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Abstract

In this paper we present two approaches to estimate the Hausdorff dimension of an invariant compact set of a dynamical system: the method of characteristic exponents (estimates of Kaplan-Yorke type) and the method of Lyapunov functions. In the first approach, using Lyapunov's first method we exploit charachteristic exponents for obtaining such estimate. A close relationship with uniform asymptotic stability is hereby established. A second bound for the Hausdorff dimension of an invariant compact set is obtained by exploiting Lyapunov's direct method and thus relies on the use of Lyapunov functions.

1 Introduction

In the study of dynamical systems demonstrating oscillatory behavior it is convenient to have some characteristics of this behavior. In case of periodic oscillations one could think

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of the magnitude, period and, sometimes, phase. The situation becomes more complex if the oscillations are neither periodic nor close to periodic (in any sense). In this case one of the possible candidates for a characteristic of this kind is the Hausdorff dimension of an invariant set of the dynamical system.

Although the notion of dimension in a common sense is intuitively clear, the first formal definition attributed to L. Brouwer appeared only in 1913, cf. [6]. Brouwer gave an inductive definition of the dimension of a (separable) space. Later, Uryson gave a definition of dimension of a compact space in terms of coverings of this space by cubes and proved a remarkable result that the value of dimension of a compact space calculated in this way coincides with the inductive dimension, see [27]. A very similar approach was used by Hausdorff ([15]) to define the dimension of arbitrary compact sets. In case of "nice" sets (a point, an arc, a cube, etc) this dimension coincides with what is intuitively expected; however there are examples of compact sets with fractional Hausdorff dimension. Probably the most famous example is the Cantor set, which has Hausdorff dimension equal to $\log 2 / \log 3$. Now it is well known that compact sets of this type may appear as invariant sets of dynamical systems. The direct computation of the Hausdorff dimension of invariant compact sets is a problem of high (numerical) complexity. Therefore it is interesting to obtain analytic estimates of this dimension. There are a number of results in this direction, see e.g. [10, 11, 16, 19, 20, 25, 26], to mention only few. It is very interesting that some analytic methods of deriving such estimates are very close to those of investigating local stability of invariant sets (see, e.g. [20], [19]). In this paper we present some new estimates for compact invariant sets. To obtain a first estimate we use the method of characteristic exponents (via Lyapunov's first method). Our particular interest is to obtain under quite general assumptions an estimate which can be computed based on only *one* "typical" trajectory of the dynamical system. It turns out that this problem is very close to the problem of uniform asymptotic stability of linear nonautonomous systems. At the same time, in general negativity of the Lyapunov exponents (i.e. Lyapunov's first method) of linear nonautonomous system does not ensure uniform asymptotic stability. In this paper we use some ideas of P.Bohl [3] to introduce a spectrum of exponents of linear systems which we call Bohl exponents. The second group of estimates is related to the second, or direct method of Lyapunov, i.e the method of Lyapunov functions. We further develop ideas from the monographs [20, 19] to obtain analytic estimates of the Hausdorff dimension based on some auxillary functions.

The paper is organized as follows. First in Section 2 the notion of Bohl exponents of systems of linear differential equations is introduced and a number of related results is proven. In Section 3 we present the definition of the Hausdorff dimension of compact sets. In Section 4 we derive upper estimates of the Hausdorff dimension of invariant sets for systems of ordinary differential equations. In Section 5 we present an approach for estimating the region of attraction of an equilibrium point by means of the system of first order approximation. This result is closely connected to the problem of estimation of the Hausdorff dimension. Section 6 contains some concluding remarks.

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In the paper we use the following notations. The Euclidean norm in $\mathbb{R}^n$ is denoted as $|\cdot|$, $|x|^2 = x^\top x$, where $^\top$ stands for the transpose. For matrices the notation $||P||$ stands for the spectral norm of P, i.e. $||P||^2$ is the largest eigenvalue of the matrix $P^\top P$. Eigenvalues of the matrix $P^\top P$ are called singular values of P. If a quadratic form $x^\top P x$ with a symmetric matrix $P = P^\top$ is positive definite then the matrix P is called positive definite. For positive definite matrices we use the notation $P > 0$. $\mathbb{R}_+$ stands for the set of nonnegative reals. The solution of the system of differential equations $\dot{x} = f(t,x)$, with $x \in \mathbb{R}^n$ starting at t_0 in x_0 and calculated at t will be denoted as $x(t,t_0,x_0)$, i.e. $x(t_0,t_0,x_0) = x_0$. Sometimes, if no confusion arises, we will omit the dependence of some arguments.

2 Lyapunov and Bohl exponents

Consider the class C^0 continuous functions $f : \mathbb{R}_+ \to (0,\infty)$. We define functionals χ_L, χ_B on C^0 via

$$\chi_L[f] = \limsup_{t\to\infty} \frac{1}{t} \log f(t),$$

$$\chi_B[f] = \limsup_{s,t-s\to\infty} \frac{1}{t-s} \log \frac{f(t)}{f(s)}.$$

The value $\chi_L[f]$ is called the *Lyapunov exponent* of the function f and the value $\chi_B[f]$ is called the *Bohl exponent* of the function f. The cases $\chi_L[f] = \pm\infty$ or $\chi_B[f] = \pm\infty$ are excluded in this paper.

Suppose continuous functions $f_i : \mathbb{R}_+ \to (0,\infty)$, $i = 1,\ldots,n$ are given. Let F be the continuous function defined as $F(t) = \max_i\{f_i(t)\}$.

Lemma 1 *The following equality is true*

$$\chi_L[F] = \max_i\{\chi_L[f_i]\}.$$

Proof: Suppose $\chi_L[F] = \lambda$. It means that for arbitrary $\varepsilon > 0$, $F(t)/\exp((\lambda+\varepsilon)t) = o(1)$. Therefore $\max_i\{f_i(t)/\exp((\lambda+\varepsilon)t)\} = o(1)$ and therefore $\max_i\{\chi_L[f_i]\} \leq \lambda$. On the other hand, since $\chi_L[F] = \lambda$, for arbitrary $\varepsilon > 0$ there is a sequence $\{t_k\}$, $t_k \to \infty$ as $k \to \infty$ such that

$$\limsup_{k\to\infty} \frac{F(t_k)}{\exp((\lambda-\varepsilon)t_k)} = \infty.$$

Therefore

$$\limsup_{k\to\infty} \max_i\{\frac{f_i(t_k)}{\exp((\lambda-\varepsilon)t_k)}\} = \infty.$$

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In other words, for some i there is a subsequence $\{t_q\}$ of $\{t_k\}$ such that

$$\limsup_{q\to\infty} \frac{f_i(t_q)}{\exp((\lambda-\varepsilon)t_q)} = \infty.$$

Consequently, for some i, $\chi_L[f_i] \geq \lambda$, but we have proved that $\max_i\{\chi_L[f_i]\} \leq \lambda$. Therefore the lemma is proved. ∎

Lemma 2 *If the continuous function $f : \mathbb{R}_+ \to (0,\infty)$ satisfies*

$$\sup_{0\leq t-s\leq 1} f(t)/f(s) \leq K < \infty \qquad (1)$$

and $\chi_B[f] = \alpha$, then for any $\varepsilon > 0$ there is a $N = N(\varepsilon) > 0$ such that the following estimate is valid for all $t \geq s \geq 0$

$$f(t) \leq Ne^{(\alpha+\varepsilon)(t-s)}f(s). \qquad (2)$$

Proof: From the definition of the Bohl exponent it follows that for any $\varepsilon > 0$ there is a $T > 0$ such that

$$f(t) \leq Ne^{(\alpha+\varepsilon)(t-s)}f(s)$$

holds for all $s \geq T$, $t - s \geq T$ and some $N > 0$ (perhaps depending on ε). From the identity

$$\frac{f(t)}{f(t-1)}\frac{f(t-1)}{f(t-2)} = \frac{f(t)}{f(t-2)}$$

and property (1) we conclude that for all $t - s \leq T$ it follows that $f(t)/f(s) \leq K^{T+1}$, i.e $f(t)/f(s)$ is bounded for all $t - s \leq T$ and therefore we can find N such that (2) is valid for all $t, s \geq T$. The set $0 \leq t \leq T, 0 \leq s \leq T$ is compact and hence we can (using larger N if necessary) obtain the estimate for all $0 \leq s \leq t$. ∎

Now we establish a property of Bohl exponents similar to that proved for Lyapunov exponents in Lemma 1. As before, consider continuous functions $f_i : \mathbb{R}_+ \to (0,\infty)$, $i = 1,\ldots,n$. Let F be the function defined as $F(t) = \max_i\{f_i(t)\}$.

