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Literature Study on Mathematical Tools for Analyzing Limit Cycling Phenomena

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Literature Study on Mathematical Tools For Analyzing Limit Cycling Phenomena

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December 2000

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Chapter 1

Introduction

Since the discovery of the limit cycle by Henri Poincaré [10] in his four-part paper *Integral curves defined by differential equations* (1881-1886), there have been many results produced by mathematicians and engineers to explain the limit cycling phenomena. In mathematical theory, they studied mainly the existence, uniqueness and stability of limit cycles, and the problems how the limit cycles were generated and disappeared.

The driving force behind the study of theory of limit cycles was furnished more by practical problems than by great mathematicians. This is the situation: during the twentieth century, applied electronics had made rapid advancement; physicists invented the triode vacuum tube which was able to produce stable self-excited oscillations of constant amplitude, thus making it possible to propagate sound and pictures through electronics. However, it was not possible to describe this oscillation phenomenon by linear differential equations. In 1926, van der Pol first obtained a differential equation, which was later named after him, to describe oscillations of constant amplitude of a triode vacuum tube:

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0 \quad (\mu \neq 0). \tag{1.1}$$

After transforming this equation into an equivalent differential system in the phase plane, he used graphical methods to prove the existence of an isolated closed trajectory. Three years later, a Russian theoretical physicist A.A. Andronov in a short paper [1] clarified that the isolated closed trajectory of the van der Pol equation was the limit cycle studied earlier by Poincaré. Thus, he has established a close relationship between pure mathematical theory and electronic technology. From that time onward, a tremendous amount of researches has been carried out on the theory of limit cycles and electronics technology.

On a way to study the limit cycling phenomena in controlled mechanical systems, it is inevitable to look at those theoretical results in order to make a link between those results and applications in controlled mechanical systems. This literature study report tries to collect important results on the theory of limit cycles. Beside this introduction, this report is organized as follow: The second chapter *Existence of Periodic Orbits* will introduce some mathematical theorems about (non)existence of limit cycle or periodic orbit in planar systems, Lienard system, and general autonomous nonlinear differential systems. In the third chapter *Stability of Limit Cycles,* the Floquet theory about stability conditions of limit cycles or periodic solutions will be explained. The chapter four will present numerical methods to compute the bifurcations of periodic orbits. Then, in the fifth chapter we will exploit an example to show how the numerical methods work and how they meet the theoretical results.

Chapter 2

Existence of Periodic Orbits

This chapter will introduce some theoretical results that provide conditions for the (non)existence of periodic orbits in an autonomous differential system. Those theorems are arranged according to the type of system where they are applicable. The first part of the theorems are applicable to planar autonomous systems. Then followed by theorems that are applicable only to a special class of planar systems, and the last part of the theorems are applicable to more general autonomous systems.

2.1 Bendixson's Criterion

The Bendixson's criterion is a criterion for non-existence of periodic orbits in two-dimensional or planar autonomous systems of the form

$$\dot{x}_1 = f_1(x_1, x_2) \text{ and } \dot{x}_2 = f_2(x_1, x_2) \tag{2.1}$$

where f_1 and f_2 are C^1 functions. The criterion is as follows: Let D be an open subset of $\mathbf{R}^2$ which is simply connected (does not contain any holes or disjoint regions). If the divergence

$$div\, f \equiv \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} \tag{2.2}$$

is of constant sign and not identically zero in D, then the system 2.1 has no periodic orbit lying entirely in the region D.

This criterion is a necessary condition but not sufficient to exclude the existence of a periodic orbit in a planar system 2.1. For example consider the following system

$$\dot{x}_1 = -x_2 + x_1^2 - x_1 x_2 \tag{2.3}$$

$$\dot{x}_2 = x_1 + x_1 x_2 \tag{2.4}$$

whose divergence, $div\, f = 3x_1 - x_2$, which changes sign whenever the system's trajectories cross the line $x_2 = 3x_1$ in the $x_2 - x_1$ plane. However, as shown in Figure 1 the system 2.3 has no periodic orbit. Dulac generalized the Bendixson's criterion such that it gives a sufficient condition to exclude the periodic orbit in the planar system 2.1.{Figure in Nayfeh 155}

2.2 Dulac's Criterion

Let D be a simply connected open subset of $\mathbf{R}^2$ and $B(x_1, x_2)$ be a real-valued C^1 function. If the function

$$divBf = \frac{\partial(Bf_1)}{\partial x_1} + \frac{\partial(Bf_2)}{\partial x_2} \tag{2.5}$$

is of constant sign and not identically zero in D, then the system 2.1 has no periodic orbit lying entirely in the region D.

The function $B(x_1, x_2)$ is called a Dulac function. This criterion reverts to Bendixson's in the special case when $B(x_1, x_2) \equiv 1$. Unfortunately, there is no general method for determining an appropriate Dulac function for a given planar system. We can use Dulac's criterion to exclude the existence of limit cycle in the system 2.3. Take a Dulac function

$$B(x_1, x_2) = (1 + x_2)^{-3}(-1 - x_1)^{-1}, \tag{2.6}$$

the we have

$$\frac{\partial(Bf_1)}{\partial x_1} + \frac{\partial(Bf_2)}{\partial x_2} = x_1^2(1 + x_2)^{-3}(-1 - x_1)^{-2}. \tag{2.7}$$

Again, this expression is not of one sign, but do not despair. Notice, that the line $x_2 = -1$ is invariant under the flow, and the vector field crosses $x_1 = -1$ in the same direction. Thus if there is a periodic orbit it must lie entirely in one of the four regions separated by these two lines. However, the function 2.7 keeps one sign in one of these four regions. Consequently, from Dulac's criterion the planar system 2.3 has no periodic orbit.

2.3 Poincaré-Bendixson Theorem

In order to state this Poincaré-Bendixson theorem, we need the definitions of an ω-limit set of autonomous planar systems [4].

