

## Controllability of non-smooth systems

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Controllability of  
Non-smooth Systems**

Report No. DCT 2005.129

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Eindhoven, The Netherlands, August 2005



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# Controllability of Non-smooth Systems

Date: August 2005

Editors: Kanat Camlibel, Henk Nijmeijer

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# 1

## Introduction

The notion of controllability has played a central role throughout the history of modern control theory. Conceived by Kalman, the controllability concept has been studied extensively in the context of finite-dimensional linear systems, nonlinear systems, infinite-dimensional systems,  $n$ -D systems, hybrid systems, and behavioral systems. One may refer for instance to Sontag's book [2] for historical comments and references.

Outside the linear context, characterizations of global controllability have been hard to obtain. In the setting of smooth nonlinear systems, results have been obtained for local controllability but there is no hope to obtain general algebraic characterizations of controllability in the large. The complexity of characterizing controllability has been studied by Blondel and Tsitsiklis [1] for some classes of hybrid systems, and these authors show that even within quite limited classes there is no algorithm to decide the controllability status of a given system.

In this report, the results on the controllability of non-smooth systems attained within Work-Package 5 of SICONOS are presented. In the first chapter, Bernard Brogliato studies the reachability within a prespecified region of planar variational inequality systems. Based on a case-by-case analysis, necessary and sufficient conditions are obtained. The second contribution is due to Kanat Camlibel, Maurice Heemels and Hans Schumacher. This chapter studies the controllability of a class of piecewise linear systems that are called conewise linear systems. Basically, these are systems for which the product of the state space and the input space is covered by a finite number of conical regions, and that on each of these regions separately we have linear dynamics, with continuous transitions between different regimes. By employing geometric control tools, as well as mathematical programming methods, the contribution presents algebraic necessary and sufficient conditions.

## References

1. V.D. Blondel and J.N. Tsitsiklis. Complexity of stability and controllability of elementary hybrid systems. *Automatica*, 35(3):479–490, 1999.
2. E.D. Sontag. *Mathematical Control Theory: Deterministic Finite Dimensional Systems*. Springer, New York, 2nd edition, 1998.



## 2

# On the controllability of planar variational inequalities

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**Abstract.** This note deals with the controllability of a class of planar complementarity dynamical systems, which can also be viewed as planar evolution variational inequalities. It is shown that the complementarity conditions influence a lot the controllability of the system.

## 1 Introduction

Hybrid dynamical systems constitute a very large class of systems [2]. It is consequently necessary to focus on specific subclasses to make their study possible, see e.g. [13] for controllability issues in piecewise-linear systems. An interesting subclass is made of so-called complementarity systems [12] [1]. Similarly to the fact that the stability of unilaterally constrained systems can significantly differ from that of their unconstrained counterpart [3] [4], it will be shown that their controllability properties can differ a lot as well. This reinforces the fact that such nonsmooth dynamical systems deserve full attention and are not a mere extension of unconstrained or bilaterally constrained systems. In this note we will restrict ourselves to a narrow class of complementarity systems, that we call planar evolution variational inequalities. These systems are also sometimes called projected dynamical systems [3] [6] and are used to model the dynamics of oligopolistic markets, spatial price equilibrium, elastic demand traffic equilibrium [3]. As illustrated at the end of the note, they can also model some electrical circuits with ideal diodes. In this note it is shown that the controllability of such systems depend a lot on the convex set within which the state is constrained to evolve.

## 2 Planar evolution variational inequalities

The linear complementarity systems (LCS) [1] we are dealing with in this study, possess the following dynamics

$$\begin{cases} \dot{z}_1(t) = z_2(t) + C_1^T \lambda \\ \dot{z}_2(t) = u(t) + C_2^T \lambda \\ 0 \leq \lambda \perp Cz(t) + d \geq 0 \end{cases} \quad (1)$$

where  $C = (C_1 \ C_2) \in \mathbb{R}^{m \times 2}$ ,  $C_1 \in \mathbb{R}^m$  and  $C_2 \in \mathbb{R}^m$  are the two columns of  $C$ ,  $d \in \mathbb{R}^m$ ,  $\lambda \in \mathbb{R}^m$ . The non-negativity is understood componentwise and  $m < +\infty$ . The LCS in (1) is equivalent to the linear evolution variational inequality (VI)

$$\begin{cases} \langle \dot{z}(t) - Az(t) - Bu(t), v - z(t) \rangle \geq 0, \forall v \in K \\ z(t) \in K, \forall t \geq 0 \end{cases} \quad (2)$$

where  $z = (z_1, z_2)^T \in \mathbb{R}^2$ ,  $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ ,  $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ ,  $K = \{z \mid Cz + d \geq 0\} = \{(z_1, z_2) \in \mathbb{R}^2 \mid C_1 z_1 + C_2 z_2 + d \geq 0\}$ . The equivalence between (1) and (2) is obtained by noting that

$$\begin{aligned}
 & \begin{cases} \dot{z}(t) = Az(t) + Bu(t) + C^T \lambda \\ 0 \leq \lambda \perp Cz(t) + d \geq 0 \end{cases} & \Leftrightarrow & -\dot{z}(t) + Az(t) + Bu(t) \in C^T \partial \psi_{(\mathbb{R}^+)^m}(Cz(t) + d) \\
 & \Leftrightarrow & \begin{cases} -\dot{z}(t) + Az(t) + Bu(t) \in N_K(z(t)) \\ z(t) \in K, \forall t \geq 0 \end{cases} & (3)
 \end{aligned}$$

where  $\psi_{(\mathbb{R}^+)^m}(\cdot)$  is the indicator function of  $(\mathbb{R}^+)^m$ ,  $\partial \psi_{(\mathbb{R}^+)^m}(\cdot)$  is its subdifferential, and  $N_K(\cdot)$  is the normal cone to  $K$ . The last formalism in (3) is exactly (2), from the definition of the normal cone. These equivalences are obtained from standard convex analysis rules and definitions [14] and are not developed further here for the sake of brevity.

It is noteworthy that, seen from an LCS point of view, the controlled dynamics in (1) is rather a narrow class. However the VI formalism in (2) shows that it is not so restrictive from an application point of view, since VI are widely used in some domains of science (see [3] for market and finance applications). VI can also represent some electrical circuits with ideal diodes [4]. The LCS in (1) is a particular gradient complementarity system [7], which is in turn equivalent to so-called projected dynamical systems [3] [5] [6]. There certainly remains a big gap between this work, and obtaining similar results for general projected dynamical systems (to say nothing for general LCS). Since the studies on controllability of this type of dynamical systems are rare, this chapter nevertheless has some interest. The following lemma is a direct consequence of [8, corollary 2.2]:

**Lemma 1.** *Consider the system in (1). For all continuous inputs  $u(\cdot)$  with locally  $L^1$ -bounded derivatives  $\frac{du}{dt}(\cdot)$ , a continuous, right differentiable solution with locally bounded derivative exists and is unique on  $[0, +\infty)$ .*

Let us now introduce a controllability definition.

**Definition 1.** *The system in (1) (equivalently in (2)) is said to be  $K$ -controllable, if any state  $z_f \in K$  can be reached from any state  $z_i \in K$ , in a finite or infinite time  $T$ , and with an admissible input  $u(\cdot)$ .*

Admissibility of the input means that the well-posedness conditions of lemma 1 are respected. We do not make the difference between finite and infinite  $T$  to simplify the presentation (as we shall see below, this allows us to consider the controllability in the whole of the closed convex set  $K$  without excluding some isolated points of the boundary  $\partial K$ ).

The objective of this work is to prove that, under some restrictions on the convex set  $K$ ,  $K$ -controllability holds. To begin with and to motivate the study, let us remark that in case  $m = 1$  and  $K = \{z | z_2 \geq -c, c < 0\}$ , then surely the system is not  $K$ -controllable. Indeed  $z_1$  can only move from the left to the right in the phase plane, since  $\dot{z}_1 = z_2 \geq -c > 0$ . This controlled VI is accessible [10] with reachable subspaces from  $(z_1(0), z_2(0))$  equal to  $\{(z_1, z_2) | z_1 \geq z_1(0), z_2 \geq -c\}$ , but not  $K$ -controllable.

Let us note that adding some “imaginary” state re-initialization rules on  $\partial K$  such that  $K$ -controllability holds, is not envisaged here since the dynamical systems in (1) or (2) are the topic of the study. However motivated by this simple example of non-controllability, one guesses that a crucial step in the study will be to prove whether or not one is able to move on  $\partial K$  in order to reach some states which are otherwise unreachable. Due to the complementarity conditions (third line in (1)) which imply that the vector field is modified when  $\partial K$  is attained, this will under certain conditions be possible.

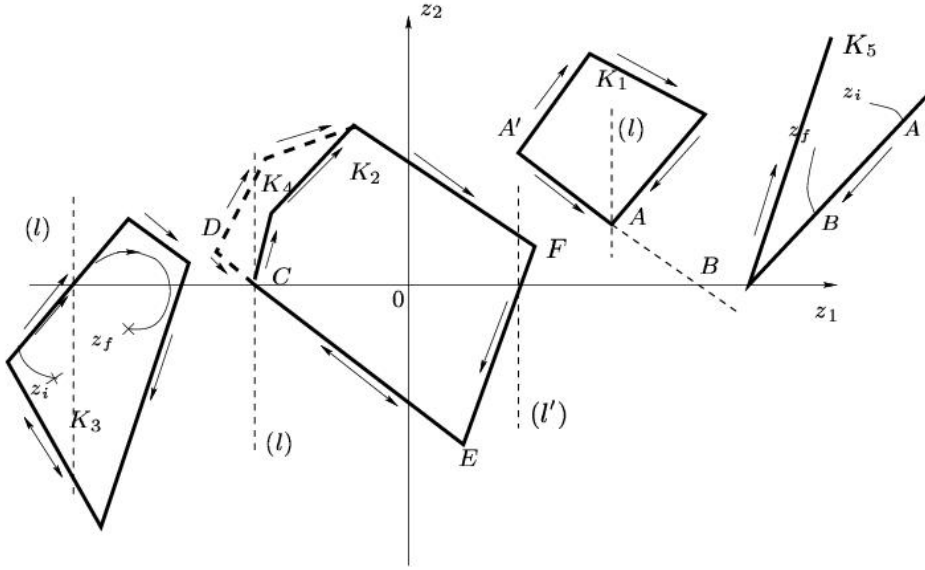
### 3 Main result

The following assumption is made and supposed to hold in the sequel:

**Assumption 1** *The set  $K$  has a positive measure in  $\mathbb{R}^2$ .*

It is easy to construct  $C$  and  $d$  in (1) such that indeed  $K = \emptyset$  or it has zero measure. Polyhedra with a positive area are an example of sets  $K$ , as well as cones (see figure 1), or half-planes.

Let  $C_1 = (a_1, \dots, a_m)^T$ ,  $C_2 = (b_1, \dots, b_m)^T$ ,  $d = (d_1, \dots, d_m)^T$  and let us denote the faces of the convex set  $K$  as  $D^i$ , such that  $D^i \subseteq \{z | a_i z_1 + b_i z_2 + d_i = 0\}$  and  $\bar{D}^i = \{z | a_i z_1 + b_i z_2 + d_i = 0\}$ . In other words the faces are segments  $D^i$  (possibly unbounded, like in the case  $K$  is a cone, or if  $K$  is defined as a half-space), and the segments can be extended to straight lines  $\bar{D}^i$  whose equations in the plane are  $a_i z_1 + b_i z_2 + d_i = 0$ ,  $1 \leq i \leq m$ . For instance on figure 1 and considering the set  $K_1$ , one has  $D^1 = A'A$  whereas  $\bar{D}^1$  is the line passing through  $A'$  and  $A$  and intersecting  $\{z | z_2 = 0\}$  at  $B$ . Let us place ourselves in the phase plane of the system, with the two axis  $(0, z_1)$  and  $(0, z_2)$ .



**Fig. 1.** Examples of  $K$ -controllable and  $K$ -uncontrollable systems.

Then the following is true

**Proposition 1.** *The system in (1) (equivalently in (2)) is  $K$ -controllable if and only if there is no face of  $K$  such that:*

- *there is a portion of  $D^i$  with finite negative slope on the right (resp. left) of the point  $\bar{D}^i \cap \{z | z_1 = 0\}$ , when  $K$  is below (resp. above)  $D^i$ .*
- *$D^i$  is vertical and above (resp. below)  $\{z | z_2 = 0\}$  if  $K$  is on the right (resp. left) of  $D^i$ .*
- *$D^i$  is horizontal and in the half-space  $\{z | z_2 < 0\}$  (resp.  $\{z | z_2 > 0\}$ ) if  $K$  is below (resp. above)  $D^i$ .*
- *$D^i = \{z | z_2 = 0\}$ .*

■

For instance on figure 1, the faces  $A'A$  of  $K_1$ , or  $DC$  of  $K_4$ , preclude controllability because they satisfy the first item.

Let us state intermediate results which characterize the motion on the boundary  $\partial K$ . The proof of proposition 1 will then be a direct consequence of lemma 2. In the next lemma we place ourselves in the case when there is a single constraint and we study the behaviour of the system

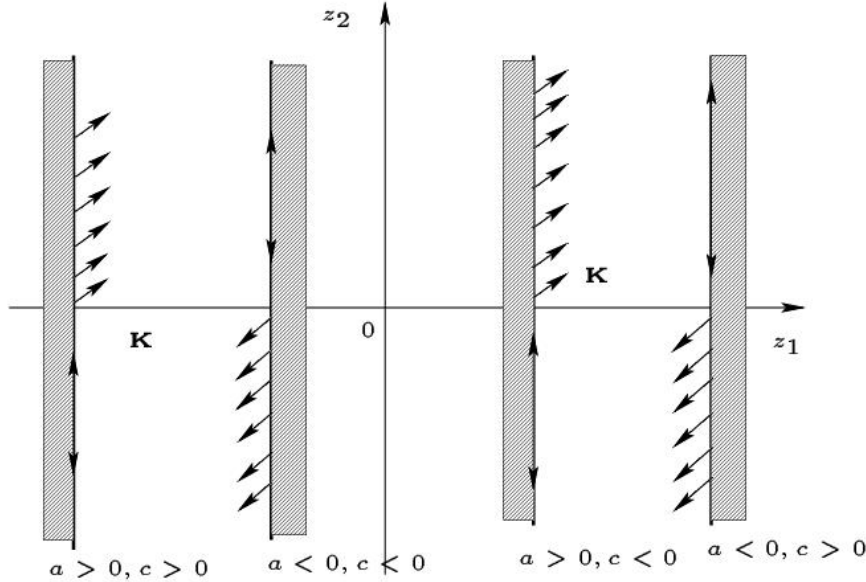


Fig. 2. Trajectories on  $\partial K$  (vertical faces).

on this constraint. When  $K$  has several faces it will suffice to consider each of them separately and apply the results of the lemma independently to each constraint. Let us consider the system in (1) or (2), with  $C_1 = a \in \mathbb{R}$ ,  $C_2 = b \in \mathbb{R}$ ,  $d = c \in \mathbb{R}$ . Let us define the coordinate change  $\begin{cases} x_1 = bz_1 - az_2 + \frac{bc}{a} \\ x_2 = az_1 + bz_2 + c \end{cases}$ . We denote as  $P$  the intersection between the line  $az_1 + bz_2 + c = 0$  and the  $z_1$ -axis, i.e.  $P$  is the origin of the new frame  $(x_1, x_2)$  and the constraint boundary is  $\{x|x_2 = 0\}$ . The new coordinate frame  $(x_1, x_2)$  is depicted on figure 3.

