}$ given in (21), is equivalent to

$${}^{A}\dot{H}_B \, {}^{A}H_B^{-1} = {}^{A}\mathrm{v}^{\wedge}_{A,B}, \tag{31}$$

by definition of ${}^{A}\mathrm{v}_{A,B}$ given in (26). The adjoint action of the group SE(3) to its algebra $\mathfrak{se}(3)$, given by (29), is linear with respect to its second argument. It is therefore possible, when representing $\mathfrak{se}(3)$ as a vector is $\mathbb{R}^6$ as done in (7), to define the adjoint action (with a slight abouse of notation) as a map $\mathrm{Ad} : \mathrm{SE}(3) \times \mathbb{R}^6 \to \mathbb{R}^6$. In this way, for $g = {}^{A}H_B$, ${}^{A}\mathrm{v}_{A,B} = \mathrm{Ad}_g \, {}^{B}\mathrm{v}_{A,B}$, with $\mathrm{Ad}_g = {}^{A}X_B$ given in (27), as straightforward computations allow to conclude.

Given the ubiquity of the velocity transformation $\mathrm{Ad}_{g_{A,B}}$ in multibody dynamics computations and algorithms (and its associate wrench transformation $\mathrm{Ad}^*_{g_{A,B}}$ that we introduce later below), it is convenient to indicate it simply with the compact notation ${}^{A}X_B$ (respectively, ${}_{B}X^{A}$). It is likewise important, however, to recall its connection with Lie group theory to be able to understand the body of literature of geometric mechanics written with the standard Ad notation. ■

We conclude this section by introducing the velocity of frame B with respect to frame A expressed in frame C, indicated with ${}^{C}\mathrm{v}_{A,B}$. The left and right trivialized velocities of B with respect to A, respectively given by (21) and (26), are special cases of this concept. Namely, we define

$${}^{C}\mathrm{v}_{A,B} = \begin{bmatrix} {}^{C}v_{A,B} \\ {}^{C}\omega_{A,B} \end{bmatrix} \in \mathbb{R}^6 \tag{32}$$

as

$${}^{C}\mathrm{v}_{A,B} := {}^{C}X_A \, {}^{A}\mathrm{v}_{A,B} = {}^{C}X_B \, {}^{B}\mathrm{v}_{A,B}, \tag{33}$$

the latter equality following from (28) and ${}^{C}X_A \, {}^{A}X_B = {}^{C}X_B$.

5.2 On the linear and angular components of a twist

As evident from (19) and (24), the angular component of the twists ${}^B\mathrm{v}_{A,B}$ and ${}^A\mathrm{v}_{A,B}$ only depends on the relative orientation between the frames A and B, given by the rotation matrix ${}^AR_B$, and its time evolution. It corresponds to the classic concept of *angular velocity* found in undergraduate physics textbooks and it can be expressed with respect to a different orientation frame simply by multipliying its coordinates by a suitable rotation matrix so that, e.g.,

$$ {}^C\omega_{A,B} = {}^CR_B {}^B\omega_{A,B} = {}^CR_A {}^A\omega_{A,B}. \tag{34} $$

The linear component of the twists ${}^B\mathrm{v}_{A,B}$ and ${}^A\mathrm{v}_{A,B}$ requires a bit more of attention. While ${}^Bv_{A,B}$ given in (18) is the time derivative of ${}^Ao_B$ (the coordinates of the origin of B with respect to the frame A) expressed in the coordinate of frame B, ${}^Av_{A,B}$ is *not* the time derivative of ${}^Ao_B$, but rather the (initially) somehow counterintuivive expression given in (23). At each instant of time, the linear velocity ${}^Av_{A,B}$ is the linear velocity of that point, thought as fixed with respect to frame B, that finds itself at the origin of frame A at the given instant of time. The right trivialized velocity ${}^A\mathrm{v}_{A,B}$ finds application in the theory of mechanical systems with symmetries and in the definition of momentum map [11, 2, 12]. It is also a key ingredient in understanding the efficient numerical algorithms for multibody dynamics described, e.g., in [7, 9, 14].

