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MATHEMATICAL ADDENDA TO HOPPER'S MODEL OF PLANE STOKES FLOW DRIVEN BY CAPILLARITY ON A FREE SURFACE

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Introduction

In an interesting and stimulating series of papers [H1], [H2], [H3], Hopper presents some special exact solutions of the shape evolution of a piece of viscous matter driven by surface tension on the free boundary.
Hopper's paper [H1] is of a conceptual nature and consists of two parts. In his first part Hopper derives an evolution equation for the change of shape in time: The unknown function in this evolution equation is a Riemann mapping function from the unit disc onto the region occupied by the fluid at time t. Hopper's evolution equation is a partial differential equation of a very special nature, requiring 'compensation of analytic singularities'. In [H1] we find, what might be called, a pseudo Lagrangian description of the piece of matter and several other innovative concepts. However, a lot of important mathematical and physical details are missing in [H1]. In my view e.g. the kinematical aspects are completely neglected in [H1] (and also in [R]).
Chapter 1 in my paper might well be called: 'Mathematical addenda to Hopper's derivation of Hopper's equation'.
In the second part of [H1] and also in [H2], [H3], Hopper finds solutions of his equation which are of type $\Omega(z, \lambda(t))$. He makes a clever guess of a parametrized set of analytic functions $\Omega(z; \lambda)$, such that substitution of them in the evolution equation leads to one ordinary differential equation for $\lambda(t)$.
In Chapter 2 of this paper I study several mathematical aspects of Hopper's equation. On the 'state space', which is a part of an ellipsoid in Hilbert space, Hopper's evolution equation can be considered as an infinite system of ordinary differential equations. For this system there are 3 'exhausting' series of finite dimensional sub systems leading to solutions which are: 1. Complex polynomials with real coefficients, 2. Complex polynomials with complex coefficients, 3. Rational functions. Some local results on these finite dimensional sub systems are presented.
For numerical solutions to the same problem which use Lorentz- Ladyzhenskaja potentials I refer to work being done in Eindhoven [VM1], [VM2], [VM3].
I wish to thank Dr. H.K. Kuiken of Philips Research Laboratories for drawing my attention to these interesting problems.

1 A shape–evolution equation

1.1 Formulation of a Stokes problem with a free boundary

On a simply connected open domain $G_t \subset I\!R^2$ with a smooth boundary ∂G_t we consider the system of Navier–Stokes equations for the unknown velocity field $\underline{v}(\underline{x},t) = (v_1(\underline{x},t), v_2(\underline{x},t))$, $\underline{x} = (x,y)$ and the unknown pressure $p(\underline{x},t)$,

$$\left.\begin{aligned} &\rho\,\frac{D\underline{v}}{dt} = \rho\,\frac{\partial \underline{v}}{\partial t} + \rho(\underline{v}.\nabla)\underline{v} = -\nabla p + \eta\Delta\underline{v} + \rho\underline{g} \\ &\nabla.\underline{v} = 0\ , \end{aligned}\right\}\ (x,y) \in G_t,\ t > 0\ ,$$

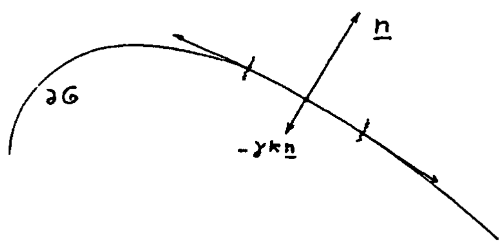
with the boundary condition

$$T\underline{n} = -\gamma(\nabla.\underline{n})\underline{n} = -\gamma\kappa\underline{n} \quad \text{on } \partial G_t\ .$$

Here T is the stress–tensor (= stress–matrix)

$$T = -pI + \eta\Big[\Big(\frac{d\underline{v}}{d\underline{x}}\Big) + \Big(\frac{d\underline{v}}{d\underline{x}}\Big)^T\Big]\ .$$

Further, $\underline{n}(\underline{x})$ and $\kappa(\underline{x})$ are the outward normal and the curvature at points $\underline{x} \in \partial G_t$.
The relevant physical constants are: The density ρ $[ML^{-3}]$, the viscosity η $[ML^{-1}T^{-1}]$ and the surface tension γ $[MT^{-2}]$.
Note that with this boundary condition the surface is supposed to behave like a membrane.



Next we introduce dimensionless quantities. Put

$$\underline{v} = \frac{\gamma}{\eta}\,\tilde{\underline{v}} \qquad \underline{x} = R\tilde{\underline{x}} \qquad t = \frac{\eta R}{\gamma}\,\tilde{t}$$

$$p = \frac{\gamma}{R}\,\tilde{p} \qquad \kappa = \frac{\tilde{\kappa}}{R} \qquad \iint_G d\sigma = \pi R^2\ .$$

Then the Navier–Stokes system becomes

$$\left.\begin{aligned} &S\,\frac{D\tilde{\underline{v}}}{d\tilde{t}} = -\tilde{\nabla}\tilde{p} + \tilde{\Delta}\tilde{\underline{v}} + \frac{R^2\rho}{\gamma}\,\underline{g} \\ &\tilde{\nabla}.\tilde{\underline{v}} = 0 \end{aligned}\right\}\ \tilde{\underline{x}} \in \tilde{G}_t$$

$$\tilde{T}\underline{n} = -\tilde{\kappa}\underline{n} \qquad \text{on } \partial\tilde{G}_t$$

with

$$\tilde{T}_{ij} = \frac{R}{\gamma}\, T_{ij} = -\tilde{p}\delta_{ij} + \left(\frac{\partial \tilde{v}_i}{\partial \tilde{x}_j} + \frac{\partial \tilde{v}_j}{\partial \tilde{x}_i}\right) .$$

If the Suratmannumber $S = \frac{\rho\gamma}{\eta^2}\, gR$ and the Bondnumber $B = \frac{R^2\rho}{\gamma}$ are very small, e.g. if R is very small, it suffices to solve Stokes' equations on G_t. (We omit the tilde $\sim$)

$$\left.\begin{aligned} \Delta\underline{v} &= \nabla p \\ \nabla.\underline{v} &= 0 \end{aligned}\right\} \quad \text{in } G_t$$

$$T\underline{n} = -\kappa\underline{n} \qquad \text{on } \partial G_t \ .$$

An equivalent formulation is

$$\left.\begin{aligned} \nabla T = \partial_i T_{ij} &= 0 \\ \nabla.\underline{v} &= 0 \end{aligned}\right\} \quad \text{in } G_t$$

$$T\underline{n} = -\kappa\underline{n} \qquad \text{on } \partial G_t \ .$$

1.2 The general solution of Stokes' equations

In this section we want to describe the general solution of the Stokes system on a fixed, simply connected open domain $G \subset I\!R^2$

$$\frac{\partial^2 v_1}{\partial x^2} + \frac{\partial^2 v_1}{\partial y^2} = \frac{\partial p}{\partial x}$$

$$\frac{\partial^2 v_2}{\partial x^2} + \frac{\partial^2 v_2}{\partial y^2} = \frac{\partial p}{\partial y}$$

$$\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} = 0 \ .$$

Suppose that the pair $(\underline{v}, p)$ solves this system on G and let T be stress tensor obtained from this solution.
Then because of the simple connectedness of G there exists a 'streamfunction' ψ and an 'Airy function' ϕ such that

$$\underline{v} = (v_1, v_2) = (\psi_y \ , -\psi_x)$$

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} = \begin{pmatrix} -\phi_{yy} & \phi_{xy} \\ \phi_{xy} & -\phi_{xx} \end{pmatrix} .$$

