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Bayesian network-based models for bridge network management

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ABSTRACT: Maintenance for highway bridges is crucial in order to keep the network in a satisfactory condition for users but is also a costly affair. This paper proposes a dynamic, Bayesian network-based model to provide cost-efficient strategies in the context of bridge network management. Characteristics related to uncertainties in both the degradation phase and subsequent maintenance strategies are handled through the desirable probabilistic dependencies properties BNs possess. The extension to a specific version of Influence diagrams allows formulating the optimization part of the problem in order to eventually provide long-term strategies as well as minimize expected costs. To that end, a case study that tackles both conditional and unconditional cases is presented.

1 INTRODUCTION

Highway bridges are subject to continuous degradation caused by various events, from environmental impact to an increasing traffic load. The Netherlands is densely populated with large bridges. An overview of the spatial distribution of large bridges (span greater than 150m) is displayed in Figure 1 for the South Holland region. Most of which are ageing as more than 60% of the stock was built prior to the 1970s. Managing and maintaining them is a costly business. Ensuring that safety levels as well as environmentally quality criteria are met as well as providing comfort to users are the top priorities. The Dutch authorities in charge of their maintenance are thus challenged to take decisions on inspection, repair, renovation/rebuilding.

The ability to model degradation behavior and subsequently schedule maintenance decisions as far in advance as possible significantly helps in saving time and money in order to build long-term viable strategies and extend their service lifetime. In practice, asset managers make use of lifetime and deterioration models parametrized with available data to develop predictions on degradation for single bridges (van Noortwijk & Frangopol (2004)). Nevertheless, bridge owners are also often responsible for a whole road network with multiple bridges and, therefore, need to make decisions on a network or stock level. Frangopol & Bocchini (2012) provide an up-to-date state-of-the-art review on this subject.

A network of bridges is comprised of elements or underlying factors that interact between each other.

Thus, dependencies are natural and important to account for. Moreover, these factors can be deterministic or random, hence a probabilistic methodology is logical choice. Another desirable characteristic is the capacity to efficiently insert available evidence in terms of computational demand. This would dynamically update the degradation model in addition to future decision plans. Optimizing costs locally on single bridges and globally with respect to the whole network as well as finding the best balance between the various maintenance decisions are the keys to success. Thus, the development of a model that encompasses these unique features of our problem is the goal of this paper.

Bayesian networks (BNs) comply very well with the requirements cited above. They offer an intuitive understanding of (un)conditional dependencies and a comprehensive visual representation. Models that rely on BNs in the area of reliability and risk-analysis are numerous (Weber et al. 2012). For bridge degradation modelling, Imran Rafiq et al. (2014) used Dynamic BNs (DBNs) that are BNs with a time-indexed sequence of nodes. For the decision-making part, recent developments provide efficient algorithms to solve optimization issues as extensions of BNs. An extended version of Influence Diagrams (IDs) that facilitates long-term plan computations is the central focus in finding optimal solutions to our problem.

In this paper, we propose a BN-based approach to optimize decision related costs in the context of bridge network maintenance management. Maintenance is regarded as imperfect in the sense that the
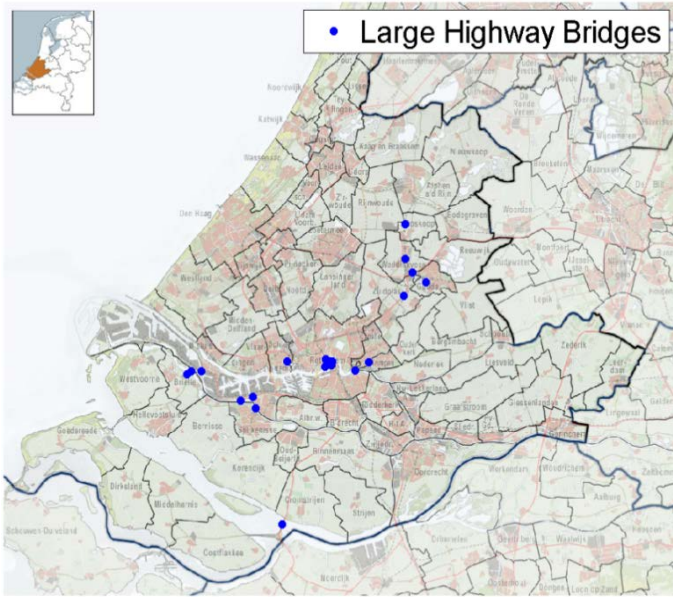
Figure 1. Map of large highway bridges in the South Holland region

system is not as good as new with probability one after performing such actions. The model depicts a method using Limited Memory Influence Diagrams (LIMIDs) which are BNs augmented with decision and utility nodes. Moreover, the quantification of the BN is carried out by an expert elicitation method associated with field measurements.