There are situations in which, however, one would like to describe the linear and angular velocity of a frame as ${}^A\dot{o}_B$ and ${}^A\omega_{A,B}$, respectively. This is possible: if we express the velocity of frame B with respect frame A in the frame $B[A] := (o_B, [A])$, that is, the frame with the same origin of B and same orientation of A, one gets

$$ {}^{B[A]}\mathrm{v}_{A,B} = {}^{B[A]}X_B {}^B\mathrm{v}_{A,B} = \begin{bmatrix} {}^AR_B & 0 \\ 0 & {}^AR_B \end{bmatrix} \begin{bmatrix} {}^BR_A {}^A\dot{o}_B \\ {}^B\omega_{A,B} \end{bmatrix} = \begin{bmatrix} {}^A\dot{o}_B \\ {}^A\omega_{A,B} \end{bmatrix}. \tag{35} $$

In [3], (35) is referred to as the hybrid velocity of frame B with respect to frame A. To avoid confusion with hybrid systems theory, we will call (35) the *mixed velocity* of frame B with respect to frame A (*mixed* as it has both the flavors of a left trivialized velocity for the linear velocity part and of a right trivialized velocity for the angular velocity part).

5.3 The cross product on $\mathbb{R}^6$ ($\times$)

Equation (21) can be rewritten as

$$ {}^A\dot{H}_B = {}^AH_B {}^B\mathrm{v}^\wedge_{A,B}. \tag{36} $$

By time differentiation of (27), it can be shown that a similar formula holds for ${}^AX_B$, namely, that

$$ {}^A\dot{X}_B = {}^AX_B {}^B\mathrm{v}_{A,B}\times \tag{37} $$

with ${}^{B}\mathrm{v}_{A,B}\times$ defined as

$$
{}^{B}\mathrm{v}_{A,B}\times := \begin{bmatrix} {}^{B}\omega^{\wedge}_{A,B} & {}^{B}v^{\wedge}_{A,B} \\ 0_{3\times 3} & {}^{B}\omega^{\wedge}_{A,B} \end{bmatrix}. \tag{38}
$$

We will refer to (38) as the matrix representation of the *cross product on* $\mathbb{R}^6$.

Technical note. In the language of Lie groups, (38) is nothing else that the matrix representation of the adjoint action of $\mathbb{R}^6$ on itself, indicated with ad, when thinking at $\mathbb{R}^6$ as the Lie algebra *induced* by the Lie algebra homeomorphism (7) between $\mathbb{R}^6$ and $\mathfrak{se}(3)$. Defining $g = {}^{A}H_B \in \mathrm{SE}(3)$, (37) is then usually written as (cf. [11, Chapter 9, equation (9.3.4)])

$$
\frac{d}{dt}\,\mathrm{Ad}_g = \mathrm{Ad}_g\,\mathrm{ad}_{g^{-1}\dot{g}}, \tag{39}
$$

where $\mathrm{Ad}_g = {}^{A}X_B$ and $\mathrm{ad}_{g^{-1}\dot{g}} = {}^{A}\mathrm{v}_{A,B}\times$, with $g^{-1}\dot{g} = {}^{B}\mathrm{v}_{A,B}$. This notation is used in the robotic literature in, e.g., [8] and [14]. This connection with Lie group theory allows to obtain immediately useful algebraic equalities such as, e.g., the identity $(\mathrm{v}\times\mathrm{w})^{\wedge} = \mathrm{v}^{\wedge}\mathrm{w}^{\wedge} - \mathrm{w}^{\wedge}\mathrm{v}^{\wedge} =: [\mathrm{v}^{\wedge}, \mathrm{w}^{\wedge}]$, valid for arbitrary vectors v and $\mathrm{w} \in \mathbb{R}^6$, that derives from the fact that the adjoint operator ad is nothing else than the matrix commutator $[\cdot,\cdot]$ when using the matrix representations ($\mathrm{v}^{\wedge}$ and $\mathrm{w}^{\wedge}$) of the Lie algebra elements. ■

5.3.1 Basic properties of the cross product

Equation (38) defines a cross product between vectors of $\mathbb{R}^6$, with the classical anticommutative property

$$
{}^{C}\mathrm{v}_{A,B} \times {}^{C}\mathrm{v}_{D,E} = -{}^{C}\mathrm{v}_{D,E} \times {}^{C}\mathrm{v}_{A,B}. \tag{40}
$$

As a direct consequence of anticommutativity is

$$
{}^{C}\mathrm{v}_{A,B} \times {}^{C}\mathrm{v}_{A,B} = 0_{6\times 1}. \tag{41}
$$

At deeper look at the cross product defined via (38) reveals that this operation turns $\mathbb{R}^6$ into a Lie algebra, a vector space with a anticommutative bilinear operation satisfying the Jacobi identity [11, Chapter 9].