The latter can be argued as follows: Since $\nabla \cdot T = 0$ the stress tensor T must be of the form

$$T = \begin{pmatrix} f_y & g_y \\ -f_x & -g_x \end{pmatrix} .$$

The symmetry of T then requires $-f_x = g_y$ which says $\nabla \cdot (f, g) = 0$. Hence $(f, g) = (-\phi_y, \phi_x)$ for some function ϕ. Note that, if $\underline{v}$ is given, the streamfunction ψ is determined up to a constant C and the Airy function is determined up to a linear function $Ax + By + C_1$.
Taking the trace of T we find

$$p = \tfrac{1}{2}(\phi_{xx} + \phi_{yy}) = \tfrac{1}{2}\Delta\phi .$$

Combining this with the equation $\Delta\underline{v} = \nabla p$ we find the Stokes equations in Cauchy–Riemann form

$$\frac{\partial}{\partial x}\,(\tfrac{1}{2}\Delta\phi) - \frac{\partial}{\partial y}\,(\Delta\psi) = 0$$

$$\frac{\partial}{\partial y}\,(\tfrac{1}{2}\Delta\phi) + \frac{\partial}{\partial x}\,(\Delta\psi) = 0 .$$

So $(\frac{1}{2}\Delta\phi) + i(\Delta\psi)$ is an analytic function on G, therefore $\Delta\Delta\phi = 0$ and $\Delta\Delta\psi = 0$, so the functions ϕ and ψ are biharmonic.
Any biharmonic function ϕ on a simply connected domain G can be represented as

$$\phi = 2Re(\overline{z} f_1 + g_1) \;, \quad z = x + iy$$

with f_1 and g_1 analytic on G. Cf. [M], pp. 106-111.
Following the same reasoning we put

$$\psi = Im(\overline{z} f_2 + g_2)$$

with f_2 and g_2 analytic on G.
From the Cauchy–Riemann representation of Stokes' equations it follows that $f_1'' = f_2''$. Consistency in the stress tensor requires $g_1'' = g_2''$. So there are constants $A, B, D, E \in \mathbb{R}$ and $C, F \in \mathbb{C}$ such that

$$f_1 = f_2 + Az + iBz + C \;, \quad g_1 = g_2 + Dz + iEz + F .$$

Define

$$\varphi = f_2 + Az \;, \quad \chi = g_2 .$$

Then

$$\psi = Im(\overline{z}\varphi + \chi)$$

$$\phi = 2Re(\overline{z}\varphi + \overline{z}(iBz + C) + \chi + (D + iE)z + F)$$

$$= 2Re(\overline{z}\varphi + \chi) + 2Re(\overline{z}C + (D + iE)z + F) .$$

Omission of the second term leads to the same stress tensor. Summarizing: The state of the system is described by the analytic functions φ and χ and

$$\tfrac{1}{2}\,\phi + i\psi = \bar{z}\varphi + \chi\,.$$

Note that φ is uniquely determined by the state $(\underline{v}, p)$ and that addition of a complex constant to χ leads to the same state. Conversely, any pair of analytic functions φ and χ leads to a solution of Stokes' equations.

1.3 Kinematic and Dynamic Quantities expressed in φ and χ

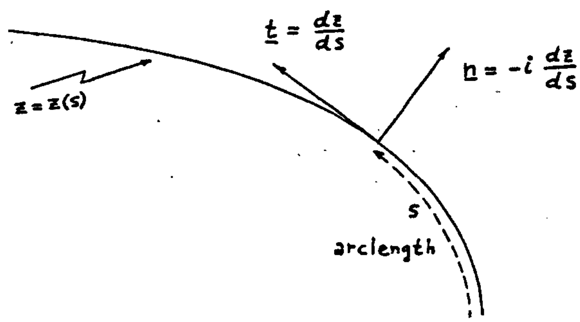
In a straightforward way the velocity field $\underline{v}(\underline{x}) = (v_1(x,y), v_2(x,y))$ and the stress tensor field $T(\underline{x}) = [T_{ij}(x,y)]$ can be expressed in the analytic potentials φ and χ. Write $z = x + iy$.

- $$\begin{aligned} v_1 + iv_2 = \psi_y - i\psi_x &= \frac{\partial}{\partial y}\, Im(\bar{z}\varphi + \chi) - i\,\frac{\partial}{\partial x}\, Im(\bar{z}\varphi + \chi) \\ &= Im(-i\varphi + i\bar{z}\varphi' + i\chi') - i\; Im(\varphi + \bar{z}\varphi' + \chi') \\ &= Re(-\varphi + \bar{z}\varphi' + \chi') - i\; Im(\varphi + \bar{z}\varphi' + \chi') \\ &= -Re\;\varphi - i\; Im\;\varphi + \overline{z\varphi' + \chi'} \\ &= -\varphi + z\overline{\varphi'} + \overline{\chi'} \end{aligned}$$

- $$\begin{aligned} T_{11} + T_{22} = -2p = -\Delta\phi &= -2\Delta Re(\bar{z}\varphi + \chi) \\ &= -8\; Re\;\varphi' = -4(\varphi' + \overline{\varphi'}) \end{aligned}$$

- $$\begin{aligned} T_{22} - T_{11} + i2T_{12} &= -\phi_{xx} + \phi_{yy} + 2i\phi_{xy} \\ &= (-2\frac{\partial^2}{\partial x^2} + 2\,\frac{\partial^2}{\partial y^2}) Re(\bar{z}\varphi + \chi) + 4i\,\frac{\partial^2}{\partial x \partial y}\; Re(\bar{z}\varphi + \chi) \\ &= 2\,\frac{\partial v_2}{\partial y} - 2\,\frac{\partial v_1}{\partial x} + 2i\phi_{xy} = -4\psi_{xy} + 2i\phi_{xy} \\ &= 4i\,\frac{\partial^2}{\partial x \partial y}\,[i\; Im(\bar{z}\varphi + \chi) + Re(\bar{z}\varphi + \chi)] = -4(\bar{z}\varphi'' + \chi'')\,. \end{aligned}$$

