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**Citation for published version (APA):**

Tran, N., Ryvo Octaviano, R., Ozkan, L., & Backx, T. (2014). Generalized predictive control tuning by controller matching. In *Proceedings of the 2014 American Control Conference* (pp. 4889-4894). Institute of Electrical and Electronics Engineers. <https://doi.org/10.1109/ACC.2014.6858951>

**DOI:**

[10.1109/ACC.2014.6858951](https://doi.org/10.1109/ACC.2014.6858951)

**Document status and date:**

Published: 01/06/2014

**Document Version:**

Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

**Please check the document version of this publication:**

- A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
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# Generalized Predictive Control tuning by controller matching

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**Abstract**—The tuning of state-space model predictive control (MPC) based on reverse engineering has been investigated in literature using the inverse optimality problem ([1] and [2]). The aim of the inverse optimality is to find the tuning parameters of MPC to obtain the same behavior as an arbitrary linear-time-invariant (LTI) controller (favorite controller). This requires equal control horizon and prediction horizon, and loop-shifting is often used to handle non-strictly-proper favorite controllers. This paper presents a reverse-engineering tuning method for MPC based on transfer function formulation, also known as generalized predictive control (GPC). The feasibility conditions of the matching of a GPC with a favorite controller are investigated. This approach uses a control horizon equal to one and does not require any loop-shifting techniques to deal with non-strictly-proper favorite controllers. The method is applied to a binary distillation column example.

## I. INTRODUCTION

Model predictive control (MPC) is a popular advanced process control technology due to its ability to handle system constraints explicitly. Apart from the obvious advantage of handling constraints, interpreting the available degrees of freedom in the MPC cost function while the constraints are inactive is not straightforward ([3]). Among various ways of selecting the cost function (the tuning parameters) of MPC in literature ([4]), the tuning methods which enable MPC to inherit the characteristics of an arbitrary LTI controller are of particular interest. When MPC operates closely to the constraints and active constraints occur frequently, the system will take advantage of the traditional ability of MPC. When MPC operates away from the constraints (e.g. at commissioning), the system can inherit the characteristics of an LTI controller, e.g. its robustness.

The matching of MPC with an LTI controller when MPC is formulated in the state space has been investigated by several authors ([5], [6], [7], [3] and [8]). In that case, the unconstrained solution of MPC can be written as a state feedback control law and the aim of the matching is to minimize the error between the state feedback gain of the favorite controller and that of MPC. The foundation of this approach is the inverse problem of linear optimal control, laid by [1] and [2]. The inverse optimality problem is extended to a more general cost function in [9] with a cross-product term between the state and the control input. In [7], a matching method based on formulating an optimization problem with linear matrix inequality (LMI) or bilinear matrix inequality (BMI)

constraints is proposed. The cost function of the optimization problem is the error between the control action of the MPC and the favorite controller. The matching methods based on the inverse problem of linear optimal control usually consider the case in which the MPC is equivalent to a linear-quadratic regulator (LQR) and the states of the system are available.

In many applications, the states of the system are not measurable and the use of a state observer is required. In [5], the observer is designed with the loop-shaping procedure introduced in [10] and the tuning parameters of MPC are found by investigating the inverse problem of the normalised left co-prime factorization (NLCPF) optimal control. In [6], separate designs of the robust observer and state feedback gain are used for the matching purpose, and non-convex optimization techniques are employed to perform the matching when the terminal weight is not used.

In [11], it is shown that robustness is not guaranteed even when one attempts to design a “good” observer and a state feedback gain. Based on this observation, [3] makes use of the observer realization techniques described in [12] to divide a favorite controller into the observer part and state feedback part before performing the matching. A major drawback of this approach is that if a favorite output feedback controller contains a feed-through term from the outputs to the control inputs (i.e. a non-strictly-proper controller), loop-shifting techniques must be used to “transfer” the feed-through term to the dynamics of the plant so that the matching is feasible. Introducing some assumptions, [8] has proposed a solution to the problem by considering the feed-through term in the framework of reference tracking.

Due to the nature of the inverse optimality problem ([1], [2] and [9]), the controller matching is often studied with a state-feedback MPC law and an observer design. Nevertheless, MPC can also be formulated by transfer functions and this formulation is also well adopted by several MPC providers in process industry ([13]). The MPC based on transfer function models (GPC) was introduced in [14], [15] and further developed in [16]. Although there is certain equivalence between the GPC and the state-feedback MPC, there are differences in formulating the cost function and computing the solution. Hence, the matching of GPC with a favorite controller is investigated to overcome the limitations of the matching problem in state space. [17] and [18] proposed a tuning method for the GPC such that the poles and zeros of the closed-loop

system approximates certain desired ones. [19] makes use of optimization techniques to find an output feedback gain that minimizes the difference between the closed-loop behavior of the GPC and the desired behavior in the frequency domain. In that work, the tuning parameters are found by solving a convex optimization problem with LMI constraints. The approach is limited to the case where the control horizon is 1.