5.3.2 Velocity transformation and the cross product

The cross product of $\mathfrak{se}(3)$ satisfies the distributive property

$$
{}^{A}X_B\,{}^{B}\mathrm{v}_{A,B}\times = ({}^{A}X_B\,{}^{B}\mathrm{v}_{A,B}) \times {}^{A}X_B = {}^{A}\mathrm{v}_{A,B} \times {}^{A}X_B. \tag{42}
$$

In the context of Lie groups, this latter property is the well known result

$$
\mathrm{Ad}_g\,\mathrm{ad}_{g^{-1}\dot{g}} = \mathrm{ad}_{\mathrm{Ad}_g\,g^{-1}\dot{g}}\,\mathrm{Ad}_g = \mathrm{ad}_{\dot{g}g^{-1}}\,\mathrm{Ad}_g, \tag{43}
$$

where $g = {}^{A}H_B \in \mathrm{SE}(3)$ as in (39) and $\dot{g}g^{-1} = {}^{A}\mathrm{v}_{A,B}$ [11, Chapter 9].

5.4 Frame acceleration and acceleration vectors

Several definitions of frame accelerations are present in the literature [6]. As an example, in [7], "coordinate free" or "absolute" frame accelerations are introduced by considering only twists with respect to an (implicitly defined) inertial frame. This particular definition is convenient for obtaining computational efficient algorithms for multibody dynamics, but it is not natural for task specification and closed-loop control, where it is common to use linear accelerations that are the derivative of the (inertial) coordinates of a point in space.

To avoid confusion and preserve the generality that will allows to us a common notation to describe numerical algorithm for multibody dynamics as well as "natural" task specifications, we define the acceleration of a frame B with respect to a frame A written in terms of a frame C simply as the time-derivative of the corresponding twist, that is

$$^{C}\dot{\mathrm{v}}_{A,B} := \frac{d}{dt}\left(^{C}\mathrm{v}_{A,B}\right). \tag{44}$$

Of particular interest, for task specification, is the acceleration

$$^{B[A]}\dot{\mathrm{v}}_{A,B} = \begin{bmatrix} ^{B[A]}\dot{v}_{A,B} \\ ^{B[A]}\dot{\omega}_{A,B} \end{bmatrix} . = \begin{bmatrix} ^{A}\ddot{o}_{B} \\ ^{A}\dot{\omega}_{A,B} \end{bmatrix}. \tag{45}$$

Using (37), one can easily prove that the acceleration of a frame B with respect to a frame A expressed in frame C satisfies

$$^{C}\dot{\mathrm{v}}_{A,B} = {^{C}X_{B}}\left(^{B}\mathrm{v}_{C,B} \times {^{B}\mathrm{v}_{A,B}} + {^{B}\dot{\mathrm{v}}_{A,B}}\right). \tag{46}$$

From equation above, for the special case $C = A$, one obtains, using (41), the fundamental relationship between left and right trivialized accelerations, that is

$$^{A}\dot{\mathrm{v}}_{A,B} = {^{A}X_{B}}{^{B}\dot{\mathrm{v}}_{A,B}}. \tag{47}$$

6 Force covectors (wrenches)

The coordinates of a wrench f with respect to frame B are indicated with

$${_{B}\mathrm{f}} := \begin{bmatrix} _{B}f \\ _{B}\tau \end{bmatrix} \in \mathbb{R}^{6}. \tag{48}$$

Note that the frame B is simply used to indicate the coordinate frame with respect to which the wrench f is expressed in coordinates and there is no necessity for the wrench f to also be applied to, e.g., the rigid body (if any) to which B is attached. Similarly to what we did for a twist, we can define a linear map to change the coordinates of a wrench from a frame B to another frame A. This coordinate transformation is indicated with ${_{A}X^{B}}$ and written as

$${_{A}\mathrm{f}} = {_{A}X^{B}}\,{_{B}\mathrm{f}}. \tag{49}$$

The mapping ${}_AX^B$ is actually induced by the velocity transformation (27) (why this is the case will be explained below) and is related to ${}^BX^A$ via the definition

$$ {}_AX^B := {}^BX_A^T. \tag{50}$$

It is important to realize that (50) is such to make the following identity (of power) hold

$$ \left\langle {}_B\mathrm{f}, {}^B\mathrm{v}_{A,B} \right\rangle = \left\langle {}_A\mathrm{f}, {}^A\mathrm{v}_{A,B} \right\rangle , \tag{51}$$

where f could be interpreted as a wrench applied to a rigid body to which the moving frame B is rigidly attached and A as the absolute inertial frame.