Stress orthogonal to a given curve



$$T\underline{n} = \begin{pmatrix} -\frac{1}{2}\Delta\phi + 2\psi_{yx} & \phi_{xy} \\ \phi_{xy} & -\frac{1}{2}\Delta\phi - 2\psi_{xy} \end{pmatrix} \begin{pmatrix} Im\ \dot{z} \\ -Re\ \dot{z} \end{pmatrix} =$$

$$= \tfrac{1}{2}\Delta\phi(-Im\ \dot{z} + i\ Re\ \dot{z}) + (2\psi_{yx} Im\ \dot{z} - \phi_{xy} Re\ \dot{z}) +$$

$$+ i(\phi_{xy} Im\ \dot{z} + 2\psi_{xy} Re\ \dot{z})$$

$$= \tfrac{1}{2} i(\Delta\phi)\dot{z} - 2\overline{(\tfrac{1}{2}\phi_{xy} + i\psi_{xy})\dot{z}}$$

$$= i\{\Delta\ Re(\bar{z}\varphi + \chi)\}\dot{z} - 2\{\frac{\partial^2}{\partial x \partial y}\ \overline{(\tfrac{1}{2}\phi + i\psi)}\}\bar{\dot{z}}$$

$$= 2i\{(\varphi' + \overline{\varphi'}\dot{z}) + (z\overline{\varphi''} + \overline{\chi''})\dot{z}\}$$

$$= 2i\ \frac{d}{ds}\ (z\overline{\varphi'} + \varphi + \overline{\chi'})\ .$$

Note that if we replace φ by $\varphi_1 = \varphi + C + i\beta z$ with $C \in \mathbb{C}$, $\beta \in \mathbb{R}$ and keep the same χ than a rigid motion is added to the velocity field $v_1 + iv_2$. However this modification does not affect T and $T\underline{n}$.

1.4 A road to Hopper's equation

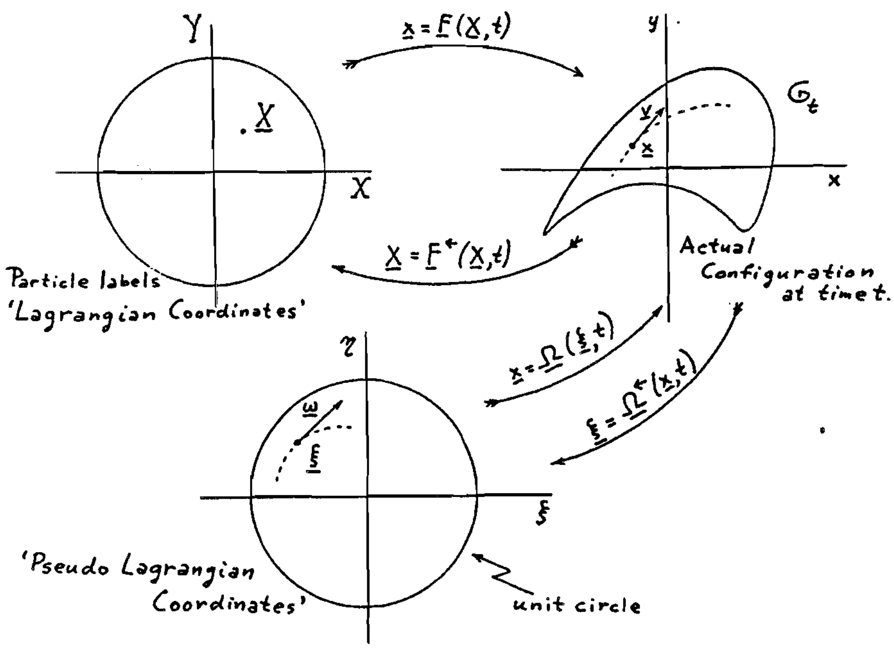
In continuum mechanics there are two conventional ways of describing the motion of matter. In the *Lagrangian* description each matter particle gets its own label, $\underline{X}$ say, and one wants to find the position $\underline{x}$ of each particle $\underline{X}$ as a function of time, i.e. one looks for the function $\underline{x} = \underline{F}(\underline{X}, t)$.
On the other hand, users of the *Eulerian* description are not so much interested in the position of each particle. In the Eulerian description one wants to calculate the velocity field

$$\underline{v}(\underline{x},t) = \dot{\underline{F}}(\underline{F}^{\leftarrow}(\underline{x},t),t)$$

with

$$\dot{\underline{F}}(\underline{x},t) = \frac{\partial}{\partial t}\,\underline{F}(\underline{x},t)\;.$$

In this paper we want to determine the evolution of the shape of a piece of matter and the positions $\underline{F}(\underline{x},t)$ of the particular and the velocity fields $\underline{v}(\underline{x},t)$ are not so relevant.
Instead of the Lagrangian or Eulerian approach we use what we call the "Pseudo–Lagrangian picture": At each time t a fixed domain D in $\underline{\xi}$-space is mapped by a function $\underline{x} = \underline{\Omega}(\underline{\xi},t)$ onto the actual configuration of the piece of matter. The function Ω is made 'more or less rigid' by requiring extreme smoothness of it. In our 2 dimensional case, following Hopper [H1], we require it to be analytic.



We now gather some convenient kinematical expressions.
The trajectory of particle $\underline{X}$ in configuration space is

$$t \mapsto \underline{x}(t) = \underline{F}(\underline{X},t)\;.$$

The trajectory of particle $\underline{X}$ in Pseudo–Lagrangian coordinates is

$$t \mapsto \underline{\xi}(\underline{X},t) = \Omega^{\leftarrow}(\underline{F}(\underline{X},t),t)\;.$$

Differentiating the identity

$$\underline{\Omega}^{\leftarrow}(\underline{\Omega}(\underline{\xi},t),t) = \underline{\xi}$$

according to t leads to

$$(D\underline{\Omega}^{\leftarrow})(\underline{\Omega}(\underline{\xi},t))\dot{\underline{\Omega}}(\underline{\xi},t) + (\underline{\Omega}^{\leftarrow})^{\bullet}(\underline{\Omega}(\underline{\xi},t),t) = \underline{0}\ .$$

Here the dot $\cdot$ denotes partial differentiation to t. So

$$(\Omega^{\leftarrow})^{\bullet}(\underline{x},t) = -(D\Omega)^{-1}(\Omega^{\leftarrow}(\underline{x},t))\dot{\Omega}(\underline{\Omega}^{\leftarrow}(\underline{x},t),t)\ .$$

For the velocity field in configuration space we find

$$\underline{v}(\underline{x},t) = \dot{\underline{F}}(\underline{F}^{\leftarrow}(\underline{x},t),t)\ .$$

