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A HETEROPOLYMER NEAR A LINEAR INTERFACE*

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ABSTRACT. We consider a quenched-disordered heteropolymer in the vicinity of an interface between two solvents. We show that the free-energy localization concept introduced in [BdH] is equivalent to pathwise localization. In particular, we prove that positivity of the excess free energy implies exponential tightness of the polymer excursions away from the interface, positive density of intersections with the interface, and convergence of ergodic averages along the polymer. We include an argument due to [G], showing that if the excess free energy is zero then there is pathwise delocalization in a certain weak sense.

1. INTRODUCTION

The model.

Heteropolymers near an interface between two solvents are intriguing because of the possibility of a localization/delocalization phase transition. A typical example is a polymer consisting of hydrophobic and hydrophylic monomers in the presence of an oil-water interface.

In the bulk of a single solvent, the polymer is subject to thermal fluctuations and therefore is rough on all space scales. However, near the interface the polymer can benefit from the fact that part of the monomers prefer to be in one solvent and part in the other. The energy gain earned by placing monomers in their preferred solvent can, at least for low temperatures, tame the entropy-driven fluctuations. The polymer becomes captured by the interface and therefore is smooth on large space scales. The two regimes of characteristic behavior are separated by a phase transition.

As in [BdH], we model the polymer by a random walk path $(i, S_i)_{i\in\mathbb{L}}$, where $\mathbb{L} \subseteq \mathbb{Z}$ indexes the monomers, $S_i \in \mathbb{Z}$ and $S_i - S_{i-1} = \pm 1$. (View the path as a directed polymer in $\mathbb{Z}^2$.) The interface is the horizontal in $\mathbb{L} \times \mathbb{Z}$. We distinguish

*Some ideas in this paper are based on a note by S. Albeverio, F. den Hollander and X.Y. Zhou [AdHZ], which never went beyond the preparatory stage due to the unfortunate deceasing of the third author.

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two cases: (1) the *semi-infinite* polymer, where $\mathbb{L} = \mathbb{N}$ and $S_0 = 0$; (2) the *doubly-infinite* polymer, where $\mathbb{L} = \mathbb{Z}$ and S_0 is arbitrary. The heterogeneity within the polymer is represented by assigning a random variable $\omega_i = \pm 1$ to monomer i for each $i \in \mathbb{L}$. For instance, if the two solvents are oil and water, then $\omega_i = +1$ means that monomer i is hydrophobic and $\omega_i = -1$ that it is hydrophylic.

Let $F(\mathbb{L})$ be the set of all finite connected subsets of $\mathbb{L}$. In the simplest model, the thermodynamics of the heteropolymer is governed by the family $(H^{\omega}_{\Lambda})_{\Lambda \in F(\mathbb{L})}$ of Hamiltonians

$$H^{\omega,\lambda,h}_{\Lambda}(S) = \lambda \sum_{i \in \Lambda} (\omega_i + h) \Delta_i(S)$$

w.r.t. the reference measure giving all paths equal probability (i.e., the measure P for simple random walk). Here, λ and h are parameters, $\omega = (\omega_i)_{i \in \mathbb{L}}$ is the disorder configuration, and

$$\Delta_i(S) = \begin{cases} \text{sign}(S_i) & \text{if} \quad S_i \neq 0 \\ \text{sign}(S_{i-1}) & \text{if} \quad S_i = 0. \end{cases}$$

The role of the Hamiltonian is that it favors the combinations $S_i > 0, \omega_i = +1$ and $S_i < 0, \omega_i = -1$, so hydrophobic monomers in the oil above the interface and hydrophylic monomers in the water below the interface. (Note that the definition of $\Delta_i(S)$ actually corresponds to a bond model.) λ plays the role of the inverse temperature and h stands for an asymmetry between the affinities of the monomer species with the solvents. The Hamiltonian is $(S, \omega, h) \to (-S, -\omega, -h)$ symmetric. In view of this, we shall henceforth take

$$\mathcal{I} = \{(\lambda, h) \colon \lambda > 0,\ h \geq 0\}$$

as our parameter space.

It is clear that the disorder configuration $\omega = (\omega_i)_{i \in \mathbb{L}}$ determines the thermodynamic features of the heteropolymer. The *annealed* case (i.e., the partition sum is averaged over ω) treated by Sinai and Spohn [SS], mimics the situation where the ω_i's are sufficiently equilibrized with the polymer's other degrees of freedom. This case turns out to be exactly solvable when the ω_i's are i.i.d. or interact via an Ising Hamiltonian. In particular, the annealed heteropolymer is delocalized even in the presence of an interface. For localization of the polymer an additional binding potential at the interface has to be superimposed.

The *quenched* case (i.e., ω is kept frozen) is mathematically much harder. The *periodic* quenched problem (e.g., ω represents some periodic constraint within the polymer) has been successfully dealt with by using a transfer-matrix approach (Grosberg et al. [GIN]). The *random* quenched problem (e.g., ω i.i.d.), however, for several years withstood investigative attempts, except those exploiting the replica trick (Garel et al. [GHLO]).

The free energy and a phase transition.

The semi-infinite quenched i.i.d. random model was recently analyzed in detail by Bolthausen and den Hollander [BdH]. In this paper, a localization/delocalization phase transition is established by estimating the *free energy*

$$\phi(\lambda, h) = \lim_{n \to \infty} \frac{1}{|\Lambda_n|} \log E\left(e^{H^{\omega,\lambda,h}_{\Lambda_n}}\right),$$

where $\Lambda_n = \{0, 1, \ldots, n\}$ and E stands for the expectation w.r.t. SRW starting at 0. The limit is shown to exists and to be ω-independent by the subadditive ergodic theorem.

It was observed in [BdH] that $\phi(\lambda, h) \geq \lambda h$, with the lower bound attained for delocalized paths. Indeed, $P(S_i \geq 0 \ \forall\ 0 \leq i \leq n) \sim C/n^{1/2}$ (as $n \to \infty$), and conditioned on this event

$$\frac{1}{|\Lambda_n|} H^{\omega,\lambda,h}_{\Lambda_n} = \frac{1}{|\Lambda_n|} \lambda \sum_{i\in\Lambda_n} (\omega_i + h) = \lambda h(1 + o(1)) \quad \omega - a.s.$$

For this reason, it is natural to work with the *excess free energy*

$$\psi(\lambda, h) = \phi(\lambda, h) - \lambda h$$

and to put forward the following concept of a phase transition:

Definition 1. [BdH] *The polymer is said to be*

(a) *localized if* $\psi > 0$,
(b) *delocalized if* $\psi = 0$.