6.1 The dual cross product on $\mathbb{R}^6$ ($\bar{\times}^*$)

The time derivative of the wrench coordinate transformation ${}_AX^B$ has an expression that is dual to velocity coordinate transformation ${}^AX_B$ given in (37). Indeed, straightforward computations lead to obtain

$$ {}_A\dot{X}^B = {}_AX^{B\,B}\mathrm{v}_{A,B}\bar{\times}^* \tag{52}$$

where the (matrix representation of the) dual cross product $\bar{\times}^*$ is defined by

$$ {}^B\mathrm{v}_{A,B}\bar{\times}^* := \begin{bmatrix} {}^B\omega^\wedge_{A,B} & 0_{3\times 3} \\ {}^Bv^\wedge_{A,B} & {}^B\omega^\wedge_{A,B} \end{bmatrix}. \tag{53}$$

It is worth noting that (53) is obtained from (38) by simply transposing this latter expression and taking the negative value of the result: a fact that is also encoded in the symbol $\bar{\times}^*$, where the overline sign has been chosen to represent the minus sign and the star the transpose operation (more formally, the adjoint of a linear map, typically indicated with a star). The dual cross product (53) takes one twist and one wrench and return one wrench (as opposed to the cross product (38) that takes as input two twists and return one twist); this is also the reason why the sub- and superscripts in (52) makes sense: when ${}_A\dot{X}^B$ is applied to a wrench ${}_B\mathrm{f}$ expressed in B, the dual cross product between ${}^B\mathrm{v}_{A,B}$ and the wrench will return a wrench expressed in B that can then be converted into a wrench expressed in A via ${}_AX^B$. It is also straighforward to prove that

$$ {}_AX^{B\,B}\mathrm{v}_{A,B}\bar{\times}^* = {}^A\mathrm{v}_{A,B}\bar{\times}^*{}_AX^B. \tag{54}$$

Technical note. In the language of differential geometry, the dual space of $\mathfrak{se}(3)$ (i.e., the space of linear applications from $\mathfrak{se}(3)$ to $\mathbb{R}$) is indicated with $\mathfrak{se}(3)^*$ and is the space where wrenches belong (as opposed to $\mathfrak{se}(3)$ where twists belong). In terms of Lie group theory, the wrench coordinate transfromation ${}_AX^B$ is written

$$ {}_AX^B = \mathrm{Ad}^*_{g^{-1}} \tag{55}$$

with $g = {}^A H_B \in \mathrm{SE}(3)$. Recall that $Ad_g = {}^A X_B$ and $Ad_{g^{-1}} = {}^B X_A$. Then, posing $\xi^\wedge = {}^B \mathrm{v}^\wedge_{A,B} \in \mathfrak{se}(3)$, one sees that

$$ {}^B \mathrm{v}_{A,B} \,\bar{\times}^* = -\,\mathrm{ad}^*_\xi \,. \tag{56} $$

Once again, note how the symbol $\bar{\times}^*$ appearing in (53) has been explicitly chosen to remind the fact that (53) is obtained from the product $(\times)$ given in (38), by computing its adjoint $(*)$ and changing its sign $(-)$. Finally, (52) is simply

$$ \frac{d}{dt}\,\mathrm{Ad}^*_{g^{-1}} = -\,\mathrm{Ad}^*_{g^{-1}}\,\mathrm{ad}^*_\xi \tag{57} $$

for $\dot{g} = g\xi$, with $g = {}^A H_B$ and $\xi = {}^B \mathrm{v}^\wedge_{A,B}$. ■

7 Generalized inertia tensor

The 6×6 generalized inertia of a link (=rigid body) L expressed with respect to a frame C centered at the center of mass of L is defined as

$$ {}_C(\mathbb{M}_L)_C = \begin{bmatrix} m_L I_{3\times 3} & 0_{3\times 3} \\ 0_{3\times 3} & {}_C(\mathbb{I}_L)_C \end{bmatrix}, \tag{58} $$

where m_L is the mass of the rigid body and ${}_C(\mathbb{I}_L)_C$ is the standard inertia tensor of the rigid body L expressed in the orientation frame $[C]$.

The generalized inertia of a rigid body L expressed with respect a generic frame B is defined according to

$$ \begin{aligned} {}_B(\mathbb{M}_L)_B &= {}_B X^C \, {}_C(\mathbb{M}_L)_C \, {}^C X_B \\ &= \begin{bmatrix} m_L I_{3\times 3} & -m_L {}^B o^\wedge_C \\ m_L {}^B o^\wedge_C & {}_B(\mathbb{I}_L)_B \end{bmatrix}, \end{aligned} \tag{59} $$

where

$$ {}_B(\mathbb{I}_L)_B = {}^C R_B^T \, {}_C(\mathbb{I}_L)_C \, {}^C R_B - m_L {}^B o^\wedge_C {}^B o^\wedge_C. \tag{60} $$

We recall that ${}^B o_C = -{}^B R_C {}^C o_B$. The term $-m_L {}^B o^\wedge_C {}^B o^\wedge_C$ appearing in the inertia matrix ${}_B(\mathbb{I}_L)_B$ is the classic correction term of the Huygens-Steiner theorem (also known as parallel axis theorem).

