

# On the connection between a symmetry condition and several nice properties of the spaces $S_{\{\Phi(A)\}}$ en $T_{\{\Phi(A)\}}$

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ON THE CONNECTION BETWEEN  
A SYMMETRY CONDITION AND SEVERAL NICE  
PROPERTIES OF THE SPACES  $S_{\Phi(A)}$  AND  $T_{\Phi(A)}$

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**ON THE CONNECTION BETWEEN  
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Abstract.

In this paper it is proved that several topological properties of the spaces  $S_{\Phi(A)}$  and  $T_{\Phi(A)}$  are equivalent with a symmetry condition on the directed set  $\Phi$ .

**0. Introduction**

This paper is based on a paper [EGK] of S.J.L. van Eijndhoven, J. de Graaf and P. Kruszyński in which the spaces  $S_{\Phi(A)}$  and  $T_{\Phi(A)}$  are introduced. In Chapter IV of that paper it is proved that a symmetry condition implies some topological properties of the spaces  $S_{\Phi(A)}$  and  $T_{\Phi(A)}$ . In the underlying paper we show that a weaker symmetry condition is equivalent with all those topological properties and a lot more.

For the terminology of locally convex topological vector spaces we refer to [Wil].

**1. Notations and some known theorems**

Let  $X$  be a separable Hilbert space,  $n \in \mathbb{N}$  and let  $A_1, \dots, A_n$  denote  $n$  self-adjoint operators whose corresponding spectral projections mutually commute. There exists a unique spectral measure  $E$  on the set  $B(\mathbb{R}^n)$  of Borel sets of  $\mathbb{R}^n$  so that for every  $k \in \{1, \dots, n\}$  the map  $\Delta \mapsto E(\mathbb{R}^{k-1} \times \Delta \times \mathbb{R}^{n-k})$ ,  $\Delta$  a Borel set in  $\mathbb{R}$ , equals the spectral measure of  $A_k$ . For every Borel measurable function  $f$  on  $\mathbb{R}^n$ , there can be defined the self-adjoint operator

$$f(A) = \int_{\mathbb{R}^n} f(\lambda) dE_\lambda$$

in a natural manner. (See [EGK], page 280.)

For every  $\Delta \in B(\mathbb{R}^n)$  let  $\chi_\Delta$  be the characteristic function of  $\Delta$ . For all  $m \in \mathbb{Z}^n$  let

$$Q_m := \{ \lambda \in \mathbb{R}^n : \forall_{k \in \{1, \dots, n\}} [\lambda_k \in [m_k - 1, m_k]] \}.$$

Let  $B_b(\mathbb{R}^n)$  be the set of all bounded Borel sets of  $\mathbb{R}^n$  and let  $G^+$  be the set of all maps  $F$  from  $B_b(\mathbb{R}^n)$  into  $X$  with the property

$$F(\Delta_1 \cap \Delta_2) = E(\Delta_2)F(\Delta_1) \quad (\Delta_1, \Delta_2 \in B_b(\mathbb{R}^n)).$$

Define  $\text{emb} : X \rightarrow G^+$  by  $[\text{emb } x](\Delta) := E(\Delta)x$ ,  $x \in X$ ,  $\Delta \in B_b(\mathbb{R}^n)$ . The map  $\text{emb}$  is injective. So the Hilbert space  $X$  is embedded in  $G^+$ .

Let  $\phi$  be a Borel measurable function on  $\mathbb{R}^n$  which is bounded on bounded Borel sets. Denote  $\phi := \{ \lambda \in \mathbb{R}^n : \phi(\lambda) \neq 0 \}$ . (Remark:  $\phi$  need not be closed.) Let  $x \in X$ . Define  $\phi(A) \cdot x \in G^+$  by  $[\phi(A) \cdot x](\Delta) = \phi(A)E(\Delta)x$ ,  $\Delta \in B_b(\mathbb{R}^n)$ . For every  $F \in \phi(A) \cdot X$  there exists a unique  $x \in E(\phi)(X)$  such that  $F = \phi(A) \cdot x$ . Hence an inner product can be defined on  $\phi(A) \cdot X$  such that the map  $\phi(A) \cdot : E(\phi)(X) \rightarrow \phi(A) \cdot X$  is a unitary map between two Hilbert spaces. The set  $B_b(\mathbb{R}^n)$  is a directed set under inclusion, so we can define the following subspace of  $G^+$ :