Then the following holds:

**Lemma 2.** – (i)

- (positive slope) If  $-\frac{a}{b} > 0$ ,  $b \neq 0$ , any point  $x_{1f}$  on the constraint can be attained from any point  $x_{1i} \geq x_{1f}$ .
- (negative slope) If  $-\frac{a}{b} < 0$ ,  $b \neq 0$ , then any point  $x_{1f}$  on the constraint can be attained from any point  $x_{1i} \leq x_{1f}$ , and any point  $x_{1f}$  on the constraint can be attained from any point  $x_{1i} \geq x_{1f}$  only on the axis  $x_1 \in [P, +\infty)$ . Moreover the point  $P$  can be attained from any  $x_{1i} > 0$  only asymptotically.
- (ii) If  $a = 0$  then the boundary is a horizontal line  $z_2 = -\frac{c}{b}$  and
  - if  $(b > 0 \text{ and } c < 0)$  or  $(b < 0 \text{ and } c > 0)$ , trajectories move from the left to the right,
  - if  $(b > 0 \text{ and } c > 0)$  or  $(b < 0 \text{ and } c < 0)$ , trajectories move from the right to the left,
  - if  $c = 0$  then the system remains stuck on  $\partial K$  at the contacting point.
- (iii) If  $b = 0$  then the boundary is a vertical line  $z_1 = -\frac{c}{a}$  and
  - if  $(a > 0 \text{ and } c > 0)$  or  $(a > 0 \text{ and } c < 0)$  then the system is controllable in the set  $\{z|z_2 < 0\}$  and any trajectory initialized in the set  $\{z|z_2 \geq 0\}$  detaches from  $\partial K$ ,
  - if  $(a < 0 \text{ and } c < 0)$  or  $(a < 0 \text{ and } c > 0)$  then the system is controllable in the set  $\{z|z_2 > 0\}$  and any trajectory initialized in the set  $\{z|z_2 \leq 0\}$  detaches from  $\partial K$ .

■

Let us note that the case  $a = b = 0$  is meaningless since the system is no longer constrained, hence it is not treated in lemma 2. We note that the two depicted cases can be rotated to obtain

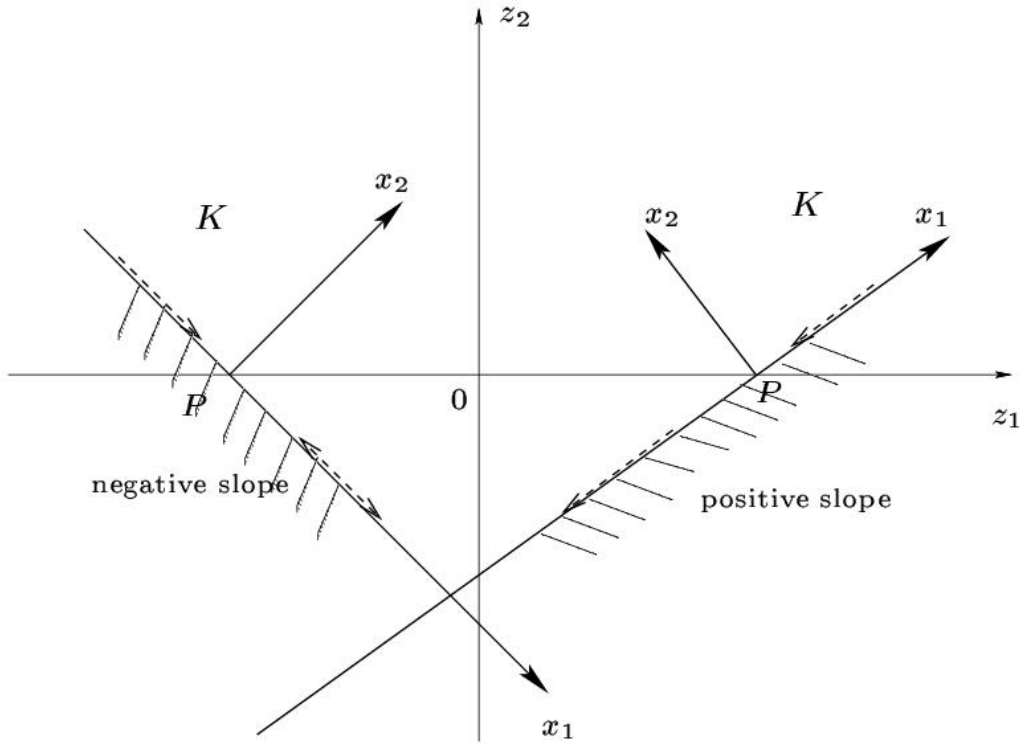


Fig. 3. The new coordinate frame.

the admissible domain below the boundary. The axis  $(P, x_2)$  points inside the admissible set  $K$ . The dashed arrows on  $\partial K$  indicate the directions in which trajectories can be controlled on  $\partial K$ . On figure 1 the arrows also indicate the possible directions of motion on  $\partial K$ . Due to the complementarity conditions, it follows that in some regions of  $\partial K$ , trajectories are restricted to move in a single direction (otherwise they leave  $\partial K$ ). The cases when the boundary is vertical, is depicted in figure 2.

**Proof of lemma 2:** (i) It is simple to calculate that the dynamics (1) in the coordinates  $(x_1, x_2)$  is

$$\begin{cases} \dot{x}_1(t) = \frac{-ab}{a^2+b^2}x_1(t) + \frac{b^2}{a^2+b^2}x_2(t) - au \\ \dot{x}_2(t) = \frac{-a^2}{a^2+b^2}x_1(t) + \frac{ab}{a^2+b^2}x_2(t) + bu + (a^2 + b^2)\lambda \\ 0 \leq \lambda \perp x_2(t) \geq 0 \end{cases} \quad (4)$$

Let us study the dynamics when the system evolves on  $\{x|x_2 = 0\}$  on an interval  $[\tau, \tau + \epsilon)$ ,  $\epsilon > 0$ . Consequently the derivatives  $x_2^{(i)} = 0$  as well for all  $i \geq 1$  on  $(\tau, \tau + \epsilon)$ . Then on  $[\tau, \tau + \epsilon)$  the complementarity condition  $0 \leq \lambda \perp x_2 \geq 0$  implies that  $0 \leq \lambda \perp \dot{x}_2 \geq 0$ . Indeed the “velocity” can point only inside  $K$  on  $[\tau, \tau + \epsilon)$ . Therefore one can replace (4) by

$$\begin{cases} \dot{x}_1(t) = \frac{-ab}{a^2+b^2}x_1(t) + \frac{b^2}{a^2+b^2}x_2(t) - au(t) \\ \dot{x}_2(t) = \frac{-a^2}{a^2+b^2}x_1(t) + \frac{ab}{a^2+b^2}x_2(t) + bu(t) + (a^2 + b^2)\lambda \\ 0 \leq \lambda \perp \dot{x}_2(t) \geq 0 \end{cases} \quad (5)$$

Now since  $x_2 = 0$  on the considered time interval one deduces from (5)

$$\begin{cases} \dot{x}_1(t) = \frac{-ab}{a^2+b^2}x_1(t) - au(t) \\ \frac{-a^2}{a^2+b^2}x_1(t) + bu(t) + (a^2 + b^2)\lambda = 0 \\ 0 \leq \lambda \perp \dot{x}_2(t) \geq 0 \end{cases} \quad (6)$$

Since  $\lambda \geq 0$  on  $\partial K$ , one deduces that  $-\frac{a^2}{a^2+b^2}x_1 + bu \leq 0$  is a necessary and sufficient condition that both  $x_1$  and  $u(\cdot)$  have to satisfy so that the system keeps moving on a face included in  $\{x|x_2 = 0\}$ . Detachment from this face occurs at a time  $t_d$  if and only if  $\dot{x}_2(t_d) > 0$ , which implies  $\lambda(t_d) = 0$  and  $\frac{-a^2}{a^2+b^2}x_1(t_d) + bu(t_d) > 0$ . We can say that the system *grazes* the surface when  $x_2 = 0$  and  $\lambda = 0$ . We conclude that the dynamics on a face included in  $\{x|x_2 = 0\}$  is given by

$$\begin{cases} \dot{x}_1(t) = -\frac{ab}{a^2+b^2}x_1(t) - au(t) \\ -\frac{a^2}{a^2+b^2}x_1(t) + bu(t) \leq 0 \end{cases} \quad (7)$$

If  $a \neq 0$  then the feedback  $u = -\frac{1}{a} \left( v + \frac{ab}{a^2+b^2}x_1 \right)$  yields from (7)

$$\begin{cases} \dot{x}_1(t) = v(t) \\ -x_1(t) - \frac{b}{a}v(t) \leq 0 \end{cases} \quad (8)$$

where  $v$  is the new input. We notice that if  $-x_1 - \frac{b}{a}v = 0$  then the system grazes  $\partial K$ . If  $-\frac{a}{b} > 0$ ,  $b \neq 0$ , then necessarily  $v \leq -\frac{a}{b}x_1$ , and  $v$  can be chosen  $< 0$  so that  $x_1$  can be made to decrease while staying on  $\partial K$ . If  $-\frac{a}{b} < 0$ ,  $b \neq 0$ , then necessarily  $v \geq -\frac{a}{b}x_1$ . If  $x_1 < 0$  then  $v > 0$ , so on  $(-\infty, 0)$ ,  $x_1$  can only increase. On  $(0, +\infty) \ni x_1$ , one can choose  $v = -\frac{a}{b}x_1$  so that  $P$  is attained only asymptotically from any  $x_{1i} > 0$ .

(ii) Now if  $a = 0$  (and consequently  $b \neq 0$ ), the dynamics on  $\partial K$  is given by

$$\begin{cases} \dot{z}_1(t) = -\frac{c}{b} \\ z_2(t) = -\frac{c}{b} \\ u(t) + b\lambda = 0 \text{ and } \lambda \geq 0 \Rightarrow bu(t) \leq 0 \end{cases} \quad (9)$$

This is obtained in a similar way as above, noting that on  $\partial K$  ones has  $b\dot{z}_2 = 0$  and  $0 \leq \lambda \perp b\dot{z}_2 \geq 0$ . The results follow. The detachment from the surface  $bz_2 + c = 0$  occurs if and only if  $b\dot{z}_2(t_d) > 0$  at some time  $t_d$ , i.e.  $bu(t_d) + b^2\lambda(t_d) = bu(t_d) > 0$  (indeed  $\lambda(t_d) = 0$  from the complementarity conditions).

(iii) If  $b = 0$  (and consequently  $a \neq 0$ ) the dynamics on  $\partial K$  is given by

$$\begin{cases} z_1(t) = -\frac{c}{a} \\ \dot{z}_2(t) = u(t) \\ z_2(t) + a\lambda = 0 \text{ and } \lambda \geq 0 \Rightarrow az_2(t) \leq 0 \end{cases} \quad (10)$$

The results stated in lemma 2 (iii) are a direct consequence of (10). ■

**Lemma 3.** *The unilateral constraint  $\{x \mid x_2 \geq 0\}$  in (1) can be activated or deactivated with a continuous input signal  $u(\cdot)$ .*

**Proof:** Let us consider (1) or equivalently (4). The contact phases, or active constraint, are characterized by  $\lambda \geq 0$  and  $x_2 = 0$  whereas the non-contact phases, or inactive constraint, correspond to  $\lambda = 0$  and  $x_2 > 0$ . When steering the state inside  $K$  (i.e. in  $K \setminus \partial K$ ) it is always possible to attain the boundary  $\partial K$ , and to remain on  $\partial K$ , with a continuous input. Indeed this amounts to finding a path in the phase plane  $(z_1, z_2)$ , parameterized by  $t$ , linking two points  $z^0 \in \text{Int}(K)$  and  $z^1 \in \partial K$ , and such that its second derivative with respect to  $t$  satisfies the inequality in (7) on  $\partial K$ . Detachment can also be forced with a continuous control input. Indeed one sees from (5) that as soon as  $\partial K$  is attained  $\lambda$  is the solution of a linear complementarity problem (LCP) with matrix  $a^2 + b^2 > 0$  (a scalar in this case) and consequently depends continuously on  $u(\cdot)$  (see e.g. [15, ex. 4.8.15]). One can speak of a *controlled LCP* in (5) which can be controlled with a continuous input. Consequently the controllability result holds with input signals  $u(\cdot)$  which are continuous and piecewise differentiable. This guarantees the existence of a unique solution of (1) or (4) as a result of lemma 1. It is noteworthy that these results still hold if the codimension of  $\partial K$  is  $\geq 2$  (activation or detachment at corners of  $\partial K$ ).

**Proof of proposition 1:** The proof is done by observing that under the stated conditions, and from lemmas 2 and 3, then any point in  $K$  can be steered by a continuous  $u(\cdot)$  to any other point in  $K$ . Indeed if a state  $z_f$  cannot be attained from  $z_i$  via a trajectory in  $K \setminus \partial K$ , then a portion of path can be tracked on  $\partial K$ . Concatenating paths in the interior of  $K$  and on  $\partial K$  allows one to construct a path linking  $z_i$  to  $z_f$ . The conditions of proposition 1 are sufficient but can also be seen to be necessary, for if one of them fails then there exists couples of states in  $K$  which cannot be joined by a controlled trajectory. ■

*Examples:* From the results of lemma 2, one sees that the boundary of the domain  $K_3$  on figure 1 can be tracked clockwise. Consequently any point  $z_f$  on the right of the line  $(l)$  can be attained from any point  $z_i$  on the left of  $(l)$ . There has to be a portion of the trajectory that evolves on  $\partial K_3$  to reach  $z_f$  from  $z_i$ . Let us consider the set  $K_1$  on figure 1. The system is not  $K_1$ -controllable because the only way to attain a point on the left of the vertical line  $(l)$  from a point on the right of  $(l)$ , is to follow the boundary  $\partial K_1$ . However once the point  $A$  has been reached, it is impossible to move on  $\partial K_1$  towards  $A'$ . The system can be steered on the line  $AA'$  only in the direction of  $B$ . Consequently all points of  $K_1$  which are situated on the left of  $(l)$ , cannot be attained from points in  $K_1$  on the right of  $(l)$ . It is noteworthy that even local controllability [16, Definition 3.7.4] may fail. For instance two arbitrarily close states  $z_i$  and  $z_f$  in  $K_1$ , with  $z_i$  on the right of  $(l)$  and  $z_f$  on the left of  $(l)$ , cannot be joined by a solution of (1) with some control  $u(\cdot)$ . Consider now  $K_2$ . Then trajectories can be controlled from  $E$  to  $C$ , though  $C$  is reachable in infinite time only. Assume that  $C$  is just below the axis  $\{z \mid z_2 = 0\}$ . It follows from lemma 2 that  $\partial K_2$  can be tracked clock-wise by applying some suitable control input. Thus, the points on the right of the vertical line  $(l')$  can be steered to anywhere in  $K_2$  by first moving on  $FE$ . One may say that the dynamics is suitably modified on the boundary  $FE$  so that  $z_1$  can decrease in the first quadrant. In the same way the system is  $K_5$ -controllable, but it is not  $K_4$ -controllable (the states on the left of the line  $(l)$  cannot be reached from the states in  $K_2$ ). The system is  $K_5$ -controllable since as illustrated a state  $z_f$  that cannot be attained from  $z_i$  via a trajectory which remains in  $K \setminus \partial K$ , can be attained via a path  $z_i AB z_f$ .

*Remark 1.* As we said after definition 1, including infinite time  $T$  in the controllability allows us to disregard some isolated points of  $K$  that may not be reachable in finite time. This is the case for the domain  $K_2$  where the point  $C$  can be attained asymptotically only.

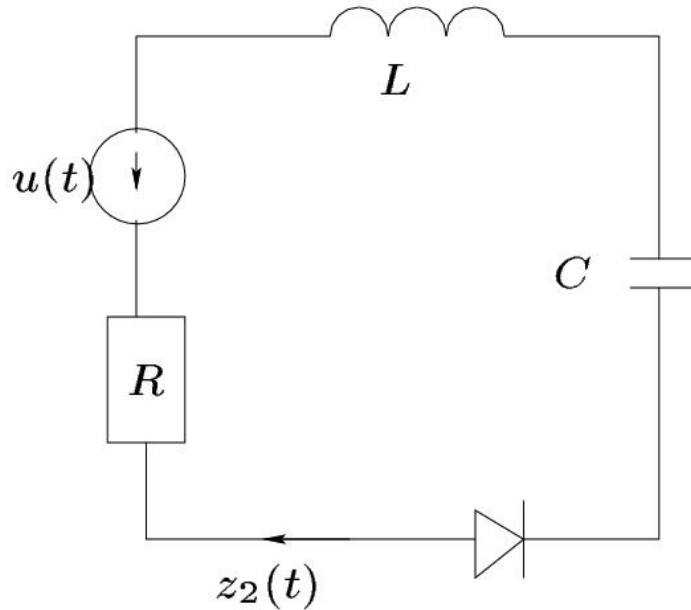


Fig. 4. A simple electrical circuit.