Lemma 3 *Assume that the functions f_i, $i = 1,\ldots,n$, satisfy condition (1). Then the following inequality is true*

$$\chi_B[F] \leq \max_i\{\chi_B[f_i]\}.$$

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Proof: Let $\chi_B[F] = \beta$. From the previous lemma it follows that for any $\varepsilon > 0$, there exist sequences $\{t_j\}, \{s_j\}$, $s_j, t_j - s_j \to \infty$ as $j \to \infty$ such that for some N (depending on ε) we have

$$F(t_j) \geq N e^{(\beta-\varepsilon)(t_j - s_j)} F(s_j).$$

Equivalently,

$$\max_i f_i(t_j) \geq N e^{(\beta-\varepsilon)(t_j - s_j)} \max_i f_i(s_j).$$

Therefore, for some i there is a subsequence $\{t_q\}$ of $\{t_j\}$ and corresponding subsequence $\{s_q\}$ of $\{s_j\}$ such that

$$f_i(t_q) \geq N e^{(\beta-\varepsilon)(t_q - s_q)} f_i(s_q).$$

This means that for some i, $\chi_B[f_i] \geq \beta$. $\blacksquare$

Consider a linear time-varying system

$$\dot{x} = A(t)x \tag{3}$$

where $x \in \mathbb{R}^n$ and the $n \times n$ matrix $A(t)$ is continuous on the interval $[t_0, \infty)$. Let $X(t)$ be a fundamental matrix of (3), i.e. $X(t)$ satisfies

$$\dot{X}(t) = A(t)X(t)$$

for some initial conditions $X(t_0)$, $\det X(t_0) \neq 0$. Denote by $\sigma_1(t_0), \sigma_2(t_0), \ldots, \sigma_n(t_0) > 0$ the singular values of $X(t_0)$, i.e. $\sigma_i(t_0) > 0$, $i = 1, \ldots, n$ are the eigenvalues of the matrix $\sqrt{X(t_0)^\top X(t_0)}$ or, that is the same, $\sqrt{X(t_0)X(t_0)^\top}$. The shift operator $x_0 \mapsto x(t, t_0, x_0)$ allows to continue the functions $\sigma_i(t)$ on the maximal interval of existence such that the functions $\sigma_i(t)$ are differentiable. More precisely, there are two differentiable orthogonal matrices $U(t), V(t)$ such that the singular value decomposition of $X(t)$ is given as $X(t) = U(t)\Lambda(t)V(t)$ with differentiable and diagonal $\Lambda(t)$ with entries being $\sigma_i(t)$.

Definition 1 *The numbers*

$$\lambda_i = \limsup_{t \to \infty} \frac{1}{t} \log \sigma_i(t)$$

are called the Lyapunov exponents *of the system (3), provided they exist.*

In the sequel, we order the exponents in decreasing order, i.e. $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$.

First notice that if the matrix $A(t)$ is bounded: $||A(t)|| \leq M < \infty$, $t \in [t_0, \infty)$ or, more generally, *integrally bounded*

$$\sup_t \int_t^{t+1} ||A(\tau)||d\tau \leq M < \infty, \quad t \in [t_0, \infty),$$

then the growth rate of all solutions of (3) is not faster than exponential and, therefore, the Lyapunov exponents exist. Indeed, any fundamental matrix of (3) satisfies the following estimate

$$||X(t)|| \leq e^{\int_s^t ||A(\tau)||d\tau} ||X(s)||, \quad t_0 \leq s \leq t < \infty \tag{4}$$

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and therefore

$$\|X(t)\| \leq e^{\int_{[t-1]}^{t} \|A(\tau)\| d\tau} \cdots e^{\int_{t_0+1}^{t_0+2} \|A(\tau)\| d\tau} e^{\int_{t_0}^{t_0+1} \|A(\tau)\| d\tau} \leq e^{M(t+1)} \|X(t_0)\|,$$

where $[t]$ stands for the maximal integer less or equal than t.

It is important to note that the numbers λ_i do not depend on $X(t_0)$ as soon as $\det X(t_0) \neq 0$. Indeed, if $X(t)$ is a fundamental matrix then the matrix $X(t)S$ where S is nonsingular, is again a fundamental matrix and for the singular values $\sigma_i'(t)$ of $X(t)S$ we have the following simple estimate $\xi_{\min}\sigma_i(t) \leq \sigma_i'(t) \leq \xi_{\max}\sigma_i(t)$ where $\xi_{\min}$ and $\xi_{\max}$ are the minimal and maximal singular values of the matrix S. From this estimate one can conclude that the Lyapunov exponents $\chi_l[\sigma_i']$ and $\chi_l[\sigma_i]$ coincide.

Definition 2 *The numbers*

$$\alpha_i = \lim_{t \to \infty} \frac{1}{t} \log \sigma_i(t)$$

are called the strict Lyapunov exponents *of the system (3), provided they exist.*

Definition 3 *A system for which all Lyapunov exponents are strict is called* regular.

It is trivial to observe that if the matrix $A(t)$ is constant then the system (3) is regular. Now we recall definitions of asymptotic stability and uniform asymptotic stability for the origin of the linear system (3).

Definition 4 *The origin of the system (3) is said to be* Lyapunov stable *if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that the condition $|x(t_0, t_0)| \leq \delta$ implies $|x(t, t_0)| \leq \varepsilon$ for all $t \geq t_0$. If δ can be chosen independently of t_0 then the origin is called* uniformly stable. *If the origin is stable and additionally $|x(t, t_0)| \to 0$ as $t \to \infty$ then the origin is said to be* asymptotically stable. *If the origin is uniformly stable and $|x(t, t_0)| \to 0$ uniformly in t_0 as $t \to \infty$ then it is* uniformly asymptotically stable.

The columns of the fundamental matrix are solutions of the system (3) and $\|X(t)\|$ is the largest singular value of $X(t)$. From this fact and using Lemma 1 one concludes that $\lambda_1 < 0$ is a sufficient condition for asymptotic stability of the trivial solution of (3). Indeed, $\lambda_1 < 0$ implies exponential decay of $|x(t, t_0, x_0)|$ and for linear systems the condition $\lim_{t \to \infty} |x(t, t_0, x_0)| = 0$ is a necessary and sufficient for asymptotic stability of the zero equilibrium (see Theorem 4.1.3 in [1]). At the same time, it is well known that for nonlinear systems, the condition $\lim_{t \to \infty} |x(t, t_0, x_0)| = 0$ is not even sufficient for Lyapunov stability of the origin. Although for the linear system (3) negativity of the Lyapunov exponents is sufficient for asymptotic stability of the origin, it is not sufficient for uniform asymptotic stability of the origin. Recall that the system (3) is uniformly asymptotically stable if and only if for any solution the following estimate is true

$$|x(t, t_0)| \leq C e^{-\alpha(t-s)} |x(s, t_0)|, \quad t_0 \leq s \leq t < \infty \tag{5}$$

for some positive C, α which can be chosen independently of s (see Theorem 4.4.2 in [1]). Motivated by this we introduce the following definition.