Since

$$\frac{\partial\underline{\xi}(\underline{X},t)}{\partial t} = (D\underline{\Omega}^{\leftarrow})(\underline{F}(\underline{X},t),t)\dot{\underline{F}}(\underline{X},t) + (\underline{\Omega}^{\leftarrow})^{\bullet}(\underline{F}(\underline{X},t),t)$$

we find for the velocity field in Pseudo–Lagrangian coordinates

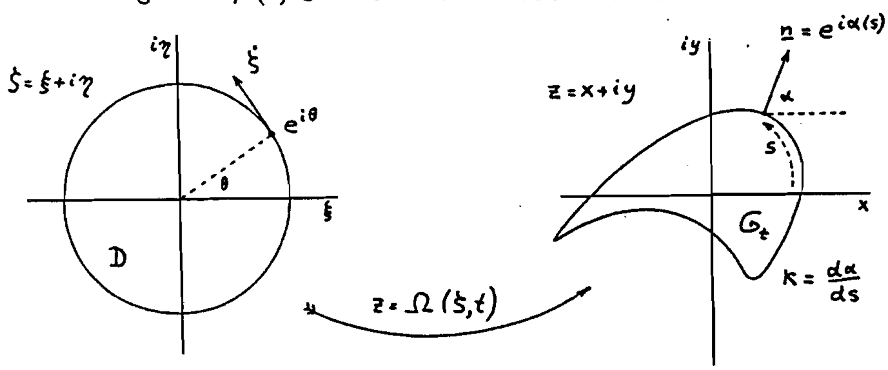
$$\underline{\omega}(\underline{\xi},t) = (D\Omega^{\leftarrow})(\underline{\Omega}(\underline{\xi},t),t)\underline{v}(\underline{\Omega}(\underline{\xi},t),t) + (\Omega^{\leftarrow})^{\bullet}(\underline{\Omega}(\underline{\xi},t),t)$$

$$= (D\underline{\Omega})^{-1}(\underline{\xi},t)[\underline{v}(\underline{\Omega}(\underline{\xi},t),t) - \dot{\underline{\Omega}}(\underline{\xi},t)]\ .$$

Our ultimate goal is to calculate

$$\underline{\Omega}(\underline{\xi},t) \quad \text{for } |\underline{\xi}| = 1$$

which represents the shape of our piece of matter.
Now suppose that at time t the fluid occupies a domain $G_t \subset I\!R^2$. Fix a point in G_t and choose $x + iy$ coordinates such that this point becomes the origin. Introduce a conformal mapping $\Omega :\ D \to G_t$, with $z = \Omega(\zeta,t)$, $\zeta = \xi + i\eta$. D is the unit disc in the complex ζ plane and $\Omega(0,t) = 0$. Note that Ω is uniquely determined if we require $\Omega'(0,t) > 0$. Suppose further that at time t the state $(\underline{v},p)$ of the Stokes system is described by the complex analytic 'potentials' φ and χ. If necessary we may add a uniform rotation velocity field to $\underline{v}(\underline{x},t)$ in order to arrange that $\varphi'(0) \in I\!R$. Cf. the remark at the end of section 1.3.



At the boundary ∂G_t of G_t, see picture, we have the boundary condition $T\underline{n} = -\kappa\underline{n}$.
With the potentials φ and χ this becomes, in complex notation,

$$2i \frac{d}{ds}(z\overline{\varphi'} + \varphi + \overline{\chi'}) = i \frac{d}{ds} e^{i\alpha(s)} ,$$

which leads to

$$2(z\overline{\varphi'} + \varphi + \chi') = e^{i\alpha(s)} + C_1 , \quad \text{at } \partial G_t .$$

The constant C_1 can be made zero by addition of a suitable constant velocity field to $\underline{v}(\underline{x}, t)$.
Then combination at ∂G_t of the latter result with

$$v_1 + iv_2 = (-\varphi + z\overline{\varphi'} + \overline{\chi'})$$

yields

$$2(v_1 + iv_2) = -4\varphi + e^{i\alpha(s)} , \quad \text{at } \partial G_t .$$

With the parametrization $\Omega(e^{i\theta}, t)$ for ∂G_t this becomes

$$2v_1(\Omega(e^{i\theta}, t), t) + 2iv_2(\Omega(e^{i\theta}, t), t) = -4\varphi(\Omega(e^{i\theta}, t), t) + \frac{e^{i\theta}\Omega'(e^{i\theta}, t)}{|\Omega'(e^{i\theta}, t)|} .$$

Rewrite

$$\underline{\omega} = (D\underline{\Omega})^{-1}\underline{v} - (D\underline{\Omega})^{-1}\dot{\underline{\Omega}}$$

in the complex ζ-plane

$$\dot{\zeta}(\zeta, t) = \frac{v_1(\Omega(\zeta, t), t) + iv_2(\Omega(\zeta, t), t)}{\Omega'(\zeta, t)} - \frac{\dot{\Omega}(\zeta, t)}{\Omega'(\zeta, t)} .$$

At ∂D we have $\zeta = e^{i\theta}$, $\theta \in I\!R$, and

$$Re\ e^{-i\theta}\dot{\zeta}(e^{i\theta}, t) = 0 .$$

With $e^{i\theta} = \sigma$ and $\tilde{\varphi}(\zeta, t) = \varphi(\Omega(\zeta, t), t)$ we arrive at

$$\frac{2\tilde{\varphi}(\sigma, t)}{\sigma\Omega'(\sigma, t)} + \frac{\dot{\Omega}(\sigma, t)}{\sigma\Omega'(\sigma, t)} = \frac{1}{2|\Omega'(\sigma, t)|} - \frac{\dot{\zeta}(\sigma, t)}{\sigma} .$$

We now make the important observation that the two terms on the right hand side are the respective real and imaginary parts of an analytic function $\mathcal{F}(|\Omega'(\zeta, t)|)$ restricted to the boundary ∂D. This analytic function $\mathcal{F}$ is uniquely defined by

$$Re\ \mathcal{F}(|\Omega'(\zeta, t)|) = \frac{1}{2|\Omega'(\zeta, t)|} , \quad \zeta \in \partial D$$

$$Im\ \mathcal{F}(|\Omega'(0, t)|) = 0 .$$

Summarizing, we find on D the relation

$$2\tilde{\varphi}(\zeta, t) = \zeta\Omega'(\zeta, t)\ \mathcal{F}(|\Omega'(\zeta, t)|) - \dot{\Omega}(\zeta, t) .$$

We now proceed to derive an evolution equation for Ω in which the unknown complex analytic potentials φ and χ play no role.
In $2(z\overline{\varphi'} + \varphi + \overline{\chi'})$ on G_t substitute $z = \Omega(\zeta, t)$ and put $\tilde{\chi}(\zeta, t) = \chi(\Omega(\zeta, t), t)$, $\tilde{\varphi}(\zeta, t) = \varphi(\Omega(\zeta, t), t)$. At the boundary ∂D this leads to

$$2\Big(\Omega\,\frac{\overline{\tilde{\varphi}'}}{\overline{\Omega}'}+\tilde{\varphi}+\frac{\overline{\tilde{\chi}'}}{\overline{\Omega}'}\Big)=\frac{\sigma\Omega')\sigma)}{|\Omega'(\sigma)|}\,,\quad \sigma\in\partial D\,.$$