### 4 An example

Let us consider the simple electrical circuit in figure 4, where  $R$  is a resistor,  $L$  is an inductor,  $C$  is a capacitor, and the diode is supposed ideal. Its dynamics is given by

$$\begin{cases} \dot{z}_1(t) = z_2(t) \\ \dot{z}_2(t) = -\frac{R}{L}z_2(t) + \frac{u(t)}{L} - \frac{1}{LC}z_1(t) - \frac{1}{L}\lambda \\ 0 \leq \lambda \perp -z_2(t) \geq 0 \end{cases} \tag{11}$$

where  $z_1(\cdot)$  is the time integral of the current across the capacitor,  $z_2(\cdot)$  is the current across the circuit and  $-\lambda$  is the voltage of the diode,  $u(\cdot)$  is a voltage control. One has  $K = \{z \mid z_2 \leq 0\}$ . One sees that this system is not controllable by simple application of proposition 1. One may transform the system in (11) into the canonical form in (1), by applying a pre-feedback  $u(z_1, z_2) = Lv(t) + \frac{R}{L}z_2 + \frac{1}{LC}z_1$ . In fact the state  $z_1(\cdot)$  can only decrease, or be controlled to a constant value on  $\partial K$ . Consequently the system in (11) is not  $K$ -controllable. This is intuitively sound since it corresponds to having the capacitor loaded with a non-positive current at all times.

### 5 Conclusion

In this note we have proposed a characterisation of the controllability properties of planar evolution variational inequalities with control input. These systems are a subclass of complementarity dynamical systems. They are nonsmooth and nonlinear. The material in this note relies heavily on the properties of the system on the boundary of the constraint set and on the behaviour of the trajectories of planar systems in their phase plane. Consequently an extension of this work should rely on the analytical tools in [11] that characterize the control capabilities of a system,



on the boundary of its admissible domain. The class of systems that is considered is a narrow class of complementarity dynamical systems. However the results show that the controllability of complementarity dynamical systems differs significantly from that of unconstrained systems.

## References

1. W.P.M.H. Heemels, J.M. Schumacher, S. Weiland, 2000 "Linear complementarity systems", *SIAM J. Applied Math.*, vol.60, no 4, pp.1234-1269.
2. A. van der Schaft, H. Schumacher, 2000 *An Introduction to Hybrid Dynamical Systems*, Springer, LNCIS 251, London.
3. A. Nagurney, D. Zhang, 1996 *Projected Dynamical Systems and Variational Inequalities with Applications*, Kluwer's International Series in Operations research and Management Science, Boston.
4. D. Goeleven, B. Brogliato, 2004 "Stability and instability matrices for linear evolution variational inequalities", *IEEE Transactions on Automatic Control*, vol.49, no 4, April.
5. D. Zhang, A. Nagurney 1995 "On the stability of projected dynamical systems", *J. of Optimization Theory and Applications*, vol.85, no 1, pp.97-124.
6. P. Dupuis, A. Nagurney, 1993 "Dynamical systems and variational inequalities", *Annals of Operations Research*, vol.44, pp.9-42.
7. W.P.M.H. Heemels, J.M. Schumacher, S. Weiland, 2000 "Projected dynamical systems in a complementary formalism", *Operations Research Letters*, vol.27, pp.83-91.
8. D. Goeleven, M. Motreanu, V. Motreanu, 2003 "On the stability of stationary solutions of evolution variational inequalities", *Advances in Nonlinear Variational Inequalities*, vol.6, pp.1-30.
9. H.K. Khalil, 1996 *Nonlinear Systems*, 2nd edition, Prentice-Hall.
10. H. Nijmeijer, A. J. van der Schaft, 1990 *Nonlinear Dynamical Control Systems*, Springer Verlag.
11. A.A. ten Dam, E. Dwarshuis, J.C. Willems, 1997 "The contact problem for linear continuous-time dynamical systems: a geometric approach", *IEEE Transactions on Automatic Control*, vol.42, no 4, pp.458-472, April.
12. B. Brogliato, 2003 "Some perspectives on the analysis and control of complementarity dynamical systems", *IEEE Transactions on Automatic Control*, vol.48, no 6, pp.918-935.
13. A. Bemporad, G. Ferrari-Trecate, M. Morari, 2000 "Observability and controllability of piecewise affine and hybrid systems", *IEEE Transactions on Automatic Control*, vol.45, no 10, pp.1854-1864, October.
14. J.B. Hiriart-Urruty, C. Lemaréchal, 2001 *Fundamentals of Convex Analysis*, Springer, Grundlehren Text Editions.
15. F. Facchinei, J.-S. Pang, 2003 *Finite-Dimensional Variational Inequalities and Complementarity Problems, vol.1*, Springer Series in Operations Research, New York.
16. E.D. Sontag, 1998 *Mathematical Control Theory*, Springer Texts in Applied Mathematics 6, New York.

# 3

## On the controllability of conewise linear systems

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**Abstract.** The problem of checking certain controllability properties of even very simple piecewise linear systems is known to be undecidable. This chapter focuses on conewise linear systems, i.e. systems for which the state space is partitioned into conical regions and a linear dynamics is active on each of these regions. For this class of systems, we present algebraic necessary and sufficient conditions for controllability. We also show that the classical results of controllability of linear systems and input-constrained linear systems can be recovered from our main result. Our treatment employs tools both from geometric control theory and mathematical programming.

### 1 Introduction

In this chapter we present algebraically verifiable necessary and sufficient conditions for global controllability of a large class of piecewise linear systems. We assume that the product of the state space and the input space is covered by a finite number of conical regions, and that on each of these regions separately we have linear dynamics, with continuous transitions between different regimes. Systems of this type do appear naturally; some examples are provided in Section 2. The systems that we consider are finite-dimensional, but beyond that there is no restriction on the number of state variables or the number of input variables.

The construction of verifiable necessary and sufficient conditions relies on the fact that, in a situation where different linear systems are obtained by applying different feedbacks to the same output, the zero dynamics of these systems are the same. On the basis of classical results in geometric control theory, the systems may therefore be decomposed in a part which is common and a part which is specific to each separate system but which, due to the invertibility assumption, has a simple structure in the sense that there exists a polynomial inverse. The latter fact may be exploited to “lift” the controllability problem from each separate mode to the common part. The reduced controllability problem in this way is still nonclassical due to the presence of a sign-dependent input nonlinearity. The controllability of such “push-pull” systems may be studied with the aid of results obtained by R.F. Brammer in 1972 [4]. By a suitable adaptation of Brammer’s results, we arrive at the desired characterization of controllability.

Controllability problems for piecewise linear systems have drawn considerable attention recently. Brogliato obtains necessary and sufficient conditions for global controllability of a class of piecewise linear systems in a recent paper [5]. This work applies only to the planar case (state space dimension equal to 2) and is based on a case-by-case analysis which does not seem to carry over easily to higher dimensions. A characterization of controllability has been provided by Lee and Arapostathis [24] for a class of “hypersurface systems”. They assume, among other things, that the number of inputs in each subsystem is equal to the number of states minus one. Their conditions are not stated in an easily verifiable form. An algorithmic approach based on optimization tools is suggested by Bemporad et al. [2]. Although this approach makes it possible to check controllability of a given (discrete-time) system, it does not allow drawing conclusions about any class of systems. The characterization that we obtain in this chapter is much more akin to classical controllability conditions. Characterizations of controllability that apply to some classes

of piecewise linear discrete-time systems have been obtained by Nesic [26]. In continuous time, there is work by Smirnov [32, Ch. 6] that applies to a different class of systems than we consider here, but that is partly similar in spirit. Habets and Van Schuppen [15] discuss “controllability to a facet”, which is a different problem from the one considered here.

The controllability result that we obtain in this chapter can be specialized to obtain a number of particular cases which may be of independent interest. For instance, earlier work in [8] and [6] on planar bimodal systems and on general bimodal systems, which in fact provided the stimulus for continued investigation, can now be recovered as special cases, as is demonstrated in Section 4 below.

The chapter is organized as follows. The class of systems that we consider is defined in Section 2, and some examples are given to show how systems in this class may arise. Some preparatory material about systems with linear dynamics but possibly a constrained input set is collected in Section 3. Section 4 presents the main results and Section 5 concludes. The bulk of the proofs is in Appendix C, which is preceded by two appendices which respectively summarize notation and recall some facts from geometric control theory.

## 2 Conewise linear systems

A special class of piecewise linear systems is of interest in this chapter. These systems consist of a number of linear dynamics that are active on some cones in the input-state space. More specifically, they are systems of the form

$$\dot{x}(t) = Ax(t) + Bu(t) + f(Cx(t) + Du(t)) \quad (1a)$$

$$u(t) \in \mathcal{U} \quad (1b)$$

$$Cx(t) + Du(t) \in \mathcal{Y} \quad (1c)$$

where  $x \in \mathbb{R}^n$  is the state,  $u \in \mathbb{R}^m$  is the input,  $A \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{n \times m}$ ,  $C \in \mathbb{R}^{p \times n}$ ,  $D \in \mathbb{R}^{p \times m}$ ,  $\mathcal{U} \subseteq \mathbb{R}^m$ ,  $\mathcal{Y} \subseteq \mathbb{R}^p$  is a cone, and  $f$  is a *conewise linear function on  $\mathcal{Y}$* , i.e. there exist an integer  $r$ , cones  $\mathcal{Y}_i$ , and matrices  $M^i \in \mathbb{R}^{n \times p}$  for  $i = 1, 2, \dots, r$  such that

$$\cup_{i=1}^r \mathcal{Y}_i = \mathcal{Y}, \quad (2a)$$

$$f(y) = M^i y \text{ if } y \in \mathcal{Y}_i. \quad (2b)$$

These systems will be called *conewise linear systems* (CLS).

Some examples, with an increasing level of generality, are in order.

*Example 1.* The simplest examples of CLSs, except the trivial case of linear systems, are the bimodal piecewise linear systems. Consider, for instance, the mechanical system shown in Figure 1. We assume that all the elements are linear. Let  $x_1$  and  $x_2$  denote the displacements of the left and right cart from the tip of the leftmost spring, respectively. Also let the masses of the carts denoted by  $m_1$  (for the left one) and  $m_2$  (for the other), the spring constants by  $k'$  (for the leftmost one) and  $k$  (for the other), and the damping constant by  $d$ . Then, the governing differential equations can be given by

$$m_1 \ddot{x}_1 + k(x_1 - x_2) + d(\dot{x}_1 - \dot{x}_2) + k' \max(-x_1, 0) = 0 \quad (3)$$

$$m_2 \ddot{x}_2 + k(x_1 - x_2) + d(\dot{x}_1 - \dot{x}_2) = F \quad (4)$$

where  $F$  is the force that is applied to the right cart. By denoting the velocities of the left and right cars, respectively, by  $x_3$  and  $x_4$ , one arrives at the following conewise linear system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k/m_1 & k/m_1 & -d/m_1 & d/m_1 \\ -k/m_2 & k/m_2 & -d/m_2 & d/m_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} F + \begin{bmatrix} 0 \\ 0 \\ 0 \\ f(x_1) \end{bmatrix} \quad (5)$$

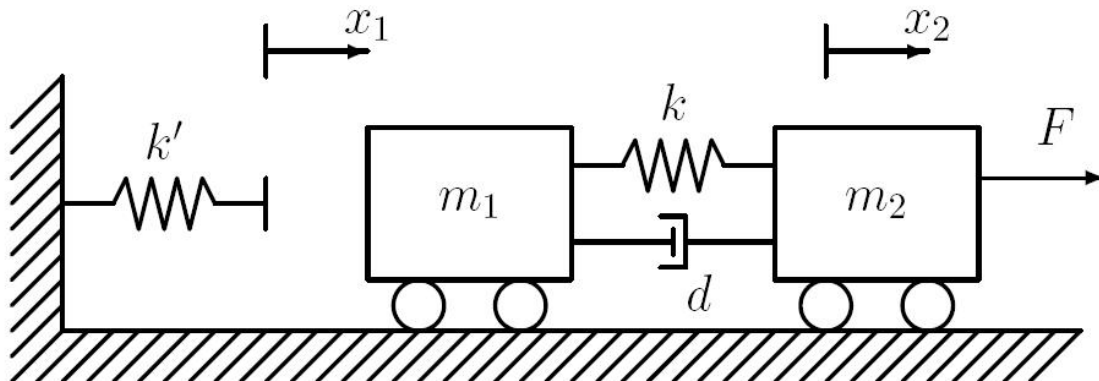


Fig. 1. Linear mechanical system with a one-sided spring

where the function  $f$  is given by

$$f(y) = \begin{cases} -\frac{k'}{m_1}y & \text{if } y \leq 0 \\ 0 & \text{if } y \geq 0 \end{cases}.$$

*Example 2.* In general, a bimodal piecewise linear system with a continuous vector field can be described in the form

$$\dot{x} = \begin{cases} A_1x + B_1u & \text{if } c^T x + d^T u \leq 0, \\ A_2x + B_2u & \text{if } c^T x + d^T u \geq 0 \end{cases} \quad (6)$$

where  $A_1, A_2 \in \mathbb{R}^{n \times n}$ ,  $B_1, B_2 \in \mathbb{R}^{n \times m}$ ,  $c \in \mathbb{R}^n$ , and  $d \in \mathbb{R}^m$  with the property that

$$c^T x + d^T u = 0 \Rightarrow A_1x + B_1u = A_2x + B_2u. \quad (7)$$

Equivalently,  $A_2 - A_1 = ec^T$  and  $B_2 - B_1 = ed^T$  for some  $n$ -vector  $e$ . To fit the system (6) into the framework of CLS (1), one can take  $A = A_1$ ,  $B = B_1$ ,  $C = c^T$ ,  $D = d^T$ ,  $r = 2$ ,  $\mathcal{Y}_1 = (-\infty, 0]$ ,  $M^1 = 0$ ,  $\mathcal{Y}_2 = [0, \infty)$ , and  $M^2 = e$ .

*Example 3.* An interesting example of CLSs arises in the context of linear complementarity systems. Consider the linear system

$$\dot{x} = Ax + Bu + Ez \quad (8a)$$

$$w = Cx + Du + Fz \quad (8b)$$

where  $x \in \mathbb{R}^n$ ,  $u \in \mathbb{R}^m$ , and  $(z, w) \in \mathbb{R}^{p+p}$ . When the external variables  $(z, w)$  satisfy the so-called complementarity relations

$$\mathcal{C} \ni z \perp w \in \mathcal{C}^* \quad (8c)$$

where  $\mathcal{C}$  is a cone and  $\mathcal{C}^*$  is its dual, the overall system (8) is called a *linear cone complementarity system* (LCCS). A wealth of examples, from various areas of engineering as well as operations research, of these piecewise linear (hybrid) systems can be found in [11, 16, 30, 31]. For the work on the analysis of general LCCSs, we refer to [7, 10, 17, 18, 28, 29]. A special case of interest emerges when  $\mathcal{C} = \mathbb{R}_+^p$  and all the principal minors of the matrix  $F$  are positive. Such matrices are called *P*-matrices in the literature of the mathematical programming. It is well-known (see for instance [12, Thm. 3.1.6 and Thm. 3.3.7]) that every positive definite matrix is in this class.

$P$ -matrices enjoy several interesting properties. One of the most well-known is in the context of linear complementarity problem, i.e. the problem of finding a  $p$ -vector  $z$  satisfying

$$0 \leq z \perp q + Fz \geq 0 \quad (9)$$

for a given  $p$ -vector  $q$  and a  $p \times p$  matrix  $F$ . It is denoted by  $\text{LCP}(q, F)$ . When the matrix  $F$  is a  $P$ -matrix,  $\text{LCP}(q, F)$  admits a unique solution for any  $q \in \mathbb{R}^p$ . This is due to a well-known theorem (see [12, Thm. 3.3.7]) of mathematical programming. Moreover, for each  $q$  there exists an index set  $\alpha \subseteq \{1, 2, \dots, p\}$  such that

1.  $-(F_{\alpha\alpha})^{-1}q_\alpha \geq 0$  and  $q_{\alpha^c} - F_{\alpha^c\alpha}(F_{\alpha\alpha})^{-1}q_\alpha \geq 0$ ,
2. the unique solution  $z$  of the  $\text{LCP}(q, F)$  is given by  $z_\alpha = -(F_{\alpha\alpha})^{-1}q_\alpha$  and  $z_{\alpha^c} = 0$

where  $\alpha^c$  denotes the set  $\{1, 2, \dots, p\} \setminus \alpha$ . This shows that the mapping  $q \mapsto z$  is a conewise linear function.