The focus of this paper is to match a GPC with a favorite controller when the constraints are inactive. Instead of using optimization techniques, we solve a set of linear equations to find the output feedback gain of GPC. To this end, the rank conditions of coefficient matrices are investigated. Once the rank conditions are fulfilled, an output feedback gain that guarantees the matching can always be found. Then, a convex optimization problem with LMI constraints similar to [7] is used to find the tuning parameters which provide the computed output feedback gain. The degrees of freedom of the problem are increased by extending the objective function of the GPC with cross-product terms between the outputs and inputs. The proposed approach does not require any loop-shifting technique to tackle the feed-through term from output to input in the controller.

This paper is organized as follows. Section II presents the formulation and notations used throughout the paper. Section III formulates the problem and Section IV provides the method to find the output feedback gain for the matching. The approach to computing the tuning parameters is given in Section V. Section VI illustrates the method with a binary distillation column model and Section VII gives conclusive comments.

## II. PRELIMINARIES: GENERALIZED PREDICTIVE CONTROL

The prediction of GPC hinges upon the Controlled autoregressive integrated moving average (CARIMA) model ([16]). Consider a linear system with  $n_u$  inputs and  $n_y$  outputs:

$$D(z)y_k = N(z)u_k + \frac{T(z)}{\Delta(z)}v_k \quad (1)$$

where  $N(z)^{n_y \times n_u}$  and  $D(z)^{n_y \times n_y}$  are the numerator and denominator matrices of the system in the backward shift operator  $z^{-1}$ , respectively;  $T(z) = t(z)I_{n_y}$  is a diagonal transfer matrix used to model the disturbance signal and usually considered as a design parameter ([14] and [15]);  $\Delta(z)$  is the difference operator  $\Delta(z) = 1 - z^{-1}$ ;  $y_k \in \mathbb{R}^{n_y}$ ,  $u_k \in \mathbb{R}^{n_u}$  and  $v_k \in \mathbb{R}^{n_y}$  represents the output, input and a zero-mean random variable at time instant  $k$ , respectively.

In the rest of the paper, a square system where  $n_u = n_y = n_d$  is considered for the sake of simplicity. It is also typical of MPC to make use of down squaring to make the system square ([16]). The output reference is assumed to be incorporated in the model and the aim of the controller is to steer the outputs of the system to zero.

From (1), it follows:

$$D(z)\Delta(z)T^{-1}(z)y_k = N(z)T^{-1}(z)\Delta u_k + v_k \quad (2)$$

$$\Rightarrow D(z)\Delta(z)\tilde{y}_k = N(z)\Delta\tilde{u}_k + v_k \quad (3)$$

where  $\tilde{y}_k = T^{-1}(z)y_k$  is the filtered outputs and  $\Delta\tilde{u}_k = T^{-1}(z)\Delta u_k$  is the filtered inputs. With the assumption that the considered system is strictly proper, let

$$D(z)\Delta(z) = I + D_1z^{-1} + D_2z^{-2} + \dots + D_{n+1}z^{-n-1} \quad (4)$$

and

$$N(z) = N_1z^{-1} + N_2z^{-2} + \dots + N_nz^{-n}. \quad (5)$$

Let  $H_p$  denote the prediction horizon, the prediction model of the system is constructed as follows (with the assumption that the best prediction of  $v_k$  is zero):

$$\begin{aligned} \tilde{y}_{k+1} + D_1\tilde{y}_k + \dots + D_{n+1}\tilde{y}_{k-n} &= \\ N_1\Delta\tilde{u}_k + \dots + N_n\Delta\tilde{u}_{k-n+1} & \\ \tilde{y}_{k+2} + D_1\tilde{y}_{k+1} + \dots + D_{n+1}\tilde{y}_{k-n+1} &= \\ N_1\Delta\tilde{u}_{k+1} + \dots + N_n\Delta\tilde{u}_{k-n+2} & \\ \vdots & \\ \tilde{y}_{k+H_p} + D_1\tilde{y}_{k+H_p-1} + \dots + D_{n+1}\tilde{y}_{k+H_p-1-n} &= \\ N_1\Delta\tilde{u}_{k+H_p-1} + \dots + N_n\Delta\tilde{u}_{k+H_p-n} & \end{aligned} \quad (6)$$