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Definition 5 *The numbers*

$$\beta_i = \limsup_{s,t-s\to\infty} \frac{1}{t-s} \log \frac{\sigma_i(t)}{\sigma_i(s)}, \quad i = 1,\ldots,n$$

are called the Bohl exponents *of the system (3), provided they exist.*

Lemma 4 *Assume that the matrix $A(t)$ is integrally bounded. Then all Bohl exponents exist and the following estimates are true*

$$\sigma_i(t) \leq N e^{(\beta_i+\varepsilon)(t-s)} \sigma_i(s), \quad t_0 \leq s \leq t < \infty \qquad (6)$$

for arbitrary $\varepsilon > 0$ and some $N = N(\varepsilon)$.

Proof: From the integral boundedness of $A(t)$ we conclude, using (4) that

$$\sup_{0\leq t-s\leq 1} ||X(t)||\,||X(s)||^{-1} \leq e^M = K. \qquad (7)$$

Moreover,

$$||X(t)||\,||X(s)||^{-1} \leq K^{t-s+1} = K e^{\log K(t-s)} = K e^{M(t-s)}$$

and therefore the largest Bohl exponent is not greater than M. Similarly, from the estimate

$$\sup_{0\leq s-t\leq 1} ||X(s)||\,||X(t)||^{-1} \leq e^M = K$$

we conclude that the smallest Bohl exponent is bounded from below by $-M$. Therefore all exponents exist. To obtain the estimates (6) one can use Lemma 2 since (7) implies that for all i, $\sup_{0\leq t-s\leq 1} \sigma_i(t)/\sigma_i(s) \leq K$. ∎

From the previous lemma we immediately obtain the following result.

Theorem 1 *Assume that the matrix $A(t)$ is integrally bounded, then negativity of the largest Bohl exponent β_1 is a necessary and sufficient condition for uniform asymptotic stability of (3).*

Proof: The estimate (5) is equivalent to the estimate

$$||X(t)|| \leq C e^{-\alpha(t-s)} ||X(s)||, \quad t_0 \leq s \leq t < \infty.$$

At the same time $||X(t)||$ is the largest singular value of $X(t)$. The sufficiency part of the theorem follows from (6) and Lemma 3. To complete the proof we need to show that if there is a nonnegative Bohl exponent then it is possible to find a solution which violates (5) and, hence, the origin is not uniformly asymptotically stable.

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Let $X(t)$ be a fundamental matrix of (3) and let $U^\top(t)\Lambda(t)V^\top(t)$ be its singular value decomposition with orthogonal and differentiable matrices $U(t), V(t)$. In other words,

$$\Lambda(t) = U(t)X(t)V(t),$$

where $\Lambda(t)$ is a diagonal matrix with positive elements. Differentiating Λ gives

$$\dot{\Lambda}(t) = \dot{U}(t)U^\top(t)\Lambda(t) + U(t)A(t)U^\top(t)\Lambda(t) + \Lambda(t)V^\top(t)\dot{V}(t).$$

Observe that

$$\dot{U}(t)U^\top(t) = \frac{d}{dt}\{(U^\smile(t))^{-1}\}U^\top(t) = -U^{-\top}(t)\dot{U}^\top(t)U^{-\top}(t)U^\top(t) = -U(t)\dot{U}^\top(t),$$

that is, the matrices $\dot{U}(t)U^\top(t)$ and $V(t)^\top\dot{V}(t)$ are skew-symmetric and hence all diagonal entries of $\dot{U}(t)U^\top(t)\Lambda(t) + \Lambda(t)V^\top(t)\dot{V}(t)$ are zero. Thus, using the diagonality of Λ,

$$\dot{\Lambda}(t) = \Gamma(t)\Lambda(t), \tag{8}$$

where the matrix $\Gamma(t)$ is diagonal and has diagonal elements $\gamma_1(t), \ldots, \gamma_n(t)$ equal to the diagonal elements of the matrix $U(t)A(t)U^\top(t)$. Let the system have Bohl exponents denoted by β_i and Lyapunov exponents λ_i (not necessarily ordered). According to the definitions of the Lyapunov and Bohl exponents, it is seen that

$$
\begin{aligned}
\lambda_i &= \limsup_{\tau \to \infty} \frac{1}{\tau} \int_{t_0}^\tau \gamma_i(t)dt \\
\beta_i &= \limsup_{s,\tau-s\to\infty} \frac{1}{\tau - s} \int_s^\tau \gamma_i(t)dt
\end{aligned}
$$

Now, integrating (8) with different initial conditions of the type $\mathrm{diag}(0,0,\ldots,1,0,\ldots,0)$ we obtain n linearly independent solutions $\Lambda_i(t)$ of (8). Then $Z_i(t) := U^\top(t)\Lambda_i(t)V^\top(t)$ $i = 1, \ldots, n$ satisfy $\chi_L[\|Z_i\|] = \lambda_i$ and $\chi_B[\|Z_i\|] = \beta_i$. Therefore, for arbitrary $\varepsilon > 0$ there are $N = N(\varepsilon)$ and sequences $\{t_j\}, \{s_j\}$ such that $t_j \to \infty$, $t_j - s_j \to \infty$ as $j \to \infty$ and

$$\|Z_i(t_j)\| \geq Ne^{(\beta_i-\varepsilon)(t_j-s_j)}\|Z_i(s_j)\|$$

Now it is clear that if there is a nonnegative Bohl exponent, then there is a solution which violates (5). ∎

Remark 1 From the proof of this theorem it is seen that any solution $x(t)$ has a Lyapunov exponent $\chi_L[x]$ from the set of Lyapunov exponents $\lambda_1, \ldots, \lambda_n$ (the same can be said about Bohl exponents). Therefore the definition of the Lyapunov exponents of the linear system given in this section is equivalent to that given by Lyapunov. However, our construction

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of Lyapunov exponents of a linear system is easier because it does not require notions of uncompressible and normal fundamental solution (see, e.g. [1]). $\triangle$

Fixing s (taking e.g. $s = t_0$) in (6) one obtains an estimate in which β_i can be replaced by λ_i. From this observation we conclude that if we order Lyapunov and Bohl exponents in decreasing order then

$$\beta_i \geq \lambda_i. \tag{9}$$

By analogy,

$$\liminf_{s,t-s\to\infty} \frac{1}{t-s} \log \frac{\sigma_i(t)}{\sigma_i(s)} \leq \liminf_{t\to\infty} \frac{1}{t} \log \sigma_i(t). \tag{10}$$

Definition 6 *The numbers*

$$\gamma_i = \lim_{s,t-s\to\infty} \frac{1}{t-s} \log \frac{\sigma_i(t)}{\sigma_i(s)}$$

are called the strict Bohl exponents *of the system (3), provided they exist.*

From the estimates (9), (10) it follows that if the system (3) has strict Bohl exponents than they are equal to the strict Lyapunov exponents. This motivates the following definition.

Definition 7 *The system (3) for which all Bohl exponents are strict is called* strongly regular.

The following example emphasizes the fact that Lyapunov exponents and Bohl exponents may be different, and moreover, may have different signs.