$$D_\phi := \{ F \in G^+ : \Delta \mapsto \phi(A)F(\Delta), \Delta \in B_b(\mathbb{R}^n), \text{ is a Cauchy net in } X \}.$$

For every  $F \in D_\phi$  define  $\phi(A) * F := \lim_{\Delta} \phi(A)F(\Delta) \in X$ . Corresponding to the same function  $\phi$  we can also define an operator  $\phi(A) : G^+ \rightarrow G^+$  by

$$[\phi(A)F](\Delta) := \phi(A)F(\Delta) \quad (F \in G^+, \Delta \in B_b(\mathbb{R}^n)).$$

Let  $\Phi$  be a set of Borel functions on  $\mathbb{R}^n$ . Suppose the set  $\Phi$  satisfies the next axiom.

#### AXIOM 1.

$\Phi$  is a directed set of real valued Borel functions on  $\mathbb{R}^n$  and every element of  $\Phi$  is bounded on bounded Borel sets. The set  $\Phi$  has the following properties:

- AI. Each  $\phi \in \Phi$  is nonnegative and the function  $\lambda \mapsto \phi(\lambda)^{-1}$ ,  $\lambda \in \phi$  is bounded on bounded Borel sets.
- AII. The sets  $\phi, \phi \in \Phi$ , cover the whole  $\mathbb{R}^n$ , i.e.  $\mathbb{R}^n = \bigcup_{\phi \in \Phi} \phi$ .
- AIII.  $\forall_{\phi \in \Phi} \exists_{\psi \in \Phi} \exists_{c > 0} \forall_{m \in \mathbb{Z}^n} [(1 + |m|) \sup_{\lambda \in Q_m} \phi(\lambda) \leq c \inf_{\lambda \in Q_m} \psi(\lambda)]$ .

The set  $\Phi$  induces a new set  $\Phi^+$ .

#### DEFINITION 2.

Let  $\Phi$  be a set which satisfies Axiom 1. Then  $\Phi^+$  will denote the set of all Borel functions  $f$  on  $\mathbb{R}^n$  so that

- i)  $f$  is a nonnegative Borel function and the map  $\lambda \mapsto f(\lambda)^{-1}$ ,  $\lambda \in \underline{f}$  is bounded on bounded Borel sets.
- ii)  $\forall_{\phi \in \Phi} [\sup_{\lambda \in \mathbb{R}^n} f(\lambda) \phi(\lambda) < \infty]$ .

LEMMA 3.

The set  $\Phi^+$  satisfies Axiom 1.

Proof. See [EGK], Lemma 1.5. □

Now we can define two subspaces  $S_{\Phi(A)}$  and  $T_{\Phi(A)}$  of  $G^+$ . Let  $\Phi$  be a set which satisfies Axiom 1.

DEFINITION 4.

Let  $S_{\Phi(A)} := \bigcup_{\phi \in \Phi} \phi(A) \cdot X$ .

The topology  $\sigma_{\text{ind}}$  for  $S_{\Phi(A)}$  is the inductive limit topology generated by the Hilbert spaces  $\phi(A) \cdot X$ ,  $\phi \in \Phi$ .

DEFINITION 5.

Let  $f \in \Phi^+$ . Then  $S_{\Phi(A)} \subset D_f$ . The seminorm  $s_f$  on  $S_{\Phi(A)}$  is defined by  $s_f(w) := \|f(A) * w\|$ ,  $w \in S_{\Phi(A)}$ .

THEOREM 6.