The remainder of the paper is organized as follows. Section 2 introduces BNs' semantics and the way to exploit them for decision-making through an extended version of Influence Diagrams known as LIMIDs. Section 3 formulates the model in detail and provides optimal strategies taking into account imperfect maintenance assumptions. Section 4 presents a case study on a set of hypothetical network of highway bridges. Section 5 draws conclusions and discusses future work.

2 BAYESIAN NETWORKS

2.1 *Basics and notations*

Formally introduced by Pearl (1988), Bayesian networks are a compact graphical way to represent probabilistic dependencies relationships (*arcs*) between elements of the graph (*nodes*) that stand for discrete or continuous random variables. BNs are directed acyclic graphs which induces a "parent-child" hierarchy for each node, the only exception being for *source* and *leaf* nodes having no parents and children respectively. Denoted by $(X_1, ..., X_n)$ a set of n nodes and $pa(X_i)$ the set of parent nodes of X_i, each variable is associated with a conditional probability table (CPT) of that variable given its parents, $P(X_i|pa(X_i))$.. From the simple chain rule of probability we can derive the multidimensional joint distribution of the whole network as

$$P(X_1, ..., X_n) = \prod_{i=1}^{n} P(X_i|pa(X_i)) \qquad (1)$$

A BN is completely determined once the graph and the entailed dependence structure are specified for the qualitative part. The quantitative component consists of feeding the BN with the (un)conditional probability distribution for every variable to completely determine the joint distribution of Eq.(1). The ability to insert evidence can then be performed throughout the graph in both a bottom-up and top-down manner. Inference is an attractive feature BNs possess and is undoubtedly enticing to update forecasts on the basis of observations. Mathematically, it consists of the use of simple Bayes' formula for conditional probability. However, according to the complexity of the graph, this operation may quickly become intractable so recent developments on more adapted algorithms were proposed (Murphy 2002). By complexity one shall understand the degree of a node, which is the number of edges incident to it, together with their number of states.

2.2 *Influence Diagrams*

BNs can be extended to IDs (Howard & Matheson 1984), that is, augmented with decision and utility nodes in addition to random variable (*chance*) nodes, displayed as square, diamond and ellipse shapes, respectively. By definition, IDs inherit BNs' semantics. Furthermore, all types of dependencies with respect to decision and utility nodes must be specified as follows. A decision node d is a parent of chance node c if the distribution of c can depend on decision d. A decision node d_1 is a parent of decision node d_2 if the choice of alternative for decision d_1 is known to the decision maker when decision d_2 is taken and may influence that decision. When chance node c is a parent of a decision node d it indicates that the value of c will be known when decision d is taken and might influence that decision. Arcs into value nodes represent the decision makers' (expected) utility given the states of its parents.

IDs have been utilized in the area of maintenance decision-making. Cai et al. (2009) proposed an ID to minimize risks costs and provide cost-effective maintenance strategy for civil aircraft maintenance. Hao (2000) used IDs and sequential hypothesis testing to minimize potential loss regarding failed inspections for bridges. However, aspects of degradation modelling and network dependencies have not been investigated.

Partially motivated to overcome the structural limitations of IDs mainly regarding the no-forgetting property (which stipulates that each decision node and its parents are parents to all subsequent decision nodes), Zhang et al. *(*1994) introduced *Decision networks.* This later was extended by Lauritzen & Nilsson (2001) who proposed LImited Memory IDs (LIMIDs). Solving a LIMID means finding a *strategy* that maximizes the expected utility over the set of

utility nodes. A *policy* is a set of rules that describes the decision to opt for as a function of available information. A strategy is simply a set of policies of all decision nodes. Similarly for BNs, LIMIDs face the challenge of complexity and this motivated scientific literature to search for more efficient ways to cope with this problem (see for instance Mauá et al. 2012).