Example 1 (Perron equation) [9, 22] Consider the following scalar linear differential equation

$$\dot{x} = (\sin\log(t+1) + \cos\log(t+1) - a)x. \tag{11}$$

Since $X(t) = e^{(t+1)(\sin\log(t+1)-a)}e^a$ is solution of (11) with initial condition $X(0) = 1$, it is easy to see that $\lambda = \limsup_{t\to\infty}(\sin\log(t+1) - a) = 1 - a$. On the other hand, we will show that $\beta = \sqrt{2} - a$. Indeed let $a = 0$ for simplicity and choose t_j, s_j as follows

$$\log(t_j + 1) = 2j\pi + \pi/4 + \varepsilon, \quad \log(s_j + 1) = 2j\pi + \pi/4.$$

Then $t_j - s_j = e^{2j\pi+\pi/4}(e^\varepsilon - 1) \to \infty$ as $j \to \infty$ and therefore

$$\begin{aligned}
\beta &\geq \limsup_{j\to\infty} \frac{(t_j + 1)\sin\log(t_j + 1) - (s_j + 1)\sin\log(s_j + 1)}{t_j - s_j} \\
&= \limsup_{j\to\infty}(\cos\log t'_j + \sin\log t'_j),
\end{aligned}$$

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where the number $t'_j \in (s_j + 1, t_j + 1)$ is defined according to the mean value theorem. Since $\log t'_j = 2j\pi + \pi/4 + \eta_j$, where $0 < \eta_j < \varepsilon$ we have

$$\beta \geq \sqrt{2} \cdot \limsup_{j \to \infty} \sin\left(\frac{\pi}{2} + \eta_j\right) \geq \sqrt{2}\sin\left(\frac{\pi}{2} - \varepsilon\right)$$

for any $\varepsilon > 0$, i.e. $\beta \geq \sqrt{2}$. On the other hand,

$$\beta \leq \sup_{t \geq 0} |\sin\log(t + 1) + \cos\log(t + 1)| = \sqrt{2},$$

i.e. $\beta = \sqrt{2}$. Similarly, for a different from zero, we obtain $\beta = \sqrt{2} - a$ and $\beta - \lambda = \sqrt{2} - 1 > 0$. Hence one concludes that the system (11) is asymptotically stable for $a > 1$. In case $1 < a < \sqrt{2}$ the system is asymptotically stable yet not uniformly asymptotically stable, and, finally, the system is uniformly asymptotically stable for $a > \sqrt{2}$. $\qquad \triangle$

The system considered in the previous example is not regular. Thus one may conjecture that if the system (3) is regular ($\lambda_i = \alpha_i$) then the Bohl and Lyapunov exponents coincide: $\beta_i = \alpha_i$. However, this is not the case as one can see from in following example.

Example 2 Consider the equation

$$\dot{x} = \pi \sin \pi \sqrt{t} x. \tag{12}$$

Its solution is given by

$$x(t) = \exp\left(\int_0^t \pi \sin \pi \sqrt{\tau} d\tau\right) x(0) = \exp\left(\frac{2}{\pi}(\sin \pi \sqrt{t} - \pi \sqrt{t} \cos \pi \sqrt{t})\right) x(0).$$

Since

$$\lim_{t \to \infty} \frac{1}{t}\frac{2}{\pi}(\sin \pi \sqrt{t} - \pi \sqrt{t} \cos \pi \sqrt{t}) = 0$$

one concludes that the system is regular and $\lambda = \alpha = 0$. Let us show that $\beta > 0$. Recall that

$$\beta = \limsup_{s, t-s \to \infty} \frac{1}{t - s}\left(\frac{2}{\pi}\sin \pi \sqrt{t} - 2\sqrt{t}\cos \pi \sqrt{t} - \frac{2}{\pi}\sin \pi \sqrt{s} + 2\sqrt{s}\cos \pi \sqrt{s}\right).$$

Choose increasing sequences $\{t_j\}, \{s_j\}$ to make $\sin \pi \sqrt{t_j}$ and $\sin \pi \sqrt{s_j}$ zero and additionally

$$\cos \pi \sqrt{t_j} = -1, \quad \cos \pi \sqrt{s_j} = 1.$$

In other words, we can take

$$t_j = ((4j + 3)/2)^2, \quad s_j = 4j^2.$$

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This means that $t_j - s_j = 6j + 9/4 \to \infty$ as $j \to \infty$. At the same time

$$2(\sqrt{t_j} + \sqrt{s_j}) = 8j + 3$$

and hence

$$\beta \geq \lim_{j \to \infty} \frac{8j + 3}{6j + 9/4} = \frac{4}{3} > 0.$$

In the same way, choosing $\{t_j\}, \{s_j\}$ to satisfy

$$\cos \pi \sqrt{t_j} = 1, \quad \cos \pi \sqrt{s_j} = -1$$

it can be shown that

$$\liminf_{s, t-s \to \infty} \frac{1}{t - s} \left(\frac{2}{\pi} \sin \pi \sqrt{t} - 2\sqrt{t} \cos \pi \sqrt{t} - \frac{2}{\pi} \sin \pi \sqrt{s} + 2\sqrt{s} \cos \pi \sqrt{s} \right) \leq -\frac{4}{3}.$$

Therefore, system (12) is regular but not strongly regular. $\triangle$

From the previous example we have seen that the class of strongly regular systems is "smaller" than the class of regular systems. However we can specify an important class of systems which are strongly regular.

Theorem 2 *If the matrix $A(t)$ is periodic then the system (3) is strongly regular.*

Proof: If the matrix $A(t)$ is periodic then according to the Lyapunov reducibility theorem (see theorem 3.2.2 in [1]) there is a time-varying coordinate transformation $y = L(t)x$ with differentiable and bounded $L(t)$ together with bounded inverse $L(t)^{-1}$ (such matrix is called *Lyapunov matrix*), such that in the new coordinates the system (3) has the form

$$\dot{y} = By \tag{13}$$

with a constant matrix B. Therefore the fundamental matrix $X(t)$ for (3) is

$$X(t) = L(t)^{-1}e^{Bt}. \tag{14}$$

Since the matrix $L(t)$ is bounded and has bounded inverse, its minimal and maximal singular values are bounded from below and above and therefore the system (3) has the same Bohl exponents as (13). It is clear, that for systems with constant coefficients Lyapunov and Bohl exponents are strict and coincide. $\blacksquare$

Recall that Erugin's reducibility theorem (see e.g. Theorem 3.2.1 in [1]) claims that the system (3) is reducible to a system with constant coefficients if and only if its fundamental matrix can be represented in the form (14), where $L(t)$ is the Lyapunov matrix (differentiable and bounded together with bounded inverse) and B is constant. Therefore we can enlarge the class of strongly regular systems.

Theorem 3 *If the system (3) is reducible to a system with constant coefficients then it is strongly regular.*

Concluding this section we make some historical remarks. While Lyapunov exponents are widely known, the notion of Bohl exponents introduced in this section is relatively new. Originally it comes from the ideas of Latvian mathematician P. Bohl [3] who studied the following property of solutions of linear systems:

$$|x(t, t_0)| \leq Ne^{(\beta+\varepsilon)(t-s)}|x(s, t_0)|.$$

He introduced the index which is, in our terminology, the upper Bohl exponent taken with opposite sign. Using this index he proved (in 1913!) a number of remarkable results some of them were rediscovered 15-50 years later. In modern terminology, he studied uniform asymptotic stability while the definition of uniform asymptotic stability as we know it now, appeared only in 1953. In the 60s, when Latvia was a part of the former Soviet Union, his selected works were published in Russian [4] with an introduction written by A.D. Myshkis and I.M. Rabinovich. Although the contribution of this great mathematician is well known (for example he founded the theory of quasiperiodic functions) some of his results were overlooked. A sensational example is that an analog of the famous Brouwer fixed point theorem published in 1910 was known to Bohl in 1904. A brief biography of P.Bohl can be found on the Web: http://www-groups.dcs.st-and.ac.uk/ history/Mathematicians/Bohl.html.

Nevertheless the ideas of Bohl on uniform asymptotic stability turned out to be very useful. For example, Ju.L. Daleckii and M.G. Krein in [9] extensively used the notion of general exponent (in our terminology - upper Bohl exponent) to study dichotomy of solutions of linear nonautonomous systems in Banach space. In their book the authors also made an attempt to establish a priority of Bohl in some results and the book has very interesting bibliographical and historical comments. The notion of "Bohl exponent" as introduced in [12] to give credit to Bohl's contribution is equivalent to the concept of general exponent used in [9]. In [5] the same functional was called singular exponent, which is a particular case of a more sophisticated construction called central exponent (see also [1]). The main contribution of this section is that we introduced the spectrum of the Bohl exponents.