The locally convex topology for  $S_{\Phi(A)}$  generated by the seminorms  $s_f$ ,  $f \in \Phi^+$  is equivalent to the topology  $\sigma_{\text{ind}}$  for  $S_{\Phi(A)}$ .

Proof. See [EGK], Theorem 1.8. □

Remark: It follows that the topology  $\sigma_{\text{ind}}$  is Hausdorff.

DEFINITION 7.

Let  $T_{\Phi(A)} := \{F \in G^+ : \forall_{\phi \in \Phi} [\phi(A)F \in \text{emb}(X)]\}$ .

The topology  $\tau_{\text{proj}}$  is the locally convex topology generated by the seminorms  $t_\phi$ ,  $\phi \in \Phi$ , defined by  $t_\phi(F) := \|\text{emb}^{-1}(\phi(A)F)\|$ , ( $F \in T_{\Phi(A)}$ ,  $\phi \in \Phi$ ).

There exists a characterisation of bounded sets in  $T_{\Phi(A)}$ .

THEOREM 8.

Let  $B \subset T_{\Phi(A)}$  be a set. Then  $B$  is bounded in  $(T_{\Phi(A)}, \tau_{\text{proj}})$  iff there exist  $f \in \Phi^+$  and a bounded set  $B_0 \subset X$  so that  $B = f(A) \cdot B_0$ .

Proof. See [EGK], Theorem 2.4.II. □

It follows that  $T_{\Phi(A)} = \bigcup_{f \in \Phi^+} f(A) \cdot X$ .

DEFINITION 9.

The topology  $\tau_{\text{ind}}$  for  $T_{\Phi(A)}$  is the inductive limit topology generated by the Hilbert spaces  $f(A) \cdot X$ ,  $f \in \Phi^+$ .

Further a duality between the spaces  $S_{\Phi(A)}$  and  $T_{\Phi(A)}$  can be introduced.

DEFINITION 10.

Define  $\langle , \rangle : S_{\Phi(A)} \times T_{\Phi(A)} \rightarrow \mathbb{C}$

$\langle \phi(A) \cdot x, F \rangle = (x, \text{emb}^{-1}(\phi(A)F))$  ( $\phi \in \Phi$ ,  $x \in E(\phi)(X)$ ,  $F \in T_{\Phi(A)}$ ).

(See [EGK], page 288.)

Note: For all  $f \in \Phi^+$ ,  $w \in S_{\Phi(A)}$  and  $x \in X$  holds:  $\langle w, f(A) \cdot x \rangle = (f(A) * w, x)$ .

THEOREM 11.

$\langle S_{\Phi(A)}, T_{\Phi(A)} \rangle$  is a dual pair and the topology  $\sigma_{\text{ind}}$  resp.  $\tau_{\text{proj}}$  is compatible with the dual pair  $\langle S_{\Phi(A)}, T_{\Phi(A)} \rangle$  resp.  $\langle T_{\Phi(A)}, S_{\Phi(A)} \rangle$ .

Proof. See [EGK], Theorem 3.1. □

2. The weak symmetry condition

Let  $\Phi$  be a set which satisfies Axiom 1. In Chapter IV of [EGK] the authors require the following strong symmetry condition on the set  $\Phi$ .

$$\text{AIV.} \quad \forall_{\zeta \in \Phi^{++}} \exists_{\phi \in \Phi} \exists_{c > 0} [\zeta \leq c \phi].$$

With this condition they prove several nice properties of the topological spaces  $(S_{\Phi(A)}, \sigma_{\text{ind}})$  and  $(T_{\Phi(A)}, \tau_{\text{proj}})$ . They note that the operator  $\zeta(A) \phi(A)^{-1} \chi_{\phi}(A)$  extends to a bounded operator on  $X$ . So the set  $\Phi$  satisfies the following weak symmetry condition.

$$\text{AIV'}. \quad \forall_{\zeta \in \Phi^{++}} \exists_{\phi \in \Phi} \exists_{c > 0} [\chi_{\{\lambda \in \mathbb{R}^n : \zeta(\lambda) > c \phi(\lambda)\}}(A) = 0].$$

A careful reading of their proofs shows that they use only condition AIV' to get the nice properties. The next theorem shows that the spaces  $S_{\Phi(A)}$  and  $T_{\Phi(A)}$  cannot have those nice properties without condition AIV'.