In LIMIDs, a rather strong assumption is asserted saying that past decisions or data have no impact whatsoever on the current ones. Lifting the no-forgetting property defines this memoryless feature for LIMIDs. The latter sharply reduces the computational burden when the time horizon becomes large and results in a more dynamic method that does not yet prevent including time-dependent mechanisms. In addition, the ability to accommodate multiple-values nodes that amounts to relaxing the single-value node constraint has to do again with conditional independence, in cases where costs can be dependent for instance. In fact these two aspects lead to a more compact set of situations that can be handled.

3 MODEL FORMULATION

In this paper we assume a network of m bridges and time horizon T, both integers. Each bridge can take up to a finite number of condition states in a set denoted by $\Omega = \{1, \ldots, n\}$, state 1 being the best state. We further assume bridges can only deteriorate sequentially between those states and newly constructed bridges are supposed to be in the best condition. Because of its adequacy with our assumptions, the Markov chain structure parametrizes the BNs. The parameters of the chain itself, namely the one-time-step transition probabilities $p_{ij}, i, j \in \Omega$, are assessed through expert judgment (see Kosgodagan et al. 2015 for the procedure). For the sake of completeness, expert assessments are used to quantify the remainder CPTs.

In combination with expert assessments, field measurements are used to obtain a statistical distribution for some of the random variables. The importance of these factors (*covariates*) is twofold. On one hand they can impact degradation - often in a causal manner (e.g. nature of the traffic, environmental conditions, loading, etc.) - and secondly we can reasonably assume dependencies between them in different locations in the network. In this sense the network is constructed and inference mechanisms update available information and propagate beliefs on the remainder of the network.

We use the expert estimates to derive a distribution that serves to model degradation. For the best state, the probability of the event that a bridge

k $(1 \leq k \leq m)$ being in this condition at time t is simply given by $P(B_t^k = 1) = (p_{11}^k)^t$. Concerning the other states $j > 1$, the same distribution is given by the following recursive formula

$$P(B_{t+1}^k = j) =$$
$$\begin{cases} P(B_t^k = j) + \phi_{1,j}^k(t+1) - \phi_{1,j+1}^k(t+1)\mathbf{1}_{j<n} \\ P(B_0^k = j) = 0 \end{cases} \quad (2)$$

where

$$\phi_{i,j}^k(t) = P(B_t^k = j, B_{t-1}^k \neq j, \ldots, B_1^k \neq j | B_0^k = i)$$

stands for the first time t that a bridge k has reached state j being in state i at time 0, and $\mathbf{1}_A = 1$ if A is true and 0 otherwise.

One time slice corresponds to one year of elapsing time. Decision nodes have finite discrete states as well as utility nodes. Furthermore decision action types include both preventive (repairs) and corrective (full renovation) maintenance. In practice, many maintenance techniques are available each having repercussions on a bridge being in a specific state and transitioning to a better one. We denote by A the set of available repairs actions in which decision nodes take values. We assume A has the same cardinality as Ω. Maintenance can thus be viewed as imperfect - except for the replacement case - if the system is not as good as new with probability one after performing such actions. For an exhaustive discus-
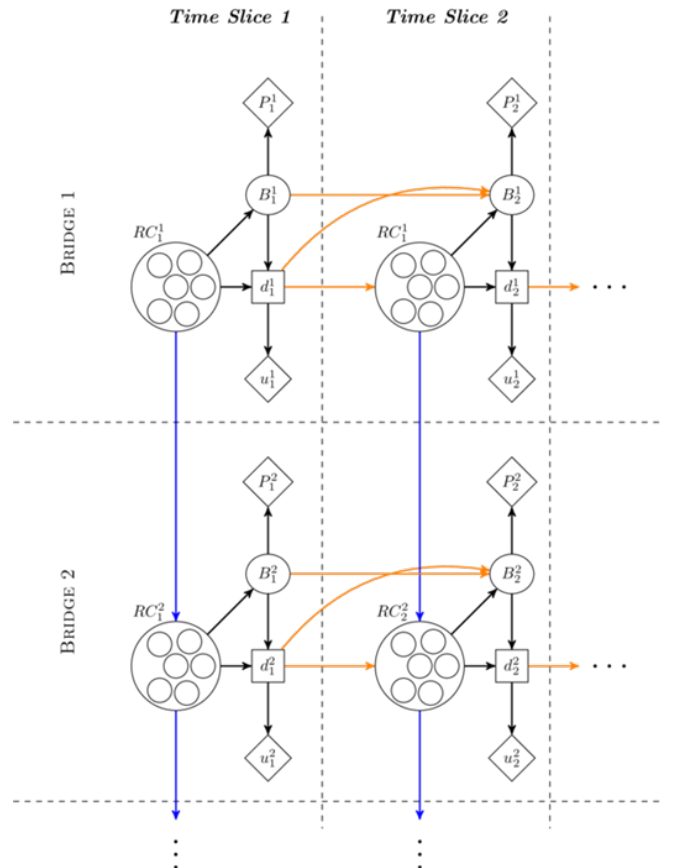
Figure 2. LIMID modelling the bridge network decision problem

