Note on the order of successive displacements

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by

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(dedicated to J.J. Seidel at the occasion of his 60-th birthday)

0. Recently a note [3] has been published, questioning the generality of the well-established fact that finite rotations as a rule do not commute. An example was provided from which might be inferred that rotations are indeed rather commutable. The authors' last remark, to scrutinize statements regarding the order of rotations, suggested me to publish this note; no theorem in it is new, but some of them seem to have escaped public recognition.

1. We denote space, three-dimensional Euclidean space that is, by $\mathbb{E}^3$, its company vector space by $\mathbb{E}^3_*$ (for details of affine spaces see e.g. [5], Ch. 4). A pair (a,b) of points $a,b \in \mathbb{E}^3$ gives rise to a vector $\underline{u} := a \to b \in \mathbb{E}^3_*$; we may also write $b = a + \underline{u}$ or $a = b - \underline{u}$. A line ℓ through a with direction $\underline{d}$ ($\underline{d} \neq \underline{0}$) may conveniently be written $a + \rho\underline{d}$; it is understook then without further comment that ρ runs through the reals $\mathbb{R}$. A displacement α in $\mathbb{E}^3$ is always a screw (see for example, to mention a recent source only, [2] p. 85) and a screw is the unique superposition of a rotation about an axis ℓ and a translation along a vector parallel to ℓ. It should be observed that in this superposition the rotational and translational components are commutable. If $a + \rho\underline{d}$ ($\underline{d} \neq \underline{0}$) is the axis and the translation vector is $h\underline{d}$, then the displacement α acts on points $x \in \mathbb{E}^3$ by

$$\alpha(x) = a + h\underline{d} + \alpha_*(a \to x) . \tag{1}$$

Here α_*, the rotation of α, is a direct orthogonal linear transformation of $\mathbb{E}^3_*$, with eigenvector $\underline{d}$ belonging to the eigenvalue 1; hence $\alpha_*(\underline{d}) = \underline{d}$. Since α_* conveys an angle of rotation, measured clockwise when seen in the direction $\underline{d}$, α can be represented by four entities in the quadruple $\{a,\underline{d},\varphi,h\}$; this representation is not unique, of course; if $\varphi = 0$ then

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α is a mere translation; if $h = 0$ then α is a rotation; if, in the latter case, $\varphi = \pi$ then α is called a halfturn.

If α and $\beta := \{b, \underline{e}, \psi, k\}$ are displacements we can superpose the actions of α and β in both orders to give displacements $\beta \circ \alpha$ (first apply α, then β) and $\alpha \circ \beta$. These superpositions have to be understood as composite mappings; hence, e.g., the axis of β is given in space and not affected by the action of α in $\beta \circ \alpha$.

α and β are immediately seen to be commutable if at least one of them is the identity map or both are translations. Complications arise when α or β involves rotation. So let us first concentrate on the rotational part of $\beta \circ \alpha$.

__Lemma 1__. The rotation $(\beta \circ \alpha)_*$ of $\beta \circ \alpha$ is equal to the composite $\beta_* \circ \alpha_*$ of the linear transformations α_* and β_*.

__Proof__: From (1) we derive

$$\alpha(a + \underline{u}) = a + h\underline{d} + \alpha_*(\underline{u})$$

hence

$$\beta \circ \alpha(a + \underline{u}) = \beta(a + h\underline{d} + \alpha_*(\underline{u})) = \beta(b + b \to a + h\underline{d} + \alpha_*(\underline{u})) =$$

$$= b + k\underline{e} + \beta_*(b \to a) + h\beta_*(\underline{d}) + \beta_* \circ \alpha_*(\underline{u}) \ ,$$

and from this we easily deduce

$$\beta \circ \alpha(a + \underline{u}) = \beta \circ \alpha(a) + \beta_* \circ \alpha_*(\underline{u}) \ ,$$

proving the assertion. □

__Lemma 2__. Two rotations α_* and β_* of $\mathbb{E}^3_*$ commute precisely in the following cases:

1. α_* or/and β_* is the identity map of $\mathbb{E}^3_*$.
2. α_* and β_* have a common axis.
3. α_* and β_* are halfturns with orthogonal axes.