THEOREM 12.

Let  $\Phi$  be a set of Borel functions on  $\mathbb{R}^n$  which satisfies Axiom 1. The following conditions are equivalent.

- I.  $\Phi$  has property AIV'.
- II.  $(T_{\Phi(A)}, \tau_{\text{proj}}) = (S_{\Phi^+(A)}, \sigma_{\text{ind}})$  as topological vector spaces.

- III.  $(T_{\Phi(A)}, \tau_{\text{proj}}) = (T_{\Phi(A)}, \tau_{\text{ind}})$  as topological vector spaces.
- IV.  $(T_{\Phi(A)}, \tau_{\text{proj}})$  is bornological.
- V.  $(T_{\Phi(A)}, \tau_{\text{proj}})$  is barrelled.
- VI.  $(T_{\Phi(A)}, \tau_{\text{proj}})$  is quasibarrelled.
- VII.  $S_{\Phi(A)}$  is complete.
- VIII.  $S_{\Phi(A)}$  is sequentially complete.
- IX.  $S_{\Phi(A)} = \bigcap_{f \in \Phi^+} D_f$ .
- X.  $S_{\Phi(A)} = S_{\Phi^+(A)}$  as sets.
- XI. For every bounded set  $B \subset S_{\Phi(A)}$  there exist  $\phi \in \Phi$  and a bounded set  $B_0 \subset X$  so that  $\phi(A) \cdot |_{B_0} : B_0 \rightarrow B$  is a homeomorphism.

Proof.

I  $\Rightarrow$  II. Theorem 4.2.I of [EGK].

II  $\Rightarrow$  III. Always  $(S_{\Phi^+(A)}, \sigma_{\text{ind}}) = (T_{\Phi(A)}, \tau_{\text{ind}})$  holds.

III  $\Rightarrow$  I. Let  $\zeta \in \Phi^+$ . Define  $W : T_{\Phi(A)} \rightarrow X$  by  $W(f(A) \cdot x) := (\zeta f)(A)x$ ,  $f \in \Phi^+$ ,  $x \in X$ . Let  $f \in \Phi^+$ . Then  $\|W(f(A) \cdot x)\| \leq \|(\zeta f)(A)\| \|x\| = \|(\zeta f)(A)\| \|f(A) \cdot x\|_{f(A)X}$  for all  $x \in E(f)(X)$ . By definition of  $\tau_{\text{ind}}$ , the map  $W$  is continuous from  $(T_{\Phi(A)}, \tau_{\text{ind}})$  into  $X$ . By assumption, the map  $W$  is continuous from  $(T_{\Phi(A)}, \tau_{\text{proj}})$  into  $X$ , so there exist  $\phi \in \Phi$  and  $c > 0$  such that  $\|W(F)\| \leq t_\phi(F)$  for all  $F \in T_{\Phi(A)}$ . In particular,  
 $\|(\zeta \chi_{Q_m})(A)x\| = \|W(\chi_{Q_m}(A) \cdot x)\| \leq c \|(\phi \chi_{Q_m})(A)x\|$   
 for all  $x \in X$  and  $m \in \mathbb{Z}^*$ . So  $\chi_{\{\lambda \in \mathbb{R}^n : \zeta(\lambda) > c \phi(\lambda)\}}(A) = 0$ .

III  $\Rightarrow$  IV  $\Rightarrow$  VI and III  $\Rightarrow$  V  $\Rightarrow$  VI are trivial.