3   Hausdorff dimension

Consider a compact subset K of a compact metric space X. Given $d \geq 0$, $\varepsilon > 0$, consider a covering of K by open spheres B_i with radii $r_i \leq \varepsilon$. Denote by

$$\mu(K, d, \varepsilon) = \inf \sum_i r_i^d \tag{15}$$

the d-measured volume of covering of the set K. Here the infimum is calculated over all ε-coverings of K. There exists a limit, which may be infinite,

$$\mu_d(K) := \sup_{\varepsilon > 0} \mu(K, d, \varepsilon).$$

It can be proved that μ_d is an outer measure on X (see, e.g. Proposition 5.3.1 in [20]).

Definition 8 *The measure μ_d is called the* Hausdorff d-measure.

The properties of the measure μ_d can be summarized as follows. There exists a single value of $d = d_*$ such that for all $d < d_*$ $\mu_d(K) = +\infty$ and for all $d > d_*$ $\mu_d(K) = 0$. Here

$$d_* = \inf\{d \ : \ \mu_d(K) = 0\} = \sup\{d \ : \ \mu_d(K) = +\infty\}.$$

Definition 9 *The value d_* is called the* Hausdorff dimension *of the set K.*

In the sequel, we will use the notation $\dim_H K$ for the Hausdorff dimension of the set K.

Now, following Douady and Oesterlé [10], we define the elliptic Hausdorff d-measure of a compact set $K \subset \mathbb{R}^n$. Let E be an open ellipsoid in $\mathbb{R}^n$. Let $a_1(E) \geq a_2(E) \geq \ldots \geq a_n(E)$ be the lengths of semiaxes of E numbered in the decreasing order. Represent an arbitrary number d, $0 \leq d \leq n$ in the form $d = d_0 + s$, where $d_0 \in \mathbb{Z}$ and $s \in [0, 1)$ and introduce the following

$$\omega_d(E) = \prod_{i=1}^{d_0} a_i(E) (a_{d_0+1}(E))^s. \tag{16}$$

Fix a certain d and $\varepsilon > 0$ and consider all kinds of finite coverings of the compact K by ellipsoids E_i for which

$$[\omega_d(E_i)]^{1/d} \leq \varepsilon$$

(if $d = 0$ we put $[\omega_d(E_i)]^{1/d} = a_1(E_i)$). Similar to the definition of Hausdorff d-measure we denote

$$\tilde{\mu}_d(K, d, \varepsilon) = \inf \sum_i \omega_d(E_i),$$

where the infimum is calculated over all coverings.

Definition 10 *The value*

$$\tilde{\mu}_d(K) = \sup_{\varepsilon > 0} \tilde{\mu}(K, d, \varepsilon)$$

is called the Hausdorff elliptical d-measure *of the compact K.*

It was proven in [10, 26] that elliptical and spherical Hausdorff d-measures are equivalent and therefore, using extremal properties of μ_d, the values of Hausdorff dimensions determined by means of spherical and elliptic coverings are equal.

4 Upper estimates for the Hausdorff dimension of invariant compact sets

Let $\{\varphi^t\}$ be a one-parameter semigroup of diffeomorphisms $\Omega \to \Omega$, $\Omega \subset \mathbb{R}^n$, $t \in \mathbb{I}$. We will only consider the case $\mathbb{I} = \mathbb{R}_+$, the case $\mathbb{I} = \mathbb{Z}_+$ can be treated in the same fashion. Let the compact set $K \subset \Omega$ be invariant under φ^t, i.e. for any $\tau \in \mathbb{R}_+$, $\varphi^\tau(K) = K$. A subset $\gamma(x_0)$ of $\mathbb{R}^n$ of the form $\gamma(x_0) = \{x \ : \ x = \varphi^t(x_0), \ \forall t \in \mathbb{R}_+\}$ is called the *trajectory, or orbit,* of the point x_0.

By $T_x\varphi^t$ we denote the derivative of φ^t at the point $x \in \mathbb{R}^n$, that is, $T_x\varphi^t$ is a linear operator $\mathbb{R}^n \to \mathbb{R}^n$. For a linear operator $L \ : \ \mathbb{R}^n \to \mathbb{R}^n$ denote by $a_1(L) \geq a_2(L) \geq \dots \geq a_n(L)$ its singular values. For an arbitrary $k \in \mathbb{Z}_+$, $0 \leq k \leq n$ we denote

$$\omega_k(L) = \begin{cases} \prod_{i=1}^{k} a_i(L), & k > 0 \\ 1, & k = 0 \end{cases}$$

As before, for arbitrary $d \in [0, n]$ we put $d = d_0 + s$, where $d_0 \in \mathbb{Z}_+$ and $s \in [0, 1)$ and introduce the following definition

$$\omega_d(L) = \omega_{d_0}^{s-1}(L)\omega_{d_0+1}^s(L).$$

Consider a compact set $\tilde{K}$ such that $K \subset \tilde{K}$.

Theorem 4 *Assume that there exists $d \in [0, n]$ such that for any $\varepsilon > 0$ there exists $t_\varepsilon > 0$ such that for all $t \geq t_\varepsilon$*

$$\sup_{x \in \tilde{K}} \omega_d(T_x\varphi^t) \leq \varepsilon. \tag{17}$$

Then $\dim_H K \leq d$.

Basically, this theorem is a reformulation of the well known Douady-Oesterlé theorem [10], an analog of this statement for arbitrary Hilbert spaces is proved in [26]. Leonov [17] (see also Theorem 5.4.1 in [20] and Theorem 8.1.2 in [19]) proved a generalization of the Douady-Oesterlé theorem: instead of (17) it is sufficient to require that

$$\sup_{x \in \tilde{K}} \left[\frac{p(\varphi^t(x))}{p(x)} \omega_d(T_x\varphi^t) \right] \leq \varepsilon \tag{18}$$

where $p \ : \ \tilde{K} \to (0, \infty)$ is a scalar positive continuous function. This approach turns out to be useful for estimates of the Hausdorff dimension in terms of auxillary (Lyapunov) functions satisfying certain partial differential inequalities. We will apply this idea in the sequel.

Remark 2 If instead of the condition $\varphi^t(K) = K$ we assume that $\varphi^t(K) \subset \tilde{K}$ then the theorem hypothesis implies that $\mu_d(K) < \infty \implies \lim_{t \to \infty} \mu_d(\varphi^t(K)) = 0$, see [17, 20, 19]. $\triangle$

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We will use Theorem 4 to obtain estimates for the Hausdorff dimension of invariant compact sets of a system of ordinary differential equations.

Consider the system

$$\dot{x} = f(x), \tag{19}$$

where $x \in \Omega$ and $f : \Omega \to \Omega$, is a smooth vector field on $\Omega \subset \mathbb{R}^n$. We will assume that there is a compact set $K \subset \Omega$ such that the semigroup of shifts $\{\varphi^t\}$, $\varphi^t : x_0 \mapsto x(t, x_0)$, leaves K invariant: $K = \varphi^t(K), \forall t \in \mathbb{R}_+$.

Assumption 1 *There is a point $x_0 \in \Omega$ such that the trajectory $\gamma(x_0)$ is bounded and K is the ω-limit set of $x(t, x_0)$.*

It is worth mentioning that we do not assume that $x_0 \in K$, in other words we do not assume that there is a dense trajectory *in* K. All we need is that for some x_0 it follows that $K \subset \mathrm{cl}\gamma(x_0)$ and $\mathrm{cl}\gamma(x_0)$ is compact. The trajectory corresponding to this solution will be denoted as γ.