As already alluded to, this definition is justified by noting that delocalized paths yield no contribution to ψ. Conversely, only those excursions that move below the interface can raise ψ above zero. Let us define

$$\mathcal{L} = \{\psi > 0\} \cap \mathcal{I}$$
$$\mathcal{D} = \{\psi = 0\} \cap \mathcal{I}$$

as the sets of parameters for which the model is localized respectively delocalized in the sense of Definition 1. Neither of these sets is trivial, as shown by the following theorem.

Theorem 1. [BdH] *There is a continuous non-decreasing function* $h_c\colon (0, \infty) \to (0, 1)$ *such that*

$$\mathcal{L} = \{(\lambda, h) \in \mathcal{I}\colon 0 \leq h < h_c(\lambda)\}.$$

Moreover,

$$\lim_{\lambda\to\infty} h_c(\lambda) = 1 \qquad \text{and} \qquad \lim_{\lambda\downarrow 0} \frac{h_c(\lambda)}{\lambda} = K_c,$$

where $0 < K_c < \infty$ *is a number related to a Brownian version of the model.*

Theorem 1 asserts that $\mathcal{L}$ and $\mathcal{D}$ are separated by a *single* phase transition line, which persists for all temperatures. Although it is relatively easy to establish the existence and uniqueness of $h_c(\lambda)$ (essentially via the convexity of ϕ) and to evaluate the limit for large λ (through an appropriate lower bound on Z_Λ), the scaling law for $\lambda \downarrow 0$ is a rather involved problem. The intuitive reason why a Brownian constant should appear for $\lambda \downarrow 0$ is that for high temperatures the polymer's excursions are very large. Therefore, from a coarse-grained point of view, both the excursions and the disorder inside the excursions can be approximated by their Brownian counterparts. However, the details of this approximation are quite delicate.

The persistence of the phase transition line for all λ seems to be a remnant of the one-dimensionality and the discreteness of our model. Namely, Grosberg et al. [GIN] consider a $3d$ SRW with Gaussian steps near a $2d$ planar interface and find that the phase transition curve diverges at a certain finite λ_c: above this value delocalization cannot be induced by any asymmetry, no matter how strong. (The mechanism enhancing localization seems to rely on the ability of the path to avoid lengthy but costly excursions, by moving in the directions parallel to the interface.)

Despite its crudeness, the free-energy localization concept has proved to be useful also in the study of higher-dimensional generalizations of the present model (work in progress by Bolthausen-Giacomin [BG]). The latter authors consider a d-dimensional Gaussian surface, stuck at the interface outside a finite box and weighted by the same type of Hamiltonian as in our case. Similarly as in [BdH], a localization/delocalization transition is found. However, the phase transition curve seems to end in a critical point at some finite λ_c when $d \geq 3$.

Pathwise properties.

Theorem 1 characterizes the phase transition in terms of the free energy rather than the path. One would like to prove that, indeed, $\mathcal{L}$ corresponds to a localized path and (the interior of) $\mathcal{D}$ to a delocalized path. Moreover, one would like to learn more about the path characteristics, e.g., the length and the height of a typical excursion. Progress in this direction has been made by Sinai [S], who proved pathwise localization in the symmetric case $h = 0$ for all values of λ.

Sinai introduces a (Gibbsian) probability distribution Q_n^ω in the volume $\Lambda_n = \{0, 1, \ldots, n\}$,

$$\frac{dQ_n^{\omega,\lambda,0}}{dP_n}(S) = \frac{e^{H_{\Lambda_n}^{\omega,\lambda,0}(S)}}{Z_{\Lambda_n}^{\omega,\lambda,0}},$$

where the reference measure P_n is the projection onto Λ_n of the $1d$ SRW measure P. His result reads:

Theorem 2. [S] *Let $h = 0$ and $\lambda > 0$. Then there exist random variables $n(\omega) \in \mathbb{N}$, $m(\omega) \in \mathbb{N}$ and a number $\zeta = \zeta(\lambda) > 0$ such that*

$$\sup_{0 \leq i \leq n} Q_n^{\omega,\lambda,0}(|S_i| > s) \leq e^{-\zeta s} \quad (n \geq n(\omega),\ s \geq m(\omega)).$$

Here σ is the right-shift operator acting on ω.

Theorem 2 states that the path measure exhibits *exponential tails*. This result was extended by Albeverio and Zhou [AZ], who showed that the length of the longest excursion in Λ_n is of order $\log n$ and so is the height of the highest excursion.

In the sequel, we shall extend Sinai's result to all of $\mathcal{L}$. We in fact adopt a more comprehensive attitude by discussing the entire Gibbsian structure associated with the above Hamiltonian (Sinai's result is in this respect a statement about the Gibbs measures generated by the free boundary condition). In particular, we shall establish uniqueness within a certain reduced (but physical) class of Gibbs measures, and prove exponential tightness in the vertical direction and ergodicity in the horizonal direction.

2. Preliminaries

Gibbsian structure.

Let $(\omega_i)_{i\in\mathbb{L}}$ be an i.i.d. sequence of ±1–valued random variables defined on a probability space $(\Omega,\mathcal{B},\mathbb{P})$. Here Ω is the space of all disorder configurations, $\mathcal{B}$ is the σ-algebra generated by the cylinder sets, and $\mathbb{P}$ is the i.i.d. measure with $\mathbb{P}(\omega_i=+1)=\mathbb{P}(\omega_i=-1)=1/2$. The expectation w.r.t. $\mathbb{P}$ will be denoted by $\mathbb{E}$.

Let $\Sigma=\{(S_i)_{i\in\mathbb{L}}\colon |S_i-S_{i-1}|=\pm1\,\forall i\in\mathbb{L}\}$ be the space of SRW-paths. Let $\mathcal{F}$ be the σ-algebra generated by the cylinder sets. For $\Lambda\in F(\mathbb{L})$, let $\mathcal{F}_\Lambda$ be the σ–algebra of the path projected onto Λ, and let $\mathcal{T}=\cap_{\Lambda\in\mathcal{S}}\mathcal{F}_{\Lambda^c}$ be the tail σ-field. Let $\mathcal{P}(\Sigma,\mathcal{F})$ be the space of all probability measures on $(\Sigma,\mathcal{F})$. Note that $\mathcal{P}(\Sigma,\mathcal{F})$ is compact in the weak topology for both the semi-infinite and the doubly-infinite case. Let P,E be probability and expectation under SRW.