Definition 1 *A point y is an ω-limit of the orbit $\gamma(x^0)$ if there is a sequence t_j with $t_j \to \beta_{x^0}$ as $j \to \infty$ such that $\varphi(t_j, x^0) \to y$ as $j \to \infty$. That is, y is an ω-limit point of the orbit $\gamma(x^0)$ if, for any $\varepsilon > 0$, there is a $t(\varepsilon)$ such that $\left\| y - \varphi(t(\varepsilon), x^0) \right\| < \varepsilon$. The set of all ω-limit points of the orbit $\gamma(x^0)$ is called the ω-limit set of $\gamma(x^0)$ and is denoted by $\omega(x^0)$. Where $\gamma(x^0)$ denotes a periodic orbit of planar systems that goes through the point x^0 in the phase plane, and $\varphi(t, x^0)$ denotes a solution of planar systems with initial condition $x(t_0) = x^0$*

Poincaré-Bendixson theorem: if $\omega(x^0)$ of a planar system is a bounded set which contains no equilibrium point, then $\omega(x^0)$ is a periodic orbit.

In order to use the Poincaré-Bendixson theorem to show the existence of a nontrivial periodic orbit, one could attempt to construct an open bounded set D in $\mathbf{R}^2$ which contains no equilibrium point and such that any solution that begins in D remains in D for all $t \geqslant 0$, that is, D is an open and bounded positively invariant set. Next, for any x^0 in D, one also shows that $\omega(x^0)$ contains no point in the boundary of D. Then, since D contains no equilibrium point, $\omega(x^0)$ must be a periodic orbit. Let us illustrate this remark on an example. Consider the following planar system

$$\dot{x}_1 = x_2\ , \quad \dot{x}_2 = -x_1 + x_2(1 - x_1^2 - 2x_2^2). \tag{2.8}$$

Observe that the origin is the only equilibrium point of 2.8. We will attempt to construct an annular region D with the desired properties mentioned above. Take a function $V(x_1, x_2) = (x_1^2 + x_2^2)/2$, and compute its derivative along the solutions of 2.8:

$$\dot{V}(x_1, x_2) = x_2^2(1 - x_1^2 - 2x_2^2).$$

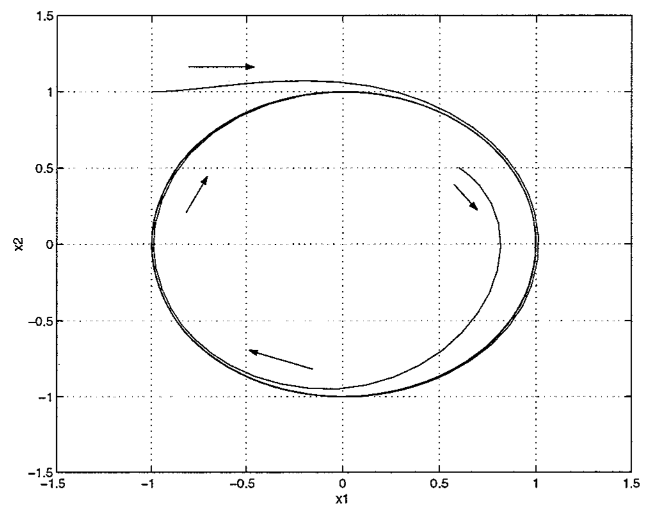

Figure 2.1: A periodic Orbit of 2.8 in the region D

Since $\dot{V}(x_1, x_2) \geq 0$ for $\frac{1}{2}x_1^2 + x_2^2 \leq \frac{1}{2}$, then for $x_1^2 + x_2^2 < \frac{1}{2}$, $\dot{V}(x_1, x_2) \geq 0$ also hold. $\dot{V}(x_1, x_2) \leq 0$ for $x_1^2 + 2x_2^2 \geq 1$, consequently $\dot{V}(x_1, x_2) \leq 0$ for $x_1^2 + x_2^2 > 1$. Thus, any solution which starts in the annulus $D \triangleq \frac{1}{2} < x_1^2 + x_2^2 < 1$ remains in this annulus for all $t \geq 0$. Since the origin is not in the closure of this annulus, following the Poincaré-Bendixson theorem, there exists at least one periodic orbit of system 2.8 in the annulus D. The periodic solution of 2.8 in the region D can be seen in Figure .

2.4 Levinson-Smith Theorem

This theorem [2] is applied for a special class of planar systems, i.e. Lienard's systems:

$$\ddot{x} + f(x)\dot{x} + g(x) = 0 \tag{2.9}$$

where $f \in C^0$, $g \in C^1$are real-value functions. The system 2.9 is equivalent to

$$\dot{x} = y - F(x), \qquad \dot{y} = -g(x) \tag{2.10}$$

where $F(x) := \int_0^x f(s)ds$.

This theorem said: Let $G(x) := \int_0^x g(s)ds$. If f is even, there exists an $x_0 > 0$ such that $F(x) < 0$ for $0 < x < x_0$, and $F(x) > 0$ for $x > x_0$, g is odd $xg(x) > 0$ for $x \neq 0$, F is monotone increasing in (x_0, ∞), and $F(x) \to \infty$, $G(x) \to \infty$ as $x \to \infty$ then 2.9 has a unique non-constant periodic solution.

A famous example that satisfies this theorem is the Van der Pol's equation, 1.1. Hence, by following this theorem the Van der Pol's equation has a unique limit cycle.

2.5 Villari's Theorem

This theorem [2], as the previous theorem, is also applied only for the Lienard's system 2.9.

Assume that

(i) $f \in C^0$, $g \in C^1$are real-value functions;

(ii) $f(0) < 0$, and there exists an $x_0 > 0$ such that $f(x) > 0$ for $|x| > x_0$;
(iii) $xg(x) > 0$, for $x \neq 0$;
(iv) $\min(\limsup_{x\to\infty}(g(x)/f(x)))$, $\min(\limsup_{x\to-\infty}(-g(x)/f(x))) < \infty$;
(v) there exists positive constants $\bar{x} > x_0$ and $b > 0$ such that $f(x) + |g(x)| > b > 0$ for $|x| > \bar{x}$.

then 2.9 has at least one non-constant periodic solution.

Compare to the Levinson-Smith theorem, this theorem has an advantage that it does not require symmetry properties from the functions f and g. On the other hand it does not guarantee the uniqueness of the periodic solution.