When a control horizon  $H_c < H_p$  is considered, the inputs become constant from time instant  $k + H_c$  if the prediction is made at time instant  $k$ :  $\Delta u_{k+H_c+l} = 0$  for  $l \geq 0$ . In this work, a control horizon of 1 is considered. The extension to the case where  $H_c > 1$  is inspired by [7] and has also been investigated but it is not presented here. It should be noted that although  $\Delta u_{k+1+l} = 0$  for  $l \geq 0$ , it does not necessarily lead to  $\Delta\tilde{u}_{k+1+l} = 0$  for  $l \geq 0$  due to the filtering effect of  $T^{-1}(z)$ . Therefore, a so-called filter horizon  $H_f$  is defined such that  $\Delta\tilde{u}_{k+H_f+l} \approx 0$  for  $l \geq 0$ . Hence:

$$\begin{aligned} C_D \begin{bmatrix} \tilde{y}_{k+1} \\ \tilde{y}_{k+2} \\ \vdots \\ \tilde{y}_{k+H_p} \end{bmatrix} &= -H_D \begin{bmatrix} \tilde{y}_k \\ \tilde{y}_{k-1} \\ \vdots \\ \tilde{y}_{k-n} \end{bmatrix} \\ &+ C_{zN} \begin{bmatrix} \Delta\tilde{u}_k \\ \Delta\tilde{u}_{k+1} \\ \vdots \\ \Delta\tilde{u}_{k+H_f-1} \end{bmatrix} &+ H_{zN} \begin{bmatrix} \Delta\tilde{u}_{k-1} \\ \Delta\tilde{u}_{k-2} \\ \vdots \\ \Delta\tilde{u}_{k-n} \\ \Delta\tilde{u}_{k-n-1} \end{bmatrix} \end{aligned} \quad (7)$$

where

$$\begin{aligned} C_D &= \begin{bmatrix} I & 0 & \dots & 0 \\ D_1 & I & \dots & 0 \\ D_2 & D_1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}; H_D = \begin{bmatrix} D_1 & D_2 & \dots & D_{n+1} \\ D_2 & D_3 & \dots & 0 \\ D_3 & D_4 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}; \\ C_{zN} &= \begin{bmatrix} N_1 & 0 & \dots & 0 \\ N_2 & N_1 & \dots & 0 \\ N_3 & N_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}; H_{zN} = \begin{bmatrix} N_2 & N_3 & \dots & N_n & 0 & 0 \\ N_3 & N_4 & \dots & 0 & 0 & 0 \\ N_4 & N_5 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}. \end{aligned}$$

The predicted output is then given by:

### III. PROBLEM FORMULATION

Let a favorite proper (but not necessarily strictly proper) controller be given by:

$$(I + A_1 z^{-1} + A_2 z^{-2} + \dots + A_p z^{-p}) u_k = (\tilde{B}_0 + \tilde{B}_1 z^{-1} + \dots + \tilde{B}_p z^{-p}) y_k. \quad (20)$$

Hence:

$$(I + A_1 z^{-1} + \dots + A_p z^{-p}) \Delta u_k = (\tilde{B}_0 + \tilde{B}_1 z^{-1} + \dots + \tilde{B}_p z^{-p}) (1 - z^{-1}) y_k \quad (21)$$

$$\Rightarrow A(z) \Delta u_k = -B(z) y_k \quad (22)$$

where  $A(z) = I + A_1 z^{-1} + \dots + A_p z^{-p}$  and  $B(z) = B_0 + B_1 z^{-1} + \dots + B_{p+1} z^{-p-1}$ .

To investigate the matching problem, two subproblems are studied:

- Matching transfer matrices: Find the controller gain  $\tilde{K}_{MPC}$  in (14) such that  $\tilde{N}_k(z) = B(z)$  and  $\tilde{D}_k(z) = A(z)$ .
- Finding the tuning parameters: Find cost function (12) such that  $(\tilde{H}^\top \mathcal{Q} \tilde{H} + \mathcal{R})^{-1} \tilde{H}^\top \mathcal{Q} = \tilde{K}_{MPC}$ .

### IV. MATCHING TRANSFER MATRICES

The aim of the matching is to equate  $B(z)$  with  $\tilde{N}_k(z)$  and  $A(z)$  with  $\tilde{D}_k(z)$ , while the order of  $B(z)$  is  $p+1$ ,  $\tilde{N}_k(z)$  is  $n$ ,  $A(z)$  is  $p$  and  $\tilde{D}_k(z)$  is  $n+1$ . Due to the orders of the transfer matrices, the prediction model must be over-parameterized if  $n < p+1$  and the controller must be over-parameterized if  $n > p+1$ . The simplest over-parametrization technique is adding zero coefficients to high order terms of the transfer functions. Assume that  $n = p+1$ , the target of the matching is to find  $\tilde{K}_{MPC}$  such that:

$$\begin{aligned} \tilde{K}_{MPC} \tilde{P} &= [A_1 \quad \dots \quad A_p \quad 0 \quad 0] - [T_1 \quad \dots \quad T_{n+1}] \\ \tilde{K}_{MPC} \tilde{Q} &= [B_0 \quad \dots \quad B_{p+1}]. \end{aligned} \quad (23)$$

The problem above is feasible for arbitrary  $[A_1 \quad A_2 \quad \dots \quad A_p]$  and  $[B_0 \quad B_1 \quad \dots \quad B_{p+1}]$  if both following conditions hold:

- Matrix  $[\tilde{P} \quad \tilde{Q}]$  is full rank.
- Matrix  $[\tilde{P} \quad \tilde{Q}]$  is square or skinny.