*Remark 1.* CLSs form a special class of linear hybrid systems (see for instance [22]). In fact, they can be cast as hybrid automata for which

- i. the vector fields in each location are linear,
- ii. the invariant sets are cones,
- iii. the guard sets are the boundaries of these cones, and
- iv. the reset maps are all identity.

*Remark 2.* Linear systems with piecewise linear input nonlinearities are of particular interest for this chapter. They can be considered as special cases of Hammerstein-type (see e.g. [14]) systems. A simple example can be given as

$$\dot{x} = Fx + \begin{cases} G_1 u & \text{if } u \leq 0 \\ G_2 u & \text{if } u \geq 0 \end{cases} \quad (10)$$

where  $x \in \mathbb{R}^n$  and  $u \in \mathbb{R}$ . By taking  $A = F$ ,  $B = 0$ ,  $C = 0$ ,  $D = 1$ , and

$$f(y) = \begin{cases} G_1 y & \text{if } y \leq 0 \\ G_2 y & \text{if } y \geq 0 \end{cases} \quad (11)$$

one can fit (10) into the framework of (1).

## 2.1 Solutions of conewise linear systems

We say that an absolutely continuous function  $x$  is a *solution* of (1) for the initial state  $x_0$  and the locally-integrable input  $u$  if  $(x, u)$  satisfies (1) almost everywhere and  $x(0) = x_0$ .

The following will be a standing assumption throughout the chapter.

**Assumption 2** *The cones  $\mathcal{Y}_i$  are polyhedral and solid.*

The cones  $\mathcal{Y}_i$  are closed due to the polyhedrality. This readily implies continuity of the function  $f$ . Obviously, continuity implies Lipschitz continuity for conewise linear functions. This, in turn, guarantees the existence and uniqueness of solutions for all initial states  $x_0 \in \mathbb{R}^n$  if  $\mathcal{U} = \mathbb{R}^m$  and  $\mathcal{Y} = \mathbb{R}^p$ . The case  $\mathcal{U} = \mathbb{R}^m$  and  $\mathcal{Y} = \mathbb{R}^p$  is one of the two cases that we are mainly interested in throughout the chapter. The other case is obtained by taking  $C = 0$ ,  $D = I$ , and  $\mathcal{U} = \mathcal{Y}$ . The existence and uniqueness of solutions for this case is straightforward. From now on, we will be looking at either one of these two cases when we refer to (1).

Let us denote the unique solution of (1) for the initial state  $x_0$  and the input  $u$  by  $x^{x_0, u}$ . We call the system (1)

- (completely) controllable with respect to  $\mathcal{U}$  if for any pair of states  $(x_0, x_f) \in \mathcal{X} \times \mathcal{X}$  there exists a locally integrable input  $u$  such that  $u(t) \in \mathcal{U}$  for almost all  $t \geq 0$  and the solution  $x^{x_0, u}$  of (1) satisfies  $x^{x_0, u}(T) = x_f$  for some  $T > 0$ .
- reachable from zero with respect to  $\mathcal{U}$  if  $0 \in \mathcal{X}$  and for any state  $x_f \in \mathcal{X}$  there exists a locally integrable input  $u$  such that  $u(t) \in \mathcal{U}$  for almost all  $t \geq 0$  and the solution  $x^{0, u}$  of (1) satisfies  $x^{0, u}(T) = x_f$  for some  $T > 0$ .

We sometimes use the term “controllable” instead of “completely controllable”. When the set  $\mathcal{U}$  is not mentioned, it is assumed to be  $\mathbb{R}^m$ . Before proceeding further, we will briefly review the controllability problem for the case of linear dynamics.

### 3 Controllability of linear systems

Consider the linear system

$$\dot{x} = Ax + Bu \tag{12}$$

where  $A \in \mathbb{R}^{n \times n}$  and  $B \in \mathbb{R}^{n \times m}$ .

Ever since Kalman’s seminal work [21] introduced the notion of controllability in the state space framework, it has been one of the central notions in systems and control theory. Tests for controllability were given by Kalman himself and many others (see e.g. [20, 33] for historical details). The following theorem summarizes the classical results on the controllability of linear systems.

**Theorem 1.** *The following statements are equivalent.*

1. The system (12) is completely controllable.
2. The controllability subspace  $\langle A \mid \text{im } B \rangle$  coincides with  $\mathbb{R}^n$ .
3. The controllability matrix  $[B \ AB \ \cdots \ A^{n-1}B]$  is of rank  $n$ .
4. The implication

$$\lambda \in \mathbb{C}, \quad z \in \mathbb{C}^n, \quad z^*A = \lambda z^*, \quad B^T z = 0 \quad \Rightarrow \quad z = 0$$

holds.

5. The rank of the matrix  $[sI - A \ B]$  is equal to  $n$  for all  $s \in \mathbb{C}$ .

Sometimes, we say that the pair  $(A, B)$  is controllable meaning that the associated linear system (12) is completely controllable.

In some situations, one may encounter controllability problems for which the input may only take values from a set  $\mathcal{U} \subset \mathbb{R}^m$ . A typical example of such constrained controllability problems would be a (linear) system that admits only nonnegative controls. Study of constrained controllability goes back to the sixties. Early results consider only restraint sets  $\mathcal{U}$  which contain the origin in their interior (see for instance [23]).

When only nonnegative controls are allowed, the set  $\mathcal{U}$  does not contain the origin in its interior. Saperstone and Yorke [27] were the first to consider such constraint sets. In particular, they considered the case  $\mathcal{U} = [0, 1]^m$ . More general restraint sets were studied by Brammer [4]. The following theorem states necessary and sufficient conditions in case the restraint set is a cone.

**Theorem 2.** *Consider the system (12) together with a solid cone  $\mathcal{U}$  as the restraint set. Then, (12) is completely controllable with respect to  $\mathcal{U}$  if, and only if, the following conditions hold.*

1. The pair  $(A, B)$  is controllable.
2. The implication

$$\lambda \in \mathbb{R}, \quad z \in \mathbb{R}^n, \quad z^T A = \lambda z^T, \quad B^T z \in \mathcal{U}^* \quad \Rightarrow \quad z = 0$$

holds.

The proof of this theorem can be obtained by applying [4, Cor. 3.3] to (12) and its time-reversed version.

Sometimes, we say that a pair  $(A, B)$  is controllable with respect to  $\mathcal{U}$  whenever the linear system (12) is completely controllable with respect to  $\mathcal{U}$ .

The special case  $\mathcal{U} = \mathbb{R}_+^m$ , i.e. the case in which only nonnegative inputs are allowed, is of particular interest. In this case, Brammer's result boils down to the following corollary.

**Corollary 1.** *Consider the system (12) for which the locally-integrable input function is constrained by  $u(t) \geq 0$  for almost all  $t$ . Then, (12) is completely controllable if, and only if, the following conditions hold.*

1. *The pair  $(A, B)$  is controllable.*
2. *The implication*

$$\lambda \in \mathbb{R}, \quad z \in \mathbb{R}^n, \quad z^T A = \lambda z^T, \quad B^T z \geq 0 \quad \Rightarrow \quad z = 0$$

*holds.*

## 4 Main results

### 4.1 Controllability of push-pull systems

The special case of CLS (1) when  $C = 0$ ,  $D = I$ , and  $\mathcal{Y} = \mathcal{U}$  is of particular interest for the controllability discussion as we shall see later. In this case, one has a CLS of the form

$$\dot{x} = Ax + f(u) \tag{13a}$$

$$u \in \mathcal{U} \tag{13b}$$

where  $x \in \mathbb{R}^n$ ,  $u \in \mathbb{R}^m$ ,  $A \in \mathbb{R}^{n \times n}$ , and  $f$  is a conewise linear function as defined in (2). Note that  $\mathcal{U} = \mathcal{Y} = \cup_{i=1}^r \mathcal{Y}_i$  for this case.

Notice that these systems are of the form of Hammerstein systems (see e.g. [14]). We prefer to call systems of the type (13) *push-pull systems*. The terminology is motivated by the following special case. Consider the system

$$\dot{x} = Ax + \begin{cases} B_1 u & \text{if } u \leq 0 \\ B_2 u & \text{if } u \geq 0 \end{cases} \tag{14}$$

where the input  $u$  is a scalar. In a sense, “pushing” and “pulling” have different effects for this system.

The following theorem presents necessary and sufficient conditions for the controllability of push-pull systems. Later, we will show that controllability problem of a conewise linear system can always be reduced to that of a corresponding push-pull system.

**Theorem 3.** *The following statements are equivalent.*

1. *The system (13) is completely controllable with respect to  $\mathcal{U}$ .*
2. *The system (13) is completely controllable with respect to  $\mathcal{U}$  with  $C^\infty$ -inputs.*
3. *The system (13) is reachable from zero with respect to  $\mathcal{U}$ .*
4. *The system (13) is reachable from zero with respect to  $\mathcal{U}$  with  $C^\infty$ -inputs.*
5. *The implication*

$$z^T \exp(At) f(u) \geq 0 \text{ for all } t \geq 0 \text{ and } u \in \mathcal{U} \quad \Rightarrow \quad z = 0 \tag{15}$$

*holds.*

6. *The pair  $(A, [M^1 \ M^2 \ \dots \ M^r])$  is completely controllable with respect to  $\mathcal{Y}_1 \times \mathcal{Y}_2 \times \dots \times \mathcal{Y}_r$ .*

## 4.2 Controllability of conewise linear systems

Consider the CLS (1) with  $m = p$  and  $\mathcal{U} = \mathcal{Y} = \mathbb{R}^m$ . Our first aim is to put it into a certain canonical form. Let  $\mathcal{V}^*$  and  $\mathcal{T}^*$ , respectively, denote the largest output-nulling controlled invariant and the smallest input-containing conditioned invariant subspaces of the system  $\Sigma(A, B, C, D)$ . Also let  $K \in \mathcal{K}(\mathcal{V}^*)$ . Apply the feedback law  $u = -Kx + v$  where  $v$  is the new input. Then, (1) becomes

$$\dot{x} = (A - BK)x + Bv + f(y) \quad (16a)$$

$$y = (C - DK)x + Dv \in \mathcal{Y}. \quad (16b)$$

Obviously, controllability is invariant under this feedback. Moreover, the systems  $\Sigma(A, B, C, D)$  and  $\Sigma(A - BK, B, C - DK, D)$  share the same  $\mathcal{V}^*$  and  $\mathcal{T}^*$  due to Proposition 1 (see Appendix B). Suppose that the transfer matrix  $D + C(sI - A)^{-1}B$  is invertible as a rational matrix. Proposition 2 implies that the state space  $\mathbb{R}^n$  admits the following decomposition

$$\mathbb{R}^n = \mathcal{V}^* \oplus \mathcal{T}^*. \quad (17)$$

Let the dimensions of the subspaces  $\mathcal{V}^*$  and  $\mathcal{T}^*$  be  $n_1$  and  $n_2$ , respectively. Also let the vectors  $\{x_1, x_2, \dots, x_n\}$  be a basis for  $\mathcal{X}$  such that the first  $n_1$  vectors form a basis for  $\mathcal{V}^*$  and the last  $n_2$  for  $\mathcal{T}^*$ . Also let  $L \in \mathcal{L}(\mathcal{T}^*)$ . One immediately gets

$$B - LD = \begin{bmatrix} 0 \\ B'_2 \end{bmatrix} \quad (18)$$

$$C - DK = [0 \ C_2] \quad (19)$$

in the coordinates that are adapted to the above basis as  $\mathcal{V}^* \subseteq \ker(C - DK)$  and  $\text{im}(B - LD) \subseteq \mathcal{T}^*$ . Here  $B'_2$  and  $C_2$  are  $n_2 \times m$  and  $p \times n_2$  matrices, respectively. Note that  $(A - BK - LC + LDK)\mathcal{V}^* \subseteq \mathcal{V}^*$  and  $(A - BK - LC + LDK)\mathcal{T}^* \subseteq \mathcal{T}^*$  according to Proposition 1. Therefore, the matrix  $(A - BK - LC + LDK)$  should be of the form  $\begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix}$  in the new coordinates where the row (column) blocks have  $n_1$  and  $n_2$  rows (columns), respectively. Let the matrices  $K$  and  $L$  be partitioned as

$$K = [K_1 \ K_2] \quad L = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}$$

where  $K_k$  and  $L_k$  are  $m \times n_k$  and  $n_k \times m$  matrices, respectively. With these partitions, one gets

$$A - BK = \begin{bmatrix} A_{11} & L_1 C_2 \\ 0 & A_{22} \end{bmatrix} \quad (20a)$$

$$B = \begin{bmatrix} L_1 D \\ B_2 \end{bmatrix} \quad (20b)$$

where  $A_{kk}$  and  $B_2$  are matrices of the sizes  $n_k \times n_k$  and  $n_2 \times m$ . Also let the matrices  $M^i$ , in the new coordinates, be partitioned as

$$M^i = \begin{bmatrix} M_1^i \\ M_2^i \end{bmatrix} \quad (21)$$

where  $M_k^i$  is a matrix of the size  $n_k \times m$  and let  $f_k$  be defined accordingly as

$$f_k(y) = M_k^i y \text{ if } y \in \mathcal{Y}_i. \quad (22)$$

Now, one can write (16) in the new coordinates as

$$\dot{x}_1 = A_{11}x_1 + g(y) \quad (23a)$$

$$\dot{x}_2 = A_{22}x_2 + B_2v + f_2(y) \quad (23b)$$

$$y = C_2x_2 + Dv \quad (23c)$$



where  $g(y) = L_1 y + f_1(y)$  is a conewise linear function.

By construction, one has

$$\mathcal{V}^*(A_{22}, B_2, C_2, D) = \{0\} \quad (24a)$$

$$\mathcal{T}^*(A_{22}, B_2, C_2, D) = \mathbb{R}^{n_2}. \quad (24b)$$

We already know from the invertibility hypothesis and Proposition 2 that the matrix  $[C_2 \ D]$  is of full row rank and the matrix  $\text{col}(B_2, D)$  is of full column rank. Therefore, Proposition 2 guarantees that the transfer matrix of the system  $\Sigma(A_{22}, B_2, C_2, D)$  has a polynomial inverse. This allows us, as stated in the following lemma, to reduce the controllability problem of the CLS (23) to that of the push-pull system (23a) where the variable  $y$  is considered as the input.

**Lemma 1.** *Consider the CLS (1) such that  $p = m$ ,  $\mathcal{U} = \mathcal{Y} = \mathbb{R}^m$ , Assumption 2 holds, and the transfer matrix  $D + C(sI - A)^{-1}B$  is invertible as a rational matrix. Then, the following statements are equivalent.*

1. The CLS (1) is completely controllable.
2. The push-pull system

$$\dot{x}_1 = A_{11}x_1 + g(y) \quad (25)$$

*is completely controllable with respect to  $\mathcal{Y}$ .*

By combining the above lemma with Theorem 3, we are in a position to present the main result of the chapter.

**Theorem 4.** *Consider the CLS (1) such that  $p = m$ ,  $\mathcal{U} = \mathcal{Y} = \mathbb{R}^m$ , Assumption 2 holds, and the transfer matrix  $D + C(sI - A)^{-1}B$  is invertible as a rational matrix. The CLS (1) is completely controllable if, and only if,*

1. the relation

$$\sum_{i=1}^r \langle A + M^i C \mid \text{im}(B + M^i D) \rangle = \mathbb{R}^n \quad (26)$$

*is satisfied and*

2. the implication

$$\lambda \in \mathbb{R}, z \in \mathbb{R}^n, w_i \in \mathbb{R}^m$$

$$\begin{bmatrix} z^T & w_i^T \end{bmatrix} \begin{bmatrix} A + M^i C - \lambda I & B + M^i D \\ C & D \end{bmatrix} = 0, w_i \in \mathcal{Y}_i^* \text{ for all } i = 1, 2, \dots, r \Rightarrow z = 0$$

*holds.*

*Remark 3.* Note that the the second condition is a statement about the real invariant zeros and the invariant left zero directions of the systems  $\Sigma(A + M^i C, B + M^i D, C, D)$ . A quick observation shows that the invariant zeros of the systems  $\Sigma(A + M^i C, B + M^i D, C, D)$  coincide. They also coincide with the invariant zeros of the system  $\Sigma(A, B, C, D)$ . Therefore, this condition comes to play only if the system  $\Sigma(A, B, C, D)$  has some real invariant zeros.