Proof: That α_* and β_* in each case commute is clear.
Now let us suppose that $\alpha_* \circ \beta_* = \beta_* \circ \alpha_*$ and prove that α_* and β_* have one of the three indicated properties.
Exclude case 1; then α_* and β_* both have a unique axis, directed by vectors $\underline{d}$ and $\underline{e}$ respectively; suppose $|\underline{d}| = |\underline{e}| = 1$.
From $\alpha_*(\underline{d}) = \underline{d}$ and $\beta_*(\underline{e}) = \underline{e}$ we infer

$$\alpha_* \circ \beta_*(\underline{d}) = \beta_* \circ \alpha_*(\underline{d}) = \beta_*(\underline{d})\,;$$

hence $\beta_*(\underline{d})$ is eigenvector of α_* to the eigenvalue 1 and, since $|\beta_*(\underline{d})| = 1$ anyway, $\beta_*(\underline{d}) = \pm\,\underline{d}$.
Analogously, $\alpha_*(\underline{e}) = \pm\,\underline{e}$.
If $\beta_*(\underline{d}) = \underline{d}$ then $\underline{d}$ is a direction of the axis of β_*, or $\underline{d} = \pm\,\underline{e}$ and the axes of α_* and β_* coincide and $\alpha_*(\underline{e}) = \underline{e}$.
The only possibility left is $\alpha_*(\underline{e}) = -\underline{e}$ and $\beta_*(\underline{d}) = -\underline{d}$, so that α_* and β_* have an eigenvalue -1, which means that they are halfturns.
Moreover, $(\underline{d},\underline{e}) = 0$, and thus the axis of α_* and β_* are orthogonal. □

Remark. This lemma is a special case of a much more general theorem about commutability of matrices; see, e.g., [4], p. 265. □

Proceeding to our theorem 1 we have, finally, to consider halfturns more in detail. So let $\lambda := \{a,\underline{d},\pi,0\}$ and $\mu := \{b,\underline{e},\pi,0\}$ be halfturns.
Points a and b and vectors $\underline{d}$ and $\underline{e}$ can be chosen such that $(a \to b,\underline{d}) = (a \to b,\underline{e}) = 0$; $a \to b$ and $\underline{d} \times \underline{e}$, when both different from zero, have the same direction; and an angle φ is defined between λ and μ (in this order) measured from $\underline{d}$ to $\underline{e}$ clockwise when seen in the direction $\underline{d} \times \underline{e}$ (observe that coincidences between $a + \rho\underline{d}$ and $b + \sigma\underline{e}$ or nuances in the choice of a, b, $\underline{d}$ or $\underline{e}$ do not disturb these definitions).
If $a + \rho\underline{d}$ and $b + \sigma\underline{e}$ coincide then $\lambda = \mu$ and $\mu \circ \lambda$ is the identity.
Otherwise, one of the vectors $\underline{d} \times \underline{e}$ and $a \to b$ is different from $\underline{0}$; take $\underline{f}$ as a unit vector in this direction.

Lemma 3. If $\lambda \neq \mu$ then $\mu \circ \lambda = \{a,\underline{f},2\varphi,2|a \to b|\}$.

<u>Proof</u>. A very neat one can be found in [1], p. 286. □

<u>Remarks</u>:

1. If $\underline{d}$ and $\underline{e}$ are linearly dependent or, geometrically speaking, if the axes of λ and μ are parallel, then $\varphi = 0$ and $\mu \circ \lambda$ is a translation.
2. If $a = b$ then the axes of λ and μ intersect and $\mu \circ \lambda$ is a rotation. □

<u>Device</u>: Every screw α can be decomposed in two halfturns λ and μ; this decomposition is not unique, but, if the respective axes are ℓ_α, ℓ_λ and ℓ_μ, then the geometric relations $a \to b$ and φ of ℓ_λ and ℓ_μ are uniquely determined by α. Moreover, if $\alpha = \mu \circ \lambda$ and also $\alpha = \kappa \circ \mu$ then $\ell_\kappa = \mu(\ell_\lambda)$. □

<u>Theorem 1</u>: Two displacements $\alpha := \{a,\underline{d},\varphi,h\}$ and $\beta := \{b,\underline{e},\psi,k\}$ commute precisely in the following cases:

i: one at least of α and β is the identity.

ii: both α and β are translations.

iii: α is a translation, β is not a translation and $\underline{d}$, $\underline{e}$ are linearly dependent.

iv: Neither of α and β is a translation and their axes coincide.

v: α and β are halfturns with orthogonally intersecting axes.

<u>Proof</u>: That α and β in each case commute is immediate. Now suppose $\alpha \circ \beta = \beta \circ \alpha$. From lemma 1 we infer $\alpha_* \circ \beta_* = \beta_* \circ \alpha_*$. From lemma 2 we are urged to consider three cases.