2.6 Dragilëv's Theorem

This theorem [10] provides necessary conditions for existence of stable limit cycles in the Lienard's systems 2.9.

Let $G(x) := \int_0^x g(s)ds$, and the following conditions hold:

1. $xg(x) > 0$ when $x \neq 0$, and $G(\pm\infty) = +\infty$.
2. $xF(x) < 0$ when $x \neq 0$, and $|x|$ is sufficiently small.
3. There exist a constant $M > 0$ and $K > K^T$ such that $F(x) \geq K$ when $x > M$ and $F(x) \leq K^T$ when $x < -M$.

Then the system 2.9 has stable limit cycles.

2.7 Yan-Qian Theorem

This Yan-Qian theorem [10] provides conditions for existence of stable or unstable limit cycles.

Let $G(x) := \int_0^x g(s)ds$, and let $xF(x) < 0$ when $|x| \neq 0$, and $|x|$ is sufficiently small, and there exist constants $M > 0,\ x_1 > 0,$ and $x_2 > 0$ such that:

1. $xg(x) > 0$ when $-x_2 < x < 0$ and $0 < x < x_1$;
2. $F(x) \geq -M$ (or $F(x) \leq M$) when $0 < x < x_1$, $and F(x_1) \geq M + \sqrt{2l}$ (or $F(x_1) \leq -M - \sqrt{2l}$);
3. $F(x) \leq M$ (or $F(x) \geq -M$) when $-x_2 < x < 0$, $and F(-x_2) \leq -M - \sqrt{2l}$ (or $F(-x_2) \geq M + \sqrt{2l}$);

where $l = \max[G(x_1), G(-x_2)]$. Then the system 2.9 has stable (unstable) limit cycles.

2.8 Filippov-Yan-Qian Theorem

Let $G(x) := \int_0^x g(s)ds$ and suppose that in 2.9 $g(x)$ satisfies $xg(x) > 0\ when\ x \neq 0,\ G(\pm\infty) = +\infty$; and suppose that after a change of variables

$$\int_0^x g(s)ds = z_1(x),\ \int_0^x f(s)ds = F(x) = F_1(z_1)\ when\ x > 0,$$
$$\int_0^x g(s)ds = z_2(x),\ \int_0^x f(s)ds = F(x) = F_2(z_2)\ when\ x < 0,$$

the function $F_1(z)$ and $F_2(z)$ satisfy the following conditions:

(i) For small z $(0 < z < \delta)$ we have $F_1(z) \leq F_2(z)$ but $F_1(z)$ does not identically equal to $F_2(z)$; and $F_1(z) < a\sqrt{z}\ and\ F_2(z) > -a\sqrt{z}$, where $0 < a < \sqrt{8}$.

(ii) There exists a number $z_0 > 0$ such that

$$\int_0^{z_0} (F_1(z) - F_2(z_2))dz > 0$$

and, when $z > z_0$, $F_1(z) \geq F_2(z)$, $F_1(z) > -a\sqrt{z}$ *and* $F_2(z) < a\sqrt{z}$.

Then the system 2.9 has stable limit cycles.

It is noted [10] that from the condition (i) of this theorem, one can establish a rule to determine nonexistence of periodic orbit.

Corollary 2 *if* $F_1(z) \leq F_2(z)$ *for all* $z > 0$, *and* $F_1(z)$ *does not identically equal to* $F_2(z)$ *in* $(0, \delta)$ *for any* $\delta > 0$, *then the system* 2.9 *does not have a closed trajectory.*

2.9 Ponso-Wax Theorem

This theorem [2] provides conditions for existence of periodic orbits of the autonomous system

$$\ddot{x} + f(x, \dot{x})\dot{x} + g(x) = 0. \tag{2.11}$$

It is interesting that 2.11 can model some realistic mechanical systems such as a rigid manipulator. In fact 2.11 is a special case of the famous Lagrangian formulation of rigid mechanical system for one degree of freedom. The equation 2.11 in its equivalent Cauchy normal form is rewritten as

$$\dot{x} = y, \quad \dot{y} = -f(x, y)y - g(x). \tag{2.12}$$

Ponso-Wax Theorem: Assume that in 2.12 $f, g \in C^1$ are real-value functions, and

1. $f(0,0) < 0$, *there are* $a < 0 < b$ *such that*
   $f(a, y) = f(b, y) = 0$ *for all* $y \in \mathbf{R}$,
   $f(x, y) > 0$ *if* $x \in (-\infty, a) \cup (b, \infty)$ *for all* $y \in \mathbf{R}$,
   for $y \geq 0$ *and for every* $x < a$ *the function* $yf(x, y)$ is increasing in y, *and* $\lim_{y\to\infty} yf(x, y) = \infty$,
   there is an $M > 0$ *such that for* $x \in [a, b] : f(x, y) \geq -M$;
2. $xg(x) > 0$ *for* $x \neq 0$, *for* $G(x) = \int_0^x g(s)ds$, $\lim_{x\to\pm\infty} G(x) = \infty$;
3. for the function $u : (-\infty, a) \to \mathbf{R}_+$ *defined implicitly by*

   $$uf(x, u) + g(x) = 0, \quad x < a$$

   assume that $\bar{u}(x) := \max_{s\leq x} u(s)$ *exists for* $x < a$.

Then 2.11 has at least one periodic solution.

2.10 Smith's Theorem

Smith's theorem [2], unlike the previous theorems, is applicable for a more general autonomous systems of the form

$$\dot{x} = f(x) \tag{2.13}$$

where $f : X \to \mathbf{R}^n$ is a C^1function on an open and connected set $X \subset \mathbf{R}^n$. Let $\varphi : [-\infty, \infty] \to X$ be a solution of 2.13 and $\gamma = \{x \in \mathbf{R}^n : x = \varphi(t), -\infty < t < \infty\}$ its path. An ω-limit set of γ is defined exactly as in the case of planar systems.

This theorem guarantees the existence of periodic orbits of 2.13 and stated as follows.