sion on this topic we refer to Pham & Wang (1996). In the area of bridge reliability, imperfect maintenance can be interpreted for instance as cracking inspection where the probability of detection depends on the crack size.

In order to account for a broad variety of outcomes, given a bridge being in a state i, we assume that each action type $r \in A$ has probability p_r to reach state $i - r + 1$, and probability $1 - p_r$ to reach state $i - r + 2$, with $1 < i, r \leq n$ and $r \leq i$. Note that for the case $i > r$, $p_r = 1$ to reach state 1. We also assume all actions take negligible time with respect to our time unit, in other words actions are considered instantaneous. Note that for a corrective maintenance action, this assumption could be relaxed in order to take into account the downtime period needed to perform the replacement that would

have an incidence on traffic disruption on other bridges for instance. The model outputs an optimum policy regarding each decision epoch. This is performed by the Single Policy Update (SPU) algorithm (Lauritzen & Nilsson 2001).

The LIMID structure is displayed in Figure 2. For a particular time slice t and bridge k, the structure is comprised of chance nodes that can represent two different entities, namely the degradation process B_t^k modelled through Eq. (2) and the random covariates, RC_t^k. The latter represents a set of nodes having an effect on degradation as mentioned before and in doing so several dependence configurations can be constructed between those. In the same way, decision and utility nodes are labelled d_t^k and u_t^k respectively. Lastly, a utility node P_t^k represents a penalty according to a bridge condition, as a consequence, for example, of traffic restrictions or disruptions.

Table 1. Variable domains and distributions

Variable	States	Conditional probability distribution
B_0^k	1 - Perfect 2 - Fair 3 - Bad 4 - Poor	$P(B_0^k = i) = \begin{cases} 1 & if\ i = 1 \\ 0 & otherwise \end{cases}$
B_t^k	1 - Perfect 2 - Fair 3 - Bad 4 - Poor	$P(B_{t+1}^k = i \mid B_t^k = j, d_t^k = l) =$ $\begin{cases} 1 & if\ i = 1; j = 1,2; l = 3,4 \\ P(B_t^k = i) & if\ i = j; l = 1 \\ 1 - P(B_t^k = i) & if\ j = i + 1; l = 1 \\ 1 & if\ i = j = 4; l = 1 \\ 1 & if\ i = 1; l = 2 \\ p_2 & if\ i = j + 1; l = 2 \\ 1 - p_2 & if\ i = j, l = 2 \\ p_3 & if\ i = j + 2, l = 3 \\ 1 - p_3 & if\ i = j + 1, l = 3 \\ 0 & otherwise \end{cases}$
$Traff_t^1$	1 - Standstill 2 - Queue 3 - Free flow	$P(Traff_t^1 = \tau)$: obtained from field measurements
$Traff_t^k$	1 - Standstill 2 - Queue 3 - Free flow	$P(Traff_t^k = \tau \mid Traff_t^1 = T)$: obtained from field measurements
$Load_t^k$	1 - Heavy 2 - Light	$P(Load_t^k = L)$: obtained from field measurements
d_t^k	1 – Do nothing 2 – Partial repair 3 – Advanced repair 4 - Renovation	Distribution is randomly chosen and iteratively updated according to the requirements of the SPU algorithm.
u_t^k	NA	$u_t^k = \begin{cases} 0 & if\ d_t^k = 1 \\ -C_{R1} & if\ d_t^k = 2 \\ -C_{R2} & if\ d_t^k = 3 \\ -C_{R3} & if\ d_t^k = 4 \end{cases}$
P_t^k	NA	$P_t^k = \begin{cases} 0 & if\ B_t^k = 1 \\ -C_{P1} & if\ B_t^k = 2 \\ -C_{P2} & if\ B_t^k = 3 \\ -C_{P3} & if\ B_t^k = 4 \end{cases}$