8 On geometric Jacobians

The goal of this section is to arrive at a precise, unambiguous notation to denote the 'geometric Jacobians' for fixed-based and, in particular, free-floating multibody systems.

In this section, A will denote the (absolute) *inertial frame* and B a frame rigidly attached to one of the bodies composing the multibody system, selected to be the *floating base*. The configuration of a free-floating multibody system will be parametrized using $q = (H, q_J) \in \mathrm{SE}(3) \times \mathbb{R}^{n_J}$, with $H = {}^A H_B \in \mathrm{SE}(3)$ representing the pose (position and orientation) of the floating base frame B and $q_J \in \mathbb{R}^{n_J}$ the internal joint displacements. The configuration space (more correctly, the configuration manifold) has correspondingly dimension $n = 6+n_J$.

Let E be a frame (rigidly) attached to an arbitrary body to be used, e.g., for the specification of a task to be executed by the robot. The frame E could represent, e.g., the pose of a specific frame rigidly attached to a *end effector* of a robot manipulator, like a hand or foot of a humanoid robot. Let

$$ {}^A H_E = {}^A H_E(q) = {}^A H_E(H, q_J) \tag{61} $$

denote the homogeneous transformation expressing E with respect to A as a function of the configuration $q = (H, q_J)$.

Let δH denote an infinitesimal perturbation of the pose of the floating base – in the language of differential geometry, $\delta H \in T_H \mathrm{SE}(3)$– and δq_J an infinitesimal perturbation of the joint displacements. Then, the corresponding infinitesimal perturbation of frame E can be computed as

$$ {}^A \delta H_E = {}^A D_1 H_E(H, q_J) \cdot \delta H + {}^A D_2 H_E(H, q_J) \cdot \delta q_J, \tag{62} $$

where we recall that H is a short form for ${}^A H_B$ and δH a short form for ${}^A \delta H_B$. Define the trivialized infinitesimal perturbations ${}^E \Delta_{A,E}$ and ${}^B \Delta_{A,B} \in \mathbb{R}^6$ such that

$$ {}^E \Delta^{\wedge}_{A,E} = {}^A H_E^{-1}\, {}^A \delta H_E \tag{63} $$

and

$$ {}^B \Delta^{\wedge}_{A,B} = {}^A H_B^{-1}\, {}^A \delta H_B. \tag{64} $$

Combining (63) and (64) together with (62), it is straightforward to show that ${}^E \Delta_{A,E}$ depends linearly on ${}^B \Delta_{A,B}$ and δq_J. Such a linear map defines the "geometric Jacobian" for a floating base system. It will be indicated with the symbol ${}^E J_{A,E/B}$ to indicate that the Jacobian allows to compute the infinitesimal perturbation of frame E relative to frame A, expressed with respect to frame E (a first left-trivialization), based on the infinitesimal perturbation of the internal joint configuration and that of the floating base B, the latest expressed with respect B (a second left-trivialization). For this reason, we will refer to ${}^E J_{A,E/B}$ as a *double left-trivialized* geometric Jacobian. In formulas, we get

$$ {}^E \Delta_{A,E} = {}^E J_{A,E/B}({}^A H_B, q_J) \begin{bmatrix} {}^B \Delta_{A,B} \\ \delta q_J \end{bmatrix}. \tag{65} $$

The infinitesimal perturbation of frames E and B can be clearly be expressed with respect to arbitrary frames, let us say, C and D. The geometric Jacobian ${}^D J_{A,E/C}$ is related to ${}^E J_{A,E/B}$ by the transformation

$$ {}^D J_{A,E/C} = {}^D X_E\, {}^E J_{A,E/B}\, {}^B Y_C, \tag{66} $$

with

$$
{}^{B}Y_{C} := \begin{bmatrix} {}^{B}X_{C} & 0_{6\times n_J} \\ 0_{n_J\times 6} & I_{n_J\times n_J} \end{bmatrix}. \tag{67}
$$

The special case ${}^{E[A]}J_{A,E/B[A]}$, obtained from the above formula with $D = E[A]$ and $C = B[A]$, will be denoted the *mixed-mixed* geometric Jacobian.

9 Free-floating rigid body dynamics

Here we introduce the notation that we use to write, in a compact form, the equations of motion of a free-floating rigid body system. As mentioned in Section 8, the configuration of a free-floating rigid body system will be parametrized as $q = (H, q_J) := ({}^{A}H_{B}, q_J)$, where A is the inertial frame, B the (selected) floating base frame and $q_J \in \mathbb{R}^{n_J}$ the joint displacements.