Smith's Theorem I: Let $K \subset X$ be a compact set and suppose that there exits a real symmetric $n \times n$ matrix P having 2 negative and $n-2$ positive eigenvalues such that for any pair of solutions $\varphi^1(t)$ and $\varphi^2(t)$ of 2.13 staying in K for every $t \in \mathbf{R}$, i.e. $\varphi^1(t)$, $\varphi^2(t) \in K, t \in \mathbf{R}$, we have $U(\varphi^1(t) - \varphi^2(t)) \leq 0$ for $t \in \mathbf{R}$ where U is the quadratic form $U(x) = x^T Px$; if for a positive semi trajectory γ^+ of 2.13, we have $\gamma^+ \subset K$, and the ω-limit set, $\omega(\gamma^+)$ of γ^+ does not contain any equilibria then $\omega(\gamma^+)$ contains at least one periodic orbit.

Compare to the Poincaré-Bendixson's theorem, even with the additional assumptions on 2.13, the Smith's theorem is not a full generalization of the Poincaré-Bendixson's theorem since it allows more than one periodic orbit in $\omega(\gamma^+)$. It is noted that the conditions imposed on 2.13 make it possible to project the omega limit set into a two dimensional plane in a homeomorphic way and draw a conclusion from the projected phase portrait.

In order to have a full generalization of the Poincaré-Bendixson's theorem further conditions are to be imposed upon system 2.13. The following hypothesis will be used:

(H) Assume that there are real symmetric $n \times n$ matrices $P_1$and P_2 such that $P_1 - P_2$ has 2 negative and $n-2$ positive eigenvalues; let $K \subset X$ be a compact set and assume that there are positive constants $\mu_1, \varepsilon_1, \mu_2, \varepsilon_2$ such that the quadratic forms $V_i(x) := x^T P_i x$ $(i = 1, 2)$ satisfy the inequalities

$$\frac{d}{dt}(V_1(\varphi^1(t) - \varphi^2(t))) + 2\mu_1 V_1(\varphi^1(t) - \varphi^2(t)) \leq -\varepsilon_1 \left|\varphi^1(t) - \varphi^2(t)\right|^2 \tag{2.14}$$

and

$$\frac{d}{dt}(V_2(\varphi^1(t) - \varphi^2(t))) - 2\mu_2 V_2(\varphi^1(t) - \varphi^2(t)) \leq -\varepsilon_2 \left|\varphi^1(t) - \varphi^2(t)\right|^2 \tag{2.15}$$

for every t at which $\varphi^1(t), \varphi^2(t) \in K$ where φ^1 and φ^2 are arbitrary solutions of 2.13.

Smith's Theorem II: Suppose that hypothesis (H) holds and 2.13 has a positive semi-trajectory $\gamma^+ \subset K$; if the omega limit set $\omega(\gamma^+)$ does not contain any equilibrium point of 2.13, then it consists of a single periodic orbit.

These two theorems show the difficulties that arise in establishing the existence of periodic orbits in higher dimensions. In these theories the results could be achieved because the conditions made it possible to project the problem into a two dimensional plane. These conditions, especially (H), may seem to be rather artificial. Nevertheless, the results can be applied , e.g., for important classes of feedback control systems [9] of the following form:

$$\dot{x} = Ax + BF(Cx) \tag{2.16}$$

where A is a $n \times n$, B is a n$\times r$, and C is a s$\times n$ constants real, and $F : \mathbf{R}^s \to \mathbf{R}^r$.

Chapter 3

Stability of Periodic Orbits

This chapter will explain stability criteria of periodic orbits using the Floquet Theory [6]. Here, we consider the stability of periodic solutions of autonomous systems

$$\dot{x} = f(x; \mu) \tag{3.1}$$

where x is an n-dimensional state vector and μ is an m-dimensional parameter vector. Let the periodic solution of (3.1) at $\mu = \mu_0$ be denoted by $x_0(t)$ with period T. Then, a disturbance y superimposed on x_0, resulting in

$$x(t) = x_0(t) + y(t). \tag{3.2}$$

Substituting (3.2) into (3.1), assuming that f is at least twice continuously differentiable (i.e. C^2), expanding the result in Taylor series about x_0 and retaining only linear terms in the disturbance, we obtain

$$\dot{y} = D_x f(x_{0;}\mu_0)y + O(\|y\|^2) \;\; or \;\; \dot{y}(t) \simeq A(t; \mu_0)y \tag{3.3}$$

where A is the Jacobian matrix of f. The stability analysis is local because we linearized in the disturbance y. The matrix A is periodic in time and has a period T, which is the period of the periodic solution $x_0(t)$. However, T may not be the minimal period of A. For instance, when f has only odd nonlinearities, the minimal period of A is $\frac{1}{2}T$. Floquet theory deals with linear systems, such as (3.3), with periodic coefficients.

The n-dimensional linear system (3.3) has n linearly independent solutions y_i, where $i = 1, 2, ..., n$. These solutions are usually called a fundamental set of solutions. This fundamental set can be expressed in the form of $n \times n$ matrix called a fundamental matrix solution as

$$Y(t) = [y_1(t) \; y_2(t) \; ... \; y_n(t)]. \tag{3.4}$$

Clearly, Y satisfies the matrix equation

$$\dot{Y} = A(t; \mu_0)Y. \tag{3.5}$$

Changing the dependent variable in (3.5) from t to $\tau = t + T$, we arrive at

$$\frac{dY}{d\tau} = A(\tau - T; \mu_0)Y = A(\tau; \mu_0)Y \tag{3.6}$$

on account of the fact that A is periodic, $A(\tau - T; \mu_0) = A(\tau; \mu_0)$. Hence, if

$$Y(t) = [y_1(t) \; y_2(t) \; ... \; y_n(t)]$$

is a fundamental solution, then

$$Y(t+T) = [y_1(t+T)\ y_2(t+T)\ \dots\ y_n(t+T)]$$

is also a fundamental matrix solution. Because (3.3) has at most n linearly independent solutions and because the $y_i(t)$ are such that n linearly independent solutions, the $y_i(t+T)$ must be linear combinations of the $y_i(t)$, $i = 1, ..., n$; that is

$$Y(t+T) = Y(t)\Phi \tag{3.7}$$

where Φ is $n \times n$ constant matrix. We note that Φ depends on the chosen fundamental matrix solution and is not unique. This matrix may be thought of as a map or a transformation that maps an initial vector in $\mathbf{R}^n$ at $t = 0$ to another vector in $\mathbf{R}^n$ at time $t = T$. Specifying the initial condition

$$Y(0) = I \tag{3.8}$$

where I is the $n \times n$ identity matrix and setting $t = 0$ in (3.7), we obtain

$$\Phi = Y(T). \tag{3.9}$$

The matrix Φ, defined by (3.7-3.9), is called the monodromy matrix.