*Remark 4.* The necessity of the first condition is rather intuitive. What might be curious is that this condition is not sufficient as shown by the following example. Consider the bimodal system

$$\dot{x}_1 = \begin{cases} x_2 & \text{if } x_2 \leq 0 \\ -x_2 & \text{if } x_2 \geq 0 \end{cases}$$

$$\dot{x}_2 = u.$$

In order to cast this system as a CLS, one can take

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = [0 \ 1], \quad D = 0 \quad (27)$$

$$\mathcal{Y}_1 = \mathbb{R}_-, \quad M^1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathcal{Y}_2 = \mathbb{R}_+, \quad M^2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}. \quad (28)$$

Straightforward calculations yield that  $\langle A + M^1 C \mid \text{im}(B + M^1 D) \rangle = \langle A + M^2 C \mid \text{im}(B + M^2 D) \rangle = \mathbb{R}^2$ . Hence, the first condition is fulfilled. However, the overall system cannot be controllable as the derivative of  $x_1$  is always nonpositive. This is in accordance with the theorem since the second condition is violated in this case for the values  $\lambda = 0$ ,  $z = [1 \ 0]$ ,  $w_1 = -1$ , and  $w_2 = 1$ .

*Remark 5.* The above remark shows that even though all the constituent linear systems are controllable the overall system may not be controllable. On the other extreme, one can find examples in which the constituent systems are not controllable but the overall system is. To construct such an example, note that the second condition becomes void if the system has no real invariant zeros. Therefore, it is enough to choose constitute linear systems such that i) they are uncontrollable, ii) they do not have any real invariant zeros, and iii) they satisfy the first condition of Theorem 4. For such an example, consider the bimodal system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \begin{cases} -x_1 & \text{if } x_5 \geq 0 \\ -x_1 + x_5 & \text{if } x_5 \leq 0 \end{cases} \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= \begin{cases} -x_3 + x_5 & \text{if } x_5 \geq 0 \\ -x_3 & \text{if } x_5 \leq 0 \end{cases} \\ \dot{x}_5 &= u. \end{aligned}$$

To cast this system as a CLS, one can take

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad C = [0 \ 0 \ 0 \ 0 \ 1], \quad D = 0 \quad (29)$$

$$\mathcal{Y}_1 = \mathbb{R}_-, \quad M^1 = 0, \quad \mathcal{Y}_2 = \mathbb{R}_+, \quad M^2 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}. \quad (30)$$

It can be verified that the system  $(A, B, C, D)$  has no real invariant zeros. So, the second condition of Theorem 4 is void. It can also be verified that  $\langle A + M^1 C \mid \text{im}(B + M^1 D) \rangle = \text{span}\{e_1, e_2, e_5\}$  and  $\langle A + M^2 C \mid \text{im}(B + M^2 D) \rangle = \text{span}\{e_3, e_4, e_5\}$  where  $e_i$  is the  $i$ th standard basis vector, i.e. all components of  $e_i$  are zero except the  $i$ th component which is equal to 1. Note that both constituent linear systems are not controllable but the overall system is since the first condition is satisfied.

In what follows, we shall establish various already known controllability results as special cases of Theorem 4.

*Remark 6.* (linear systems) Take  $C = 0$  and  $D = I$ . Take  $r = 1$ . Let  $\mathcal{Y}_1 = \mathbb{R}^m$  and  $M^1 = 0$ . With these choices, the CLS (1) boils down to a linear system of the form

$$\dot{x} = Ax + Bu.$$

In this case, condition 1 is equivalent to saying that  $\langle A \mid \text{im } B \rangle = \mathbb{R}^n$ , i.e. the pair  $(A, B)$  is controllable whereas the left hand side of the implication 2 can be satisfied only with  $w_1 = 0$  as  $\mathcal{Y}_1^* = \{0\}$ . This means, however, that the second condition is readily satisfied provided that the first one is satisfied. Therefore, the system is controllable if, and only if,  $\langle A \mid \text{im } B \rangle = \mathbb{R}^n$ .

*Remark 7.* (linear systems with positive controls) Take  $C = 0$  and  $D = I$ . For an index set  $\alpha \subseteq \{1, 2, \dots, m\}$ , define the cone  $\mathcal{Y}_\alpha := \{y \in \mathbb{R}^m \mid y_i \geq 0 \text{ if } i \in \alpha, y_i \leq 0 \text{ if } i \notin \alpha\}$ . Obviously,  $\cup_\alpha \mathcal{Y}_\alpha = \mathbb{R}^m$  and Assumption 2 is automatically satisfied as the cones  $\mathcal{Y}_\alpha$  are polyhedral and solid. Let  $N^\alpha$  be a diagonal matrix such that the  $(i, i)$ th element is 1 if  $i \in \alpha$  or  $-1$  otherwise. Note that  $\mathcal{Y}_\alpha = \{y \mid N_\alpha y \geq 0\}$ . Also note that  $N^\alpha y = |y|$  whenever  $y \in \mathcal{Y}_\alpha$ . Here  $|y|$  denotes the componentwise absolute value of the vector  $y$ . Define  $M^\alpha = B(N^\alpha - I)$ . Note that  $Bu + f(Cx + Du) = B|u|$  with the above choices of  $C, D, N_\alpha$ , and  $\mathcal{Y}_\alpha$ . Hence, the CLS (1) boils down to a linear system of the form

$$\dot{x} = Ax + Bu$$

where the input is restricted to be nonnegative. Note that  $A + M^\alpha C = A$  and  $B + M^\alpha D = BN^\alpha$ . Thus,  $\langle A + M^\alpha C \mid \text{im}(B + M^\alpha D) \rangle = \langle A \mid \text{im } BN^\alpha \rangle = \langle A \mid \text{im } B \rangle$  as  $N^\alpha$  is nonsingular. This shows that the condition 1 is equivalent to 1 of Corollary 1. Let  $\lambda \in \mathbb{R}$ ,  $z \in \mathbb{R}^n$ , and  $w_\alpha \in \mathbb{R}^m$  be as in 2, i.e. be such that

$$[-z^T \ w_\alpha^T] \begin{bmatrix} A - \lambda I & BN^\alpha \\ 0 & I \end{bmatrix} = 0 \quad (31a)$$

$$w_\alpha \in \mathcal{Y}_\alpha^* \quad (31b)$$

for all  $\alpha \subseteq \{1, 2, \dots, m\}$ . It immediately follows from (31a) that

$$z^T A = \lambda z^T \quad (32a)$$

$$w_\alpha = N^\alpha B^T z. \quad (32b)$$

Note that  $\mathcal{Y}_\alpha$  is self-dual, i.e.  $\mathcal{Y}_\alpha^* = \mathcal{Y}_\alpha$ . So, (32b) implies that  $B^T z \geq 0$  as  $N^\alpha N^\alpha = I$ . Together with (32a), this proves the equivalence of the condition 2 to 2 of Corollary 1.

As a consequence of above analysis, Corollary 1 can be seen as a special case of Theorem 4. More generally, Theorem 2 becomes a special case whenever the set  $\mathcal{U}$  is a closed convex solid cone in  $\mathbb{R}^m$  such that  $B\mathcal{U}$  is closed.

*Remark 8.* (push-pull systems) To show that Theorem 3 is a special case of Theorem 4, one can take  $B = 0, C = 0, D = I$ . In this case, one gets  $A + M^i C = A$  and  $B + M^i D = M^i$ . Then,  $\langle A + M^i C \mid \text{im}(B + M^i D) \rangle = \langle A \mid \text{im } M^i \rangle$ . Note that

$$\sum_{i=1}^r \langle A \mid \text{im } M^i \rangle = \langle A \mid \text{im } [M^1 \ M^2 \ \dots \ M^r] \rangle. \quad (33)$$

Let  $\lambda \in \mathbb{R}$ ,  $z \in \mathbb{R}^n$ , and  $w_i \in \mathbb{R}^m$  be as in 2, i.e. be such that

$$[-z^T \ w_i^T] \begin{bmatrix} A - \lambda I & M^i \\ 0 & I \end{bmatrix} = 0 \quad (34a)$$

$$w_i \in \mathcal{Y}_i^* \quad (34b)$$

for all  $i$ . It immediately follows from (34a) that

$$z^T A = \lambda z^T \quad (35a)$$

$$w_i^T = z^T M^i. \quad (35b)$$

Then, (34b) can be written as  $(M^i)^T z \in \mathcal{Y}_i^*$  in view of (35b). Consequently, the implication

$$\lambda \in \mathbb{R}, \quad z \in \mathbb{R}^n, \quad z^T A = \lambda z^T, \quad (M^i)^T z \in \mathcal{Y}_i^* \text{ for all } i \quad \Rightarrow \quad z = 0$$

is equivalent to the implication in 2. Together with (33), this shows that 1 and 2 hold if, and only if, the pair  $(A, [M^1 \ M^2 \ \dots \ M^r])$  is completely controllable with respect to  $\mathcal{Y}_1 \times \mathcal{Y}_2 \times \dots \times \mathcal{Y}_r$ .

*Remark 9.* (bimodal systems) In [6], necessary and sufficient conditions for the controllability of single-input bimodal piecewise linear systems of the form

$$\dot{x} = \begin{cases} A'x + bu & \text{if } c^T x \leq 0, \\ (A' + ec^T)x + bu & \text{if } c^T x \geq 0 \end{cases} \quad (36)$$

are presented. It was shown, under the assumption that the transfer matrix  $c^T(sI - A')^{-1}b$  is nonzero, that necessary and sufficient conditions for controllability of the systems of the form (36) are

1. the pair  $(A', [b \ e])$  is controllable, and
2. the implication

$$[z^T \ w_i] \begin{bmatrix} A_i - \lambda I & b \\ c^T & 0 \end{bmatrix} = 0, \quad \lambda \in \mathbb{R}, \quad z \neq 0, \quad i = 1, 2 \Rightarrow w_1 w_2 > 0,$$

where  $A_1 := A'$  and  $A_2 := A' + ec^T$  holds. One can recover this result from Theorem 4 as follows. To fit the system (36) into the framework of CLS (1), take  $m = 1$ ,  $r = 2$ ,  $A = A'$ ,  $B = b$ ,  $C = c^T$ ,  $D = 0$ ,  $\mathcal{Y}_1 = \mathbb{R}_-$ ,  $M^1 = 0$ ,  $\mathcal{Y}_2 = \mathbb{R}_+$ , and  $M^2 = e$ . Note that  $A + M^1 C = A'$ ,  $A + M^2 C = A' + ec^T$ , and  $B + M^1 D = B + M^2 D = b$  in this case. With these choices, it can be verified that the implication 2 of Theorem 4 is equivalent to the one given by 2. Therefore, it is enough to show that the condition 1 of Theorem 4 is equivalent to the one given by 1. Note that  $\langle A + M^1 C \mid \text{im}(B + M^1 D) \rangle + \langle A + M^2 C \mid \text{im}(B + M^2 D) \rangle = \langle A' \mid \text{im } b \rangle + \langle A' + ec^T \mid \text{im } b \rangle$ . We claim that the latter equivalence holds if the transfer function  $c^T(sI - A')b$  is nonzero (hence invertible), i.e. it holds that

$$\langle A' \mid \text{im } b \rangle + \langle A' + ec^T \mid \text{im } b \rangle = \mathbb{R}^n \Leftrightarrow \text{the pair } (A', [b \ e]) \text{ is controllable.} \quad (37)$$

Note that  $\langle A' \mid \text{im } b \rangle \subseteq \langle A' \mid \text{im } [b \ e] \rangle$  and  $\langle A' + ec^T \mid \text{im } b \rangle \subseteq \langle A' \mid \text{im } [b \ e] \rangle$ . This immediately shows that the pair  $(A', [b \ e])$  is controllable if  $\langle A' \mid \text{im } b \rangle + \langle A' + ec^T \mid \text{im } b \rangle = \mathbb{R}^n$ . For the rest, we use the following well-known identity

$$(sI - X)^{-1} - (sI - Y)^{-1} = (sI - X)^{-1}(X - Y)(sI - Y)^{-1}. \quad (38)$$

Now, suppose that the pair  $(A', [b \ e])$  is controllable. To show that  $\langle A' \mid \text{im } b \rangle + \langle A' + ec^T \mid \text{im } b \rangle$  is the entire  $\mathbb{R}^n$ , assume  $z \in \mathbb{R}^n$  such that  $z^T(A')^k b = z^T(A' + ec^T)^k b = 0$  for all integers  $k$ , i.e.  $z$  is orthogonal to the subspace  $\langle A' \mid \text{im } b \rangle + \langle A' + ec^T \mid \text{im } b \rangle$ . Stated differently, we have  $z^T(sI - A')^{-1}b \equiv z^T(sI - A' - ec^T)^{-1}b \equiv 0$ . By using (38), we get

$$0 \equiv z^T[(sI - A' - ec^T)^{-1} - (sI - A')^{-1}]b = z^T(sI - A' - ec^T)^{-1}ec^T(sI - A')^{-1}b.$$

As the transfer function  $c^T(sI - A')^{-1}b$  is nonzero, we get  $z^T(sI - A' - ec^T)^{-1}e \equiv 0$ . Now, we can use (38) once more to obtain

$$z^T(sI - A' - ec^T)^{-1}e = z^T(sI - A' - ec^T)^{-1}ec^T(sI - A')^{-1}e + z^T(sI - A')^{-1}e.$$

Hence,  $z^T(sI - A')^{-1}e \equiv 0$ . This means, however, that  $z^T(sI - A')^{-1}[b \ e] \equiv 0$ . As the pair  $(A', [b \ e])$  is controllable, this can happen only if  $z = 0$ .

*Remark 10.* (bimodal planar systems) In [8], controllability of the systems (36) is investigated for the planar case, i.e.  $x \in \mathbb{R}^2$ . Under the assumption that  $(c^T, A')$  is observable, it was shown that the system (36) is controllable if, and only if,

$$f \neq 0, \quad f^T b = 0 \quad \Rightarrow \quad f^T A' b \cdot f^T (A' + ec^T) b > 0. \quad (39)$$

Our aim is to show that this result is a special case of Theorem 4. As the case  $b = 0$  is obviously uninteresting, we assume that  $b \neq 0$ . For a single-input single-output observable planar linear system, this means that the transfer function  $c^T(sI - A')b$  is not identically zero, and hence is invertible. Let  $A_1 = A'$ ,  $A_2 = A' + ec^T$ , and consider the following statements:

1.  $0 \neq f \in \mathbb{R}^2$ ,  $f^T b = 0 \Rightarrow f^T A' b \cdot f^T (A' + e c^T) b > 0$ ,
2.  $\langle A' \mid \text{im} \begin{bmatrix} b \\ e \end{bmatrix} \rangle = \mathbb{R}^2$ ,
3.  $\begin{bmatrix} z^T & w_i \end{bmatrix} \begin{bmatrix} A_i - \lambda I & b \\ c^T & 0 \end{bmatrix} = 0$  for  $i = 1, 2$  and for some  $0 \neq z \in \mathbb{R}^2$  and  $\lambda \in \mathbb{R} \Rightarrow w_1 w_2 > 0$ .

In view of Remark 9, it is enough to show that  $1 \Leftrightarrow 2 \wedge 3$ . To do so, we distinguish two cases.

1.  $c^T b = 0$ : We claim that all of the statements 1, 2, and 3 hold in this case. To see the first one, let  $0 \neq f \in \mathbb{R}^2$  be such that  $f^T b = 0$ . Due to planarity,  $f$  and  $c$  must be linearly dependent, i.e.,  $f = \rho c$  for some nonzero real number  $\rho$ . Hence,  $f^T A' b \cdot f^T (A' + e c^T) b = \rho^2 (c^T A' b)^2$ . Then, statement 1 follows from the fact that  $c^T A' b \neq 0$  due to the observability. To see the second one, note that  $c^T A' b \neq 0$  implies that  $b$  and  $A' b$  are linearly independent. Hence, the pair  $(A', b)$  is controllable. Then, statement 2 is obvious. To see the last one, note that  $z$  and  $c$  should be linearly dependent if  $z$  satisfies the left hand side of the statement 3. In other words,  $z = \rho c$  for some real number  $\rho$ . This would result in  $\rho c^T A' + (w_1 + \rho \lambda) c^T = 0$ . Since  $(c^T, A')$  is observable, this can only happen if  $\rho = 0$ , i.e.,  $z = 0$ . Therefore, the statement 3 holds trivially.
2.  $c^T b \neq 0$ : In this case, one can verify that the following two statements are equivalent:
  - (a)  $\begin{bmatrix} z^T & w_i \end{bmatrix} \begin{bmatrix} A_i - \lambda I & b \\ c^T & 0 \end{bmatrix} = 0$  for  $i = 1, 2$  and for some  $\lambda \in \mathbb{R}$
  - (b)  $z = \rho f$ ,  $w_i = \frac{-\rho}{c^T b} f^T A_i b$  for some  $\rho \in \mathbb{R}$  and  $0 \neq f \in \mathbb{R}^2$  with  $f^T b = 0$ .
This readily shows that the statements 1 and 3 are equivalent. To conclude the proof, one can show that the statement 2 follows from 1. To see this, note that  $f^T b = 0$  and  $f^T A' b \neq 0$  imply that  $b$  and  $A' b$  are linearly independent, i.e.,  $(A', b)$  is a controllable pair. Consequently, the statement 2 follows from (37).