As the number of rows of  $[\tilde{P} \quad \tilde{Q}]$  is  $H_p n_d$  and the number of columns is  $2(n+1)n_d$ , the second condition can be satisfied if  $H_p \geq 2(n+1)$ , which is typically the case since  $H_p$  is usually chosen long enough to cover the main dynamics of the system. Given  $\tilde{P}$  and  $\tilde{Q}$  as:

$$\begin{aligned} \tilde{P} &= C_{Ty} P - C_{Ty} H C_{Tu}^{-1} H_{Tu} \\ &= C_{Ty} C_D^{-1} H_{zN} - C_{Ty} C_D^{-1} C_{zN} C_{Tu}^{-1} H_{Tu} \end{aligned} \quad (24)$$

$$\tilde{Q} = H_{Ty} - C_{Ty} C_D^{-1} H_D. \quad (25)$$

we investigate the rank of  $[\tilde{P} \quad \tilde{Q}]$ . Since  $C_{Ty}$  and  $C_D$  are square and full-rank, it follows that:

$$\text{rank}(\tilde{P}) = \text{rank}(H_{zN} - C_{zN} C_{Tu}^{-1} H_{Tu}) \quad (26)$$

$$\text{rank}(\tilde{Q}) = \text{rank}(C_D C_{Ty}^{-1} H_{Ty} - H_D). \quad (27)$$

$$\tilde{y}_{\rightarrow k} = H \Delta \tilde{u}_{\leftarrow k-1} + P \Delta \tilde{u}_{\leftarrow k-1} + Q \tilde{y}_{\leftarrow k} \quad (8)$$

where  $H = C_D^{-1} C_{zN}$ ,  $P = C_D^{-1} H_{zN}$  and  $Q = -C_D^{-1} H_D$ . To compute the solution of the MPC, a prediction model based on  $\tilde{y}_{\rightarrow k}$  and  $\Delta \tilde{u}_{\leftarrow k-1}$  is needed. Define  $T(z) = I + T_1 z^{-1} + T_2 z^{-2} + \dots + T_{n+1} z^{-n-1}$  where  $T_i = t_i I_{n_d}$ ,  $t_i \in \mathbb{R}$  and  $t_{n+1} \neq 0$ . It implies that:

$$\tilde{y}_{\rightarrow k} = \underbrace{\begin{bmatrix} I & 0 & \dots & 0 \\ \vdots & I & \ddots & \vdots \\ T_{n+1} & & \ddots & 0 \\ 0 & T_{n+1} & \dots & I \end{bmatrix}}_{C_{Ty}(H_p n_d \times H_p n_d)} \tilde{y}_{\rightarrow k} + \underbrace{\begin{bmatrix} T_1 & \dots & T_{n+1} \\ \vdots & \ddots & 0 \\ T_{n+1} & \ddots & \vdots \\ 0 & \dots & 0 \end{bmatrix}}_{H_{Ty}(H_p n_d \times (n+1)n_d)} \tilde{y}_{\leftarrow k} \quad (9)$$

and

$$\Delta \tilde{u}_{\leftarrow k-1} = C_{Tu} \Delta \tilde{u}_{\leftarrow k-1} + H_{Tu} \Delta \tilde{u}_{\leftarrow k-1} \quad (10)$$

where  $C_{Tu} \in \mathbb{R}^{H_f n_d \times H_f n_d}$  and  $H_{Tu} \in \mathbb{R}^{H_f n_d \times (n+1)n_d}$  are constructed similarly to  $C_{Ty}$  and  $H_{Ty}$ . From (8), (9) and (10), it follows:

$$\tilde{y}_{\rightarrow k} = \tilde{H} \Delta u_k + \tilde{P} \Delta \tilde{u}_{\leftarrow k-1} + \tilde{Q} \tilde{y}_{\leftarrow k} \quad (11)$$

where  $\tilde{H} = C_{Ty} H C_{Tu}^{-1} \Phi$  with  $\Phi = [I_{n_d \times n_d} \quad 0 \quad \dots \quad 0]^\top$ ,  $\tilde{P} = C_{Ty} P - C_{Ty} H C_{Tu}^{-1} H_{Tu}$  and  $\tilde{Q} = H_{Ty} + C_{Ty} Q$ . The model in (11) is used to compute the optimal input sequence that minimizes a cost function:

$$J_k = \tilde{y}_{\rightarrow k}^\top \mathcal{Q} \tilde{y}_{\rightarrow k} + \Delta \tilde{u}_{\leftarrow k-1}^\top \mathcal{R} \Delta \tilde{u}_{\leftarrow k-1} \quad (12)$$

where  $\mathcal{Q} \in \mathbb{R}^{H_p n_d \times H_p n_d}$  and  $\mathcal{R} \in \mathbb{R}^{n_d \times n_d}$  are the weighting matrices penalizing the outputs and input increments. It is common that  $\mathcal{Q}$  and  $\mathcal{R}$  are chosen to be positive definite. The unconstrained solution to the MPC at time instant  $k$  is then given by:

$$\Delta u_k = -\tilde{N}_k \tilde{y}_{\leftarrow k} - \tilde{D}_k \Delta \tilde{u}_{\leftarrow k-1} \quad (13)$$

where

$$\begin{cases} \tilde{D}_k = (\tilde{H}^\top \mathcal{Q} \tilde{H} + \mathcal{R})^{-1} \tilde{H}^\top \mathcal{Q} \tilde{P} = \tilde{K}_{MPC} \tilde{P} \\ \tilde{N}_k = (\tilde{H}^\top \mathcal{Q} \tilde{H} + \mathcal{R})^{-1} \tilde{H}^\top \mathcal{Q} \tilde{Q} = \tilde{K}_{MPC} \tilde{Q}. \end{cases} \quad (14)$$

Define

$$\tilde{N}_k(z) = \tilde{N}_k [I \quad z^{-1} \quad \dots \quad z^{-n}]^\top \quad (15)$$

$$\tilde{D}_k(z) = \tilde{D}_k [z^{-1} \quad z^{-2} \quad \dots \quad z^{-n-1}]^\top. \quad (16)$$

The transfer matrix representation of the control law is given by:

$$(I + T^{-1}(z) \tilde{D}_k(z)) \Delta u_k = -T^{-1}(z) \tilde{N}_k(z) y_k \quad (17)$$

$$\Rightarrow (T(z) + \tilde{D}_k(z)) \Delta u_k = -\tilde{N}_k(z) y_k \quad (18)$$

$$\Rightarrow \tilde{D}_k(z) \Delta u_k = -\tilde{N}_k(z) y_k. \quad (19)$$

**Theorem 1.** Given model (1) and matrices  $\tilde{P}$ ,  $\tilde{Q}$  in (11),  $\text{rank}([\tilde{P} \ \tilde{Q}]) \leq (2n+1)n_d$  if  $H_f = 1$  or  $H_p - H_f < n+1$ .

*Proof:* If  $H_f = 1$ ,

$$C_{zN} = \begin{bmatrix} N_1 \\ \vdots \\ N_n \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \quad (28)$$

It implies that

$$\begin{aligned} & H_{zN} - C_{zN}C_{Tu}^{-1}H_{Tu} \\ &= \begin{bmatrix} N_2 & \cdots & N_n & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ N_n & \ddots & & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 & 0 \\ \vdots & & & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 & 0 \end{bmatrix} - \begin{bmatrix} N_1 \\ \vdots \\ N_n \\ 0 \\ \vdots \\ 0 \end{bmatrix} C_{Tu}^{-1}H_{Tu} \quad (29) \end{aligned}$$

$$= \begin{bmatrix} \Gamma_{nn_d \times (n+1)n_d} \\ 0_{(H_p-n)n_d \times (n+1)n_d} \end{bmatrix} \quad (30)$$

$$\Rightarrow \text{rank}(\tilde{P}) \leq nn_d \quad (31)$$

$$\Rightarrow \text{rank}([\tilde{P} \ \tilde{Q}]) \leq (2n+1)n_d. \quad (32)$$

The rest of the proof shows that  $\text{rank}([\tilde{P} \ \tilde{Q}]) \leq (2n+1)n_d$  if  $H_p - H_f < n+1$ . For simplicity, assume  $n = 1$ ,  $H_f = 3$  and  $H_p = 4$ ,

$$\begin{aligned} & \text{rank}([\tilde{P} \ \tilde{Q}]) \\ &= \text{rank}([\ H_{zN} - C_{zN}C_{Tu}^{-1}H_{Tu} \quad C_D C_{Ty}^{-1}H_{Ty} - H_D \ ]). \quad (33) \end{aligned}$$

$$\text{Note that } C_{Ty}^{-1} = \begin{bmatrix} I & 0 & 0 & 0 \\ -T_1 & I & 0 & 0 \\ -T_2 + T_1^2 & -T_1 & I & 0 \\ X & -T_2 + T_1^2 & -T_1 & I \end{bmatrix},$$

matrix  $V := C_D C_{Ty}^{-1}H_{Ty} - H_D$  is:

$$\begin{bmatrix} T_1 - D_1 & T_2 - D_2 \\ (D_1 - T_1)T_1 + T_2 - D_2 & (D_1 - T_1)T_2 \\ (-T_2 + T_1^2 - T_1 D_1 + D_2)T_1 + (D_1 - T_1)T_2 & (-T_2 + T_1^2 - T_1 D_1 + D_2)T_2 \\ Y & Z \end{bmatrix}.$$

It is obvious that  $-T_2 V_{1j} - T_1 V_{2j} = V_{3j}$  for  $j = 1; 2$ . The calculation of  $H_{zN} - C_{zN}C_{Tu}^{-1}H_{Tu}$  also shows identical linear dependency. Hence,  $\text{rank}([\tilde{P} \ \tilde{Q}]) \leq 3n_d$  while  $\text{size}([\tilde{P} \ \tilde{Q}]) = 4n_d$ . The proof can be extended to general  $n$  by employing the expression of the inverse of Toeplitz matrices given in [20] for  $C_{Ty}$  and  $C_{Tu}$  but this is omitted here due to limited space. ■

Theorem 1 shows that  $H_f$  and  $H_p$  must satisfy  $1 < H_f$  and  $H_p - H_f \geq n+1$  so that matrix  $[\tilde{P} \ \tilde{Q}]$  is full rank.