The kinematics of the (floating base) rigid body system are written treating the configuration manifold $Q = \mathrm{SE}(3) \times \mathbb{R}^{n_J}$ as the Lie group defined by the group direct product $\mathrm{SE}(3) \times \mathbb{R}^{n_J}$. Seeing Q as a Lie group, employing the mixed velocity of the floating based B, allows one to write the derivative of the configuration $\dot{q} = (\dot{H}, \dot{q}_J)$ as

$$
\dot{q} = q\, Y \nu, \tag{68}
$$

where $\nu = (\mathrm{v}, \dot{q}_J)$ denotes the *mixed generalized velocity* of the floating-base rigid-body system, i.e., $\nu := {}^{B[A]}\nu = ({}^{B[A]}\mathrm{v}_{A,B}, q_J)$, and $Y := {}^{B}Y_{B[A]}$ is the natural extension of the velocity transformation X, introduced in Section 5.1, to the Lie algebra of Q. The structure of the generalized velocity transformation Y has been introduced previously in (67) and it is formally defined in such as a way that

$$
\begin{aligned}
Y\nu &= {}^{B}Y_{B[A]}\,{}^{B[A]}\nu \\
&= {}^{B}Y_{B[A]}\,({}^{B[A]}\mathrm{v}_{A,B}, q_J) \\
&= ({}^{B}X_{B[A]}\,{}^{B[A]}\mathrm{v}_{A,B}, q_J) \\
&= ({}^{B}\mathrm{v}_{A,B}, q_J) =: {}^{B}\nu.
\end{aligned} \tag{69}
$$

Note how Y acts on the floating base velocity v, leaving the joint velocities $\dot{q}_J$ unaltered. The configuration q and the mixed generalized velocity ν can then be used to write the dynamics of the floating-based rigid-body system as

$$
M(q)\dot{\nu} + C(q,\nu)\nu + G(q) = S\,\tau + \sum_{i\in\mathcal{I}_N} J_i^T(q)\mathrm{f}_i \tag{70}
$$

with M the (mixed) mass matrix, C the (mixed) Coriolis matrix, G the (mixed) potential force vector, $S := [0_{6\times n_J}; I_{n_J\times n_J}]$ the *joint selection matrix* (see, e.g., [5]), τ the joint torques, $\mathcal{I}_N$ the set of closed contacts, $\mathrm{f}_i := {}_{C_i[A]}\mathrm{f}_i$ the $i-th$ contact wrench, and

$$
J_i(q) := {}^{C_i[A]}J_{A,L_i/B[A]}(q) \tag{71}
$$

the mixed-mixed geometric Jacobian associated to the contact frame C_i, where we used the notation introduced in Section 8, where L_i is the frame *rigidly attached* to the link that is experiencing the i-th contact. To better understand where (71) comes from, recall that the infinitesimal power injected by a contact wrench ${}_C\mathrm{f}$ is given by

$$\begin{aligned} \left\langle {}_L X^C {}_C\mathrm{f}, {}^L J_{A,L/B[A]}(q)\,\nu \right\rangle &= \left\langle {}_C\mathrm{f}, {}^C X_L {}^L J_{A,L/B[A]}(q)\,\nu \right\rangle \\ &= \left\langle {}_C\mathrm{f}, {}^C J_{A,L/B[A]}(q)\,\nu \right\rangle . \end{aligned} \tag{72}$$

Note, in particular, that ${}^C J_{A,L/B[A]} \neq {}^C J_{A,C/B[A]}$ because ${}^C X_L$ is, typically, *time varying* as the contact frame is allowed to move with respect to the link which is experiencing the contact and therefore with respect to the link frame L. The twist ${}^C J_{A,L/B[A]}(q(t))\,\nu(t)$ can be interpreted as the velocity of that frame rigidly attached to the link that has, at time t, the same position and orientation of frame C.