We shall define Gibbs measures by means of the Gibbsian specification

$$\gamma_\Lambda^{\omega,\lambda,h}(S_\Lambda|\tilde S_{\Lambda^c})=1_\Sigma(S_\Lambda\vee\tilde S_{\Lambda^c})\frac{e^{H_\Lambda^{\omega,\lambda,h}(S_\Lambda\vee\tilde S_{\Lambda^c})}}{Z_\Lambda^{\omega,\lambda,h}(\tilde S_{\Lambda^c})}P(S_\Lambda|\tilde S_{\Lambda^c})\quad(\Lambda\in F(\mathbb{L})).$$

This specification is a measure on paths $S_\Lambda=(S_i)_{i\in\Lambda}$, absolutely continuous w.r.t. the SRW-bridge measure $P(S_\Lambda|\tilde S_{\Lambda^c})$, and a measurable function of the boundary condition $\tilde S_{\Lambda^c}=(\tilde S_i)_{i\in\Lambda^c}$. The partition function $Z_\Lambda^{\omega,\lambda,h}(\tilde S_{\Lambda^c})$ is the normalizing constant (which only depends on $\tilde S_{\partial\Lambda}$, with $\partial\Lambda$ the outer boundary of Λ). The specifications $(\gamma_\Lambda^{\omega,\lambda,h})_{\Lambda\in F(\mathbb{L})}$ form a consistent family.

Given $\omega\in\Omega$ and $(\lambda,h)\in\mathcal{I}$, the Gibbs measures are defined as follows:

$$\mathcal{G}_\omega^{\lambda,h}=\left\{\mu\in\mathcal{P}(\Sigma,\mathcal{F})\colon \mu=\mu\gamma_\Lambda^{\omega,\lambda,h}\,\forall\Lambda\in F(\mathbb{L})\right\}.$$

By compactness, when taking a weak limit of $\gamma_\Lambda^{\omega,\lambda,h}(\tilde S_{\Lambda^c})$ for a fixed boundary condition $\tilde S=(\tilde S_i)_{i\in\mathbb{L}}$ we obtain a Gibbs measure (because the specifications are consistent). Hence $\mathcal{G}_\omega^{\lambda,h}\neq\emptyset$.

As is typical in the theory of Gibbs measures, the boundary condition may strongly determine the properties of a Gibbs measure, in some cases even more decisively than the interaction itself. In our setting, for the semi-infinite case and any $(\lambda,h)\in\mathcal{I}$, there is a whole class of Gibbs measure (of at least countably-infinite cardinality) under which the path departs from the interface at a linear speed: namely, when $\tilde S_i$ grows linearly with i. The delocalized behavior of the path under these Gibbs measures clearly is enforced by the boundary condition. Similarly for the doubly-infinite case. One can analyze this situation by looking at the *lower free energy* $\phi_{\tilde S}$ generated by $\tilde S$, defined by

$$\phi_{\tilde S}(\lambda,h)=\liminf_{n\to\infty}\frac{1}{|\Lambda_n|}\mathbb{E}\Big(\log Z_{\Lambda_n}^{\omega,\lambda,h}(\tilde S_{\Lambda_n^c})\Big)$$
$$\psi_{\tilde S}(\lambda,h)=\phi_{\tilde S}(\lambda,h)-\lambda h.$$

Lemma 1. *(a) Consider the semi-infinite case. Let* $\limsup_{i\to\infty}|\tilde{S}_i|/i = c$, *and when* $c>0$ *let* $\lim_{i\to\infty}\operatorname{sign}(\tilde{S}_i)=\eta$. *Then*

$$\psi_{\tilde{S}}(\lambda,h) \ge \psi(\lambda,h)(1-c) + \lambda hc(1-\eta) + \frac{1}{2}\log\Big(2^{-2c}(1-c)^{1-c}(1+c)^{1+c}\Big).$$

(b) Consider the doubly-infinite case. Define constants c_+, c_- *and* η_+, η_- *analogously. Then*

$$\psi_{\tilde{S}}(\lambda,h) \ge \psi(\lambda,h)\big(1-\frac{c_+ + c_-}{2}\big) + \lambda h\big[(1-\eta_+)\frac{c_+}{2} + (1-\eta_-)\frac{c_-}{2}\big] + \frac{1}{2}\log\Big(2^{-(c_+ + c_-)}\big(1-\frac{c_+ - c_-}{2}\big)^{1-\frac{c_+ - c_-}{2}}\big(1+\frac{c_+ - c_-}{2}\big)^{1+\frac{c_+ - c_-}{2}}\Big).$$

Proof. To find a lower bound on $\phi_{\tilde{S}}(\lambda,h)$, we pick Λ_{2n} and restrict the summation in $Z^{\omega,\lambda,h}_{\Lambda_{2n}}(\tilde{S}_{\Lambda^c_{2n}})$ to paths that end by hitting the interface and then moving at maximal speed. In other words, if $c_n = |\tilde{S}_{2n}|/2n$, then the path moves from height 0 at position $2n(1-c_n)$ to height $2nc_n$ at position $2n$. This gives

$$Z^{\omega,\lambda,h}_{\Lambda_{2n}}(\tilde{S}_{\Lambda^c_{2n}}) \ge Z^{\omega,\lambda,h}_{\Lambda_{2n(1-c_n)}}(0)\exp\Big[\lambda\sum_{i=2n(1-c_n)+1}^{2n}(\omega_i+h)\eta\Big]\frac{\binom{2n(1-c_n)}{n(1-c_n)}}{\binom{2n}{n(1-c_n)}},$$

where the binomial factors come from the fact that the path must match the boundary condition. Now, it was shown in [BdH] that the partition function with zero boundary condition appearing in the r.h.s. differs by at most a factor of order n from the partition function with free boundary condition, which was used to define $\phi(\lambda,h)$. Therefore the claim follows after taking logarithms, dividing by $2n$, letting $n\to\infty$, and using the relation between ϕ and ψ.

The doubly-infinite case is completely analogous. The computation is left to the reader. □

For small c the c-dependent terms in the r.h.s. of the formula in Lemma 1(a) are of order c. So for every $(\lambda,h)\in\mathcal{L}$ there exists some $c_{\lambda,h}(\eta)>0$ such that if $\psi(\lambda,h)>0$ then $\psi_{\tilde{S}}(\lambda,h)>0$ for all $\tilde{S}$ with $c\le c_{\lambda,h}(\eta)$.In other words, any boundary condition that is sublinear or that is linear with a sufficiently small constant cannot destroy localization in the sense of Definition 1. Similarly for Lemma 1(b).

Lemma 1 thus makes a natural distinction between *good* and *bad* boundary conditions. Since the Gibbs measures can be generated as limits of specifications with different boundary conditions, the distinction between good and bad also applies to them. This leads us to the following definition.

Definition 2. *Given* $(\lambda,h)\in\mathcal{I}$, *the regular Gibbs measures are those* $\mu\in\mathcal{G}^{\lambda,h}_\omega$ *for which* $\limsup_{i\to\pm\infty}|\tilde{S}_i|/|i|\le c_{\lambda,h}$ *μ-a.s. The set of regular Gibbs measures is denoted by* $\mathcal{G}^{R,\lambda,h}_\omega$.