VI  $\Rightarrow$  III. Always  $\tau_{\text{proj}} \subset \tau_{\text{ind}}$ . Let  $\Omega \subset T_{\Phi(A)}$  be a  $\tau_{\text{ind}}$ -neighbourhood of 0. Because  $\tau_{\text{ind}}$  is regular, there exist absolutely convex  $\tau_{\text{ind}}$ -open  $\Omega_1 \subset T_{\Phi(A)}$  so that  $0 \in \Omega_1 \subset \overline{\Omega_1} \subset \Omega$ . Assertion:  $\Omega_1$  is a bornivore in  $(T_{\Phi(A)}, \tau_{\text{proj}})$ . Let  $B \subset T_{\Phi(A)}$  be a  $\tau_{\text{proj}}$ -bounded set. By Theorem 8 there exist  $f \in \Phi^+$  and a bounded set  $B_0 \subset X$  so that  $B_0 = f(A) \cdot B_0$ . Let  $M > 0$  be so that  $\|x\| \leq M$  for all  $x \in B_0$ . Since  $\Omega_1$  is  $\tau_{\text{ind}}$ -open, there exists  $\varepsilon > 0$  so that for all  $x \in X$ ,  $\|x\| < \varepsilon$  holds  $f(A) \cdot x \in \Omega_1$ . Then for all  $t \in \mathcal{C}$ ,  $|t| < \varepsilon M^{-1}$  we get  $tB \subset \Omega_1$ . This proves the assertion. Hence  $\overline{\Omega_1}$  is a bornivore barrel and by assumption a  $\tau_{\text{proj}}$ -neighbourhood of 0. So  $\tau_{\text{ind}} \subset \tau_{\text{proj}}$ .

I  $\Rightarrow$  VII. See [EGK], Corollary 4.3.III.

VII  $\Rightarrow$  VIII. Trivial.

VIII  $\Rightarrow$  IX. Always  $S_{\Phi(A)} \subset \bigcap_{f \in \Phi^+} D_f$ . Let  $F \in \bigcap_{f \in \Phi^+} D_f$ . For  $p \in \mathbb{N}$  let  $\Delta_p := \{\lambda \in \mathbb{R}^n : |\lambda| \leq p\}$ ,  $x_p := F(\Delta_p)$  and  $F_p := \chi_{\Delta_p}(A) \cdot x_p$ . Then  $F_p \in S_{\Phi(A)}$ . Assertion:  $(F_p)_{p \in \mathbb{N}}$  is a Cauchy sequence in  $S_{\Phi(A)}$ . Let  $f \in \Phi^+$  and  $\varepsilon > 0$ . There exists  $\Delta_0 \in B_b(\mathbb{R}^n)$  so that for all  $\Delta, \Delta' \in B_b(\mathbb{R}^n)$ ,  $\Delta \supset \Delta_0$ , and  $\Delta' \supset \Delta_0$  holds

$\|f(A)F(\Delta) - f(A)F(\Delta')\| \leq \varepsilon$ . Let  $p_0 \in \mathbb{N}$  be so that  $\Delta_{p_0} \supset \Delta_0$ . Let  $p \in \mathbb{N}$ ,  $p \geq p_0$ . For all  $\Delta \in B_b(\mathbb{R}^n)$ ,  $\Delta \supset \Delta_0$  we obtain  $\|f(A)F(\Delta) - f(A)F(\Delta_p)\| \leq \varepsilon$ , so  $\|f(A) * F - f(A) * F_p\| = \|f(A) * F - f(A)F(\Delta_p)\| \leq \varepsilon$ . So  $p \mapsto f(A) * F_p$  is a Cauchy sequence in  $X$  with limit  $f(A) * F$  and the assertion is proved (Theorem 6). Let  $F_0 \in S_{\Phi(A)}$  be the limit of the sequence  $(F_p)_{p \in \mathbb{N}}$ . Let  $\Delta \in B_b(\mathbb{R}^n)$ . Then  $\chi_\Delta \in \Phi^+$  and  $F_0(\Delta) = \chi_\Delta(A) * F_0 = \lim_{p \rightarrow \infty} \chi_\Delta(A) * F_p = \chi_\Delta(A) * F = F(\Delta)$ . So  $F = F_0 \in S_{\Phi(A)}$ .

