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Symbolic Calculation of Zero Dynamics for Nonlinear Control Systems

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ABSTRACT

The calculation of the zero dynamics of a nonlinear system is of advantage in the design of controllers for this system. Because the calculation is difficult to do by hand, symbolic algebra programs are used. To access the usefulness of these programs and to solve some design problems, a MAPLE procedure, ZERODYN, is written to calculate the zero dynamics symbolically. The procedure can, however, not solve all problems, mainly because general symbolic algebra programs have insufficient capabilities to solve sets of nonlinear equations and partial differential equations. A realistic analysis problem shows this.

INTRODUCTION

To fulfill the increasing requirements on the dynamic behavior of systems, it may be necessary to control the system. In the design process of controllers a model of the system is often used. A nonlinear model is often necessary to describe the system adequately. Models can be characterized by structural properties. For the design of controllers these structural properties can be used to advantage. Their calculation for linear models is straightforward. For nonlinear models a numerical calculation usually only gives an approximation of the properties of the model (along a trajectory or in a working point). Among the interesting properties of a nonlinear model, from the viewpoint of control system design, are

- controllability and observability
- the relative degree
- the zero dynamics of the model.

The linear equivalent of a model with asymptotically stable zero dynamics is a minimum phase model. These properties play a role in design problems for nonlinear models like

- exact linearization with state feedback
- exact tracking
- disturbance rejection
- matching of linear models
- stabilization of "minimum phase" models.

The theoretical foundation of the calculation of the relative degree and the zero dynamics is given in, e.g., [1, 2]. Here, we discuss the calculation of the zero dynamics of nonlinear models, with emphasis on the implementation of the algorithms within the symbolic algebra program MAPLE. For the symbolic calculation of controllability and observability for nonlinear models, see [3].

First an overview of the underlying theory will be given, followed by a discussion of the actual implementation of some of the

algorithms. Then an overview of the problems encountered follows, based on the use of the analysis procedure ZERODYN, for problems that are representative for some small scale industrial applications. Finally some conclusions and recommendations are given.

THEORY OF THE ZERO DYNAMICS

The theory in [2] is the basis for the following discussion. The *zero dynamics* of the nonlinear model

$$\dot{x}(t) = f(x(t)) + \sum_{i=1}^{m} g_i(x(t))u_i(t) \quad \text{with} \quad x \in \Re^n, \; u_i \in \Re$$

and

$$y(t) = h(x(t)) \quad \text{with} \quad y \in \Re^l$$

is the dynamics of the submodel, which results after choosing the inputs $u_i(t)$ and initial conditions x_0 so that the outputs $y(t)$ of the model are 0 for all t. Alternatively, the zero dynamics is the dynamics of the submodel of maximal dimension that can be made unobservable by state feedback.

For simplicity the remaining discussion is restricted to a single–input ($m = 1$), single–output ($l = 1$) model

$$\dot{x} = f(x) + g(x)u, \quad y = h(x). \tag{1}$$

The calculation of the zero dynamics of model (1) at $x = x_0$ consist of two steps

- bring the model in a *local normal form* with a nonlinear invertible change of coordinates $z = \Phi(x)$,
- extract the zero dynamics equations from this form.

First, calculate r components of Φ as

$$
\begin{aligned}
\phi_1(x) &= h(x) \\
\phi_2(x) &= L_f h(x) \\
&\;\;\vdots \\
\phi_r(x) &= L_f^{r-1} h(x)
\end{aligned}
$$

where $L_f h(x) = \frac{\partial h(x)}{\partial x} f(x)$, $L_f^2 h(x)$ etc. are recursively defined, and r is the *relative degree* at x_0, i.e., the smallest r for which $L_g L_f^{r-1} h(x) \neq 0$, with $L_g(\cdot) = \frac{\partial (\cdot)}{\partial x} g(x)$. Choose the remaining $n - r$ new coordinates z_i, $i = r + 1, \ldots, n$ so that $\Phi(x)$ is invertible at x_0. Additionally, select [1] ϕ_i, $i = r + 1, \ldots, n$ so that

$$L_g z_i = L_g \phi_i(x) = 0. \tag{2}$$

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The normal form in the new coordinates z is then particularly simple

$$\dot{z}_1 = z_2$$
$$\vdots$$
$$\dot{z}_{r-1} = z_r$$
$$\dot{z}_r = b(z) + a(z)u$$
$$\dot{z}_{r+1} = q_{r+1}(z)$$
$$\vdots$$
$$\dot{z}_n = q_n(z)$$

and

$$y = z_1$$

where $b(x) = L_f^r h(x)$, $a(x) = L_g L_f^{r-1} h(x)$ and $q_i(x) = L_f \phi_i(x)$, $i = r+1, \ldots, n$. Use the relation

$$x = \Phi^{-1}(z) \tag{3}$$

to express $b(x)$, $a(x)$ and $q_i(x)$ as functions of z.