**Corollary 1.** Assume  $H_f > 1$ ,  $H_p - H_f \geq n+1$  and matrix  $[\tilde{P} \ \tilde{Q}]$  is full rank. If  $H_p = 2n+2$ , a solution  $\tilde{K}_{MPC}$  to

(23) is unique. If  $H_p > 2n+2$ , there are infinite number of solutions  $\tilde{K}_{MPC}$  since matrix  $[\tilde{P} \ \tilde{Q}]$  has more rows than columns.

**Proposition 1.**  $H_f$  is chosen based on the settling time of  $T^{-1}(z)$  and  $H_p \geq \max(2(n+1); H_f + n + 1)$  so that matrix  $[\tilde{P} \ \tilde{Q}]$  is full rank and to obtain a decent prediction model.

## V. FINDING THE TUNING PARAMETERS

Section IV explains how to match  $\tilde{K}_{MPC}$  to favorite controller (20). This section describes how to find the tuning parameters after finding  $\tilde{K}_{MPC}$ . When cost function (12) is used, the problem of finding the tuning parameters can be formulated as a convex optimization problem with LMI constraints as shown in [7], [17], [18] and [19]. The problem is given by:

$$\min_{\mathcal{Q}, \mathcal{R}} \left\| (\tilde{H}^\top \mathcal{Q} \tilde{H} + \mathcal{R}) \tilde{K}_{MPC} - \tilde{H}^\top \mathcal{Q} \right\|_2 \quad (34)$$

s.t.  $\mathcal{Q} \geq 0$  and  $\mathcal{R} \geq 0$ . In [7], raising the prediction horizon is a method to increase the degrees of freedom in the LMI to obtain a lower error when matching the state feedback gain. Nevertheless, this work as well as [17], [18] and [19] solve the matching of the output feedback gain  $\tilde{K}_{MPC}$ , whose size depends on the prediction horizon. This is a fundamental difference that limits the benefit gained by increasing the prediction horizon to obtain more degrees of freedom. As shown in Corollary 1, when  $H_p > 2n+2$ , there are infinite number of  $\tilde{K}_{MPC}$  that satisfies (23). Let  $M$  be the set of all  $\tilde{K}_{MPC}$ 's that satisfy (23). One method to increase the degrees of freedom is to use  $\tilde{K}_{MPC}$  as an optimization variable in the optimization problem, subject to  $\tilde{K}_{MPC} \in M$ . However, this approach will lead to a bilinear optimization problem, which is difficult to solve and not considered in this work.

Alternatively, the use of a cross term  $S \in \mathbb{R}^{H_p n_d \times H_c n_d}$  in the cost function is proposed to gain more degrees of freedom in the optimization. The cost function of the MPC is then given by:

$$\begin{aligned} J_k &= \underline{y}_k^\top \mathcal{Q} \underline{y}_k + \Delta \underline{u}_{k-1}^\top \mathcal{R} \Delta \underline{u}_{k-1} \\ &+ \underline{y}_k^\top S \Delta \underline{u}_{k-1} + \Delta \underline{u}_{k-1}^\top S^\top \underline{y}_k \end{aligned} \quad (35)$$

The unconstrained control law is given by:

$$\Delta u_k = -\tilde{K}_{MPC} \begin{pmatrix} \tilde{P} \Delta \underline{y}_{k-1} + \tilde{Q} \underline{y}_{k-1} \end{pmatrix} \quad (36)$$

where

$$\tilde{K}_{MPC} = \left( \tilde{H}^\top \mathcal{Q} \tilde{H} + \mathcal{R} + \tilde{H}^\top S + S^\top \tilde{H} \right)^{-1} \left( \tilde{H}^\top \mathcal{Q} + S^\top \right). \quad (37)$$

The optimization problem with LMI constraints is then given by:

$$\min_{\mathcal{Q}, \mathcal{R}, S} \left\| \left( \tilde{H}^\top \mathcal{Q} \tilde{H} + \mathcal{R} + \tilde{H}^\top S + S^\top \tilde{H} \right) \tilde{K}_{MPC} - \left( \tilde{H}^\top \mathcal{Q} + S^\top \right) \right\|_2 \quad (38)$$

s.t.  $\begin{bmatrix} \mathcal{Q} & S \\ S^\top & \mathcal{R} \end{bmatrix} > 0$ . When the resulting error is not sufficiently small, the constraint can also be relaxed by forcing

Weights	Low-frequency gain	Crossover frequency [rad/min]	High-frequency gain
$w_u(z)$	$10^{-3}$	0.628	1.0005
$w_e(z)$	$10^{3.5}$	0.015	0.5

TABLE I: Input and output weights of the  $H_\infty$  controller.

only the Hessian matrix  $\tilde{H}^\top Q \tilde{H} + \mathcal{R} + \tilde{H}^\top \mathcal{S} + \mathcal{S}^\top \tilde{H}$  to be positive definite as shown in [6]. However, the interpretation of a non-positive-definite  $\begin{bmatrix} Q & \mathcal{S} \\ \mathcal{S}^\top & \mathcal{R} \end{bmatrix}$  is still under investigation.