The eigenvalues $\rho_i, i = 1, ..., n$ of the monodromy matrix Φ are called Floquet multipliers. These Floquet multipliers provide a measure of the local orbital divergence or convergence along particular direction over one period of the closed orbit of (3.1). Thus, they determine the local stability of the periodic solutions of (3.1). *It is important to note that one of the Floquet multipliers associated with a periodic solution $x_0(t)$ of an autonomous system, such as (3.1), is always unity.* In order to show this, differentiate (3.1) once with respect to time t and obtain

$$\ddot{x} = D_x f(x;\mu)\dot{x}. \tag{3.10}$$

Consequently, if x is a solution of (3.1) then $\dot{x}$ is a solution of (3.10) and hence of (3.3). Moreover, since $x_0(t) = x_0(t+T)$ then $\dot{x}_0(t) = \dot{x}_0(t+T)$ and hence

$$\dot{x}_0(0) = \dot{x}_0(T). \tag{3.11}$$

Furthermore, because $\dot{x}_0(t)$ is a solution of (3.3), it must be a linear combination of $y_1(t), ..., y_n(t)$; that is

$$\dot{x}_0(t) = Y(t)\alpha \tag{3.12}$$

where α is a constant vector. Evaluating (3.12) at $t = 0$ and at $t = T$ yields

$$\dot{x}_0(0) = Y(0)\alpha \ \ and\ \ \dot{x}_0(T) = Y(T)\alpha. \tag{3.13}$$

Considering (3.11) and (3.13), we obtain

$$Y(T)\alpha = Y(0)\alpha. \tag{3.14}$$

Using (3.8) and (3.9), we rewrite (3.14) as

$$\Phi\alpha = \alpha. \tag{3.15}$$

Therefore, 1 is an eigenvalue of Φ corresponding to the eigenvector $\alpha = \dot{x}_0(0) = f(x_0(0);\mu)$.

A periodic solution of (3.1) is known as a *hyperbolic periodic solution* if only one of its Floquet multipliers is located on the unit circle in the complex plane. A hyperbolic periodic solution is *asymtotically stable* if there is no Floquet multiplier outside unit circle. All neighboring positive orbits are attracted to this periodic orbit. Hence, this periodic solution is called a *stable limit cycle.* A hyperbolic periodic solution is *unstable* if one or more Floquet multipliers lie outside the unit circle. In this case, all neighboring positive trajectories are repelled from this periodic solution. Hence, this solution is called an *unstable limit cycle.*

If two or more Floquet multipliers are located on the unit circle, the periodic solution is called a nonhyperbolic periodic solution. A nonhyperbolic periodic solution is unstable if one or more the associated Floquet multipliers lie outside the unit circle. If none of the Floquet multiplier lies outside the unit circle, a nonlinear analysis is necessary to determine the stability of a nonhyperbolic periodic solution.

Chapter 4

Computational Method for Bifurcations of Limit Cycles

Bifurcations of limit cycles refer to any qualitative changes with respect to limit cycles, i.e. the birth, the disappearance, the multiplicity, and the change of stability of the limit cycles. There are four kinds of bifurcations of limit cycles, namely: Hopf, Fold, Flip, and Neimark-Sacker bifurcations. For further information about these bifurcations see [7]. This chapter will concern about computational methods to determine the type of bifurcations of limit cycles. This computational methods are needed because generally it is very hard if not impossible to determine a type of bifurcations of limit cycles analytically. Since the bifurcations have a qualitative flavor whereas the branches of solutions represent qualitative elements, then the computational for bifurcations analysis consists of computing branches of solutions and to determine the bifurcation points of those branches.

4.1 Tracing a Branch of Periodic Solutions

Tracing a branch of solutions is also known as path following (continuation) technique. In this technique [3] the system is parameterized as

$$\dot{x} = f(x, r) \tag{4.1}$$

and the solutions are computed for some values of the parameter r such that they form a branch of solutions. It is very important to determine the range, the step size and the direction of increments of the parameter r in order to have a complete branch of solutions. Since our interest is bifurcation of limit cycles then we are interested in finding branches of periodic solutions of system (4.1). The periodic solutions of (4.1) are computed by using shooting algorithm, for further information about shooting algorithm see [7]. The shooting algorithm finds periodic solutions of (4.1) by solving two points boundary value problem, in which the solutions are sought of

$$h(z, r) \equiv \phi_T(z, r) - x_0 = 0 \tag{4.2}$$

where x_0 is the states of (4.1) on the periodic solution, T is the period of the periodic solution, $z = [\ x_0 \quad T\ ]^T$ is the extended states because the algorithm needs to find the periodic solution and its period simultaneously.

The path following algorithm starts with a periodic solution $z_{s,1}^T$ for a given parameter value $r_{s,1}$ - the subscript s indicates a solution - which is computed by the shooting algorithm.