Along with the system (19) consider the first order approximation

$$\dot{y} = J(x(t, x_0))y, \tag{20}$$

where $y \in \mathbb{R}^n$, $x_0 \in \gamma$ and

$$J(x(t, x_0)) = \frac{\partial f}{\partial x}(x(t, x_0)).$$

Denote by $\beta_1 \geq \beta_2 \geq \ldots \geq \beta_n$ the Bohl exponents of the system (20) and for some d where $d = d_0 + s$, $d_0 \in \mathbb{Z}_+$, $s \in [0, 1)$ consider the value

$$\mathcal{B}_d = \sum_{i=1}^{d_0} \beta_i + s\beta_{d_0+1}.$$

Theorem 5 *If for some d it follows that $\mathcal{B}_d < 0$ then $\dim_H K \leq d$.*

Proof: Consider the one-parameter semigroup of diffeomorphisms $\{\varphi^t\}$, $\varphi^t : x_0 \mapsto x(t, x_0)$. The derivative of φ^t with respect to x_0 is the function denoted by $H(t, x_0)$:

$$H(t, x_0) = T_{x_0}\varphi^t = \frac{\partial x(t, x_0)}{\partial x_0}.$$

Calculation of the time derivative of $H(t, x_0)$ gives

$$\frac{d}{dt}H(t, x_0) = \frac{\partial \dot{x}(t, x_0)}{\partial x_0} = \frac{\partial f(x(t, x_0))}{\partial x_0} = J(x(t, x_0))H(t, x_0)$$

and $H(0, x_0) = I_n$. In other words, $H(t, x_0)$ is the fundamental matrix of (20). In order to use Theorem 4 we should find such d that $\omega_d(H(t, x_0)) \to 0$ as $t \to \infty$ uniformly in $x_0 \in \tilde{K}$

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where $\tilde{K} = \text{cl}\gamma$. Since $T_x\varphi^t$ is continuous in x, for any fixed t the function $\omega_d(H(t,x_0))$ approaches its maximum on $\tilde{K}$ which is equal to the supremum over $x_0 \in \gamma$. Hence, it is sufficient to prove that $\omega_d(H(t,x_0)) \to 0$ as $t \to \infty$ uniformly in $x_0 \in \gamma$. Since γ is a trajectory of (19) we should find d such that $\omega_d(H(t,x(\tau,x_0))) \to 0$ as $t \to \infty$ uniformly in $\tau \geq 0$. Notice that

$$H(t + \tau, x_0) = \frac{\partial x(t+\tau, x_0)}{\partial x_0} = \frac{\partial x(t, x(\tau, x_0))}{\partial x_0} = H(t, x(\tau, x_0))H(\tau, x_0) \qquad (21)$$

and hence

$$H(t, x(\tau, x_0)) = H(t + \tau, x_0)H(\tau, x_0)^{-1} \qquad (22)$$

is the evolutional operator of the system (20).

Since for any two linear operators L' and L'' the relation

$$\omega_d(L'L'') \leq \omega_d(L')\omega_d(L'')$$

is true (see [13]), we infer from (21)

$$\omega_d(H(t + \tau, x_0)) \leq \omega_d(H(t, x(\tau, x_0)))\omega_d(H(\tau, x_0)). \qquad (23)$$

From Lemma 3 we conclude that $\chi_B[\omega_d(H(\cdot, x_0))] \leq \mathcal{B}_d$. On the other hand, from the properties of the Bohl exponents, it follows that for arbitrary $\varepsilon > 0$ there is a positive $N = N(\varepsilon)$ such that that

$$\omega_d(H(t + \tau, x_0)) \leq Ne^{(\mathcal{B}_d+\varepsilon)t}\omega_d(H(\tau, x_0)). \qquad (24)$$

Therefore, comparing (23) and (24) we see that if $\mathcal{B}_d < 0$ for some d, then for any fixed $x_0 \in \gamma$ $\omega_d(H(t, x(\tau, x_0)))$ tends to zero as $t \to \infty$ uniformly in $\tau \geq 0$. In other words, $\omega_d(H(t, x_0)) \to 0$ uniformly in $x_0 \in \gamma$ and the result follows from Theorem 4. ∎

The estimate given by the previous theorem is based on the computation of the Bohl exponents of the first order approximation along a "typical" solution. To compute the Bohl exponents we should know the fundamental matrix of the first order approximation, i.e. we should find a set of linearly independent solutions of this system. Analytically, this problem can be solved only in simplest cases and therefore this estimate is only useful in numerical approximation. Our next goal is to obtain an estimate which can be used in analytical calculations.

Consider a matrix function $G : \tilde{K} \to \mathbb{R}^{n \times n}$ which is smooth and invertible in $\tilde{K}$. For any $t \in \mathbb{R}_+$ and $x \in \tilde{K}$, $G(\varphi^t(x))$ defines a linear operator $\mathbb{R}^n \to \mathbb{R}^n$. For any $x \in \tilde{K}$ the singular values of $G(\varphi^t(x))$ are bounded from above and below. Given an arbitrary nonsingular matrix $S(t)$ which is bounded from below and above for all t, for singular values $\sigma_i'(t)$ of the matrix $X(t)S(t)$ we have the following simple estimate $\xi_{\min}\sigma_i(t) \leq \sigma_i'(t) \leq \xi_{\max}\sigma_i(t)$ where $\xi_{\min}$ and $\xi_{\max}$ are the lower and upper bounds for the singular values of the matrix $S(t)$ and $\sigma_i(t)$ are the singular values of the matrix $X(t)$. Therefore using Theorem 4 we arrive at the following result.

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Theorem 6 *Assume that there exists $d \in [0, n]$ such that*

$$\sup_{x \in \tilde{K}} \omega_d \left[G(\varphi^t(x)) T_x \varphi^t \right] \to 0 \quad \text{as} \quad t \to \infty. \tag{25}$$

Then $\dim_H K \leq d$.

Proof: Since

$$\omega_d \left[G(\varphi^t(x)) T_x \varphi^t \right] \geq \xi^d \omega_d(T_x \varphi^t),$$

where ξ is the minimal singular value of G on $\tilde{K}$, the result follows from Theorem 4. ∎

Consider some symmetric positive definite matrix $P(x)$ which is continuously differentiable in $\tilde{K}$, and which therefore is bounded from above and below in $\tilde{K}$ and the symmetric matrix $Q(x(t, x_0))$ defined by

$$Q(x(t, x_0)) = \dot{P}(x(t, x_0)) + P(x(t, x_0))J(x(t, x_0)) + J^\top(x(t, x_0))P(x(t, x_0)).$$

Here $\dot{P}(x(t, x_0)) = \frac{d}{dt}P(x(t, x_0))$ stands for the matrix with entries equal to

$$\left(\frac{\partial p_{ij}(x(t, x_0))}{\partial x} f(x(t, x_0)) \right)_{ij}.$$

Consider the equation

$$\det[Q(x) - \lambda(x)P(x)] = 0. \tag{26}$$

For any $x \in \tilde{K}$ the equation (26) has n *real* solutions $\lambda_i(x)$ since the matrix Q is symmetric and P is positive definite. Indeed, (26) can be rewritten as

$$\det[G(x)^\top (G(x)^{-\top} Q(x)G(x)^{-1} - \lambda(x)I_n)G(x)] = 0$$

or, equivalently,

$$\det[G(x)^{-\top} Q(x)G(x)^{-1} - \lambda(x)I_n] = 0,$$

where $P(x) = G(x)^\top G(x)$ and the matrix $G(x)^{-\top} Q(x)G(x)^{-1}$ is symmetric. Order the solutions of (26) in the decreasing order for all x: $\lambda_1(x) \geq \lambda_2(x) \geq \ldots \geq \lambda_n(x)$.

Theorem 7 *Suppose that for some $P(x)$ satisfying the above assumptions there exist numbers $d_0 \in \mathbb{Z}_+$, $s \in [0, 1)$ such that*

$$\limsup_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau [\lambda_1(x(t, x_0)) + \ldots + \lambda_{d_0}(x(t, x_0)) + s\lambda_{d_0+1}(x(t, x_0))]dt < 0 \tag{27}$$

for any $x_0 \in \tilde{K}$. Then $\dim_H K \leq d_0 + s$.

Before the proof of this theorem we formulte the following result due to Smith.