## VI. EXAMPLE

The controller matching approach is applied to a binary distillation column model. A detailed description of the model is provided in [21]. The considered distillation column consists of 110 trays and the feed to the column is located at tray 39. The relative volatility and the liquid holdup are assumed to be constant at  $\alpha = 1.35$  and  $M = 30$  Kmole, respectively. The column operates at a feed flow of  $F = 219$  Kmole/min with a light component composition of  $z_F = 0.65$ . The model has 2 control variables (top ( $y_{top}$ ) and bottom ( $y_{bot}$ ) purity) and 2 manipulated variables (liquid ( $LF$ ) and vapor ( $VF$ ) flow rates). The objective of the controller is to keep the bottom and top compositions at their set points. The initial set-points of the compositions are 0.9506 [mole fraction] for the top product and 0.0529 [mole fraction] for the bottom product. The model of the column is of second order and obtained from open-loop identification ([22]). The sampling rate of the model is 5 minutes. The transfer matrix of the model is:

$$Y(k) = \frac{1}{D(z)} \begin{bmatrix} N_{11}(z) & N_{12}(z) \\ N_{21}(z) & N_{22}(z) \end{bmatrix} U(k) \quad (39)$$

where  $D(z) = 1 - 1.735z^{-1} + 0.7514z^{-2}$ ,  $N_{11}(z) = 0.0009617z^{-1} - 0.0008408z^{-2}$ ,  $N_{12}(z) = -0.001445z^{-1} + 0.001283z^{-2}$ ,  $N_{21}(z) = 0.0004101z^{-1} - 0.0003359z^{-2}$  and  $N_{22}(z) = -0.0002351z^{-1} + 0.0001847z^{-2}$ . An  $H_\infty$  controller is designed for the distillation column as a favorite controller. Then the controller matching method is applied in order to match the MPC with the  $H_\infty$  controller when the constraints are inactive. The prediction horizon of the MPC is chosen 10 samples to cover the main dynamics of the model. The filter horizon  $N_f$  is 5, which renders matrix  $\begin{bmatrix} \tilde{P} & \tilde{Q} \end{bmatrix}$  full rank.

The weights of the  $H_\infty$  controller are both diagonal:

$$W_e(z) = \begin{bmatrix} w_e(z) & 0 \\ 0 & w_e(z) \end{bmatrix}, W_u(z) = \begin{bmatrix} w_u(z) & 0 \\ 0 & w_u(z) \end{bmatrix} \quad (40)$$

where  $w_e(z)$  and  $w_u(z)$  are scalar weights on outputs and inputs, respectively, with the characteristics given in Table I.

The sensitivity functions  $S(z)$  and  $K(z)S(z)$  together with their templates are given in Fig. 1. It is shown that the functions are below their upper bound with an  $H_\infty$  cost of  $\gamma = 1.1614$ .

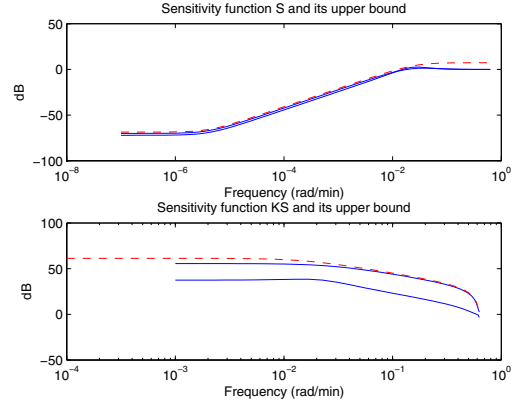


Fig. 1: Sensitivity functions S and KS

The comparison of performance between the  $H_\infty$  controller and the MPC is conducted by a simulation of 800 samples. At sample 50, a step change of 2 [Kmol/min] in the feed rate is made and at sample 300, a step change of -0.01 in the feed composition is made. The step signals are filtered by a first-order low-pass filter before entering the column. The cutoff frequency of the filter associated with the feed rate is 0.0628 [rad/min] and that of the other filter is 0.0314 [rad/min]. The reason for this choice is that the feed composition tends to change more slowly than the feed rate. At sample 500, the set-point of the bottom composition is changed to 0.0429 [mole fraction] and that of the top composition is changed to 0.9606 [mole fraction].