Then, a branch of solutions can be followed by means of a predictor-corrector mechanism. In predictor step k, the tangent $[p_{z,k}^T \; p_{r,k}]^T$ to the solution branch at $[z_{s,k}^T \; r_{s,k}]^T$ is determined by

$$\frac{\partial h}{\partial z} p_{z,k} + \frac{\partial h}{\partial r} p_{r,k} = 0. \tag{4.3}$$

In the first predictor step, $p_{r,1}$ is set to 1 if r must be increased initially or $p_{r,1}$ is set to -1 if r must be decreased initially. In subsequent steps, $p_{r,k}$ can be set to 1 such that (4.3) can be solved for $p_{z,k}$. The tangent is scaled by a factor $\sigma_{p,k}$, which is derived from the elliptical constraint

$$\sigma_{p,k}^2 (p_{z,k}^T p_{z,k} + p_{r,k}^2) = \sigma_k^2, \tag{4.4}$$

where σ_k is the step size of parameter r, which lies in a user defined interval

$$0 < \sigma_{\min} \leq \sigma_k \leq \sigma_{\max}. \tag{4.5}$$

In step $k > 1$, the sign of $\sigma_{p,k}$ is chosen such that the scaled tangent of two succeeding steps form an actual angle. It can be achieved by requiring

$$sign(\sigma_{p,k}) = sign(\sigma_{p,k-1}(p_{z,k-1}^T p_{z,k} + p_{r,k-1} p_{r,k})]. \tag{4.6}$$

This ensures that a solution branch is followed in the same direction. The prediction $[z_{p,k}^T \; r_{p,k}]^T$ is given by

$$\begin{bmatrix} z_{p,k} \\ r_{p,k} \end{bmatrix} = \begin{bmatrix} z_{s,k} \\ r_{s,k} \end{bmatrix} + \sigma_{p,k} \begin{bmatrix} p_{z,k} \\ p_{r,k} \end{bmatrix}. \tag{4.7}$$

In general, this prediction will not meet the convergence criterion that is used, hence an iterative correction process is needed.

A corrector step m is given by

$$\begin{bmatrix} z_{c,k,m+1} \\ r_{c,k,m+1} \end{bmatrix} = \begin{bmatrix} z_{c,k,m} \\ r_{c,k,m} \end{bmatrix} + \begin{bmatrix} c_{z,k} \\ c_{r,k} \end{bmatrix}. \tag{4.8}$$

In the first corrector step, the first term on the right-hand side of (4.8) will be set equals to the prediction in (4.7). Corrections are calculated by solving the following system of equations, which is similar to the Newton-Raphson algorithm

$$\begin{bmatrix} \frac{\partial h}{\partial z} & \frac{\partial h}{\partial r} \\ -((\frac{\partial h}{\partial z})^{-1} \frac{\partial h}{\partial r})^T & 1 \end{bmatrix} \begin{bmatrix} c_{z,k} \\ c_{r,k} \end{bmatrix} = \begin{bmatrix} -h \\ 0 \end{bmatrix}. \tag{4.9}$$

In this equations, h, $\partial h/\partial z$, $\partial h/\partial r$ are evaluated at $\begin{bmatrix} z_{c,k,m}^T & r_{c,k,m} \end{bmatrix}^T$ and it will force the corrections to be orthogonal to the solution space. The corrector term $\begin{bmatrix} z_{c,k,m+1}^T & r_{c,k,m+1} \end{bmatrix}^T$ from (4.8) is accepted as the next solution $\begin{bmatrix} z_{s,k+1}^T & r_{s,k+1} \end{bmatrix}^T$ if the convergence criterion is met. Since this algorithm is similar to the Newton-Raphson algorithm then the same convergence property holds.

An adaptation mechanism of the step size σ_k is needed to ensure that the algorithm follows the branch of solutions correctly. The changing of the step size is determined by the ratio between the Euclidean norm of the correction at step $k-1$ and step $k-2$. If this ration is lower than a user defined minimum, the step size will be increased. As soon as it exceeds a user defined maximum the step size will be decreased and the last prediction will be recalculated using the new step size. Furthermore, during the iterative correction process, it is required that the norm of the residue is decreased monotonically, that is

$$||h(z_{c,k,m+1}, r_{c,k,m+1})|| < ||h(z_{c,k,m}, r_{c,k,m})||. \tag{4.10}$$

If this inequality is violated, the last prediction is rejected and a new prediction will be calculated by using a smaller step size. The path following algorithm fails if the required step size σ_k is smaller than $\sigma_{\min}$, thus $\sigma_{\min}$ must be refined.

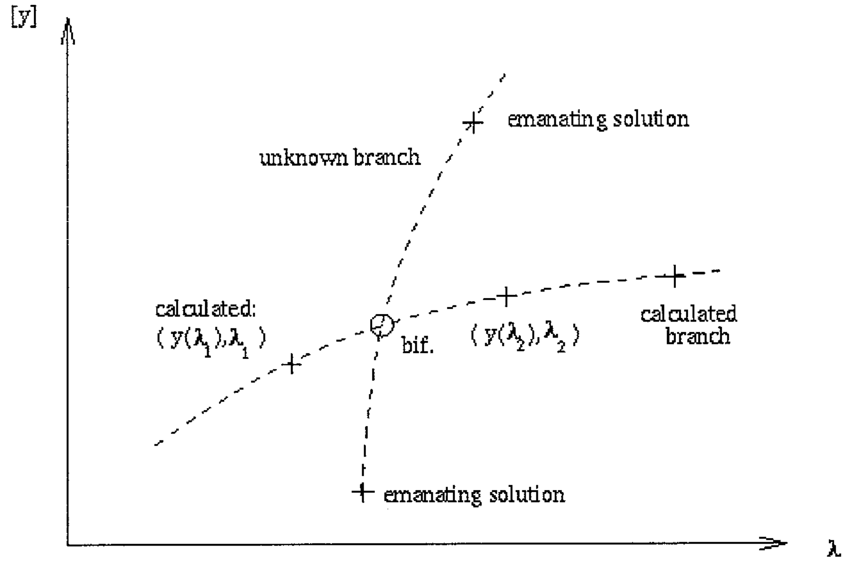

Figure 4.1: Branch switching

4.2 Computational Bifurcation

Since in the path following technique the calculated branch consists of a chain of discrete solutions. It is most unlikely that the continuation happens to hit a bifurcation exactly. Rather a bifurcation will be hidden in the space between the calculated solutions. Hence, the tasks of a computational bifurcation analysis [8] are

1. detect a bifurcation point. The minimum requirement is to straddle the bifurcation, that is to calculate one solution on either "side." This information can be easily condensed to a rough approximation to the bifurcation. For some applications it is necessary to

2. calculate the bifurcation point accurately. The occurrence of bifurcation of limit cycles can be detected by checking whether the Floquet multipliers associated to computed limit cycles crossing unit circle: Fold Bifurcation is indicated by one real Floquet multiplier crosses the unit circle at $(+1, 0)$, Flip Bifurcation (period doubling) is indicated by one real Floquet multiplier crosses the unit circle at $(-1, 0)$, and Neimark-Sacker Bifurcation is indicated by a pair of complex Floquet multipliers crosses the unit circle. After having carried out steps 1 and 2, enough data may be available to

3. determine the type of bifurcation. Depending on the type of bifurcation, a new branch may bifurcate off distinct from the branch that was calculated during the continuation. Then the completing step is to

4. switch branches.