Because of their 'unphysical' behavior, the non-regular Gibbs measures will henceforth be discarded.

Measurable Gibbsian sections.

As we have already noted, $\mathcal{G}_\omega^{\lambda,h} \neq \emptyset$ for all ω by compactness. However, although (by the axiom of choice) we can arrange the $\mu_\omega \in \mathcal{G}_\omega^{\lambda,h}$ into a measure-valued function of ω, it is not *a priori* clear that this can be done in a measurable way. Formally, if we put $\mathcal{G}^{R,\lambda,h} = \bigcup_{\omega\in\Omega}(\omega, \mathcal{G}_\omega^{R,\lambda,h})$, then the question is whether or not there are *measurable* sections $(\omega, \mu_\omega)_{\omega\in\Omega} \in \mathcal{G}^{R,\lambda,h}$. We shall answer this question affirmatively. This will be important because later on we shall want to integrate over ω.

Lemma 2. *For $(\lambda, h) \in \mathcal{I}$, let $\mathcal{G}^{\lambda,h}$ be the set of regular measurable Gibbsian sections.*
(a) $\mathcal{G}^{\lambda,h}$ is non-empty both for the semi-infinite and the doubly-infinite case.
(b) For the doubly-infinite case there is a $\mathcal{B}$-measurable measure-valued function $\mu_{(\,\cdot\,)} : \Omega \to \mathcal{G}^{R,\lambda,h}$ such that

(1) *μ_ω is Gibbsian, i.e., $\mu_\omega \in \mathcal{G}_\omega^{\lambda,h}$,*
(2) *$\mu_{\sigma\omega}(\sigma^{-1}A) = \mu_\omega(A)$,*

for $\mathbb{P}$-almost all ω and all $A \in \mathcal{F}$.

Proof. Fix $(\lambda, h) \in \mathcal{I}$ and suppress these parameters from the notation. We shall consider the doubly-infinite case and construct a function μ_ω with the desired properties. The existence proof in the semi-infinite case is analogous.

First assume Λ is a finite string of sites from $\mathbb{Z}$. Let $\tilde\gamma_\Lambda^\omega(S_\Lambda)$ be the specification in Λ defined by

$$\tilde\gamma_\Lambda^\omega(S_\Lambda) = \frac{1}{2^{|\Lambda|}} \frac{e^{H_\Lambda^\omega(S)}}{\tilde Z_\Lambda^\omega} 1_{\{\exists i\in\Lambda: S_i=0\}} 1_{\{S_{\max\Lambda} - S_{\min\Lambda} = \pm 1\}}.$$

Then $\tilde Z_\Lambda^\omega$ is a kind of partition function, containing the condition that the path has to intersect the interface somewhere and be periodic. Clearly, $\tilde\gamma_\Lambda^{\sigma\omega}(\sigma^{-1}A) = \tilde\gamma_\Lambda^\omega(A)$ for any $A \in \mathcal{F}_\Lambda$.

Pick a sequence (Λ_n) of such volumes with $|\Lambda_n| = 2n$. Now define $\mu_B^{(n)}(A)$ by

$$\mu_B^{(n)}(A) = \int_\Omega \mathbb{P}(d\omega) 1_B(\omega) \tilde\gamma_{\Lambda_n}^\omega(A) \;\; (A \in \mathcal{F}_{\Lambda_n}, B \in \mathcal{B}_{\Lambda_n}).$$

Along some subsequence (n_k) we have $\mu_B^{(n_k)}(A) \to \mu_B(A)$ for all $A \in \cup_n \mathcal{F}_{\Lambda_n}$, $B \in \cup_n \mathcal{B}_{\Lambda_n}$ and some $\mu_B(A)$. Since $\mu_B(A)$ is σ-additive on $\cup_n\mathcal{F}_{\Lambda_n} \times \cup_n \mathcal{B}_{\Lambda_n}$, it has a unique extension $\bar\mu$ to $\mathcal{F}\times\mathcal{B}$. Moreover, $\mu_B(A) \le \mathbb{P}(B)$ implies $\bar\mu_B(A) \le \mathbb{P}(B)$, so by the Radon-Nikodym theorem there exists a μ_ω such that

$$\bar\mu_B(A) = \int_\Omega \mathbb{P}(d\omega) 1_B(d\omega) \mu_\omega(A).$$

The uniqueness of this representation implies that μ_ω is a σ-additive probability measure and that $\mu_\omega \in \mathcal{G}_\omega^{\lambda,h}$ for $\mathbb{P}$–almost all ω. The latter property, which is claim (1), holds because for all $C \in \mathcal{B}$, $D \in \mathcal{F}$ and $\tilde\omega \in \Omega$ the choice $B = \{\omega \in \Omega : \omega|_\Lambda = \tilde\omega|_\Lambda\} \cap C$ and $A = D$ or $A = \gamma_\Lambda^{\tilde\omega}(D|\,\cdot\,)$ yields

$$\int_\Omega \mathbb{P}(d\omega) 1_{\{\omega\in\Omega:\omega|_\Lambda = \tilde\omega|_\Lambda\}\cap C} \left[\mu_\omega(D) - E_{\mu_\omega} \gamma_\Lambda^{\tilde\omega}(D|\,\cdot\,)\right] = 0.$$

Here we have been able to substitute the original specification $\gamma_\Lambda^{\tilde\omega}$ into μ_ω by Gibbsianness, because ω is fixed at $\tilde\omega$ inside Λ. Finally, claim (2) follows from the periodicity of the specification $\tilde\gamma_\Lambda^\omega(S_\Lambda)$: it is trivially jointly translation invariant. □

3. Uniqueness and positive density in the localization regime

For the semi-infinite regular Gibbsian sections $\psi > 0$ implies recurrence, i.e., the path hits the interface infinitely often. (Namely, for any regular boundary condition $\tilde S$ we have $Z_\Lambda^\omega(\tilde S)e^{-\lambda h|\Lambda|} \to \infty$ as $\Lambda \to \mathbb{L}$. But this implies that all situations where the path leaves the interface in finite time have zero probability.) Below we shall in fact prove more, namely that all regular measures are *positively recurrent*, i.e., the path visits every height with a certain positive frequency.

For $a \in \mathbb{Z}$, let

$$\varrho_a^-(S) = \liminf_{\Lambda\to\mathbb{L}} \frac{1}{|\Lambda|}\sum_{i\in\Lambda} 1_{\{S_i=a\}}.$$

We shall say that $(\omega,\mu_\omega) \in \mathcal{G}^{\lambda,h}$ is *localized* if $\mathbb{E}\mu_\omega(\varrho_0^- > 0) = 1$.