In this example,  $T(z) = t(z)I_2$  with  $t(z) = (1 - 0.7z^{-1})^5$ . Since the order of the model is 2 and that of the  $H_\infty$  is  $p = 3$ , the prediction model is over-parameterized with zero coefficients in high-order terms so that  $n = p + 1 = 4$ .

The first step of the matching is to find  $\tilde{K}_{MPC}$  such that the numerator and denominator of the MPC match those of the  $H_\infty$  controller. Since matrix  $\begin{bmatrix} \tilde{P} & \tilde{Q} \end{bmatrix}$  is full rank, the command `\` of MATLAB<sup>®</sup> is used to solve the set of equations given in (23) to obtain  $\tilde{K}_{MPC}$ , which is a  $2 \times 20$  matrix. With this  $\tilde{K}_{MPC}$ , the convex optimization problem (38) is solved subject to  $\tilde{H}^\top Q \tilde{H} + \mathcal{R} + \tilde{H}^\top \mathcal{S} + \mathcal{S}^\top \tilde{H} > 0$ . The resulting weighting matrices are not shown here due to limited space. The 2-norm of the error between  $\tilde{K}_{MPC}$  and

$$\left( \tilde{H}^\top Q \tilde{H} + \mathcal{R} + \tilde{H}^\top \mathcal{S} + \mathcal{S}^\top \tilde{H} \right)^{-1} \left( \tilde{H}^\top Q + \mathcal{S}^\top \right) \quad (41)$$

is 3.7193e-04. The performance of the  $H_\infty$  controller and the MPC is given in Fig. 2. It can be seen that two controllers match well when the constraints are inactive.

## VII. CONCLUSION AND FUTURE WORK

A tuning method for MPC based on controller matching is proposed. The MPC is formulated in the transfer function form (GPC). The main idea of the method is to match the transfer function of an arbitrary favorite controller and the MPC. Providing that some rank conditions are satisfied, perfect matching between the two controllers can be obtained

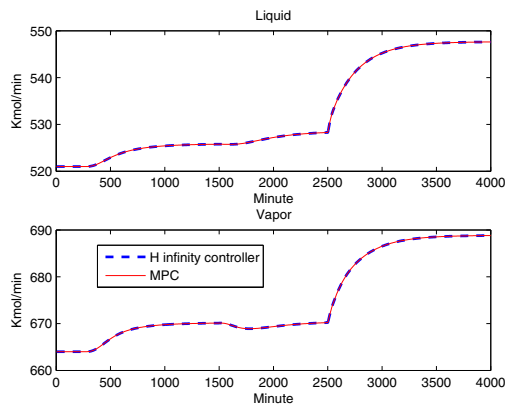


Fig. 2: The response of top and bottom compositions to  $H_\infty$  and MPC controller.

when the constraints are inactive. The use of  $T(z)$  and over-parametrization helps to fulfill the rank conditions while  $T(z)^{-1}$  remains a low-pass filter. Two main steps are followed in the matching. The first step is to find a gain  $\tilde{K}_{MPC}$  that matches the transfer function given. The second step is to find the tuning parameters by formulating a convex optimization problem with LMI constraints.

The use of  $T(z)$  helps to satisfy the rank condition of the coefficient matrices. Moreover,  $T(z)^{-1}$  is usually a low-pass filter whose bandwidth is preferred to be higher than that of the model. In this work,  $T(z)$  is selected based on engineering rules in [16]. It should be noted that  $H_f$  also affects the quality of the prediction. A high  $H_f$  is needed for a slow  $T(z)$  but leads to rank deficiency in  $[\tilde{P} \quad \tilde{Q}]$  as shown in Theorem 1, which makes the matching infeasible. A fast  $T(z)$  will allow a low  $H_f$  but may cause numerical issues since the coefficients of  $T(z)$  will become small. Therefore, a more analytical approach to the computation of  $T(z)$  should be considered in future work.

As for the finding of the tuning parameters, the convex optimization problem is easy to solve and some constraints on the weighting matrices must be removed to obtain a low error. Unlike the matching of the state feedback gain in [7], the impact of increasing the prediction horizon on the degrees of freedom is not clear. One solution to this problem is to consider the gain  $\tilde{K}_{MPC}$  as a decision variable as well, based on the results of Corollary 1. Nevertheless, this will lead to an optimization problem with bilinear constraints, which is more difficult to solve.

Since feed-forward terms are often considered in the prediction model of MPC when measurable disturbances are present, the inclusion of feed-forward terms in the controller matching will also be investigated in future work. Scaling the gain  $\tilde{K}_{MPC}$  will also be considered when optimization problem (38) is solved in order to satisfy the definite positiveness of  $\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix}$ . The case of a control horizon higher than 1 will also be investigated based on the approach presented in [7].

## ACKNOWLEDGMENT

The research leading to these results has received funding from the European Union's Seventh Framework Programme (FP7/2007-2013) under grant agreement n° 257059, the 'Autoprofit' project ([www.fp7-autoprofit.eu](http://www.fp7-autoprofit.eu)).

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