Furthermore we emphasize the role played by the arcs defined in section 2.2 according to three types:
− Orange arcs represent "time links" between two consecutive epochs. It is reasonable to assume that a decision affects the bridge condition at the next time step. In addition, an incidence from decision nodes to some of covariates can also be possible.
− Blue arcs stand for dependence within the stock; as previously assumed, those are translated through physical covariates in a consecutive fashion.
− Black arcs by default represent dependencies in determined time slice and bridge.

As decisions depend intrinsically on the degradation process and the fact that maintenance is chosen to be imperfect, strategies are optimized taking into account these two characteristics to output the best trade-off minimizing costs.

4 NUMERICAL APPLICATION

For this numerical example, the following parameter values are taken: $m = 3$ bridges, $T = 20$ years and $n = 4$ condition states for Ω and type of actions for the set A. As emphasized previously, we assume maintenance to be imperfect and we assume $p_2 = 0.85$ and $p_3 = 0.75$. In addition, we set penalty and costs values as follows. Whenever a bridge reaches state 2, 3 and 4, penalties are set to $C_{P1} = 1$, $C_{P2} = 3$, $C_{P3} = 15$ and $C_{R1} = 1$, $C_{R2} = 5$, $C_{R3} = 25$ respectively.

The covariates chosen in this example represent traffic density and loading such that the distribution of loading depends on traffic density, the latter being a source node. In fact, the choice of opting for these two factors is first because they often happen to be the most relevant ones influencing degradation, for instance regarding fatigue for steel bridges. Second, as outlined in Figure 1, a spatial concentration of bridges entailing dependencies in terms of traffic motivates the usage of BNs. Traffic density is understood here as the number of vehicles per kilometer per lane averaged over the total number of lanes. In doing so, traffic density can take three different values, namely *Standstill, Queue* and *Free flow*. Nodes providing information on loading have two states, *Heavy* and *Light*, separated by a threshold related to a proportion of the maximum load solicitation capacity a bridge can bear. These maintenance assumptions together with domains and distributions of each variable are detailed in Table 1.

Decisions directly depend on loading and degradation. Implementing the values introduced above, the SPU outputs unconditional policies for each decision node (60 in our example). In addition, the total expected utility is equal to 101.34. The SPU al-gorithm outputs 3 main different policies which in this case are all the same regardless of the bridge considered. Those are distinguished according to the following time frames. The first time frame ranges from year 0 to year 3 and the policy prescribes action type *1*, meaning "Do Nothing", to be performed. This is in line with the model's assumption since for a worst-case scenario a bridge can reach state *4* only in year 3. The next time frame starts from year 3 to year 16, a more elaborated policy is prescribed. In fact, we first observe that the nature of loading does not shape the policy. Whenever a bridge is in condition *1*, action type *1* is chosen. Furthermore, if a bridge is in condition *2* or *3*, action type *2* is prescribed and for a bridge being in state *4* maintenance type *3* is selected. This specific policy is displayed in Table 2. We can assert that this strategy is intended to be sound as the occupied time frame is wide and thus entails a balanced long-term strategy should no new information becomes available. Policies for the remaining time frame, i.e. $18 \leq t \leq 20$, consist essentially in choosing maintenance type *1*. As the horizon is getting close the algorithm opts for null costs for the few years left.

Conditional on inserting various types of pieces of evidence that increase or decrease probabilistic posterior beliefs for degradation (for instance in observing state *High* or *Low* of node *Load*), we end up obtaining policies that differ from the unconditional case. For illustration purposes, we condition on evidence that probabilistically worsen degradation as it is crucial to investigate. However the opposite event or mixture of both can be envisaged as well in order to have a broader views on other possible scenarios.