## 5 Conclusions

In this chapter we studied the controllability problem for a class of conewise linear systems. The class of conewise linear systems is closely related to many other well-known hybrid model classes like piecewise linear systems, linear complementarity systems and others. As such, previous studies of controllability for these systems indicated the hard nature of the problem. Due to additional structure implying continuity of the vector field of the conewise linear systems under study, necessary and sufficient conditions for controllability could be given. To the best of the authors' knowledge it is the first time that a full *algebraic* characterization of controllability of a class of piecewise linear systems appears in the literature. The proofs of the main results combine ideas from geometrical control theory and controllability results for constrained linear systems. As such, the original results of controllability of linear systems and input-constrained linear systems were recovered as special cases. Also, the preliminary work by the authors on bimodal continuous piecewise linear systems [6, 8] form special cases of the main result of the current chapter. Moreover, the controllability of so-called "push-pull systems" was completely characterized. Interestingly, the algebraic characterization of controllability also showed that the overall conewise linear system can be controllable although the subsystems are not. Vice versa, it can happen that all linear subsystems are controllable but the overall system is not. This work revealed the use of geometrical control theory and constrained control of linear systems in the field of piecewise linear systems. Some structure on the piecewise linear system enabled the application of this well-known theory. We believe that this opens the path to solving problems like controller design, stabilization, observability, detectability, and other system and control theoretic problems of interest for this class of systems. This investigation forms one of the major issues of our future research.

## A Appendix: Notation

In this chapter, the following conventions are in force.

*Numbers and sets.* The Cartesian product of two sets  $S$  and  $T$  is denoted by  $S \times T$ . For a set  $S$ ,  $S^n$  denotes the  $n$ -tuples of elements of  $S$ , i.e. the set  $S \times S \times \dots \times S$  where there are  $n - 1$  Cartesian products. The symbol  $\mathbb{R}$  denotes the real numbers,  $\mathbb{R}_+$  the nonnegative real numbers (i.e. the set  $[0, \infty)$ ),  $\mathbb{C}$  the complex numbers. For two real numbers  $a$  and  $b$ , the notation  $\max(a, b)$  denotes the maximum of  $a$  and  $b$ .

*Vectors and matrices.* The notations  $v^T$  and  $v^*$  denote the transpose and conjugate transpose of a vector  $v$ . When two vectors  $v$  and  $w$  are orthogonal, i.e.  $v^T w = 0$ , we write  $v \perp w$ . Inequalities for real vectors must be understood componentwise. The notation  $\mathbb{R}^{n \times m}$  denotes the set of  $n \times m$  matrices with real elements. The transpose of  $M$  is denoted by  $M^T$ . The identity and zero matrices are denoted by  $I$  and  $0$ , respectively. If their dimensions are not specified, they follow from the context. Let  $M^{n \times m}$  be a matrix. We write  $M_{ij}$  for the  $(i, j)$ th element of  $M$ . For  $\alpha \subseteq \{1, 2, \dots, n\}$ , and  $\beta \subseteq \{1, 2, \dots, m\}$ ,  $M_{\alpha\beta}$  denotes the submatrix  $\{M_{ij}\}_{i \in \alpha, j \in \beta}$ . If  $n = m$  and  $\alpha = \beta$ , the submatrix  $M_{\alpha\alpha}$  is called a *principal submatrix* of  $M$  and the determinant of  $M_{\alpha\alpha}$  is called a *principal minor* of  $M$ . For two matrices  $M$  and  $N$  with the same number of columns,  $\text{col}(M, N)$  will denote the matrix obtained by stacking  $M$  over  $N$ . For a square matrix  $M$ , the notation  $\exp(M)$  denotes the exponential of  $M$ , i.e.  $\sum_{k=0}^{\infty} \frac{M^k}{k!}$ . All linear combinations of the vectors  $\{v_1, v_2, \dots, v_k\} \subset \mathbb{R}^n$  is denoted by  $\text{span}\{v_1, v_2, \dots, v_k\}$ .

*Cones and dual cones.* A set  $\mathcal{C}$  is said to be a *cone* if  $x \in \mathcal{C}$  implies that  $\alpha x \in \mathcal{C}$  for all  $\alpha \geq 0$ . A cone is said to be *solid* if its interior is not empty. A cone  $\mathcal{C} \subseteq \mathbb{R}^n$  is said to be *polyhedral* if it is of the form  $\{v \in \mathbb{R}^n \mid Mv \geq 0\}$  for some  $m \times n$  matrix  $M$ . For a nonempty set  $\mathcal{Q}$  (not necessarily a cone), the dual cone of  $\mathcal{Q}$  is the set  $\{v \mid u^T v \geq 0 \text{ for all } u \in \mathcal{Q}\}$ . It is denoted by  $\mathcal{Q}^*$ .

*Functions.* For a function  $f : \mathbb{R} \rightarrow \mathbb{R}$ ,  $f^{(k)}$  stands for the  $k$ th derivative of  $f$ . By convention, we take  $f^{(0)} = f$ . If  $f$  is a function of time, we use the notation  $\dot{f}$  for the derivative of  $f$ . The set of all arbitrarily many times differentiable functions is denoted by  $C^\infty$ . The *support* of a function  $f$  is defined by  $\text{supp}(f) := \{t \in \mathbb{R} \mid f(t) \neq 0\}$ .

## B Appendix: Some facts from geometric control theory

Consider the linear system  $\Sigma(A, B, C, D)$

$$\dot{x} = Ax + Bu \tag{40a}$$

$$y = Cx + Du \tag{40b}$$

where  $x \in \mathbb{R}^n$  is the state,  $u \in \mathbb{R}^m$  is the input,  $y \in \mathbb{R}^p$  is the output, and the matrices  $A, B, C, D$  are of appropriate sizes.

We define the *controllable subspace* and *unobservable subspace* as  $\langle A \mid \text{im } B \rangle := \text{im } B + A \text{im } B + \dots + A^{n-1} \text{im } B$  and  $\langle \ker C \mid A \rangle := \ker C \cap A^{-1} \ker C \cap \dots \cap A^{1-n} \ker C$ , respectively. It follows from these definitions that

$$\langle A \mid \text{im } B \rangle = \langle \ker B^T \mid A^T \rangle^\perp \tag{41}$$

where  $\mathcal{W}^\perp$  denotes the orthogonal space of  $\mathcal{W}$ .

We say that a subspace  $\mathcal{V}$  is *output-nulling controlled invariant* if for some matrix  $K$  the inclusions  $(A - BK)\mathcal{V} \subseteq \mathcal{V}$  and  $\mathcal{V} \subseteq \ker(C - DK)$  hold. As the set of such subspaces is non-empty and closed under subspace addition, it has a maximal element  $\mathcal{V}^*(\Sigma)$ . Whenever the system  $\Sigma$  is clear from the context, we simply write  $\mathcal{V}^*$ . The notation  $\mathcal{K}(\mathcal{V})$  stands for the set  $\{K \mid (A - BK)\mathcal{V} \subseteq \mathcal{V} \text{ and } \mathcal{V} \subseteq \ker(C - DK)\}$ .

One can compute  $\mathcal{V}^*$  as a limit of the subspaces

$$\mathcal{V}^0 = \mathbb{R}^n, \quad \mathcal{V}^i = \{x \mid Ax + Bu \in \mathcal{V}^{i-1} \text{ and } Cx + Du = 0 \text{ for some } u\}. \tag{42}$$

In fact, there exists an index  $i \leq n - 1$  such that  $\mathcal{V}^j = \mathcal{V}^*$  for all  $j \geq i$ .

Dually, we say that a subspace  $\mathcal{T}$  is *input-containing conditioned invariant* if for some matrix  $L$  the inclusions  $(A - LC)\mathcal{T} \subseteq \mathcal{T}$  and  $\text{im}(B - LD) \subseteq \mathcal{T}$  hold. As the set of such subspaces is non-empty and closed under the subspace intersection, it has a minimal element  $\mathcal{T}^*(\Sigma)$ . Whenever

the system  $\Sigma$  is clear from the context, we simply write  $\mathcal{T}^*$ . The notation  $\mathcal{L}(\mathcal{T})$  stands for the set  $\{L \mid (A - LC)\mathcal{T} \subseteq \mathcal{T} \text{ and } \text{im}(B - LD) \subseteq \mathcal{T}\}$ .

We sometimes write  $\mathcal{V}^*(A, B, C, D)$  or  $\mathcal{T}^*(A, B, C, D)$  to make the dependence on  $(A, B, C, D)$  explicit.

We quote some standard facts from geometric control theory in what follows. The first one presents certain invariants under state feedbacks and output injections. Besides the system  $\Sigma$  (40), consider the linear system  $\Sigma_{K,L}$  given by

$$\dot{x} = (A - BK - LC + LDK)x + (B - LD)v \quad (43a)$$

$$y = (C - DK)x + Dv. \quad (43b)$$

This system can be obtained from  $\Sigma$  (40) by applying both state feedback  $u = -Kx + v$  and output injection  $-Ly$ .

**Proposition 1.** *Let  $K \in \mathbb{R}^{m \times n}$  and  $L \in \mathbb{R}^{n \times p}$  be given. The following statements hold.*

1.  $\langle A \mid \text{im } B \rangle = \langle A - BK \mid \text{im } B \rangle$ .
2.  $\langle \ker C \mid A \rangle = \langle \ker C \mid A - LC \rangle$ .
3.  $\mathcal{V}^*(\Sigma_{K,0}) = \mathcal{V}^*(\Sigma_{0,L}) = \mathcal{V}^*(\Sigma_{K,L}) = \mathcal{V}^*(\Sigma)$ .
4.  $\mathcal{T}^*(\Sigma_{K,0}) = \mathcal{T}^*(\Sigma_{0,L}) = \mathcal{T}^*(\Sigma_{K,L}) = \mathcal{T}^*(\Sigma)$ .

The next proposition relates the invertibility of the transfer matrix to the controlled and conditioned invariant subspaces.

**Proposition 2** (cf. [1]). *The transfer matrix  $D + C(sI - A)^{-1}B$  is invertible as a rational matrix if, and only if,  $\mathcal{V}^* \oplus \mathcal{T}^* = \mathcal{X}$ ,  $[C \ D]$  is of full row rank, and  $\text{col}(B, D)$  is of full column rank. Moreover, the inverse is polynomial if, and only if,  $\mathcal{V}^* \cap \langle A \mid \text{im } B \rangle \subseteq \langle \ker C \mid A \rangle$  and  $\langle A \mid \text{im } B \rangle \subseteq \mathcal{T}^* + \langle \ker C \mid A \rangle$ .*

We define the *invariant zeros* of the system (40) to be the zeros of the non-zero polynomials on the diagonal of the Smith form of

$$P_{\Sigma}(s) = \begin{bmatrix} A - sI & B \\ C & D \end{bmatrix}. \quad (44)$$

The matrix  $P_{\Sigma}(s)$  is sometimes called *the system matrix*.

We know from [1, Thm. 2] that the invariant zeros coincide with the eigenvalues of the mapping that is obtained by restricting  $A - BK - LC + LDK$  to the subspace  $\mathcal{V}^*/(\mathcal{V}^* \cap \mathcal{T}^*)$  where  $K \in \mathcal{K}(\mathcal{V}^*)$  and  $L \in \mathcal{L}(\mathcal{T}^*)$  such that  $\langle \ker C \mid A \rangle \subseteq \ker K$  and  $\text{im } L \subseteq \langle A \mid \text{im } B \rangle$ .

It is known, for instance from [34, Cor. 8.14], that the transfer matrix  $D + C(sI - A)^{-1}B$  is invertible as a rational matrix if, and only if, the system matrix  $P_{\Sigma}(\lambda)$  is of rank  $n + m$  for all but finitely many  $\lambda \in \mathbb{C}$ . In this case, the values of  $\lambda \in \mathbb{C}$  such that

$$\text{rank } P_{\Sigma}(\lambda) < n + m \quad (45)$$

coincide with the invariant zeros.

If  $\lambda \in \mathbb{C}$  is an invariant zero, then the elements of the kernel of the matrix  $P_{\Sigma}(\lambda)$  are called *invariant (right) zero directions* (see e.g. [25]). They enjoy the following dynamical interpretation. Let  $\lambda \in \mathbb{C}$  be an invariant zero and  $\text{col}(\bar{x}, \bar{u})$  be an invariant zero direction, i.e.

$$\begin{bmatrix} A - \lambda I & B \\ C & D \end{bmatrix} \begin{bmatrix} \bar{x} \\ \bar{u} \end{bmatrix} = 0. \quad (46)$$

Then, the output  $y$  of (40) corresponding to the initial state  $\bar{x}$  and the input  $t \mapsto \bar{u} \exp(\lambda t)$  is identically zero.

The following proposition presents sufficient conditions for the absence of invariant zeros. It can be proved by using (42).

**Proposition 3.** Consider the linear system (40) with  $p = m$ . Suppose that  $\mathcal{V}^* = \{0\}$  and the matrix  $\text{col}(B, D)$  is of full column rank. Then, the system matrix

$$\begin{bmatrix} A - \lambda I & B \\ C & D \end{bmatrix}$$

is nonsingular for all  $\lambda \in \mathbb{C}$ .

Systems that have transfer functions with a polynomial inverse are of particular interest for our treatment.

**Proposition 4.** Consider the linear system (40). Suppose that the transfer matrix  $D + C(sI - A)^{-1}B$  has a polynomial inverse. Let  $H(s) = H^0 + sH^1 + \dots + s^h H^h$  be this inverse. For a given  $p$ -tuple of  $C^\infty$ -functions  $\bar{y}$ , take

$$x(0) = \sum_{\ell=0}^h \sum_{j=0}^{\ell-1} A^j B H^\ell \bar{y}^{(\ell-1-j)}(0) \tag{47a}$$

$$u(t) = H\left(\frac{d}{dt}\right)\bar{y}(t). \tag{47b}$$

Then, the output  $y$ , corresponding to the initial state  $x(0)$  and the input  $u$ , of the system (40) is identical to  $\bar{y}$ .