Suppress $\sim$ and substitute $2\varphi=\zeta\Omega'\mathcal{F}-\dot{\Omega}$ and its derivative. Then, at ∂D

$$-\zeta\Omega'\overline{\Omega'}(\mathcal{F}-\frac{1}{|\Omega'|})+\frac{d}{dt}(\Omega\overline{\Omega'})-\Omega\overline{(\zeta\Omega'\mathcal{F})'}=2\overline{\chi'}\,.$$

After complex conjugation and writing $\zeta=\sigma,\ |\sigma|=1$,

$$\overline{\sigma}\,\overline{\Omega'}\,\Omega'\,\mathcal{F}-\overline{\Omega}(\sigma\Omega'\mathcal{F})'+\frac{d}{dt}(\overline{\Omega}\,\Omega')=2\chi'\,.$$

With $\sigma=e^{i\theta}$ and $\dfrac{d}{dz}=-ie^{-i\theta}\dfrac{d}{d\theta}$ this becomes

$$ie^{-i\theta}\,\frac{d}{d\theta}(e^{i\theta}\,\overline{\Omega}\,\Omega'\,\mathcal{F})+\frac{d}{dt}(\overline{\Omega}\,\Omega')=2\chi'\quad\text{on }\partial D\,.$$

Hopper [H1], writes $\frac{d}{d\sigma}=-ie^{-i\theta}\,\frac{d}{d\theta}$ for 'differentiation along the unit circle'. Then

$$\frac{d}{dt}\,(\overline{\Omega}\,\Omega')-\frac{d}{d\sigma}(\sigma\overline{\Omega}\,\Omega'\,\mathcal{F}(|\Omega'|))=2\chi'$$

which is Hopper's evolution equation for the shape of a piece of viscous matter driven by surface tension.

2 Some mathematical analysis on Hopper's equation

2.1 Mathematical generalities on Hopper's evolution equation

On the closed unit disk $\overline{D}\subset C$ we look for solutions $\Omega(\zeta,t),\ \zeta\in\overline{D},\ t\geq 0$ of the evolution equation

$$\text{(H)}\qquad \frac{d}{dt}(\overline{\Omega}\,\Omega')-\frac{d}{d\zeta}(\zeta\overline{\Omega}\,\Omega'\,\mathcal{F}(|\Omega'|))=-2\chi'=\ \text{analytic on }D\,.$$

Solutions Ω are required to be (at least) analytic on D and continuous on $\overline{D}$. Remind that, by definition,

$$\overline{\Omega}(\zeta,t)=\overline{\Omega(\frac{1}{\overline{\xi}}\,,t)}$$

and also that $\mathcal{F}(|\Omega'|;\zeta)$ is analytic on D and uniquely defined by

$$\text{(F)}\qquad \begin{cases} Re\ \mathcal{F}(|\Omega'(\zeta,t)|)=\dfrac{1}{2|\Omega'(\zeta,t)}|\,,\quad \zeta\in\partial D\\[2ex] Im\ \mathcal{F}(|\Omega'(0,t)|)=0\end{cases}\qquad.$$

The righthand side $-2\chi'$ of the evolution equation is an unknown analytic function. Therefore the question:
Does the cancellation of singularities inside D determine the shape evolution?

Note that if $\Omega(\zeta,t)$ solves (H) then also $e^{i\varphi(t)}\Omega(\zeta,t)$, $\varphi:\ \mathbb{R}\to\mathbb{R}$ arbitrary, is a solution. This type of nonuniqueness can be resolved by requiring $\Omega'(0,t)>0$.

DEFINITION. (Set of states $\Sigma\subset\overline{\Sigma}$)

$\Sigma=\{\Omega\}$, with

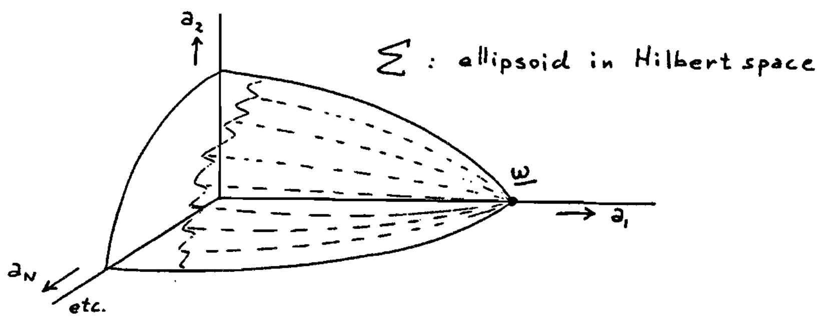
- $$\Omega(\zeta)=\sum_{n=1}^{\infty} n\zeta^n\ ;\quad \overline{D}\to\mathbb{C}\ ,\quad \text{analytic}$$

- $$\sum_{n=1}^{\infty} n|a_n|^2=1\ ,\quad \text{means: Area }\Omega(\overline{D})=\pi$$

- $$\forall\zeta,\ |\zeta|\le 1\qquad \Omega'(\zeta)\neq 0\ .$$

$\Omega\in\overline{\Sigma}$ means: Ω is analytic on D, Ω is continuous on $\overline{D}$ and $\Omega'(\zeta)\neq 0$ for $|\zeta|<1$.

Note that Σ is a part of an ellipsoid in the Hilbert space.



Calculate

$$\Omega'\overline{\Omega}=\sum_{k\ge 0,\ \ell\ge 0} k a_k\overline{a}_\ell\zeta^{k-\ell-1}=$$

$$=\sum_{m=-\infty}^{\infty}\left\{\sum_{\ell=\max(0,m)}^{\infty}(\ell-m+1)\overline{a}_\ell a_{\ell-m+1}\right\}\zeta^{-m}=$$

$$=\sum_{m=-\infty}^{\infty} u_m\zeta^{-m}\ .$$

Note:

$$m \geq 1 \quad \Rightarrow \quad u_m = \sum_{\ell=m}^{\infty} (\ell - m + 1)\overline{a}_\ell a_{\ell-m+1} \; .$$

There are two matrix forms for this expression:

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ \vdots \end{pmatrix} = \begin{pmatrix} \overline{a}_1 & 2\overline{a}_2 & 3\overline{a}_3 & 4\overline{a}_4 & \cdots \\ \overline{a}_2 & 2\overline{a}_3 & 3\overline{a}_4 & \cdot & \cdots \\ \overline{a}_3 & 2\overline{a}_4 & \cdot & \cdot & \cdots \\ \overline{a}_4 & \cdot & \cdot & \cdot & \cdots \\ \vdots & \vdots & \vdots & \vdots & \cdots \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_3 \\ \vdots \end{pmatrix} =$$

$$= \begin{pmatrix} a_1 & 2a_2 & 3a_3 & 4a_4 & \cdots \\ 0 & a_1 & 2a_2 & 3a_3 & \cdots \\ 0 & 0 & a_1 & 2a_2 & \cdots \\ 0 & 0 & 0 & a_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \cdots \end{pmatrix} \begin{pmatrix} \overline{a}_1 \\ \overline{a}_2 \\ \overline{a}_3 \\ \overline{a}_4 \\ \vdots \end{pmatrix} .$$