Theorem 3. *Let $(\lambda,h) \in \mathcal{L}$.*
(a) $\mathcal{G}^{\lambda,h}$ is a singleton both for the semi-infinite and the doubly-infinite case.
(b) The unique doubly-infinite Gibbsian section $(\omega,\mu_\omega)_{\omega\in\Omega}$ is localized and is jointly translation invariant (i.e., $\mu_{\sigma\omega}(\sigma^{-1}A) = \mu_\omega(A)$ for $\mathbb{P}$-almost all ω).
(c) The unique semi-infinite Gibbsian section $(\omega,\nu_\omega)_{\omega\in\Omega}$ is localized and is asymptotically equal to $(\omega,\mu_\omega)_{\omega\in\Omega}$:

$$\lim_{n\to\infty}\sup_{A\in\mathcal{F}}\left|\nu_\omega(\sigma^n A) - \mu_\omega(\sigma^n A)\right| = 0 \quad \mathbb{P}-a.s.$$

(d) Both have a.s. constant densities, i.e., for $\mathbb{P}$–almost all ω

$$\lim_{\Lambda\to\mathbb{L}}\frac{1}{|\Lambda|}\sum_{i\in\Lambda} 1_{\sigma^i A}(S) = \mathbb{E}\mu_\omega(A) \quad \textit{for} \quad \mu_\omega - \textit{ almost all} \quad S.$$

(e) Both exhibit exponential tails, namely

$$\nu_\omega(S_{2n} = 2s) \le \mathcal{O}(1)e^{-2\zeta_s|s|} \ (n \ge n_0(\omega,s)),$$

with $\zeta_s \ge \psi - 0^+$ when $s > 0$ and $\zeta_s \ge \psi + 2\lambda h - 0^+$ when $s < 0$.

Corollary. *Let $(\lambda,h) \in \mathcal{L}$. Then all semi-infinite Gibbs measures generated by the free boundary condition are identical to (ν_ω).*

4. Three preparatory lemmas

In order to prove the above theorem and corollary we first have to state a couple of technical lemmas. Throughout the sequel we assume $(\lambda,h) \in \mathcal{L}$ and suppress these parameters from the notation.

Lemma 3. *Let $Z^{\omega}_{2n} = Z^{\omega}_{\Lambda_{2n}}(0)$ be the partition function for the boundary condition $\tilde{S}_{2n} = 0$. Then for each $\varepsilon \in (0, \psi)$ there is a $\delta_\varepsilon > 0$ such that*

$$\mathbb{P}\Big(\frac{1}{2n} \log Z^{\omega}_{2n} < \psi + \lambda h - \varepsilon\Big) \le \mathcal{O}(1) e^{-\delta_\varepsilon 2n}.$$

Proof. Given $\varepsilon > 0$ there is an m large enough such that

$$\frac{1}{2m}\mathbb{E}(\log Z^{\omega}_{2m}) \in (\psi + \lambda h - \varepsilon/2,\, \psi + \lambda h + \varepsilon/2).$$

This follows from the fact that a sublinear boundary condition does not alter the free energy (see Lemma 1). Put $k = \lfloor n/m \rfloor$. Then, by restricting the path to return to 0 at positions $2m, 4m, \dots, 2km$ $(\le 2n)$, we obtain

$$Z^{\omega}_{2n} \ge \frac{\binom{2m}{m}^k \binom{2(n-km)}{n-km}}{\binom{2n}{n}} \Big[\prod_{j=0}^{k-1} Z^{\sigma^{-jm}\omega}_{2m}\Big] Z^{\sigma^{-km}\omega}_{2n-2km}.$$

After taking logarithms we get

$$\mathbb{P}\Big(\frac{1}{2n}\log Z^{\omega}_{2n} < \psi + \lambda h - \varepsilon\Big) \le \mathbb{P}\Big(\frac{1}{k}\sum_{j=0}^{k-1} \frac{1}{2m} \log Z^{\sigma^{-jm}\omega}_{2m} < \psi + \lambda h - 3\varepsilon/4\Big),$$

where we have assumed n, m so large that the binomial factors give rise to a correction less than $\varepsilon/4$. Now, $(1/2m)\log Z^{\sigma^{-jm}\omega}_{2m}$ $(j = 0, \dots, k-1)$ are i.i.d. bounded random variables. Therefore a standard large deviation estimate gives that the last expression is bounded by $e^{-\delta_\varepsilon 2k}$ for some $\delta_\varepsilon > 0$. From this the claim easily follows. □

It will be convenient to use the notion of *arrival times*, defined as the positions where the path hits the interface. Let

$$\dots < N_{-1} < N_0 \le 0 < N_1 < N_2 < \dots,$$

be specified by $S_{2N_k} = 0$ and $S_{2i} \ne 0$ if $i \notin (N_k)$. (The sequence (N_k) ends when no further arrival occurs.) Let $\xi_k = N_{k+1} - N_k$ be the *interarrival* times (whenever these exist).

Lemma 4. *Consider K interarrival times in a row: $(\xi_{i+j})_{j=0}^{K-1}$. If μ_ω is a (finite-volume or infinite-volume) Gibbs measure corresponding to the disorder ω, then there is a $\kappa > 0$ such that for any $L \in \mathbb{Z}$*

$$\mathbb{E}\mu_\omega\big(\{\xi_{i+j} = m_{i+j}\ \forall j = 0, \dots, K-1\}\big|\{N_i = L\}\big) \le \mathcal{O}(1) \prod_{j=0}^{K-1} e^{-\kappa m_{i+j}}.$$

Proof. Since μ_ω is Gibbsian we can apply conditioning. The event

$$A = \{\xi_{i+j} = m_{i+j}\ \forall j = 0, \dots, K-1\} \cap \{N_i = L\}$$

means that $S_{2k_j} = 0$ for $k_j = L + \sum_{k=0}^{j-1} m_{i+k}$ and $S_{2l} \neq 0$ for $k_j < l < k_{j+1}$ $(j = 0, \ldots, K-1)$. Therefore (recall the definition of the Hamiltonian)

$$\mu_\omega(A|\{N_i = L\}) = \Big[\prod_{j=0}^{K-1} \frac{1 + e^{-2\lambda \sum_{l \in I_j}(\omega_l + h)}}{2Z^\omega_{J_j} e^{-\lambda h |I_j|}} P_{J_j}\Big] \mu_\omega(S_{2k_{j+1}} = 0),$$

where $I_j = (2k_j + 1, 2k_{j+1}] \cap \mathbb{Z}$, $J_j = I_j \cup \{2k_j\}$, and P_{J_j} is the probability that SRW conditioned on $S_{2k_j} = 0 = S_{2k_{j+1}}$ never touches the interface in between. By neglecting the last factor we obtain

$$\mu_\omega(A|\{N_i = L\}) \leq \prod_{j=0}^{K-1} \frac{1 + e^{-2\lambda \sum_{l \in I_j}(\omega_l + h)}}{2Z^\omega_{J_j} e^{-\lambda h |I_j|}} P_{J_j}.$$