*Proof.* Since  $H(s)$  is the inverse of  $G(s)$ , one has  $G(s)H(s) = I$ . This yields

$$\sum_{i=0}^j G^i H^{h-j+i} = 0 \text{ for } j = 0, 1, \dots, h-1 \tag{48a}$$

$$\sum_{i=0}^h G^i H^i = I \tag{48b}$$

$$\sum_{i=0}^h G^{j+i} H^i = 0 \text{ for } j = 1, 2, \dots \tag{48c}$$

where  $G(s) = D + s^{-1}CB + s^{-2}CAB + \dots = G^0 + s^{-1}G^1 + s^{-2}G^2 + \dots$ . Note that the following identity follows from integration by parts

$$\begin{aligned} \int_0^t \exp(A(t-s))BP\bar{y}^{(k)}(s) ds &= -\exp(At) \sum_{j=0}^{k-1} A^j BP\bar{y}^{(k-1-j)}(0) + \sum_{j=0}^{k-1} A^j BP\bar{y}^{(k-1-j)}(t) + \\ &+ A^k \int_0^t \exp(A(t-s))BP\bar{y}(s) ds \end{aligned} \tag{49}$$



where  $\bar{y}$  is any  $p$ -tuple  $C^\infty$ -function,  $P \in \mathbb{R}^{p \times p}$ , and  $k > 0$  is an integer. Then, one has

$$y(t) = Cx(t) + Du(t) \quad (50a)$$

$$\stackrel{(47b)}{=} C \exp(At)x(0) + \int_0^t C \exp(A(t-s))B \sum_{\ell=0}^h H^\ell \bar{y}^{(\ell)}(s) ds + D \sum_{\ell=0}^h H^\ell \bar{y}^{(\ell)}(t) \quad (50b)$$

$$\stackrel{(49)}{=} C \exp(At)x(0) - C \exp(At) \sum_{\ell=0}^h \sum_{j=0}^{\ell-1} A^j B H^\ell \bar{y}^{(\ell-1-j)}(0) + \sum_{\ell=0}^h \sum_{j=0}^{\ell-1} C A^j B H^\ell \bar{y}^{(\ell-1-j)}(t) +$$

$$+ \sum_{\ell=0}^h \int_0^t C A^\ell \exp(A(t-s)) B H^\ell \bar{y}(s) ds + D \sum_{\ell=0}^h H^\ell \bar{y}^{(\ell)}(t) \quad (50c)$$

$$\stackrel{(47a)}{=} \underbrace{\sum_{\ell=0}^h \sum_{j=0}^{\ell-1} C A^j B H^\ell \bar{y}^{(\ell-1-j)}(t)}_{=:y_1(t)} + D \sum_{\ell=0}^h H^\ell \bar{y}^{(\ell)}(t) + \underbrace{\sum_{\ell=0}^h \int_0^t C A^\ell \exp(A(t-s)) B H^\ell \bar{y}(s) ds}_{=:y_2(t)}. \quad (50d)$$

First, we look at  $y_1$ . It can be checked that

$$y_1(t) = \sum_{\ell=0}^h \sum_{j=0}^{\ell-1} C A^j B H^\ell \bar{y}^{(\ell-1-j)}(t) + D \sum_{\ell=0}^h H^\ell \bar{y}^{(\ell)}(t) \quad (51a)$$

$$= \sum_{\ell=0}^h \sum_{j=0}^{\ell-1} G^{j+1} H^\ell \bar{y}^{(\ell-1-j)}(t) + G^0 \sum_{\ell=0}^h H^\ell \bar{y}^{(\ell)}(t) \quad (51b)$$

$$= \sum_{\ell=0}^h \sum_{j=0}^{\ell} G^j H^\ell \bar{y}^{(\ell-j)}(t) = \sum_{\ell=0}^h \sum_{m=0}^{\ell} G^{\ell-m} H^\ell \bar{y}^{(m)}(t) \quad (51c)$$

$$= \sum_{m=0}^{\ell} \sum_{\ell=m}^h G^{\ell-m} H^\ell \bar{y}^{(m)}(t) \quad (51d)$$

$$= \bar{y}(t) \quad (\text{from (48a) and (48b)}). \quad (51e)$$

One can also check that

$$y_2(t) = \sum_{\ell=0}^h \int_0^t C A^\ell \exp(A(t-s)) B H^\ell \bar{y}(s) ds \quad (52a)$$

$$= \int_0^t \sum_{\ell=0}^h \sum_{m=0}^{\infty} C A^\ell A^m \frac{(t-s)^m}{m!} B H^\ell \bar{y}(s) ds. \quad (52b)$$

Note that

$$\sum_{\ell=0}^h \sum_{m=0}^{\infty} C A^\ell A^m \frac{(t-s)^m}{m!} B H^\ell = \sum_{\ell=0}^h \sum_{m=0}^{\infty} G^{\ell+m+1} H^\ell \frac{(t-s)^m}{m!}$$

$$\stackrel{(48c)}{=} 0. \quad (53)$$

Hence,  $y_1 + y_2 = \bar{y}$ . ■

The last proposition presents sufficient conditions under which the values of the output and its higher order derivatives at a certain time instant uniquely determine the state at the same time instant.

**Proposition 5.** *Consider the linear system (40) with  $p = m$ . Suppose that  $\mathcal{V}^* = \{0\}$ . Let the triple  $(u, x, y)$  satisfy the equations (40) with the pair  $(u, y)$  being  $(n - 1)$ -times differentiable. If  $y^{(k)}(t) = CA^k \bar{x}$  for  $k = 0, 1, \dots, n - 1$  for some  $t$  and  $\bar{x} \in \mathbb{R}^n$  then  $x(t) = \bar{x}$ .*

*Proof.* Note that  $y(t) = C\bar{x}$  results in

$$Cx(t) + Du(t) = C\bar{x}$$

and hence  $x(t) - \bar{x} \in \mathcal{V}^1$  in view of (42). Similarly,  $y^{(1)}(t) = CA\bar{x}$  results in

$$CAx(t) + CBu(t) + Du^{(1)}(t) = CA\bar{x}.$$

This would mean that  $A(x(t) - \bar{x}) + Bu(t) \in \mathcal{V}^1$  and hence  $x(t) - \bar{x} \in \mathcal{V}^2$ . By continuing in this way, one can show that  $x(t) - \bar{x} \in \mathcal{V}^k$  for all  $k = 0, 1, \dots, n - 1$ . This, however, means that  $x(t) - \bar{x} \in \mathcal{V}^*$ . Therefore  $x(t) = \bar{x}$  by the hypothesis. ■

## C Appendix: Proofs

### C.1 Proof of Theorem 3

We will show that the following implications hold:

$$\begin{array}{ccccc} 2 & \Rightarrow & 1 & \Rightarrow & 3 \Leftarrow 4 \\ \uparrow & & \uparrow & & \downarrow \\ 4 & & 6 & \Leftrightarrow & 5 \Rightarrow 4 \end{array}$$

Note that the three implications in the first line are evident.

**3  $\Rightarrow$  5 :** Suppose that 3 holds. Let  $z \in \mathbb{R}^n$  be such that

$$z^T \exp(At) f(u) \geq 0 \tag{54}$$

for all  $t \geq 0$  and for all  $u \in \mathcal{U}$ . Then, for any solution  $x$  of (13) with  $x(0) = 0$  one has

$$z^T x(T) = z^T \int_0^T \exp(A(T - s)) f(u(s)) ds \geq 0. \tag{55}$$

As 3 holds,  $x(T)$  may take any arbitrary value by choosing a suitable input function. Therefore,  $z$  must be zero.

**5  $\Rightarrow$  6 :** Suppose that 5 holds. Due to Theorem 2, it is enough to show that

- i. the pair  $(A, [M^1 \ M^2 \ \dots \ M^r])$  is controllable, and
- ii. the implication

$$\lambda \in \mathbb{R}, \quad z \in \mathbb{R}^n, \quad z^T A = \lambda z^T, \quad (M^i)^T z \in \mathcal{Y}_i^* \text{ for all } i = 1, 2, \dots, r \quad \Rightarrow \quad z = 0$$

holds.

- i. Let  $s' \in \mathbb{C}$  and  $v \in \mathbb{C}^n$  be such that  $v^* [s'I - A \ M^1 \ M^2 \ \dots \ M^r] = 0$ . This means that

$$s'v^* = v^* A \tag{56a}$$

$$v^* M^i = 0 \tag{56b}$$

for all  $i = 1, 2, \dots, r$ . Let  $\sigma$  and  $\omega$  be, respectively, the real and imaginary parts of  $s'$ . Also let  $v_1$  and  $v_2$  be, respectively, the real and imaginary parts of  $v$ . One can write (56) in terms of  $\sigma$ ,  $\omega$ ,  $v_1$ , and  $v_2$  as

$$\begin{bmatrix} v_1^T \\ v_2^T \end{bmatrix} A = \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix} \begin{bmatrix} v_1^T \\ v_2^T \end{bmatrix} \tag{57a}$$

$$v_1^T M^i = v_2^T M^i = 0 \tag{57b}$$

for all  $i = 1, 2, \dots, r$ . Note that (57a) results in

$$\begin{bmatrix} v_1^T \\ v_2^T \end{bmatrix} \exp(At) = \exp\left(\begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix} t\right) \begin{bmatrix} v_1^T \\ v_2^T \end{bmatrix}. \quad (58)$$

Together with (57b), this implies that  $v_j^T \exp(At)M^i = 0$  for all  $t$ , for all  $i$ , and for all  $j \in \{1, 2\}$ . In view of 5, both  $v_1$  and  $v_2$  must be zero. Hence, so is  $v$ . Consequently, the pair  $(A, [M^1 \ M^2 \ \dots \ M^r])$  is controllable.

ii. Let  $z \in \mathbb{R}^n$  and  $\lambda \in \mathbb{R}$  be such that

$$z^T A = \lambda z^T \quad (59a)$$

$$(M^i)^T z \in \mathcal{Y}_i^* \quad (59b)$$

for all  $i = 1, 2, \dots, r$ . Then,  $z^T M^i v$  is nonnegative for any  $v \in \mathcal{Y}_i$ . Since  $\mathcal{U} = \cup_{i=1}^r \mathcal{Y}_i$ , one even has  $z^T f(v) \geq 0$  for all  $v \in \mathcal{U}$ . Note that  $z^T \exp(At) = \exp(\lambda t) z^T$  due to (59a). Then,  $z^T \exp(At) f(v) \geq 0$  for all  $v \in \mathcal{U}$ . In view of 5, this implies that  $z = 0$ .

Now, the statement 6 follows from i, ii, and Theorem 2.

**5  $\Rightarrow$  4 :** This implication follows from the following lemma.

**Lemma 2.** *Consider the system (13). Suppose that the implication*

$$z^T \exp(At) f(u) \geq 0 \text{ for all } t \geq 0 \text{ and } u \in \mathcal{U} \quad \Rightarrow \quad z = 0 \quad (60)$$

*holds. Then, there exist a positive real number  $T$  and an integer  $\ell$  such that for a given state  $x_f$  one can always find vectors  $\eta^{i,j} \in \mathcal{Y}_i$  for  $i = 1, 2, \dots, r$  and  $j = 0, 1, \dots, \ell - 1$  such that the state  $x_f$  can be reached from the zero state in time  $T$  by the application of the input*

$$\bar{u}(t) = \eta^{i,j} \theta^{\Delta_\ell} (t - (jr + i - 1)\Delta_\ell) \quad \text{for } (jr + i - 1)\Delta_\ell \leq t \leq (jr + i)\Delta_\ell \quad (61)$$

*where  $\Delta_\ell = T/(\ell r)$  and  $\theta^\Delta : \mathbb{R} \rightarrow \mathbb{R}$  is a nonnegative valued  $C^\infty$ -function with  $\text{supp}(\theta^\Delta) \subseteq (\frac{\Delta}{4}, 3\frac{\Delta}{4})$  and*

$$\int_0^\Delta \theta^\Delta(t) dt = 1. \quad (62)$$

*Proof.* First, we show that if (60) holds then there exists a positive real number  $T$  such that the implication

$$z^T \exp(At) f(u) \geq 0 \text{ for all } t \in [0, T] \text{ and } u \in \mathcal{U} \quad \Rightarrow \quad z = 0 \quad (63)$$

holds. To see this, suppose that the above implication does not hold for any  $T$ . Therefore, for all  $T$  there exists  $0 \neq z_T \in \mathbb{R}^n$  such that

$$z_T^T \exp(At) f(u) \geq 0 \text{ for all } t \in [0, T] \text{ and } u \in \mathcal{U}. \quad (64)$$

Without loss of generality, we can assume that  $\|z_T\| = 1$ . Then, the sequence  $\{z_T\}_{T \in \mathbb{N}}$  admits a convergent subsequence due to the well-known Bolzano-Weierstrass theorem. Let  $z_\infty$  denote its limit. Note that  $\|z_\infty\| = 1$ . We claim that

$$z_\infty^T \exp(At) f(u) \geq 0 \quad (65)$$

for all  $t \geq 0$  and  $u \in \mathcal{U}$ . To show this, suppose that  $z_\infty^T \exp(At') f(u') < 0$  for some  $t'$  and  $u' \in \mathcal{U}$ . Then, for some sufficiently large  $T'$ , one has  $z_{T'}^T \exp(At') f(u') < 0$  and  $t' < T'$ . However, this cannot happen due to (64). In view of (60), (65) yields  $z_\infty = 0$ . Hence, by contradiction, there exists a positive real number  $T$  such that the implication (63) holds.

Now, consider the input function in (61). Note that

$$f(\bar{u}(t)) = M^i \bar{u}(t) \quad \text{if } (jr + i - 1)\Delta_\ell \leq t \leq (jr + i)\Delta_\ell.$$

The solution of (13) corresponding to  $x(0) = 0$  and  $u = \bar{u}$  is given by

$$x(T) = \int_0^T \exp[A(T-s)]f(u(s)) ds \quad (66a)$$

$$= \sum_{j=0}^{\ell-1} \sum_{i=1}^r \int_{(jr+i-1)\Delta_\ell}^{(jr+i)\Delta_\ell} \exp[A(T-s)]M^i \eta^{i,j} \theta^{\Delta_\ell}(s - (jr+i-1)\Delta_\ell) ds. \quad (66b)$$

Straightforward calculations yield that

$$x(T) = \Lambda(\Delta_\ell) \sum_{j=0}^{\ell-1} \sum_{i=1}^r \exp[A(T - (jr+i-1)\Delta_\ell)]M^i \eta^{i,j} \quad (67)$$

where  $\Lambda(\Delta) = \int_0^\Delta \exp(-As)\theta^\Delta(s) ds$ . Then, it is enough to show that there exists an integer  $\ell$  such that the above equation is solvable in  $\eta^{i,j} \in \mathcal{Y}_i$  for  $i = 1, 2, \dots, r$  and  $j = 0, 1, \dots, \ell - 1$  for any  $x(T) \in \mathbb{R}^n$ . To do so, we invoke a generalized Farkas' lemma (see e.g. [13, Thm. 2.2.6]).

**Lemma 3.** *Let  $H \in \mathbb{R}^{P \times N}$ ,  $q \in \mathbb{R}^P$ , and a closed convex cone  $\mathcal{C} \subseteq \mathbb{R}^N$  be given. Suppose that  $HC$  is closed. Then, either the primal system*

$$Hv = q, \quad v \in \mathcal{C}$$

*has a solution  $v \in \mathbb{R}^N$  or the dual system*

$$w^T q < 0, \quad H^T w \in \mathcal{C}^*$$

*has a solution  $w \in \mathbb{R}^P$  but never both.*

An immediate consequence of this lemma is that if the implication

$$w^T H v \geq 0 \text{ for all } v \in \mathcal{C} \quad \Rightarrow \quad w = 0 \quad (68)$$

holds then the primal system has a solution for all  $q$ . Consider, now, (67) as the primal system. As  $\mathcal{Y}_i$  is polyhedral cone,  $M^i \mathcal{Y}_i$  must be polyhedral and hence closed. Also note that  $\Lambda(\Delta_\ell)$  is nonsingular for all sufficiently large  $\ell$  as it converges to the identity matrix as  $\ell$  tends to infinity. Then,  $\Lambda(\Delta_\ell) \exp(A\tau)M^i \mathcal{Y}_i$  is closed for all sufficiently large  $\ell$  and for all  $\tau$  since  $\exp(A\tau)$  is nonsingular for all  $\tau$ . Therefore, in view of (68), in order to show that for an integer  $\ell$  (67) has a solution for arbitrary  $x(T)$ , it is enough to show that the relation

$$z^T \Lambda(\Delta_\ell) \sum_{j=0}^{\ell-1} \sum_{i=1}^r \exp[A(T - (jr+i-1)\Delta_\ell)]M^i \eta^{i,j} \geq 0 \quad (69)$$

for all  $\eta^{i,j} \in \mathcal{Y}_i$ ,  $i = 1, 2, \dots, r$ , and  $j = 0, 1, \dots, \ell - 1$  can only be satisfied by  $z = 0$ . To see this, suppose, on the contrary, that for each integer  $\ell$  there exists  $z_\ell \neq 0$  such

$$z_\ell^T \Lambda(\Delta_\ell) \sum_{j=0}^{\ell-1} \sum_{i=1}^r \exp[A(T - (jr+i-1)\Delta_\ell)]M^i \eta^{i,j} \geq 0 \quad (70)$$

for all  $\eta^{i,j} \in \mathcal{Y}_i$ ,  $i = 1, 2, \dots, r$ , and  $j = 0, 1, \dots, \ell - 1$ . Clearly, we can take  $\|z_\ell\| = 1$ . In view of Bolzano-Weierstrass theorem, we can assume, without loss of generality, that the sequence  $\{z_\ell\}$  converges, say to  $z_\infty$ , as  $\ell$  tends to infinity. Now, fix  $i$  and  $t \in [0, T]$ . It can be verified that there exists a subsequence  $\{\ell_k\} \subset \mathbb{N}$  such that the inequality  $(j_{\ell_k} r + i - 1)\Delta_{\ell_k} \leq T - t \leq (j_{\ell_k} r + i)\Delta_{\ell_k}$  holds for some  $j_{\ell_k} \in \{1, 2, \dots, \ell_k\}$ . It is a standard fact from distribution theory that  $\theta^\Delta$  converges to a Dirac impulse as  $\Delta$  tends to zero. Hence,  $\Lambda(\Delta_\ell)$  converges to the identity matrix as  $\ell$  tends to infinity. Let  $\ell = \ell_k$  and  $j = j_{\ell_k}$  in (70). By taking the limit, one gets

$$z_\infty^T \exp(At)M^i \eta \geq 0$$

for all  $t \in [0, T]$ ,  $\eta \in \mathcal{Y}_i$ , and  $i = 1, 2, \dots, r$ . Consequently, one has

$$z_\infty^T \exp(At) f(u) \geq 0 \quad (71)$$

for all  $t \in [0, T]$  and  $u \in \mathcal{U}$ . Hence,  $z_\infty$  must be zero due to (63). Contradiction! ■

**6  $\Rightarrow$  5 :** Suppose that 6 holds. It follows from Theorem 2 that

- i. the pair  $(A, [M^1 \ M^2 \ \dots \ M^r])$  is controllable, and
- ii. the implication

$$\lambda \in \mathbb{R}, \quad z \in \mathbb{R}^n, \quad z^T A = \lambda z^T, \quad (M^i)^T z \in \mathcal{Y}_i^* \text{ for all } i = 1, 2, \dots, r \quad \Rightarrow \quad z = 0$$

holds.