IX  $\Rightarrow$  IV. Let  $W : T_{\Phi(A)} \rightarrow \mathcal{C}$  be a linear map which is bounded on  $\tau_{\text{proj}}$ -bounded sets. For all  $f \in \Phi^+$  the map  $x \mapsto W(f(A) \cdot x)$  from  $X$  into  $\mathcal{C}$  is bounded on bounded sets by Theorem 8, so this map is continuous. In particular: for every  $\Delta \in B_b(\mathbb{R}^n)$  there exists unique  $F(\Delta) \in X$  so that for all  $x \in X$  holds  $(x, F(\Delta)) = W(\chi_\Delta(A) \cdot x)$ . Then  $F \in G^+$ . Assertion:  $F \in \bigcap_{f \in \Phi^+} D_f$ . Let  $f \in \Phi^+$ . There exists  $y \in X$  so that for all  $x \in X$  holds  $W(f(A) \cdot x) = (x, y)$ . Let  $x \in X$ . Then  $\lim_{\Delta} (x, f(A)F(\Delta)) = \lim_{\Delta} (f(A)\chi_\Delta(A)x, F(\Delta)) = \lim_{\Delta} W(\chi_\Delta(A) \cdot f(A)\chi_\Delta(A)x) = \lim_{\Delta} W(f(A) \cdot \chi_\Delta(A)x) = \lim_{\Delta} (\chi_\Delta(A)x, y) = (x, y)$ . So weak  $\lim_{\Delta} f(A)F(\Delta) = y$ . But also  $\lim_{\Delta} \|f(A)F(\Delta)\| = \lim_{\Delta} \sup_{\|x\| \leq 1} |W(f(A) \cdot \chi_\Delta(A)x)| = \sup_{\|x\| \leq 1} |W(f(A) \cdot x)| = \|y\|$ . So strong  $\lim_{\Delta} f(A)F(\Delta) = y$ . Hence  $F \in D_f$  and  $y = f(A) * F$ . So  $F \in \bigcap_{f \in \Phi^+} D_f = S_{\Phi(A)}$ . Let  $H \in T_{\Phi(A)}$ . There are  $f \in \Phi^+$  and  $x \in X$  so that  $H = f(A) \cdot x$ . Then  $W(H) = (x, f(A) * F) = \langle \overline{F}, f(A) \cdot x \rangle = \langle \overline{F}, H \rangle$ . By Theorem 11 it follows that  $W$  is continuous.

IX  $\Leftrightarrow$  X. By equivalence of I and IX:  $S_{\Phi(A)} \subset S_{\Phi^{++}(A)} = \bigcap_{f \in \Phi^{++}} D_f = \bigcap_{f \in \Phi^+} D_f$ .

I  $\Rightarrow$  XI. See [EGK], Corollary 4.3.IV.

XI  $\Rightarrow$  VIII. Let  $w_1, w_2, \dots$  be a Cauchy sequence in  $S_{\Phi(A)}$ . Then  $\{w_n : n \in \mathbb{N}\}$  is bounded, so there exist  $\phi \in \Phi$  and a Cauchy sequence  $x_1, x_2, \dots$  in  $X$  so that  $w_n = \phi(A) \cdot x_n$ ,  $n \in \mathbb{N}$ . Let  $x := \lim_{n \rightarrow \infty} x_n$ . Then  $\lim_{n \rightarrow \infty} w_n = \phi(A) \cdot x$  in  $S_{\Phi(A)}$ .  $\square$

Remark: It is trivial by now that property AIV' is equivalent with  $(T_{\Phi(A)}, \tau_{\text{proj}})$  is reflexive and also with  $(S_{\Phi(A)}, \sigma_{\text{ind}}) = (T_{\Phi^+(A)}, \tau_{\text{proj}})$  as topological vector spaces. If  $(T_{\Phi(A)}, \tau_{\text{proj}})$  happens to be metrizable, then property AIV' holds.



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