Next, by Lemma 3 we have

$$Z_{J_j} e^{-\lambda h |I_j|} \geq e^{-(\psi - \varepsilon)|I_j|},$$

with probability at least $1 - \mathcal{O}(1) e^{-\delta_\varepsilon |I_j|}$. Moreover, a standard large deviation estimate gives

$$\mathbb{P}\Big(\sum_{l \in I_j} \omega_l < -\varepsilon |I_j|\Big) \leq \mathcal{O}(1) e^{-\delta'_\varepsilon |I_j|}.$$

Hence, by putting the three preceding estimates together we get

$$\mathbb{E}\Bigg(\frac{1 + e^{-2\lambda \sum_{l \in I_j}(\omega_l + h)}}{2Z^\omega_{J_j} e^{-\lambda h |I_j|}} P_{J_j}\Bigg) \leq \mathcal{O}(1)\Big[e^{-\delta_\varepsilon |I_j|} + e^{-\delta'_\varepsilon |I_j|} + e^{-(\psi-\varepsilon)|I_j|}\Big].$$

If we now set $\kappa = \sup_\varepsilon \min\{\delta_\varepsilon, \delta'_\varepsilon, \psi - \varepsilon\} > 0$, then the desired exponential estimate is established. □

The assertion of Lemma 4 means that the interarrival times are dominated by an i.i.d. exponential 'process'. If the r.h.s. of the formula in Lemma 4 were normalized, then we could immediately conclude that ϱ_0^- is uniformly bounded away from zero, just by using the law of large numbers. Since it is not normalized, a little more work is required.

Lemma 5. *Let $(\omega, \mu_\omega)_{\omega \in \Omega} \in \mathcal{G}^{\lambda,h}$ be a Gibbsian section. Then there is a number $\hat{\varrho} > 0$ such that $\mu_\omega(\varrho_0^- \geq \hat{\varrho}) = 1$ for $\mathbb{P}$–almost all ω. Moreover, $\hat{\varrho}$ can be chosen uniformly for all Gibbsian sections.*

Proof. Let us concentrate on the doubly-infinite case. (The semi-infinite case can be handled analogously.) Let

$$A_{n;k} = \Big\{\sum_{j=-n}^{n} 1_{\{S_{2j} = 0\}} \leq k\Big\}.$$

Let further $-2(n+n_-)$ denote the rightmost arrival in $(-\infty,-2n)$ and, similarly, $2(n+n_+)$ the leftmost arrival in $(2n,\infty)$. Since Lemma 4 provides an estimate for interarrival times in a row, we have

$$\mathbb{E}\mu_\omega(A_{n;k}) \le \mathcal{O}(1)\binom{2n+1}{k}\sum_{n_+,n_-=1}^{\infty} e^{-\kappa(2n+n_-+n_+)},$$

where the binomial factor accounts for all possible positions of the k arrivals within $[-2n,2n]$.

Pick $0<\hat{\varrho}<1$ and pick $k=k(n)$ such that $k(n)/(4n+1)\to\hat{\varrho}$ as $n\to\infty$. Then, using Stirling's formula, we obtain

$$\mathbb{E}\mu_\omega(A_{n;k(n)}) \le \mathcal{O}(1)\big[e^{-\kappa/2}\hat{\varrho}^{-\hat{\varrho}}(1-\hat{\varrho})^{-(1-\hat{\varrho})}\big]^{4n}.$$

So if $\hat{\varrho}$ satisfies $\hat{\varrho}\log\hat{\varrho}+(1-\hat{\varrho})\log(1-\hat{\varrho})+\kappa/2>0$, then the r.h.s. is summable on n. Hence, using that $\{\sum_{j=-n}^{n} 1_{\{S_{2j}=0\}} < 2n\hat{\varrho}\}\subseteq A_{n;k(n)}$ for large n, we find

$$\mathbb{E}\mu_\omega\Big(\bigcap_{m=1}^{\infty}\bigcup_{n\ge m}\big\{\sum_{j=-n}^{n} 1_{\{S_{2j}=0\}} < 2n\hat{\varrho}\big\}\Big)=0$$

by the Borel-Cantelli lemma. Therefore the claim follows (recall the definition of ϱ_0^-). □

5. Proof of Theorem 3

The proof will come in four steps:

STEP 1. The idea is to prove, by coupling to a doubly-infinite translation-invariant regular Gibbs section $(\omega,\mu_\omega)_{\omega\in\Omega}$, that the tail-behaviour of all regular Gibbs sections is unique. In order to apply the coupling theory, we have to show that paths intersect under the joint measure.

Pick $(\omega,\nu_\omega)_{\omega\in\Omega}\in\mathcal{G}^{\lambda,h}$, semi-infinite or doubly-infinite. Label the paths under μ_ω by 1, the paths under ν_ω by 2. Let

$$C_\infty=\{(S^1,S^2)\colon \exists (n_k)_{k\in\mathbb{N}},\, n_k\in\mathbb{L},\, \lim_{k\to\infty}|n_k|=\infty,\, S^1_{n_k}=S^2_{n_k}\}$$

be the set of pairs of paths that intersect infinitely often. We shall show that $(\mu_\omega\times\nu_\omega)(C_\infty)=1$ for $\mathbb{P}$-almost all ω. The proof goes as follows.

As was shown in Lemma 5, both measures have a strictly positive lower density of intersections with the interface. Hence the function $f_M=1_{\{N_1-N_0\ge M\}}$, i.e., the characteristic function of the event that 0 belongs to an excursion larger than M, is well defined on a set of full measure. Since $f_M\in L^1(\Sigma,\mathcal{F},\mathbb{E}\mu_\omega)$, we have

$$\lim_{n\to\infty}\frac{1}{2n+1}\sum_{j=-n}^{n}\sigma^j f_M=\bar{f}_M \quad \mathbb{E}\mu_\omega - a.s.$$

by the ergodic theorem. The function $\bar{f}_M$ is translation invariant, hence $\mu_\omega(\bar{f}_M > a)$ is $\mathbb{P}$-a.s. constant (by ergodicity w.r.t. the disorder) and

$$\mu_\omega(\bar{f}_M > a) = \mathbb{E}\mu_\omega(\bar{f}_M > a) \le \frac{\mathbb{E}E_{\mu_\omega}(\bar{f}_M)}{a} = \frac{\mathbb{E}E_{\mu_\omega}(f_M)}{a}.$$

The r.h.s. can be further estimated with the help of Lemma 4:

$$\mathbb{E}E_{\mu_\omega}(f_M) \le \mathcal{O}(1)\sum_{n=M}^{\infty}(2n+1)e^{-\kappa n} = \mathcal{O}(1)Me^{-\kappa M}.$$

Therefore

$$\mu_\omega(\bar{f}_M > a) \le \frac{\mathcal{O}(1)}{a}Me^{-\kappa M}$$

for $\mathbb{P}$–almost all ω.