In the branch switching, it is needed to calculate one solution on each emanating branch. This "first" solution provides information on the quality of the solutions on that new branch, and on its direction. The four basic tasks of the computational bifurcation analysis are summarized in Figure 4.1.

A qualitative bifurcation analysis involves even more tasks. For example, the linear stability of at least one solution on either side of a bifurcation needs to be tested. To obtain a more global picture, the approximate domain of attraction of a stable solution will be explored by selecting initial vectors in a larger neighborhood, and by integrating the initial-value problems until it becomes clear to which attractor the trajectory is approaching. This kind of expensive exploration by simulation frequently will be based on a trial-and-error-basis. The final aim is to explore the diameter of the domain of attraction to get a feeling for the sensitivity of a stable solution. The question is, how large a perturbation of a stable solution is allowed to be such that the response to the perturbation decays to zero. Naturally, the various kinds of bifurcation have required to develop various different solution strategies to the above-mentioned tasks.

Chapter 5

Examples

In this chapter, we will demonstrate how the described tools can be used to analyze the limit cycling phenomena in the following two examples.

5.1 A Planar System

Consider the following planar system [2]

$$\dot{x} = -\mu x + (\mu^2 - 1)y + (1 - x^2 - y^2)^2((1 - \mu^2)x - \mu y) \tag{5.1}$$
$$\dot{y} = (1 - \mu^2)x - \mu y + (1 - x^2 - y^2)^2(\mu x + (1 - \mu^2)y) \tag{5.2}$$

where μ is a real parameter, and $|\mu|$ is small. It can be checked easily that the only equilibrium point of this system is $(x, y) = (0, 0)$. Performing the polar transformation $x = r\cos\theta,\ y = r\sin\theta$, the above system becomes

$$\dot{r} = r((1 - r^2)^2(1 - \mu^2) - \mu) \tag{5.3}$$
$$\dot{\theta} = (1 - r^2)^2\mu + 1 - \mu^2. \tag{5.4}$$

This transformation helps us to define the annulus region D where the ω-limit set of (5.1-5.2) lies . The equilibrium of (5.3) will the radius of the periodic orbit given $\dot{\theta} \neq 0$. For $|\mu| < 1,\ \mu \neq 0$, we have $\dot{\theta} = 0$ iff $(1 - r^2)^2 = (\mu^2 - 1)/\mu$ which has solutions only if $\mu < 0$. It is because $\mu^2 < 1$ for $|\mu| < 1$. On the other hand, for $\dot{r} = 0$ apart from $r = 0$ we have

$$r = \sqrt{1 \pm (\mu/(1 - \mu^2))^{1/2}}, \tag{5.5}$$

i.e. for $-1 < \mu < 0$ the system has no periodic orbit, and since $\dot{r} > -\mu r$, all solutions tend to infinity as $t \to \infty$. At $\mu = 0$, from (5.5) we have $r = 1$ and from (5.4) we have $\dot{\theta} = 1$, thus the system has a single periodic orbit which is given by

$$x^2 + y^2 = 1. \tag{5.6}$$

For $0 < \mu < 1$, following (5.5) the system has two periodic orbits whose equations are

$$x^2 + y^2 = 1 - (\mu/(1 - \mu^2))^{1/2}, \text{ and} \tag{5.7}$$
$$x^2 + y^2 = 1 + (\mu/(1 - \mu^2))^{1/2}, \tag{5.8}$$

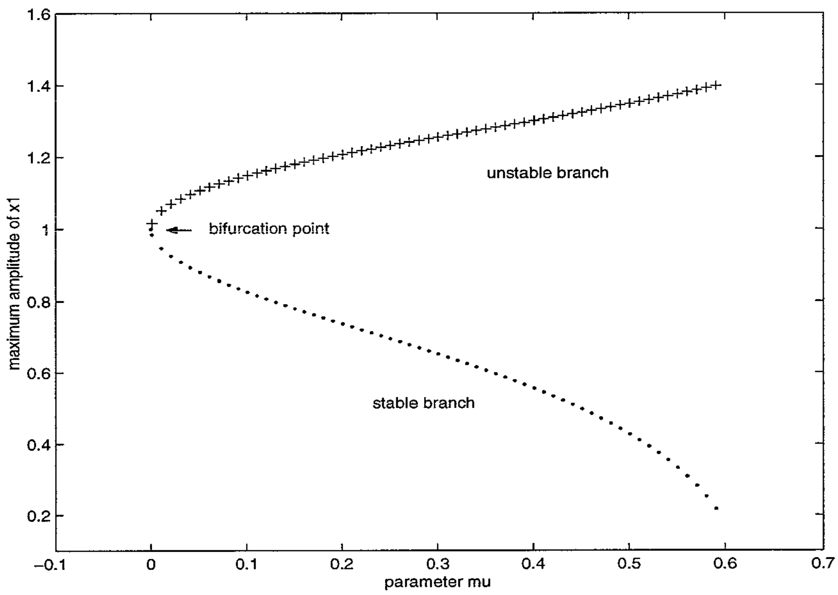

Figure 5.1: Bifurcation Diagram of Limit Cycles of 5.1 and 5.2

provided the right-hand side of (5.7) is positive. The right-hand side of (5.7) is positive if $0 < \mu < (-1+\sqrt{5})/2 \approx 0.618$. The eigenvalues of the Jacobian matrix of (5.1-5.2) at $(0,0)$, i.e. linearization around the equilibrium, are

$$\lambda_{1,2} = -\mu^2 - \mu + 1 \pm i(\mu^2 - \mu - 1). \tag{5.9}$$

It follows that the equilibrium is unstable for $-1.6 < \mu < 0.618$. This gives us insight that the small periodic orbit (5.7) should be a stable limit cycle and the bigger periodic orbit (5.8) is an unstable limit cycle. It agrees with the fact that the sign of $\dot{r}$ is positive inside the smaller periodic orbit, $\dot{r}$ is negative between the two periodic orbits, and $\dot{r}$ is positive outside the bigger periodic orbit, which confirms that the smaller orbit is stable and the bigger one is unstable.