Technical note. In the robotics literature, the equations of motions (70) are often referred to generically as forced Euler-Lagrange equations. While, indeed, the variational principle and the Lagrangian play a central role in obtaining the unforced equations

$$M(q)\dot{\nu} + C(q,\nu)\nu + G(q) = 0, \tag{73}$$

it is important to realize that the Lagrangian is a mapping defined on the tangent bundle of $Q = SE(3) \times \mathbb{R}^{n_J}$, that is $L : TQ \to \mathbb{R}$, $(q, \dot{q}) \mapsto L(q, \dot{q})$. The classical Euler-Lagrange equations, typically written in coordinates as

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0, \quad i = \{1, 2, \dots, n\}, \tag{74}$$

with n the configuration space dimension, do not apply (globally) because Q is not $\mathbb{R}^n$. Indeed, to obtain (73), one needs to resolve to geometric mechanics. In this context, one first defines a trivialized Lagrangian as the difference between kinetic and potential energy, where the velocity is parameterized via the trivialized velocity. For example, one can define the trivialized Lagrangian as $l(q,\xi) = L(q, q\xi) = 1/2\,\langle I(q)\xi, \xi\rangle - V(q)$, where $\xi = {}^B\nu$, V denotes the potential energy, and $I(q)$ the inertia matrix (defined as $Y(q)^T M(q) Y(q)$ with $Y(q)$ as in (70) and $M(q)$ as in (68)). One then applies a modified version of the Euler-Lagrange equations, typically referred to as Hamel equations to this trivialized Lagrangian l (see, e.g., [11, Section 13.6]), obtaining a differential equation for updating the velocity ξ. Since different Lie group structures can be associated to the configuration manifold Q, different parametrization are actually possible to describe the trivialized velocity. The matrix form of the Hamel equations given in (73), in particular, is the one obtained using the trivialized Lagrangian $l(q,\xi) = 1/2\,\langle M(q)\xi, \xi\rangle - V(q)$, with $M(q)$ and $V(q)$ as above but where we take $\xi = \nu = {}^{B[A]}\nu$ and where Q has been assigned the direct product structure $\mathbb{R}^3 \times \mathrm{SO}(3) \times \mathbb{R}^{n_J}$, with $\mathbb{R}^3$ and $\mathbb{R}^{n_J}$ seen as Lie groups with standard vector sum as group operation. Using this group structure, the right trivialization of $\dot{q}$ is simply $\nu := {}^{B[A]}\nu = ({}^{B[A]}\mathrm{v}_{A,B}, q_J)$ used in (70). The forced equations of motion (70) are then obtained by employing the geometric version of the Lagrange-d'Alembert principle (whose details we will not provide here) employing the appropriate Jacobians (i.e., the mixed-mixed Jacobians for (70)).

A Comparison with existing notation

In this section, we compare the notation introduced in this document with similar notations previously appeared in the literature.

A.1 Featherstone's Notation

In [7] and in the second chapter of [15], based on it, the concept of link *spatial* velocity and acceleration is used to explain the rigid body algorithms. It is worth noting that in [7] the term *spatial* has a totally different meaning with respect to how it is used in [13]. In particular, in [7] *spatial* is used to indicate a 6D vector, being it a twist, a link acceleration, a wrench, or momentum, while in [13], the term spatial is used to indicate a 6D vector expressed with respect to an *inertial* reference frame.

In [7], 6D vectors are composed using the angular-linear serialization. In this report, we use instead the linear-angular serialization. In the remining of this section, we explicitly show the difference between this report's and Featherstone's notation (disregarding the difference in angular-linear serialization).

Homogeneous transformations. In Featherstone's notation, the homogeneous transformation is seldom used, as most of the theory is introduced using directly 6D vectors. For this reason there is not direct equivalent of the notation.

Velocities. In Featherstone's notation, the 6D rigid body velocity of a body-frame B expressed in a frame C is indicated as

$$ {}^{C}\boldsymbol{v}_{B}. $$

All velocities in Featherstone's are always relative to an *implicitly defined* inertial frame A. In this report's notation, we prefer to explictly indicate this dependency, and therefore the equivalent expression for this velocity is

$$ {}^{C}\mathrm{v}_{A,B}. $$

Accelerations. Featherstone [7, 15] uses the dot notation $(\dot{\cdot})$ to indicate the differentiation with respect to an implicitly defined inertial frame, and the ring notation $(\mathring{\cdot})$ to indicate the differentiation with respect to the frame in which the quantity is expressed. As we do not implicitly assume the existence of an absolute inertial frame, we just use the $(\dot{\cdot})$ to indicate the differentiation in coordinates. Using this defition, it is easy to see that the body (*spatial*) acceleration defined in Featherstone as

$$ {}^{C}\dot{\boldsymbol{v}}_{B} = {}^{C}\boldsymbol{a}_{B} $$

is equivalent, in this report's notation, to

$$ {}^{C}X_{A}{}^{A}\dot{\mathrm{v}}_{A,B}, \tag{75} $$

where A is the inertial frame implicitly used in Featherstone's. Note that from (47), using this report's notation, we get

$$ {}^{B}\dot{\mathrm{v}}_{B} = {}^{B}X_{A}{}^{A}\dot{\mathrm{v}}_{B}, \tag{76} $$

that in Featherstone's notation is written

$$ {}^{B}\mathring{\boldsymbol{v}}_{B} = {}^{B}\dot{\boldsymbol{v}}_{B}. \tag{77} $$