1) α_* is the identity map in $\mathbb{E}^3_*$; hence α is a translation; if α is the identity we have case (i); if β also is a translation then we have case (ii); so consider the case that α is not the zero-translation and β is no translation at all. Then $\alpha = \{a,\underline{d},0,h\}$,

$$\alpha \circ \beta(b + \underline{u}) = \alpha(b + k\underline{e} + \underline{\beta_*}(\underline{u})) = b + k\underline{e} + \underline{\beta_*}(\underline{u}) + h\underline{d} ,$$

$$\beta \circ \alpha(b + \underline{u}) = \beta(b + \underline{u} + h\underline{d}) = b + k\underline{e} + \beta_*(\underline{u}) + h\underline{\beta_*}(\underline{d}),$$

hence $\underline{d} = \underline{\beta_*}(\underline{d})$ or $\underline{d}$,$\underline{e}$ linearly dependent, which is case (iii).

2) Neither α_* nor β_* is the identity, α_* and β_* have a common axis. This implies that the axes ℓ_α and ℓ_β of α and β are parallel.

2.1) Let $\ell_\alpha \| \ell_\beta$, but $\ell_\alpha \neq \ell_\beta$. Since the translational parts of α and β commute with each other and with the rotational parts of both α and β it is no loss of generality to suppose α and β to be pure rotations. We take the points $a \in \ell_\alpha$ and $b \in \ell_\beta$ such that $a \to b$ is orthogonal to ℓ_α and ℓ_β. If α and β are halfturns then $\beta \circ \alpha$ and $\alpha \circ \beta$ are translations with vectors $2(a \to b)$ and $2(b \to a)$ respectively, implying $a = b$, contrary to the assumption $\ell_\alpha \neq \ell_\beta$; hence at least one of α and β is not a halfturn. Take ℓ_μ through a and b; now decompose α and β in halfturns according to the device and in two ways, $\mu \circ \lambda = \alpha = \kappa \circ \mu$ and $\nu \circ \mu = \beta = \mu \circ \omega$, with axes ℓ_κ, ℓ_λ, ℓ_μ, ℓ_ν and ℓ_ω all lying in the same plane orthogonal to ℓ_α (fig. 1); ℓ_μ is seen to be an axis of symmetry of the configuration.

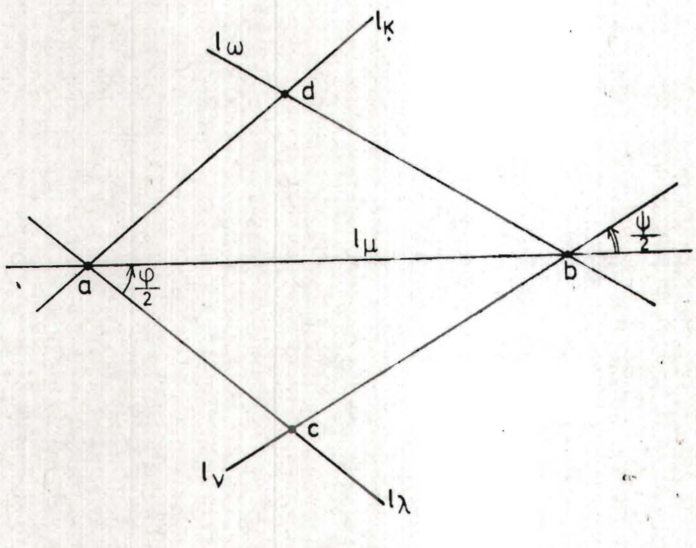

fig. 1.

The relation

$$\nu \circ \lambda = \nu \circ \mu \circ \mu \circ \lambda = \beta \circ \alpha = \alpha \circ \beta = \kappa \circ \mu \circ \mu \circ \omega = \kappa \circ \omega$$

implies that the intersection c of ℓ_λ and ℓ_ν coincides with the intersection d of ℓ_κ and ℓ_ω. This can clearly only happen on ℓ_μ; consequently one at least of ℓ_λ and ℓ_ν must coincide with ℓ_μ, implying that α or β

is the identity and violating our original assumption. (If ℓ_λ and ℓ_ν do not intersect then $\nu \circ \lambda$ defines a translation and the argument needs a minor modification.)

2.2) $\ell_\alpha = \ell_\beta$. This is precisely case (iv).

3) α_* and β_* are halfturns and their axes are orthogonal. α and β then have orthogonal axes ℓ_α and ℓ_β.