Since the system (5.1-5.2) has one stable limit cycle and one unstable limit cycle for $0 < \mu < 0.618$, and one periodic orbit at $\mu = 0$, then periodic solutions of (5.1-5.2) experience fold bifurcation where the bifurcation point is obtained at $\mu = 0$.

By using equations (5.6), (5.7), and (5.8) we can compute the bifurcation diagram of the system (5.1-5.2) which is presented in Figure 5.1.

5.2 Higher Dimensional System

For planar system we can solve its periodic solutions analytically and compute its bifurcation diagram from these solutions. It is very difficult if not impossible to solve the periodic solutions of higher dimensional systems analytically. Fortunately we have computational method, which is described in Chapter 5, to find periodic solutions and computes its bifurcation diagram of higher dimensional systems.

Consider a model of the dynamics of the photon-excitation interaction of an optical

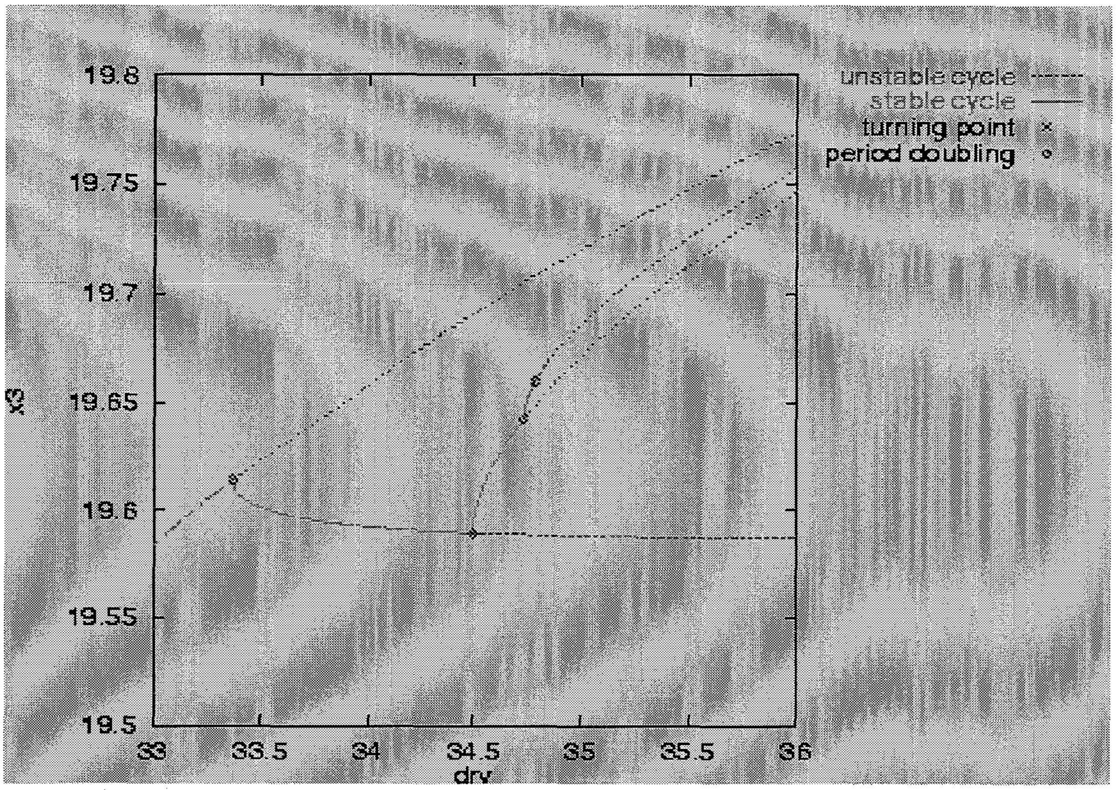

Figure 5.2: Period doubling bifurcations of system (5.10-5.14)

semiconductor, which is described by 5-dimensional autonomous differential equations [5]:

$$\dot{x}_{1r} = -x_{1r} + (g_{1i} + g_{2i}x_3)x_{2r} + (g_{1r} + g_{2r}x_3)x_{2i} + g_3(x_{1r}^2 + x_{1i}^2)x_{1i} \quad (5.10)$$

$$\dot{x}_{1i} = -x_{1i} - (g_{1r} + g_{2r}x_3)x_{2r} + (g_{1i} + g_{2i}x_3)x_{2i} - g_3(x_{1r}^2 + x_{1i}^2)x_{1r} \quad (5.11)$$

$$\dot{x}_{2r} = -(g_{1i} + g_{2i}x_3)x_{1r} + (g_{1r} + g_{2r}x_3)x_{1i} - \sigma_2 x_{2r} + d \quad (5.12)$$

$$\dot{x}_{2i} = -(g_{1r} + g_{2r}x_3)x_{1r} - (g_{1i} + g_{2i}x_3)x_{1i} - \sigma_2 x_{2i} \quad (5.13)$$

$$\dot{x}_3 = -2\sigma_2 x_3 + g_4(x_{1r}^2 + x_{1i}^2) \quad (5.14)$$

where $g_{1r} = 0.5$, $g_{1i} = 40.0$, $g_{2r} = 0.01$, $g_{2i} = -2.0$, $g_3 = 5.0$, $g_4 = 0.5$, $\sigma_2 = 1$. From the physical point of view, the crucial bifurcation parameter is d which acts as a constant driving force of the system.

The periodic solutions and their bifurcations with respect to the parameter d of the system (5.10-5.14) are computed by using the continuation software package CANDYS/QA. The resulting bifurcation diagram for $33 \leq d \leq 36$ is depicted in Figure 5.2, which shows a cascade of period doubling bifurcation of limit cycles.

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