Adjoint transformations. The adjoint transform that maps a motion vector expressed in a frame B in one expressed in a frame C is indicated in this report as ${}^{C}X_{B}$. This notiation is directly take from Featherstone's, where it is indicated with ${}^{C}\boldsymbol{X}_{B}$. However, the transformation matrix for a 6D force vector is indicated with ${}^{C}\boldsymbol{X}^{*}_{B}$ in Featherstone's, while in this report's we use ${}_{C}X^{B}$. The main reasons behind this choice are: a) the star is typically used to indicate the adjoint (in the sense of adjoint linear transformation in linear algebra) and indeed, in this report's notation we get ${}_{C}X^{B} = {}^{B}X^{*}_{C}$, which is not the case in Featherstone's; b) ${}_{C}X^{B}$ maps wrenches into wrenches while ${}^{B}X_{C}$ maps twists into twist and we use a right superscript to indicate a twist and a right subscript to indicate a wrench.

6D Cross Product. In Featherstone's, the 6D Cross product of a 6D motion vector $\boldsymbol{v}$ and a 6D motion vector $\boldsymbol{u}$ is indicated as

$$ \boldsymbol{v} \times \boldsymbol{u}. $$

A very similar notation is used in this report, namely

$$ \mathrm{v} \times \mathrm{u}. $$

The 6D cross product of a 6D motion vector $\boldsymbol{v}$ and a 6D motion vector $\boldsymbol{f}$ is indicated in Featherstone's as

$$ \boldsymbol{v} \times^{*} \boldsymbol{f}. $$

To indicate explicitly that $\times^{*}$ is nothing else that minus the adjoint representation of the Lie algebra of $SE(3)$ to itself, we write the same operation as

$$ \mathrm{v}\bar{\times}^{*}\mathrm{f}. $$

Further details are given in the explanation of (53).

Recap on this report's and Featherstone's notation comparison. Summarizing, the main difference and similarities of the two notations are the following.

This report	Featherstone [7]
${}^{C}\mathrm{v}_{A,B}$	${}^{C}\boldsymbol{v}_B$
${}^{C}X_A {}^{A}\dot{\mathrm{v}}_{A,B} = {}^{C}X_B {}^{B}\dot{\mathrm{v}}_{A,B}$	${}^{C}\dot{\boldsymbol{v}}_B$
${}^{C}X_B$	${}^{C}\boldsymbol{X}_B$
${}_{C}X^{B}$	${}^{C}\boldsymbol{X}^{*}_B$
$\mathrm{v}\times$	$\boldsymbol{v}\times$
$\mathrm{v}\bar{\times}^{*}$	$\boldsymbol{v}\times^{*}$

A.2 Siciliano's Notation

In this section, we compare this report's notation the notation used in the classical book of Siciliano et al. [16].

Homogenous transformation. In [16], the homogeneous transformation that maps the coordinates of a point from a frame A to a frame B is indicated with

$$\mathbf{T}^{A}_{B} = \begin{bmatrix} \mathbf{R}^{A}_{B} & \mathbf{o}^{A}_{B} \\ \mathbf{0}_{3\times1} & 1 \end{bmatrix}. \tag{78}$$

Comparing it with (12), we obtain the following comparison table.

This report	Siciliano [16]
${}^{A}H_B$	$\mathbf{T}^{A}_{B}$
${}^{A}R_B$	$\mathbf{R}^{A}_{B}$
${}^{A}o_B$	$\mathbf{o}^{A}_{B}$

Note that, in Siciliano et al.'s notation, $\mathbf{o}^{A}_{B}$ is simply denoted $\mathbf{p}_B$ whenever A is an inertial frame.

Velocity of a frame. In [16], the velocity of a frame B is denoted

$$\mathbf{v}_B = \begin{bmatrix} \dot{\mathbf{p}}_B \\ \boldsymbol{\omega}_B \end{bmatrix}. \tag{79}$$

Comparing it with (35), indicating with A the inertial frame implicitly assumed by the Siciliano notation, we have

This report	Siciliano [16]
${}^{B[A]}\mathrm{v}_{A,B}$	$\mathbf{v}_B$
${}^{A}\dot{o}_B$	$\dot{\mathbf{p}}_B$
${}^{A}\omega_{A,B}$	$\boldsymbol{\omega}_B$

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