Now, on $\{\bar{f}_M < \hat{\varrho}/2\}$ at least half of the arrivals under the measure ν_ω occur within the μ_ω-excursions of length at most M ($\hat{\varrho}$ is a lower bound for ϱ_0^-). This means that if the two paths (S^1, S^2) are to avoid each other, then one has to stay either above or below the other during these (infinitely many) excursions. Let $\Lambda_n^\circ = \Lambda_n \setminus \partial\Lambda_n$, and define

$$p_M = \sup_{n<M}\sup_{\omega}\max\bigl\{\gamma^\omega_{\Lambda_n}(S_i > 0\,\forall i \in \Lambda_n^\circ|\tilde{S} = 0),\ \gamma^\omega_{\Lambda_n}(S_i < 0\,\forall i \in \Lambda_n^\circ|\tilde{S} = 0)\bigr\}.$$

This the least price to pay if the path S^1 is swapped to $-S^1$ during the excursions containing an intersection of S^2 with the interface. Clearly, $p_M < 1$. Let $\mathcal{N}(n)$ denote the random number of sites i with $|i| \le n$ where $S_i^2 = 0$ and $f_M(\sigma^{-i}S^1) = 0$. Then we get

$$(\mu_\omega \times \nu_\omega)\Bigl(\{\bar{f}_M < \bar{\varrho}/2\} \cap [C_\infty]^c\Bigr) \le (\mu_\omega \times \nu_\omega)(p_M^{\mathcal{N}(n)}) \to 0 \quad \text{as } n \to \infty,$$

since $\mathcal{N}(n) \to \infty$ as $n \to \infty$, $\mu_\omega \times \nu_\omega$–a.s. for $\mathbb{P}$-almost all ω. Above we have again used Gibbsianness of μ_ω to extract interarrival probabilities (dominated by p_M), and neglected the terms $\mu_\omega(S_\cdot = 0)$, as in the proof of Lemma 4. Since

$$\mu_\omega\Bigl(\bigcup_{M=1}^{\infty}\{\bar{f}_M < \bar{\varrho}/2\}\Bigr) = 1$$

by the Borel-Cantelli lemma and our earlier estimate, we find that the paths S^1 and S^2 *must* intersect infinitely often $(\mu_\omega \times \nu_\omega)$–a.s.

STEP 2. We show by a coupling inequality that the measures have to agree on the tail σ–field. Besides other things, this implies uniqueness. The proof is done for ν_ω semi-infinite, the doubly-infinite case requiring only formal alterations.

Given $k, l \in \mathbb{N}$ such that $k \le l$, let $\Lambda_l = \{0, \dots, l\}$ and $A \in \mathcal{F}_{\Lambda_l} \cap \mathcal{F}_{\Lambda_k^c}$ (A should be thought of as approximating a tail event). Define

$$\tau = \inf\{n \ge 0\colon S_n^1 = S_n^2\}$$
$$\tau_l = \inf\{n > l\colon S_n^1 = S_n^2\}.$$

Let Λ be the random volume $[\tau, \tau_l] \cap \mathbb{N}$ and denote by E_ω the expectation w.r.t. the product measure $\mu_\omega \times \nu_\omega$. Then we can write

$$\begin{aligned}\big|\mu_\omega(A) - \nu_\omega(A)\big| &= \big|E_\omega\big(\gamma_\Lambda^\omega(A|\cdot) \times 1\big) - E_\omega\big(1 \times \gamma_\Lambda^\omega(A|\cdot)\big)\big| \\ &= \big|E_\omega\big(1_{\{\tau>k\}}\,[\gamma_\Lambda^\omega(A|\cdot) \times 1]\big) - E_\omega\big(1_{\{\tau>k\}}\,[1 \times \gamma_\Lambda^\omega(A|\cdot)]\big)\big| \\ &\le 2E_\omega(1_{\{\tau>k\}}),\end{aligned}$$

where the second equality follows because μ_ω and ν_ω have the same conditional probabilities in finite volumes. This estimate survives in the limit as $l \to \infty$, so we have

$$\sup_{A \in \mathcal{F}_{\Lambda_k^c}} \big|\mu_\omega(A) - \nu_\omega(A)\big| \le 2E_\omega(1_{\{\tau>k\}}).$$

By Step 1 the r.h.s. tends to 0 as $k \to \infty$. Consequently, μ_ω and ν_ω agree on the tail σ–field $\mathcal{T}$. In particular,

$$\lim_{k\to\infty} |\nu_\omega(\sigma^n A) - \mu_\omega(\sigma^n A)| = 0.$$

STEP 3. The a.s. convergence of ergodic averages for ν_ω can be proved through a comparision with the a.s. convergence for $\mathbb{E}\mu_\omega$, which is translation invariant. Namely, given a set $A \in \mathcal{F}$, let

$$A_> = \big\{\limsup_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} 1_{\sigma^{-k}A} > \mathbb{E}\mu_\omega(A)\big\}.$$

Clearly, $A_>$ is a tail event, and $\mathbb{E}\mu_\omega(A_>) = 0$ by the translation invariance of $\mathbb{E}\mu_\omega$. But this implies $\mathbb{E}\nu_\omega(A_>) = 0$, since $\mathbb{E}\nu_\omega$ coincides with $\mathbb{E}\mu_\omega$ on $\mathcal{T}$. So

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} 1_{\sigma^{-k}A} \le \mathbb{E}\mu_\omega(A) \quad \nu_\omega - a.s.$$

for $\mathbb{P}$–almost all ω. The same argument works for the limes inferior and so the limit holds with equality.