In short $\underline{u} = N(\underline{\overline{a}})\underline{a} = M(\underline{a})\underline{\overline{a}}$.
Note that if $\Omega(\zeta,t) = \sum_{n=1}^{\infty} a_n(t)\zeta^n$ satisfies (H) then

$$\frac{d}{dt} \int\limits_{|z|=1} \overline{\Omega}(z)\Omega'(z)dz = \frac{d}{dt} \sum_{\ell=1}^{\infty} \ell |a_\ell(t)|^2 = \frac{d}{dt}\, u_1(t) = 0 \; .$$

Hence

$$\pi \sum_{\ell=1}^{\infty} \ell |a_\ell(t)^2 = \text{constant} = \pi \; ,$$

which means 'conservation of area'.

In the next theorem we gather some results on a Taylor series representation of $\mathcal{F}(|\Omega'|)$.

THEOREM.

- For all $\Omega \in \Sigma$ the function

$$\tfrac{1}{2}[\Omega'(\zeta)\overline{\Omega'(\tfrac{1}{\overline{\zeta}})}]^{-\frac{1}{2}} = \sum_{n=-\infty}^{\infty} \alpha_n \zeta^n$$

is analytic and single valued on an annulus $\frac{1}{R} < |\zeta| < R, \; R > 1$.

- Define

$$\mathcal{F}(|\Omega'(\zeta,t)|) = \alpha_0 + 2\sum_{n=1}^{\infty} \alpha_n\zeta^n = \sum_{n=0}^{\infty} \beta_n \zeta^n$$

then $\mathcal{F}$ satisfies the conditions (F)

- $$\alpha_n = \frac{1}{4\pi}\int_0^{2\pi}\left\{\sum_{m=1}^{\infty}\sum_{\ell=1}^{\infty} m\ell\; a_m\overline{a}_\ell e^{i(m-\ell)s}\right\}^{-\frac{1}{2}} e^{-ins}ds$$

- $$\alpha_0 = \beta_0 > 0 \qquad \alpha_{-n} = \overline{\alpha_n}$$

- $$\{a_m\} \subset \mathbb{R} \quad \Rightarrow \quad \{\alpha_n\} \subset \mathbb{R}\,. \qquad \square$$

We now calculate the second term in (H)

$$\frac{d}{d\zeta}(\overline{\zeta\Omega(\tfrac{1}{\overline{\zeta}})}\;\Omega'(\zeta)\;\mathcal{F}(|\Omega'(\zeta)|) =$$

$$\frac{d}{d\zeta}\{\zeta\sum_{m=-\infty}^{\infty} u_m\zeta^{-m}\cdot\sum_{n=0}^{\infty}\beta_n\zeta^n\} =$$

$$\sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty}(n-m+1)\beta_n u_m\zeta^{n-m} = \sum_{k=-\infty}^{\infty}(-k+1)\Big[\sum_{n=0}^{\infty}\beta_n u_{n+k}\Big]\zeta^{-k}\,.$$

The singular part is

$$\sum_{k=1}^{\infty}(-k+1)\Big[\sum_{n=0}^{\infty}\beta_n u_{n+k}\Big]\zeta^{-k}\,.$$

Now the condition of cancellation of singularities leads to the following infinite system of ordinary differential equations

$$\frac{d}{dt}\begin{pmatrix} a_1 & 2a_2 & 3a_3 & 4a_4 & \cdots \\ 0 & a_1 & 2a_2 & 3a_3 & \cdots \\ 0 & 0 & a_1 & 2a_2 & \cdots \\ 0 & 0 & 0 & a_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \cdots \end{pmatrix}\begin{pmatrix}\overline{a}_1\\ \overline{a}_2\\ \overline{a}_3\\ \overline{a}_3\\ \vdots\end{pmatrix} =$$

$$-\begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 0 & \beta_0 & \beta_1 & \beta_2 & \cdots \\ 0 & 0 & 2\beta_0 & 2\beta_1 & \cdots \\ 0 & 0 & 0 & 3\beta_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \cdots \end{pmatrix}\begin{pmatrix} a_1 & 2a_2 & 3a_3 & 4a_4 & \cdots \\ 0 & a_1 & 2a_2 & 3a_3 & \cdots \\ 0 & 0 & a_1 & 2a_2 & \cdots \\ 0 & 0 & 0 & a_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \cdots \end{pmatrix}\begin{pmatrix}\overline{a}_1\\ \overline{a}_2\\ \overline{a}_3\\ \overline{a}_4\\ \vdots\end{pmatrix}\,.$$

In short

$$= \frac{d}{dt}\{M(\underline{a})\underline{\overline{a}}\} = -B(\underline{a})\underline{\overline{a}}\}\;,\quad \underline{a}(0) = \underline{a}_0\,.$$

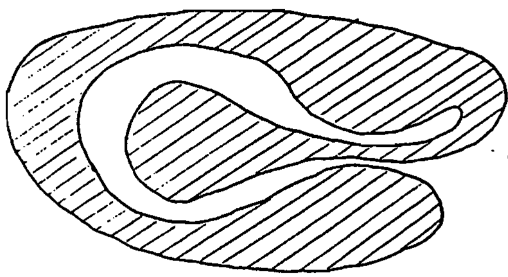
Let there be given an initial condition $\underline{a}(0) = \underline{a}_0$ and put $\underline{u}_0 = M(\underline{a}_0)\underline{\overline{a}}_0$. Now if from the infinite system of quadratic equations

$$M(\underline{a})\overline{\underline{a}} = \underline{u} \ .$$

$\underline{a}$ is locally solvable as a function of $\underline{u}$ around $\underline{u} = \underline{u}_0$ then the initial value problem is reduced to an initial value problem for the infinite system of quasi–linear differential equations

$$\frac{d}{dt}\,\underline{u} = -B(\underline{a}(\underline{u}))\underline{u} \ , \quad \underline{u}(0) = \underline{u}_0 \ .$$

Not much can be said about the solvability of this dynamical system at this moment. If every solution is a trajectory on the above mentioned ellipsoid in Hilbert space then the shape would remain simply connected if the initial domain is simply connected. Most probably such a deep result does not have a simple proof.