Theorem 8 (Smith, [25]) *Let X be a fundamental matrix on $[0,\tau]$ of the linear system*

$$\dot{x} = A(t)x,$$

with singular values $\sigma_i(t), i = 1, \ldots, n$ ordered in decreasing order for all $0 \le t \le \tau$. For all $i, 1 \le i \le n$ the following relations are true:

$$\sigma_1(\tau)\sigma_2(\tau)\cdots\sigma_i(\tau) \le \exp\frac{1}{2}\left(\int_0^\tau (\lambda_1(t) + \lambda_2(t) + \ldots + \lambda_i(t))dt\right),$$

$$\sigma_n(\tau)\sigma_{n-1}(\tau)\cdots\sigma_{n-i+1}(\tau) \ge \exp\frac{1}{2}\left(\int_0^\tau (\lambda_n(t) + \lambda_{n-1}(t) + \ldots + \lambda_{n-i+1}(t))dt\right),$$

where $\lambda_i(t)$ are the eigenvalues of the symmetrized matrix $(A(t) + A(t)^\top)$ ordered in decreasing order for all $0 \le t \le \tau$.

Proof of Theorem 7: Since the matrix $P(x) = P(x)^\top$ is positive definite it can be represented in the form $P(x) = G(x)^\top G(x)$ where $G(x)$ is continuously differentiable and bounded together with its inverse in $\tilde{K}$. For the system (20) consider the time-varying coordinate change

$$z = G(x(t, x_0))y. \tag{28}$$

From now for brevity we will omit the dependence of $x(t, x_0)$ and t. In the new coordinates the system (20) can be rewritten as

$$\dot{z} = \dot{G}G^{-1}z + GJG^{-1}z. \tag{29}$$

In view of (28) the fundamental matrix of (29) has the form GH, where H is the fundamental matrix of (20). In order to use Theorem 6 we should estimate the singular values of the matrix GH.

Denote $A = \dot{G}G^{-1} + GJG^{-1}$ and consider the equation

$$\det((A + A^\top) - \lambda I_n) = 0.$$

This equation is equivalent to

$$\det(G^\top G A + A^\top G^\top G + G^\top \dot{G} + \dot{G}^\top G - \lambda G^\top G) = 0,$$

which is the same as (26) since $P = G^\top G$.

Estimates of the singular values $\sigma_1, \sigma_2, \ldots$ of the fundamental matrix for the system (29) can be obtained using Smith's theorem formulated above.

From the equality

$$\sigma_1(\tau)\sigma_2(\tau)\cdots\sigma_{d_0+1}^s(\tau) = (\sigma_1(\tau)\cdots\sigma_{d_0}(\tau))^{1-s}(\sigma_1(\tau)\cdots\sigma_{d_0+1}(\tau))^s$$

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for the singular values of GH it follows from Theorem 8 that

$$\sigma_1(\tau)\sigma_2(\tau)\cdots\sigma_{d_0}(\tau)\sigma_{d_0+1}^s(\tau) \le \exp\frac{1}{2}\left(\int_0^\tau(\lambda_1(t)+\lambda_2(t)+\ldots+\lambda_{d_0}(t)+s\lambda_{d_0+1}(t))dt\right)$$

where λ_i are the solutions of (26) and hence for all $\tau > 0$ and arbitrary positive ε there is $N = N(\varepsilon)$ such that

$$\sup_{x_0\in K}\sigma_1(\tau)\sigma_2(\tau)\cdots\sigma_{d_0}(\tau)\sigma_{d_0+1}^s(\tau) \le Ne^{(-\alpha/2+\varepsilon)\tau}$$

where

$$-\alpha = \sup_{x_0\in K}\limsup_{\tau\to\infty}\frac{1}{\tau}\int_0^\tau(\lambda_1(t)+\lambda_2(t)+\ldots+\lambda_{d_0}(t)+s\lambda_{d_0+1}(t))dt$$

and therefore the conditions of Theorem 6 are satisfied. ∎

Corollary 1 (Leonov [17, 20]) *Let $\lambda_i(x)$, $i = 1,\ldots,n$ be the eigenvalues of the matrix $(J(x) + J(x)^\top)/2$ ordered in decreasing order. Suppose there exist numbers $d_0 \in \mathbb{Z}_+$, $s \in [0,1)$, and a continuously differentiable in $\tilde{K}$ function $v : \tilde{K} \to \mathbb{R}$ such that*

$$\lambda_1(x) + \ldots + \lambda_{d_0}(x) + s\lambda_{d_0+1}(x) + \frac{\partial v}{\partial x}f(x) < 0 \tag{30}$$

for any $x_0 \in \tilde{K}$. Then $\dim_H K \le d_0 + s$.

Proof: The result directly follows from Theorem 7 if one takes $P(x) = p^2(x)I_n$ where $p(x) > 0$ is a scalar differentiable function bounded from below and above in $\tilde{K}$ and denote $v(x) = (\log p(x))/(d_0 + s)$. In this case (26) is equivalent to the equation

$$\det[J(x)^\top + J(x) + \frac{2\dot{p}}{p}I_n - \lambda I_n] = 0$$

Since

$$2\frac{\dot{p}}{p(d_0 + s)} = 2\frac{dv}{dt} = 2\frac{\partial v}{\partial x}f(x)$$

the result follows from Theorem 7. ∎

Example 3 Consider the Rössler system

$$\begin{cases} \dot{x} = -y - z \\ \dot{y} = x + ay \\ \dot{z} = c + z(x - b) \end{cases} \tag{31}$$

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Computer simulation shows that for some values of parameters $a, b, c > 0$ the system (31) has an invariant compact set K. Let us compute an upper bound for its Hausdorff dimension. First, notice that $z_0 \geq 0$ implies that $z(t, z_0) > 0$ for all $t > 0$ (it follows from the third equation since $c > 0$). Therefore we will asssume that $K \subset \Omega$, where $\Omega = \mathbb{R} \times \mathbb{R} \times (0, \infty)$.

Denote $q = (x, y, z)^\top$. Let the function v_+ be defined as

$$v_+(t, q_0) = \begin{cases} x(t, x_0) - b - c/z(t, z_0) & \text{if } x(t, x_0) - b - c/z(t, z_0) \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Let $v_-(t, q_0)$ be so that $v_+(t, q_0) + v_-(t, q_0) = (x(t, x_0) - b - c/z(t, z_0))$.

Claim 1 *If system (31) has an invariant compact set $K \subset \Omega$ then*

$$\dim_H K \leq 2 + \frac{2a + \lambda_+}{|\lambda_-|}, \tag{32}$$

where

$$\lambda_+ = \sup_{q_0 \in K} \limsup_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau v_+(t, q_0) dt,$$

$$\lambda_- = \sup_{q_0 \in K} \limsup_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau v_-(t, q_0) dt.$$

Proof: The Jacobian of (31) has the following form

$$J(q) = \begin{pmatrix} 0 & -1 & -1 \\ 1 & a & 0 \\ z & 0 & x - b \end{pmatrix} \tag{33}$$

Choose the matrix $P(q)$ as

$$P(q) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & z^{-1} \end{pmatrix} \tag{34}$$

Since $z(t, z_0) > 0$ for all $z_0 \in K$ this matrix is well defined together with its inverse in K. As one can easily check the equation (26) has the following solutions

$$\lambda_1 = 2a, \quad \lambda_2 = 0, \quad \lambda_3 = 2(x - b) - \frac{\dot{z}}{z} = x - b - \frac{c}{z},$$

And, finally (32) follows from Theorem 7 with $\tilde{K} = K$. ∎

Approximate calculation using (32) shows that for $a = 0.2, b = 5.7, c = 0.2$ $\dim_H K < 2.066$. A typical trajectory of the Rosslër system is shown in Fig. 1.
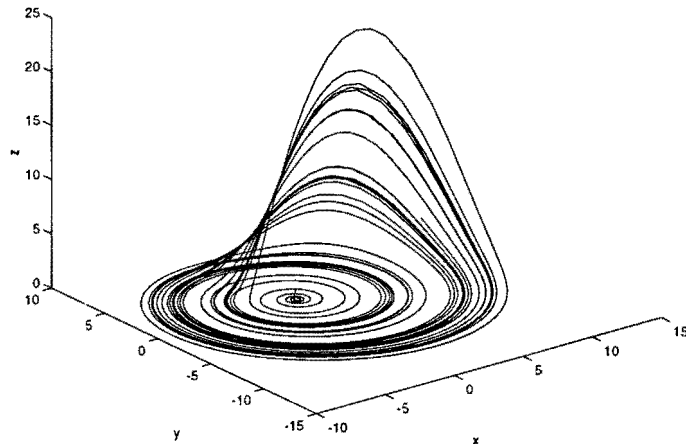
$\triangle$

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Figure 1: A typical solution of the Rössler system for $z_0 > 0$.