To get $y = 0$, choose the input u as $-b(z)/a(z)$ and the initial conditions x_0 so that $z_i = 0$, $i = 1, \ldots, r$. By definition, the zero dynamics of the model is then given by the equations for $\dot{z}_{r+1}, \ldots, \dot{z}_n$, with the coordinates $z_1, \ldots, z_r$ set to 0. With $\xi = [z_1, \ldots, z_r]'$, $\eta = [z_{r+1}, \ldots, z_n]'$ the zero dynamics can be written as $\dot{\eta} = q(\xi, \eta) = q(0, \eta)$.

For multivariable models the theory is more involved.

CALCULATION OF THE ZERO DYNAMICS

The solution of the partial differential equations (2) and of the system of nonlinear equations (3) is the main computational problem in the calculation of the zero dynamics. General computer algebra programs like REDUCE, MACSYMA, MAPLE and MATHEMATICA do not always find a solution, although theoretically a computable solution exists. Here MAPLE is used. The reasons for this choice are: availability, easy programming, low run times and previous experience. The class of problems for which a solution can be obtained is limited. The availability of a program that can tackle more problems will be welcomed.

The actual code of the MAPLE procedure is a straightforward implementation of the algorithm discussed in [2] and the previous section, although not for the general multivariable case. It is not given here.

APPLICATION

The analysis procedure is applied to several models. The model presented here is a mechanical system with two rotary joints [2], the first driven by a motor and with a spring for the second joint.
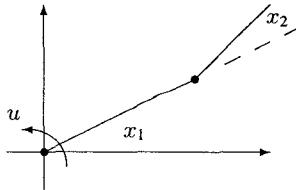
The equations of this model are

$$f(x) = \begin{bmatrix} x_3 \\ x_4 \\ \dfrac{bc(x_3+x_4)^2 \sin x_2 + (b+c\cos x_2)Kx_2 + c^2 x_3^2 \sin x_2 \cos x_2}{d-(c\cos x_2)^2} \\ \dfrac{(b+c\cos x_2)(x_4+2x_3)cx_4 \sin x_2 + (a+2c\cos x_2)(cx_3^2 \sin x_2 + Kx_2)}{-d+(c\cos x_2)^2} \end{bmatrix}$$

$$g(x) = \begin{bmatrix} 0 \\ 0 \\ \dfrac{b}{d-(c\cos x_2)^2} \\ \dfrac{-(b+c\cos x_2)}{d-(c\cos x_2)^2} \end{bmatrix}$$

$$h(x) = x_1.$$

The model parameters are a, b, c, d and K, with $d - c^2 > 0$. The relative degree r is 2. MAPLE was unable to solve the partial differential equation (2), so a solution for ϕ_3, ϕ_4 is calculated by hand, resulting in the following map Φ, invertible for all x_0

$$\left\{ z_1 = x_1, z_2 = x_3, z_3 = x_2, z_4 = x_4 + \left(1 + \frac{c}{b}\cos x_2\right)x_3 \right\}.$$

The inverse map Φ^{-1} is

$$\left\{ x_1 = z_1, x_2 = z_3, x_3 = z_2, x_4 = -\frac{z_2 b - z_4 b + z_2 c \cos z_3}{b} \right\}.$$

The normal form of the system is

$$\dot{z}_1 = z_2$$
$$\dot{z}_2 = b(z) + a(z)u$$
$$\dot{z}_3 = q_3(z)$$
$$\dot{z}_4 = q_4(z)$$

with

$$a(z) = -\frac{b}{(c\cos z_3)^2 - d}$$

$$q_3(z) = -\frac{z_2 b - z_4 b + z_2 c \cos z_3}{b}$$

and $b(z)$ and $q_4(z)$ too complex to print. The zero dynamics is now given by q_3 and q_4 with z_1 and z_2 set to 0

$$\dot{z}_3 = q_3(z) = z_4$$

$$\dot{z}_4 = q_4(z) = \frac{-Kz_3 b^2 - (c\cos z_3)^2 Kz_3 + baKz_3}{b((c\cos z_3)^2 - d)}.$$

This shows that the answer is not trivial, although the output of MAPLE could be simplified further. In this calculation the only intervention was for the solution of (2). Further analysis shows that the zero dynamics is not asymptotically stable.

CONCLUSIONS

For simple models the general symbolic algebra program MAPLE can calculate the zero dynamics. Human intervention may be needed.

The availability of a symbolic algebra program with more powerful capabilities for solving sets of nonlinear equations and partial differential equations should make it possible to tackle more problems. Such a program is welcome.

The ZERODYN procedure should be expanded with a calculation of the zero dynamics for multivariable systems.

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