When locking evidence on state *Standstill* for node *Traffic*, differences are observed in policies occupying the longest time frame, namely the one displayed in Table 2. Instead of ranging from year 3 to year 17, the same policy now ranges from year 3 to year 10. The remainder of the interval displayed in Table 3 now differs regarding the maintenance action to be taken when a bridge is in state *3*, that is, type *3* is picked while type *2* was chosen for the unconditional case. Thus when traffic density is in a *Standstill* state, it further leads to an increase in terms of loading and risk-related to poor condition as mentioned earlier. Given so, the model reacts in the sense that a more costly policy is selected to preserve a satisfactory reliability level. In this particular case, the total expected utility rises to 110.02.

Table 2. Unconditional decision policy for bridge 1

$3 \leq t \leq 17$	$Load_t^1 = 1$				$Load_t^1 = 2$			
B_t^1	1	2	3	4	1	2	3	4
$d_t^1 = 1$	1	0	0	0	1	0	0	0
$d_t^1 = 2$	0	1	1	0	0	1	1	0
$d_t^1 = 3$	0	0	0	1	0	0	0	1
$d_t^1 = 4$	0	0	0	0	0	0	0	0

Table 3. Decision policy for bridge 1 conditionally on available evidence on state *Standstill* for node Traffic for bridge 1

$11 \leq t \leq 17$	$Load_t^1 = 1$				$Load_t^1 = 2$			
B_t^1	1	2	3	4	1	2	3	4
$d_t^1 = 1$	1	0	0	0	1	0	0	0
$d_t^1 = 2$	0	1	0	0	0	1	1	0
$d_t^1 = 3$	0	0	1	1	0	0	0	1
$d_t^1 = 4$	0	0	0	0	0	0	0	0

Table 4. Decision policy for bridge 2 conditionally on available evidence on node Traffic for bridge 1

$15 \leq t \leq 17$	$Load_t^2 = 1$				$Load_t^2 = 2$			
B_t^2	1	2	3	4	1	2	3	4
$d_t^2 = 1$	1	0	0	0	1	0	0	0
$d_t^2 = 2$	0	1	0	0	0	1	1	0
$d_t^2 = 3$	0	0	1	1	0	0	0	1
$d_t^2 = 4$	0	0	0	0	0	0	0	0

From the network level perspective, conditionality has also an impact yet small on other bridges. Policies remain unchanged except for those lying in the time frame ranging from year 15 to year 17 as indicated in

Table 4. While loading does not influence the decision type to be chosen, as bridges age and their reliability decrease, repair type *3* is preferred over type *2* in the unconditional case. Note that the same policy is also selected for the remainder of the bridges. Observe that maintenance type *4,* assimilated to a full renovation, has not been taken into account by the model in any of the above situations. This is likely due to the fact that time horizon is yet too small to account for such maintenance that involves much larger costs.

5 CONCLUSIVE REMARKS

This paper has proposed a model to optimize maintenance decisions in the context of network bridge management. Various aspects have been introduced including imperfect maintenance and both conditionality and unconditionality characteristics in regards to availability of information. The latter proves to be a key component of the model in order to dynamically incorporate knowledge so that the whole network benefits from it. Furthermore, different combination of type of pieces of information to be inserted could also be performed in order to investigate further impacts on policies. Also, when examining the case related to Table 4, the model shows how imperfect maintenance is optimized. In fact, only for a short amount of time such a policy is prescribed which guarantees a cost-efficient strategy. This should be regarded with importance as a general incentive when dealing with uncertainty around maintenance actions but as to properly evaluate the relevance of inserted information as well.

The attractiveness of inference has to be mitigated though since the quantification of the BN plays a crucial role in the sensitivity of propagated evidence. However the ability to propagate available information that would come from several bridges at once or a subnetwork of such would further strengthen beliefs at a bigger scale of the stock. The optimization part could be complexified too. Integrating other types of dependencies, on the decision process for instance, such as limited or shared maintenance resources or even budget constraint would further lead to prioritize maintenance interventions on certain elements of the networks.

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