STEP 4. The last property to prove is that μ_ω and ν_ω have an exponential tail. Since we know that $\nu_\omega(\sigma^n A) \to \mu_\omega(\sigma^n A)$ as $n \to \infty$, it suffices to study the tail of μ_ω. To that end, pick $s \in \mathbb{N}$. We have from Gibbsianness

$$\mu_\omega(S_0 = 2s) = \sum_{n_+,n_-=s}^{\infty} \frac{P_{n_+,n_-}(S_0 = 2s)}{2Z^\omega_{J_{n_+,n_-}} e^{-\lambda h |I_{n_+,n_-}|}} \mu_\omega(S_{-2n_-} = S_{2n_+} = 0),$$

where $I_{n_+,n_-} = (-2n_-, 2n_+] \cap \mathbb{Z}$, $J_{n_+,n_-} = I_{n_+,n_-} \cup \{-2n_-\}$, and $P_{n_+,n_-}(S_0 = 2s)$ is the probability that SRW hitting the interface at $-2n_-$ and $2n_+$ climbes to height $2s$ at 0 without ever touching the interface in between. By Lemma 3 (after using the Borel-Cantelli lemma to ged rid of $\mathbb{E}$) we have

$$\big(Z^\omega_{J_{n_+,n_-}} e^{-\lambda h |I_{n_+,n_-}|}\big)^{-1} \le \mathcal{O}(1) e^{-2(n_++n_-)(\psi-\varepsilon)},$$

so the above series is $\mathbb{P}$-a.s. absolutely summable and of order $e^{-2s(\psi-\varepsilon)}$ (as $s \to \infty$). After letting $\varepsilon \downarrow 0$ we obtain that the tail property in Theorem 3 is proved for $s > 0$, with $\zeta_s = \psi - 0^+$. For $s < 0$ there is an additional factor

$$\exp\Big[-2\lambda \sum_{l \in I_{n_-,n_+}} (\omega_l + h)\Big]$$

in the numerator of each summand. This raises ζ_s by $2\lambda h$. □

6. Proof of Corollary

Due to the fact that our regularity concept comprises also a mild linear growth of the boundary condition (see Definition 2), all measures defined by the free boundary condition are regular. Namely, linearly growing paths have an exponentially small probability under the measure P_n and do not contribute in the thermodynamic limit. But according to Theorem 3, there is but one regular Gibbs measure, hence it is the one generated by the free boundary condition. □

7. Zero density in the delocalization regime

In this section we present an argument due to G. Giacomin, showing that in the interior of the delocalization regime the path is delocalized in the following sense:

Theorem 4. [G] *Let $(\lambda, h) \in int(\mathcal{D})$ and let $\mu_\omega \in \mathcal{G}_\omega$ be an arbitrary semi-infinite Gibbs measure. Then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} 1_{\{S_i = a\}} = 0 \quad \mu_\omega - a.s. \quad \text{for all } a \in \mathbb{Z}.$$

Proof. We shall show that for any boundary condition $\tilde{S}$ and any $\varepsilon > 0$ the event

$$A^a_{\lfloor \varepsilon n \rfloor, n} = \Big\{ \sum_{i=1}^{n} 1_{\{S_i = a\}} \geq \lfloor \epsilon n \rfloor \Big\}$$

has a probability decaying to zero under the finite volume specification $\gamma^\omega_{\Lambda_n}(\,\cdot\,|\tilde{S})$ in the limit as $n \to \infty$. The key ingredient is the entropy inequality

$$\gamma^\omega_{\Lambda_n}(A^a_{\lfloor \varepsilon n \rfloor, n}|\tilde{S}) \leq \frac{\log 2 + \mathcal{H}_n}{\log\big(1 + P_n(A^a_{k,n}|\tilde{S})^{-1}\big)},$$

where $P_n(\,\cdot\,|\tilde{S})$ is the SRW-bridge probability between 0 and $\tilde{S}_n$, and

$$\mathcal{H}_n = \mathcal{H}\big(\gamma^\omega_{\Lambda_n}(\,\cdot\,|\tilde{S})\big| P_n(\,\cdot\,|\tilde{S})\big)$$

denotes relative entropy. We note that the specific relative entropy $\mathcal{H}_n/n$ vanishes in the thermodynamic limit:

$$\lim_{N\to\infty} \frac{\mathcal{H}_n}{n} = \lambda \frac{\partial \phi}{\partial \lambda}(\lambda, h) - \phi(\lambda, h) = 0.$$

Indeed, the first equality follows from the convexity and regularity of ϕ inside $int(\mathcal{D})$, while the second equality holds because $\phi(\lambda, h) = \lambda h$ on $\mathcal{D}$. Hence, after we show that

$$\limsup_{n\to\infty} \frac{1}{n} \log P_n(A^a_{\lfloor \varepsilon n \rfloor, n} | \tilde{S}) < 0 \quad \text{for all } \varepsilon > 0,$$

for all $\varepsilon > 0$, the claim will be proved.

Let τ_1 (τ_2) be the leftmost (rightmost) site i with $0 \le i \le n$ such that $S_i = a$. Then

$$P_n(A^a_{\lfloor \varepsilon n \rfloor, n} | \tilde{S}) = \sum_{0 \le l_1 \le l_2 \le n} P_n(\tau_1 = l_1,\, \tau_2 = l_2 | \tilde{S}) P_{l_2 - l_1}(A^0_{\lfloor \varepsilon n \rfloor, n} | 0),$$

where the last factor can be further estimated by

$$P_{l_2 - l_1}(A^0_{\lfloor \varepsilon n \rfloor, n} | 0) \le \frac{P\big(A^0_{\lfloor \varepsilon n \rfloor, n} \cap \{S_{2(l_2 - l_1)} = 0\}\big)}{P(S_{2(l_2 - l_1)} = 0)} \le \mathcal{O}(\sqrt{n}) P(A^0_{\lfloor \varepsilon n \rfloor, n}),$$

since $l_2 - l_1 \le n$. Thus

$$P_n(A^a_{\lfloor \varepsilon n \rfloor, n} | \tilde{S}) \le \mathcal{O}(\sqrt{n}) P(A^0_{\lfloor \varepsilon n \rfloor, n}).$$

Next let us, similarily as in the proof of Theorem 3, define the interarrival times ξ_i as the duration between the i-th and the $(i+1)$-st intersection with the interface. Then we can write $A^0_{\lfloor \varepsilon n \rfloor, n} = \{\sum_{i=1}^{\lfloor \varepsilon n \rfloor} \xi_i \le n\}$. Now, the ξ_i are i.i.d. with distribution function satisfying

$$\sum_{l=1} P(\xi_i = l) z^l = 1 - \sqrt{1 - z^2} \quad \text{for all } 0 \le z < 1.$$

By the exponential Chebyshev inequality we have

$$P\Big(\sum_{i=1}^{\lfloor \varepsilon n \rfloor} \xi_i \le n\Big) \le z^{-n} (1 - \sqrt{1 - z^2})^{\lfloor \varepsilon n \rfloor}.$$

The right hand side attains its minimum at $z^2 = (1 - 2\varepsilon')(1 - \varepsilon')^{-2}$ with $\varepsilon' = \lfloor \varepsilon n \rfloor / n$, which for ε small enough produces

$$P(A^0_{\lfloor \varepsilon n \rfloor, n}) \le e^{-n\varepsilon}.$$

This completes the proof of the claim. □

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