Now

$$\beta \circ \alpha(a) = \beta(a + h\underline{d}) = \beta(a) + h\beta_*(\underline{d}) = a + 2(a \to b) - h\underline{d} ,$$

$$\alpha \circ \beta(a) = \alpha \circ \beta(b + b \to a) = \alpha(b + k\underline{e} + \beta_*(b \to a)) =$$

$$= \alpha(b + k\underline{e} + a \to b) = \alpha(b) + k\alpha_*(\underline{e}) + \alpha_*(a \to b) =$$

$$= a - (a \to b) - k\underline{e} - (a \to b)$$

and equating these results we find

$$4(a \to b) - h\underline{d} + k\underline{e} = \underline{0}$$

implying $a \to b = \underline{0}$, $h = k = 0$; α and β are halfturns as was to be proved. □

2. Before tackling the paradox of the foregoing theorems as compared to the example of a Cardan-suspended gear, see [3] and further on in this note, let us answer another question first.

If α and β are rotations about different but intersecting axes ℓ_α and ℓ_β then obviously there must exist a rotation χ such that $\alpha \circ \beta = \chi \circ \alpha$. Since usually χ cannot be β, what can be said about χ? Formally $\chi = \alpha \circ \beta \circ \alpha^{-1}$, where α^{-1} denotes the inverse transformation of α, with axis ℓ_α and angle opposite to the angle of α. But then a point $b \in \alpha(\ell_\beta)$ is invariant under χ, since $\chi(b) = \alpha \circ \beta \circ \alpha^{-1}(b) = \alpha \circ \beta(\alpha^{-1}(b)) = \alpha \circ \alpha^{-1}(b) = b$; this implies that $\ell_\chi = \alpha(\ell_\beta)$. Moreover, since $\mathrm{trace}(\chi_*) = \mathrm{trace}(\beta_*)$, we have reason to expect the angles of χ and β to be equal. Since these are defined in relation to directions of ℓ_χ and ℓ_β we have to produce a precise statement. We need first:

Lemma 4. The linear part $(\alpha^{-1})_*$ of α^{-1} is $(\alpha_*)^{-1}$ (so writing α_*^{-1} is without ambiguity).

Proof: $a + \underline{u} = \alpha^{-1} \circ \alpha(a + \underline{u}) = \alpha^{-1}(\alpha(a) + \alpha_*(\underline{u})) =$

$$= \alpha^{-1} \circ \alpha(a) + (\alpha^{-1})_* \circ \alpha_*(\underline{u}) = a + (\alpha^{-1})_* \circ \alpha_*(\underline{u}) ,$$

hence $(\alpha^{-1})_* \circ \alpha_* =$ identity and $(\alpha^{-1})_* = (\alpha_*)^{-1}$. $\square$

Theorem 2: If $\alpha := \{a,\underline{d},\varphi,0\}$ and $\beta = \{a,\underline{e},\psi,0\}$ then $\alpha \circ \beta \circ \alpha^{-1} = \{a,\underline{f},\psi,0\}$, where $\underline{f} := \alpha_*(\underline{e})$.

Proof: If $\underline{d}$ and $\underline{e}$ are linearly dependent then the assertion is trivial; otherwise, suppose $|\underline{d}| = |\underline{e}| = 1$. Write $\chi = \alpha \circ \beta \circ \alpha^{-1}$ as before. Since $\alpha \circ \beta \circ \alpha^{-1}(a) = a$ and $\alpha_* \circ \beta_* \circ \alpha_*^{-1}(\underline{f}) = \underline{f}$ it is clear that $\ell_\chi = a + \rho\underline{f}$.
Let the angle of χ, seen in the direction $\underline{f}$, be θ; we shall prove that $\theta = \psi$. We noticed already that trace(χ_*) = trace(β_*), implying $\cos\theta = \cos\psi$.
From vector algebra we know that $\sin\psi$ is the volume of the parallelepiped spanned by $\underline{e}$, $\underline{d} \times \underline{e}$ and $\beta_*(\underline{d} \times \underline{e})$, or $\sin\psi = \det[\underline{e}, \underline{d} \times \underline{e}, \beta_*(\underline{d} \times \underline{e})]$.
Analogously, $\sin\theta = \det[\underline{f}, \underline{d} \times \underline{f}, \chi_*(\underline{d} \times \underline{f})]$.
By the very nature of rotations and of α_* in particular,

$$\underline{d} = \alpha_*(\underline{d}) \text{ and } \alpha_*(\underline{d} \times \underline{e}) = (\alpha_*\underline{d}) \times (\alpha_*\underline{e}) ;$$

hence

$$\begin{aligned} \sin\theta &= \det[\alpha_*(\underline{e}), \alpha_*(\underline{d} \times \underline{e}), \chi_* \circ \alpha_*(\underline{d} \times \underline{e})] = \\ &= \det[\alpha_*(\underline{e}), \alpha_*(\underline{d} \times \underline{e}), \alpha_* \circ \beta_*(\underline{d} \times \underline{e})] = \\ &= \det[\underline{e}, \underline{d} \times \underline{e}, \beta_*(\underline{d} \times \underline{e})] = \sin\psi ; \end{aligned}$$

we conclude that $\theta = \psi$. $\square$

To make the notation slightly more transparent we write $\beta^\alpha := \alpha \circ \beta \circ \alpha^{-1}$; thus $\beta^\alpha \circ \alpha = \alpha \circ \beta$.