2.2 The real polynomial Hopper problem

If we substitute the Ansatz

$$\Omega(\zeta, t) = \sum_{n=1}^{N} a_n(t)\zeta^n \ , \quad n \in \mathbb{N}, \ a_n : \ [0,\infty) \to \mathbb{R}$$

in Hopper's equation (H) we find, e.g. for $N = 4$, the following finite system of ordinary differential equations

$$\frac{d}{dt}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_3 \end{pmatrix} = -\begin{pmatrix} 0 & 0 & 0 & 0 \\ - & \beta_0 & \beta_1 & \beta_2 \\ 0 & 0 & 2\beta_0 & 2\beta_1 \\ 0 & 0 & 0 & 3\beta_0 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix},$$

in short $\frac{du}{dt} = -B(\underline{a})\underline{u}$ with

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} a_1 & 2a_2 & 3a_3 & 4a_4 \\ 0 & a_1 & 2a_2 & 3a_3 \\ 0 & 0 & a_1 & 2a_2 \\ 0 & 0 & 0 & a_1 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}$$

$$= \begin{pmatrix} a_1 & 2a_2 & 3a_3 & 4a_4 \\ a_2 & 2a_3 & 3a_4 & 0 \\ a_3 & 2a_4 & 0 & 0 \\ a_4 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}$$

in short $\underline{u} = M(\underline{a})\underline{a} = N(\underline{a})\underline{a}$ and

$$\beta_n(\underline{a}) = (2 - \delta_{n0}) \frac{1}{4\pi} \int_0^{2\pi} \left\{ \sum_{m=1}^{N} \sum_{\ell=1}^{N} m\ell a_m a_\ell e^{i(m-\ell)s} \right\}^{-\frac{1}{2}} e^{ins} ds \, .$$

Denote $\underline{a} = (1, 0, 0, \ldots, 0) = \omega$.

The following properties are straightforward

PROPERTIES of β_n, $1 \leq n \leq N$

- $\beta_0(\underline{a}) > \frac{1}{N}$, for all $\underline{a} \in \Sigma$
- $\beta_n(\underline{\omega}) = \frac{1}{2}\delta_{n0}$, $1 \leq n \leq N$
- $\beta_n(\underline{a}) \to 0$ als $\underline{a} \to \underline{\omega}$, $1 \leq n \leq N$
- $\beta_0(\underline{a}) \uparrow \infty$ if $\underline{a} \to \partial\Sigma$.

The derivative of $\underline{u}$, e.g. is $N = 4$, is found to be

$$\frac{d\underline{u}}{d\underline{a}} = \begin{pmatrix} a_1 & 2a_2 & 3a_3 & 4a_4 \\ a_2 & 2a_3 & 3a_4 & 0 \\ a_3 & 2a_4 & 0 & 0 \\ a_4 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} a_1 & 2a_2 & 3a_3 & 4a_4 \\ 0 & a_1 & 2a_2 & 3a_3 \\ 0 & 0 & a_1 & 2a_2 \\ 0 & 0 & 0 & a_1 \end{pmatrix} .$$

So

$$\frac{d\underline{u}}{d\underline{a}}(\underline{\omega}) = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} .$$

Applying the Inverse Function Theorem we find that $\underline{a}$ can be solved locally as a function of $\underline{u}$ in a neighbourhood of $\underline{u} = \underline{\omega}$.
Via the method of variation of constants we find for the components u_j, $2 \leq j \leq N$,

$$|u_j(t)| \leq C_j e^{-\frac{(j-1)}{n} t} \, .$$

So $\underline{\omega}$ is a local attractor. Near $\underline{\omega}$ there is exponential decay: $|\underline{a} - \underline{\omega}| \leq Ce^{-\frac{1}{2}t}$.

Note that there are special solutions $\underline{a}(t)$ with $a_1(t) \neq 0$, $a_N(t) \neq 0$ and $a_2(t) = \ldots = a_{N-1}(t) = 0$. These are the typical solutions in Hopper's work. He 'guesses' shapes with one parameter and then solves an ordinary differential equation for this parameter as a function of t, cf. [H1], [H2], [H3].

EXAMPLE (Hopper 1990).
Try to solve Hopper's equation by the function

$$\Omega(\zeta,t) = a(t)\zeta - \frac{b(t)}{N+1}\,\zeta^N$$

with $a(t) \geq b(t) > 0$ and $N \in I\!N$ fixed.
Calculate

- $$\Omega' = a - b\zeta^N$$

- $$\overline{\Omega} = a\,\frac{1}{\zeta} - \frac{b}{N+1}\,\zeta^{-(N+1)}$$

- $$\overline{\Omega'} = a - b\zeta^{-N}$$

- $$\overline{\Omega}\Omega' = (a^2 + \frac{b^2}{N+1})\zeta^{-1} - \frac{ab}{N+1}\,\zeta^{-(N+1)} - ab\zeta^{-(N-1)}$$

- $$\frac{1}{2}(\Omega'\overline{\Omega'})^{-\frac{1}{2}} = \frac{1}{2}(a^2+b^2)^{-\frac{1}{2}}[1 - \frac{ab}{a^2+b^2}(\zeta^N + \zeta^{-N})]^{-\frac{1}{2}}$$

if $a > b$ then $\frac{ab}{a^2+b^2} < \frac{1}{2}$

- $$\mathcal{F} = \alpha_0 + 2\alpha_N\zeta^N + 2\alpha_{2N}\zeta^{2N} + \cdots,$$

$$\alpha_0(a,b) = \frac{1}{4\pi}(a^2+b^2)^{-\frac{1}{2}}\int_0^{2\pi}(1 - \frac{2ab}{a^2+b^2}\,\cos\theta)^{-\frac{1}{2}}d\theta$$

- $$\frac{d}{dt}(\overline{\Omega}\Omega') = 2(a\dot{a} + \frac{b\dot{b}}{N+1})\zeta^{-1} + (\frac{\dot{a}b + a\dot{b}}{N+1})\zeta^{-(N+1)} + \text{ Taylorseries}$$

- $$\frac{d}{d\zeta}[\zeta\overline{\Omega}\Omega'(\alpha_0 + 2\alpha_N\zeta^N)] = ab\alpha_0(a,b)\frac{N}{N+1}\,\zeta^{-(N+1)}\,.$$

The system of two ordinary differential equations

$$\begin{cases} a\dot{a} + \dfrac{b\dot{b}}{N+1} = 0 \\ \\ \dot{a}b + a\dot{b} = -ab\alpha_0(a,b)N \end{cases}$$

can be written explicitely

$$\begin{cases} \dot{a} = \alpha_0(a,b)\dfrac{ab^2N}{(N+1)a^2-b^2} \\ \\ \dot{b} = -\alpha_0(a,b)\dfrac{a^2bN(N+1)}{(N+1)a^2-b^2} \end{cases} .$$

This system is singular if $a = b$. At this point the decay of b is faster than exponential. For small b there is exponential decay

$$b(t) \approx b_0 e^{-\frac{1}{2}tN} \quad (e^{-\frac{1}{2}\frac{\gamma}{\eta}\frac{N}{R}t}) .$$