5 Estimation of the region of attraction by means of first order approximation

The results obtained in the previous section allow to find conditions based on the system of first order approximation ensuring convergence of any solution to an equilibrium point.

Consider again system (19). We assume that in Ω there exists a bounded open simply connected positively invariant set D, the boundary ∂D of which transversally intersects any trajectory originating in ∂D. This assumption means that positive invariance of D is preserved under small perturbations of the vector field f. Moreover we will assume that the set D is diffeomorphic to an open ball B^n. The existence of such a set can be established by the direct Lyapunov method. Suppose finally that D contains a finite number of equilibria.

Let $P(x)$ be a continuously differentiable positive definite matrix function defined in $\mathrm{cl}D$. As before, let λ_i be the roots of the equation (26) ordered in the decreasing order for all $x \in \mathrm{cl}D$.

The following theorem can be treated as a generalization of theorems due to Hartman-Olech [14] and Leonov [18].

Theorem 9 *Suppose that for some $P(x)$ satisfying the above assumptions*

$$\lambda_1(x) + \lambda_2(x) < 0 \tag{35}$$

for any $x \in \mathrm{cl}D$. Then any solution originating in D tends to some equilibrium point.

In the proof of Theorem 9 we need the well-known closing lemma due to Pugh [24]. Let us formulate this lemma in a form convenient for us.

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Lemma 5 (Closing lemma, [23]) *Let $\tilde{x}$ be an ω-limit point of a bounded solution $x(t, x_0)$, $t \geq 0$, lying in D, that is not an equilibrium. For any $\delta > 0$ there is a vector field $g \in C^1$ such that*

$$\max_{clD} |f(x) - g(x)| + \max_{clD} \left\| \frac{\partial f}{\partial x}(x) - \frac{\partial g}{\partial x}(x) \right\| < \delta$$

and a closed trajectory γ of the system $\dot{x} = g(x)$ passes through $\tilde{x}$.

The proof of Theorem 9 follows the same lines as the proof of Leonov theorem (Theorem 8.3.1 in [19]), the difference is that the condition (35) is weaker than that imposed in [18, 19] and we added the assumption that D is diffeomorphic to B^n not explicitly stated in [18, 19].

Proof of theorem 9: Let $x_0 \in clD$. Since D is bounded and positively invariant, the ω-limit set Ω_{x_0} of $x(t, x_0)$ is nonempty. Let $\tilde{x} \in \Omega_{x_0}$. If a closed trajectory $\gamma \subset clD$ passes through a point $\tilde{x}$ then γ is an invariant set. We put on γ some smooth two-dimensional surface $K \subset \mathbb{R}^n$ having finite area. The existence of such a surface for a smooth curve is shown, for example, in [8]. Moreover, since D is diffeomorphic to a ball we can suppose that $K \subset clD$ (any closed smooth curve in clD is diffeomorphic to some closed smooth curve in clB^n). As before, we denote by φ^t the shift operator along trajectories of system (19). Let $\mu(S)$ be the Hausdorff 2-measure of a smooth 2-dimensional surface S. Since γ is invariant under φ^t and $K \subset clD$ for any $t \geq 0$ we have

$$\inf_{t \geq 0} \mu\left(\varphi^t(K)\right) > 0. \tag{36}$$

At the same time, using (35) and repeating the proof of Theorem 7, applying Remark 2, it follows that

$$\lim_{t \to \infty} \mu\left(\varphi^t(K)\right) = 0, \tag{37}$$

which contradicts (36).

When $\tilde{x} \subset \Omega_{x_0}$ is not an equilibrium point and the trajectory passing through this point is not closed, we use Pugh's lemma. We slightly perturb f such that the conditions of the Pugh lemma are satisfied and the theorem hypothesis holds for the perturbed f with the same $P(x)$. Then there is a closed trajectory of the perturbed system passing through $\tilde{x}$. Using the above arguments we again arrive at a contradiction.

Thus $\tilde{x}$ is an equilibrium. By assumption, the equilibria are isolated in D and the result follows. ∎

Corollary 2 (Leonov [18, 19]) *Let $\lambda_i(x)$, $i = 1, \ldots, n$ be the eigenvalues of the matrix $(J(x) + J(x)^\mathsf{T})/2$ ordered in decreasing order. Suppose that there exists a continuously differentiable function $v : clD \to \mathbb{R}$ such that*

$$\lambda_1(x) + \lambda_2(x) + \frac{\partial v}{\partial x} f(x) < 0 \tag{38}$$

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for any $x \in$ cl$D$. Then any solution of the system (19) originating in D tends to some equilibrium.

The proof of this statement is based on the choice $P(x) = p^2(x)I_n$ similar to the proof of Corollary 1.

Remark 3 If we take $P(x) = \alpha I_n, \alpha > 0$, or, equivalently, $v(x) = $ const, conditions (35), (38) are equivalent to the condition imposed in the Hartman-Olech theorem [14].

6 Conclusion

In this paper we have presented two different approaches to estimate the Hausdorff dimension of invariant compact sets of dynamical systems. These two approaches are closely related to the two Lyapunov methods for testing stability (method of characteristic exponents and method of Lyapunov functions). Our goal was to obtain an estimate in the most general situation. This generality forced us to tackle possible nonuniformity of the convergence of the estimate of the elliptical Hausdorff measure to zero. To cope with this problem we used characteristic exponents different from the Lyapunov exponents which we called the Bohl exponents. Those exponents made it possible to obtain an upper estimate of the Hausdorff dimension in terms of the Bohl exponents for only one "typical" (i.e. satisfying Assumption 1) trajectory. It is interesting to compare this result with the Kaplan-Yorke type estimate presented in [11] (see Theorem 4.2). Particularly Corollary 4.1 in [11] claims that if in the definition of $\mathcal{B}_d$ we replace the Bohl exponents with the Lyapunov exponents (as defined in [11]) then there is *some* point $x_0 \in K$ such that $\dim_H K \leq d$ if $\mathcal{B}_d < 0$. However in this case there is a question if the trajectory $\gamma(x_0)$ (critical trajectory in terminology of [11]) is "typical", i.e. cl$\gamma(x_0) = K$?

The second estimate of the Hausdorff dimension is given in terms of an auxillary matrix function satisfying certain partial differential inequalities along the system solutions. The class of functions we deal with is larger then that proposed by Leonov (see [20, 19] and Corollary 1) that allows us to obtain less conservative estimates. However, here we face a problem which is typical for nonlocal stability analysis: there is no general method to find a Lyapunov function. One of the existing methods to find a Lyapunov function is based on the famous Kalman-Yakubovich lemma which gives necessary and sufficient conditions for the existence of a symmetric matrix satisfying a certain matrix inequality. Those conditions are given in a very convenient form of frequency domain inequalities. Estimates of the Hausdorff dimension based on the Kalman-Yakubovich lemma are presented in the monographs [20, 19]. The main advantage of this approach is that it gives conditions which can be easily verified. The main disadvantage is a possible conservatism due to the fact that the matrix P should be constant. Our estimate allows to reduce this conservatism.

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