Remark: This theorem, by the way, confirms the earlier ones on commutability. If $\alpha \circ \beta = \beta \circ \alpha$ then $\beta^\alpha = \beta$ or $\alpha(\ell_\beta) = \ell_\beta$. This, by the nature of rotations, means that $\ell_\beta = \ell_\alpha$ or ℓ_β intersects ℓ_α orthogonally.
Since also $\alpha_*(\underline{e}) = \pm\underline{e}$, one can deduce the further peculiarities of lemma 2. $\square$

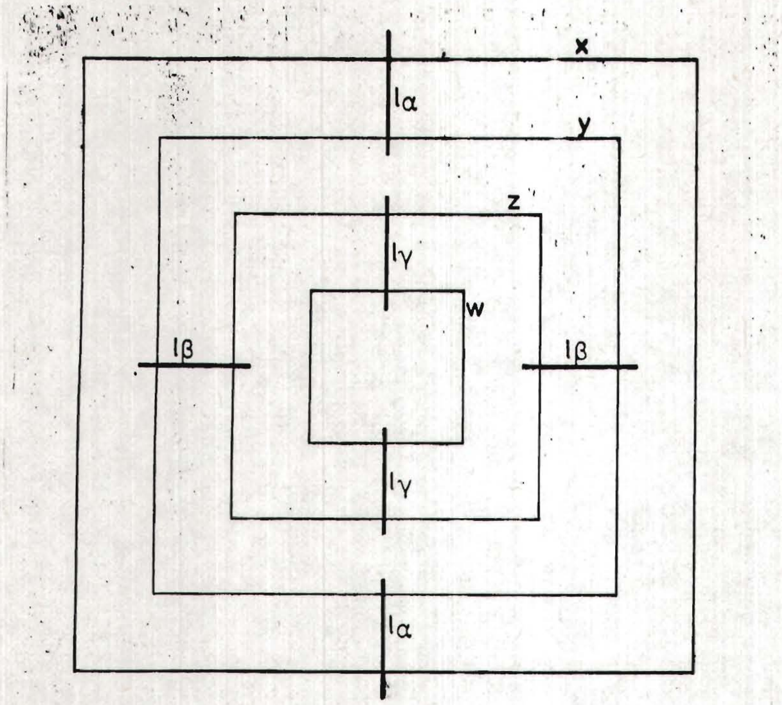

fig. 2.

3. A gyroscope in a Cardan suspension can be thought of as an assemblage of four moving spaces X, Y, Z, W; X and Y are connected by an axis ℓ_α, Y and Z by ℓ_β, Z and W by ℓ_γ (terminology slightly differing from [3]; fig. 2). X can be thought to be fixed. If Y (with its appendages) is moved to another position by a rotation about ℓ_α then ℓ_β and ℓ_γ take positions $\alpha(\ell_\beta)$ and $\alpha(\ell_\gamma)$ respectively. If, however, Z is moved through a rotation β about ℓ_β then ℓ_α does not change its position. Hence $\alpha \circ \beta$ denotes again a rotation as seen from our viewpoint in space X. Performing the other way round, we get $\beta^\alpha \circ \alpha$, which, by lemma 4, is equal to $\alpha \circ \beta$. The same argument applies to more intricate combinations; for example first α, then "β", then "γ":
α takes ℓ_β and ℓ_γ into positions $\alpha(\ell_\beta)$ and $\alpha(\ell_\gamma)$; then β^α takes $\alpha(\ell_\gamma)$ to $\beta^\alpha \circ \alpha(\ell_\gamma) = \alpha \circ \beta(\ell_\gamma)$; finally, when $\gamma^{\alpha\circ\beta}$ is brought in action, the resulting transformation is $\gamma^{\alpha\circ\beta} \circ \beta^\alpha \circ \alpha = \gamma^{\alpha\circ\beta} \circ (\alpha \circ \beta) = (\alpha \circ \beta) \circ \gamma$.

Conclusion. Superposition of displacements is as a rule not commutative; the exceptions to the rule can be clearly specified; and the Cardan construction is a typical example of the rule, not of its exceptions.

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