2.3 The Complex polynomial Hopper problem

If we substitute the Ansatz

$$\Omega(\zeta,t) = \sum_{n=1}^{N} a_n(t)\zeta^n , \quad N \in I\!N , \quad n : [0,\infty) \to \mathbb{C}$$

in Hopper's equation (H) we find, again, the finite system of ordinary differential equations

$$\frac{d}{dt}\underline{u} = -B(\underline{a})\underline{u} ,$$

but now with

$$\beta_n(\underline{a}) = (2-\delta_{n0})\frac{1}{4\pi}\int_0^{2\pi}\left\{\sum_{m=1}^{N}\sum_{\ell=1}^{N} m\ell a_m\overline{a}_\ell e^{i(m-\ell)s}\right\}^{-\frac{1}{2}} e^{-ins}ds ,$$

and

$$\underline{u} = M(a)\underline{\overline{a}} = N(\underline{\overline{a}})\underline{a} \qquad (S) .$$

Note that $u_1 = \overline{a}_1a_1 + 2\overline{a}_2a_2 + \ldots + N\overline{a}_Na_N \in I\!R$. So in order to make quasi linearization possible, at least locally, we require $u_1 \in I\!R$. Then (unlike Hopper in [H1]) we find that the system (S) consists of $2n-1$ real equations with $(2n-1)$ unknowns.

Now define

$$H : I\!R^{2n-1} \to I\!R^{2n-1} \quad : \quad \underline{a} \mapsto \underline{u} = M(\underline{a})\underline{\overline{a}}$$

with $a_1 \in I\!R$, $u_1 \in I\!R$.
The *real* derivative DH of H at $\underline{a}$ is, with complex notation and $z = (z_1,\ldots,z_N)$, $z_1 = \overline{z}_1$

$$\underline{a} \mapsto DH(\underline{a})\underline{z} = \begin{bmatrix} a_1 & 2a_2 & 3a_3 & 4a_4 \\ 0 & a_1 & 2a_2 & 3a_3 \\ 0 & 0 & a_1 & 2a_2 \\ 0 & 0 & 0 & a_1 \end{bmatrix} \begin{bmatrix} \overline{z}_1 \\ \overline{z}_2 \\ \overline{z}_3 \\ \overline{z}_4 \end{bmatrix}$$

$$+\begin{bmatrix} \overline{a}_1 & 2\overline{a}_2 & 3\overline{a}_3 & 4\overline{a}_4 \\ \overline{a}_2 & 2\overline{a}_3 & 3\overline{a}_4 & 0 \\ \overline{a}_3 & 2\overline{a}_4 & 0 & 0 \\ \overline{a}_4 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix}.$$

The real linear mapping $DH(\underline{a})$ is invertible at $\underline{a} = \underline{\omega}$. So also in the complex case $\underline{\omega}$ turns out to be a local attractor. With the modifications mentioned in this section, the complex polynomial Hopper problem can be attacked with the same methods as the real polynomial Hopper problem in the preceding section.

2.4 The Rational Hopper Problem

The Hopper equation

$$\frac{d}{dt}(\overline{\Omega}\Omega') - \frac{d}{d\zeta}(\zeta\overline{\Omega}\Omega'\mathcal{F}) = -2\chi' = \text{ analytic on } \overline{D}$$

can be written

$$\overline{\Omega}[(1+\zeta\frac{\Omega''}{\Omega'})\mathcal{F} + \zeta\mathcal{F}' - \frac{\dot{\Omega}'}{\Omega'}] - \zeta\overline{\Omega}'\mathcal{F} - \dot{\overline{\Omega}} = -\frac{2\chi'}{\Omega'} = \text{ analytic on } \overline{D}\,.$$

Following Hopper we take the Ansatz

$$\Omega = \zeta\sum_{n=1}^{N} \frac{A_n}{1-\alpha_n\zeta}\,, \qquad \overline{\Omega} = \sum_{n=1}^{N} \frac{\overline{A}_n}{\zeta - \overline{\alpha}_n}\,,$$

$$\overline{\Omega}' = -\sum_{n=1}^{N} \frac{\overline{A}_n}{(\zeta-\overline{\alpha}_n)^2}\,,$$

$$\dot{\overline{\Omega}} = \sum_{n=1}^{N} \frac{\dot{\overline{A}}_n}{(\zeta-\overline{\alpha}_n)} + \sum_{n=1}^{N} \frac{\overline{A}_n\dot{\overline{\alpha}}_n}{(\zeta-\overline{\alpha}_n)^2}\,.$$

After substitution in Hopper's equation and rearranging

$$\Big(\sum_{n=1}^{N} \frac{\overline{A}_n}{\zeta-\overline{\alpha}_n}\Big)\Big[(1+\zeta\,\frac{\Omega''}{\Omega'})\mathcal{F} + \zeta\mathcal{F}' - \frac{\dot{\Omega}'}{\Omega'}\Big] +$$

$$-\Big[\sum_{n=1}^{N} \frac{\overline{A}_n}{(\zeta-\overline{\alpha}_n)^2}\Big]\zeta\mathcal{F} - \sum_{n=1}^{N} \frac{\dot{\overline{A}}_n}{(\zeta-\overline{\alpha}_n)} - \sum_{n=1}^{N} \frac{\overline{A}_n\dot{\overline{\alpha}}_n}{(\zeta-\overline{\alpha}_n)^2} = \text{ analytic on } D\,.$$

Since 2^{nd} order and 1^{st} order poles have to compensate each other on D we find the following two sets of ordinary differential equations

$$\begin{cases} \dfrac{\dot{\alpha}_n}{\alpha_n} = -\overline{\mathcal{F}(|\Omega'(\overline{\alpha}_n,t)|} \\[2ex] \dfrac{\dot{A}_n}{A_n} = -\Big[\alpha_n\dfrac{\overline{\Omega''(\overline{\alpha}_n)}}{\overline{\Omega'(\overline{\alpha}_n)}}\,\dfrac{\dot{\alpha}_n}{\alpha_n} + \dfrac{\overline{\dot{\Omega}'(\overline{\alpha}_n)}}{\overline{\Omega'(\overline{\alpha}_n)}}\Big] \end{cases}.$$

Remind that, in this special case, $\mathcal{F}$ is a function of $\alpha_1, \ldots, \alpha_N, A_1, \ldots, A_N$.

So we have $2N$ complex, coupled, explicit ordinary differential equations which are locally solvable.

EXAMPLE (Hopper 1989).

Exact solution of the problem of coalescence of 2 equal cylinders.
Take

$$z = \Omega(\zeta, \nu(t)) = \frac{1-\nu^2}{(1+\nu^2)^{\frac{1}{2}}} \frac{\zeta}{1+\nu\zeta^2} .$$

The inverse of the 'parameter function' $\nu(t)$ is

$$t = \tfrac{1}{2}\nu \int_{\nu}^{1} [k(1+k^2)^{\frac{1}{2}} K(k)]^{-1} \, dk$$

with K an elliptic integral. See [H1].

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