At this point, we invoke the following lemma.

**Lemma 4.** *Let  $G \in \mathbb{R}^{N \times N}$  and  $H \in \mathbb{R}^{N \times M}$  be given. Also let  $\mathcal{W} \subseteq \mathbb{R}^M$  be such that its convex hull has nonempty interior in  $\mathbb{R}^M$ . Suppose that the pair  $(G, H)$  is controllable and the implication*

$$\lambda \in \mathbb{R}, \quad z \in \mathbb{R}^N, \quad z^T G = \lambda z^T, \quad H^T z \in \mathcal{W}^* \quad \Rightarrow \quad z = 0$$

*holds. Then, also the implication*

$$z^T \exp(Gt) H v \geq 0 \text{ for all } t \geq 0 \text{ and } v \in \mathcal{W} \quad \Rightarrow \quad z = 0$$

*holds.*

The proof can be found in the sufficiency proof of [4, Thm. 1.4]. Take  $G = A$ ,  $H = [M^1 \ M^2 \ \dots \ M^r]$ , and  $\mathcal{W} = \mathcal{Y}_1 \times \mathcal{Y}_2 \times \dots \times \mathcal{Y}_r$ . It follows from i and ii that the hypothesis of the above lemma is satisfied. Therefore, the implication

$$z^T \exp(At) [M^1 \ M^2 \ \dots \ M^r] v \geq 0 \text{ for all } t \geq 0 \text{ and } v \in \mathcal{Y}_1 \times \mathcal{Y}_2 \times \dots \times \mathcal{Y}_r \quad \Rightarrow \quad z = 0$$

holds. In particular, the implication

$$z^T \exp(At) f(u) \geq 0 \text{ for all } t \geq 0 \text{ and } u \in \mathcal{U} \quad \Rightarrow \quad z = 0$$

holds.

**6  $\Rightarrow$  1 :** Note that if the statement 6 holds for the system (13) so does it for the time-reversed version of the system (13). Therefore the statement 4 holds for both (13) and its time-reversal. This means that one can steer any initial state first to zero and then to any final state. Thus, complete controllability is achieved.

**4  $\Rightarrow$  2 :** As the statement 4 holds for both (13) and its time-reversal, one can steer any initial state first to zero and then to any final state. ■

## C.2 Proof of Lemma 1

We need the following auxiliary results. The first one guarantees the existence of smooth functions lying in a given polyhedral cone.

**Lemma 5.** *Let  $\mathcal{Y} \subseteq \mathbb{R}^p$  be a polyhedral cone and  $y$  be a  $C^\infty$ -function such that  $y(t) \in \mathcal{Y}$  for all  $t \in [0, \epsilon]$  where  $0 < \epsilon < 1$ . Then, there exists a  $C^\infty$ -function  $\bar{y}$  such that*

- i.  $\bar{y}(t) = y(t)$  for all  $t \in [0, \epsilon]$
- ii.  $\bar{y}^{(k)}(1) = 0$  for all  $k = 0, 1, \dots$ , and

iii.  $\bar{y}(t) \in \mathcal{Y}$  for all  $t \in [0, 1]$ .

*Proof.* We only prove the case  $p = 1$  and  $\mathcal{Y} = \mathbb{R}_+$ . The rest is merely a generalization to the higher dimensional case. Let  $\bar{y}$  be a  $C^\infty$ -function such that  $\bar{y}(t) = 1$  for  $t \leq \epsilon/4$ ,  $\bar{y}(t) > 0$  for  $\epsilon/4 < t < 3\epsilon/4$ , and  $\bar{y}(t) = 0$  for  $3\epsilon/4 \leq t$ . Such a function can be derived from the so-called bump function (e.g. the function  $\varphi$  in [19, Lemma 1.2.3]) by integration and scaling. It can be checked that the product of  $y$  and  $\bar{y}$  proves the claim. ■

The second auxiliary result concerns the existence of solutions of CLS with certain properties. It follows from [9, Lem. II.10, Lem. II.11, and Thm. II.12].

**Proposition 6.** *Consider the CLS (1) with  $\mathcal{U} = \{0\}$  and  $\mathcal{Y} = \mathbb{R}^p$ . Then, for each initial state  $x_0$  there exist an index set  $i$  and a positive number  $\epsilon$  such that  $y(t) \in \mathcal{Y}_i$  for all  $t \in [0, \epsilon]$ .*

We turn to the proof of Lemma 1. Obviously, 1 implies 2. For the rest, it is enough to show that the system (23) is controllable if 2 holds.

Note that  $\mathcal{V}^*(A_{22} + M_2^i C_2, B_2 + M_2^i D, C_2, D) = \{0\}$  and  $\mathcal{T}^*(A_{22} + M_2^i C_2, B_2 + M_2^i D, C_2, D) = \mathbb{R}^{n_2}$  for all  $i = 1, 2, \dots, r$  due to (24) and Proposition 1. Further, the matrices  $[C_2 \ D]$  and  $\text{col}(B_2, D)$  are of full, respectively, row and column rank. According to Proposition 2, the transfer matrix  $D + C_2(sI - A_{22} + M_2^i C_2)^{-1}(B_2 + M_2^i D)$  has a polynomial inverse for all  $i = 1, 2, \dots, r$ .

Take any  $x_{10}, x_{1f} \in \mathbb{R}^{n_1}$  and  $x_{20}, x_{2f} \in \mathbb{R}^{n_2}$ . Consider the system (23). Apply  $v = 0$ . By applying Proposition 6, we can find an index  $i_0$  and an arbitrarily small positive number  $\epsilon$  such that  $y(t) \in \mathcal{Y}_{i_0}$  for all  $t \in [0, \epsilon]$ . By applying Lemma 5, we can get a  $C^\infty$ -function  $y_{\text{in}}$  such that

- i.  $y_{\text{in}}(t) = y(t)$  for all  $t \in [0, \epsilon]$
- ii.  $y_{\text{in}}^{(k)}(1) = 0$  for all  $k = 0, 1, \dots$ , and
- iii.  $y_{\text{in}} \in \mathcal{Y}_{i_0}$  for all  $t \in [0, 1]$ .

Then, by applying Propositions 4 and 5 to the system  $\Sigma(A_{22} + M_2^{i_0} C_2, B_2 + M_2^{i_0} D, C_2, D)$ , we can find an input  $v_{\text{in}}$  such that the output  $y$  of (23b)-(23c) is identically  $y_{\text{in}}$  and the state  $x_2$  satisfies  $x_2(0) = x_{20}$ . Note that the input  $v_{\text{in}}$  should be zero on the interval  $[0, \epsilon]$  by the construction of  $y_{\text{in}}$  and invertibility. Moreover,  $x_2(1) = 0$  due to ii and Proposition 5. Therefore, the input  $v_{\text{in}}$  steers the state  $\text{col}(x_{10}, x_{20})$  to  $\text{col}(x'_{10}, 0)$  where  $x'_{10} := x_1(1)$ . By employing the very same ideas in the reverse time, we can come up with an input  $v_{\text{out}}$  such that it steers a state  $\text{col}(x'_{1f}, 0)$  to  $\text{col}(x_{1f}, x_{2f})$ . Now, we will show that the state  $\text{col}(x'_{10}, 0)$  can be steered to  $\text{col}(x'_{1f}, 0)$ . To see this, apply Theorem 3. This gives a positive number  $T > 0$  and a  $C^\infty$ -function  $y = y_{\text{mid}}$  such that the solution  $x_1$  of (25) satisfies  $x_1(0) = x'_{10}$  and  $x_1(T) = x'_{1f}$ . According to Lemma 2,  $y_{\text{mid}}$  function can be chosen such that  $y_{\text{mid}}^{(j)}(0) = y_{\text{mid}}^{(j)}(T) = 0$  for all  $j = 0, 1, \dots$ . Moreover, one can find a finite number of points, say  $0 = t_0 < t_1 < \dots < t_Q = T$ , such that  $y_{\text{mid}}(t) \in \mathcal{Y}_{i_q}$  whenever  $t \in [t_q, t_{q+1}]$ . Since the transfer matrix  $D + C_2(sI - A_{22} + M_2^i C_2)^{-1}(B_2 + M_2^i D)$  has a polynomial inverse for all  $i = 1, 2, \dots, r$ , repeated application of Proposition 4 to the systems  $\Sigma(A_{22} + M_2^{i_q} C_2, B_2 + M_2^{i_q} D, C_2, D)$  yields an input  $v_{\text{mid}}$  and a state trajectory  $x_2$  such that (23b)-(23c) are satisfied for  $y = y_{\text{mid}}$ . Moreover,  $x_2(0) = x_2(T) = 0$  due to Proposition 5. Consequently, the concatenation of  $v_{\text{in}}$ ,  $v_{\text{mid}}$ , and  $v_{\text{out}}$  steers the state  $\text{col}(x_{10}, x_{20})$  to the state  $\text{col}(x_{1f}, x_{2f})$ . ■

### C.3 Proof of Theorem 4

In view of Lemma 1 and Theorem 3, it is enough to show that the controllability of the pair

$$(A_{11}, [L_1 + M_1^1 L_1 + M_1^2 \cdots L_1 + M_1^r])$$

with respect to  $\mathcal{Y}_1 \times \mathcal{Y}_2 \times \dots \times \mathcal{Y}_r$  is equivalent to the conditions presented in Theorem 4. Note that the former is equivalent to the following conditions.

- i. the pair  $(A_{11}, [L_1 + M_1^1 L_1 + M_1^2 \cdots L_1 + M_1^r])$  is controllable, and

ii. the implication

$$z^T A_{11} = \lambda z^T, \quad \lambda \in \mathbb{R}, \quad (L_1 + M_1^i)^T z \in \mathcal{Y}_i^* \text{ for all } i = 1, 2, \dots, r \quad \Rightarrow \quad z = 0$$

holds.

Our aim is to prove the equivalence of i to 1 and of ii to 2.

**i  $\Leftrightarrow$  1:**

Note that  $\langle A + M^i C \mid \text{im}(B + M^i D) \rangle = \langle (A - BK) + M^i(C - DK) \mid \text{im}(B + M^i D) \rangle$  for any  $K$  due to Proposition 1. Take  $K \in \mathcal{K}(\mathcal{V}^*)$ . Note that the condition in 1 of Theorem 4 is invariant under state space transformations. Therefore, one can, without loss of generality, take

$$(A - BK) + M^i(C - DK) = \begin{bmatrix} A_{11} & (L_1 + M_1^i)C_2 \\ 0 & A_{22} + M_2^i C_2 \end{bmatrix} \quad B + M^i D = \begin{bmatrix} (L_1 + M_1^i)D \\ B_2 + M_2^i D \end{bmatrix}. \quad (72)$$

Let  $\mathcal{R}_i$  denote  $\langle (A - BK) + M^i(C - DK) \mid \text{im}(B + M^i D) \rangle$ . Note that  $\mathcal{R}_i$  is an input-containing conditioned invariant subspace of the system  $\Sigma(A, B, C, D)$ . Hence,  $\mathcal{T}^*$ , the smallest of the input-containing conditioned invariant subspaces, must be contained in  $\mathcal{R}_i$ . In the coordinates that we chose, this is equivalent to the inclusions

$$\text{im} \begin{bmatrix} 0 \\ I_{n_2} \end{bmatrix} \subseteq \mathcal{R}_i. \quad (73)$$

At this point, we need the following auxiliary lemma.

**Lemma 6.** *Let  $\mathcal{O}$ ,  $\mathcal{P}$ , and  $\mathcal{Q}$  be vector spaces such that  $\mathcal{O} = \mathcal{P} \oplus \mathcal{Q}$ . Also let  $\pi_{\mathcal{P}}(\pi_{\mathcal{Q}}) : \mathcal{O} \rightarrow \mathcal{O}$  be the projection on  $\mathcal{P}$  ( $\mathcal{Q}$ ) along  $\mathcal{Q}$  ( $\mathcal{P}$ ). Suppose that the linear maps  $F : \mathcal{O} \rightarrow \mathcal{O}$ ,  $G : \mathcal{S} \rightarrow \mathcal{O}$ , and  $\tilde{F} : \mathcal{O} \rightarrow \mathcal{O}$  satisfy the following properties:*

- $\mathcal{P}$  is  $F$ -invariant,
- $\pi_{\mathcal{P}} F \pi_{\mathcal{P}} = \tilde{F}$ , and
- $\mathcal{Q} \subseteq \langle F \mid \text{im } G \rangle$ .

Then,  $\langle \tilde{F} \mid \text{im}(\pi_{\mathcal{P}} F \pi_{\mathcal{Q}}) + \text{im}(\pi_{\mathcal{P}} G) \rangle \subseteq \langle F \mid \text{im } G \rangle$ .

*Proof.* Note that

$$\tilde{F} \langle F \mid \text{im } G \rangle = \pi_{\mathcal{P}} F \pi_{\mathcal{P}} \langle F \mid \text{im } G \rangle \quad (74a)$$

$$= \pi_{\mathcal{P}} F (\mathcal{P} \cap \langle F \mid \text{im } G \rangle) \quad (74b)$$

$$\subseteq \pi_{\mathcal{P}} (\mathcal{P} \cap \langle F \mid \text{im } G \rangle) \quad (74c)$$

$$\subseteq (\mathcal{P} \cap \langle F \mid \text{im } G \rangle) \subseteq \langle F \mid \text{im } G \rangle. \quad (74d)$$

This shows that the subspace  $\langle F \mid \text{im } G \rangle$  is  $\tilde{F}$ -invariant. Note also that

$$\text{im } \pi_{\mathcal{P}} F \pi_{\mathcal{Q}} = \pi_{\mathcal{P}} F \mathcal{Q} \subseteq \pi_{\mathcal{P}} F \langle F \mid \text{im } G \rangle \subseteq \pi_{\mathcal{P}} \langle F \mid \text{im } G \rangle \subseteq \langle F \mid \text{im } G \rangle \quad (75)$$

and

$$\text{im } \pi_{\mathcal{P}} G \subseteq \text{im } G \subseteq \langle F \mid \text{im } G \rangle \quad (76)$$

These two inclusions show that the subspace  $\langle F \mid \text{im } G \rangle$  contains  $\text{im}(\pi_{\mathcal{P}} F \pi_{\mathcal{Q}}) + \text{im}(\pi_{\mathcal{P}} G)$ . Since  $\langle \tilde{F} \mid \text{im}(\pi_{\mathcal{P}} F \pi_{\mathcal{Q}}) + \text{im}(\pi_{\mathcal{P}} G) \rangle$  is the smallest  $\tilde{F}$ -invariant subspace that contains  $\text{im}(\pi_{\mathcal{P}} F \pi_{\mathcal{Q}}) + \text{im}(\pi_{\mathcal{P}} G)$ , the inclusion

$$\langle \tilde{F} \mid \text{im}(\pi_{\mathcal{P}} F \pi_{\mathcal{Q}}) + \text{im}(\pi_{\mathcal{P}} G) \rangle \subseteq \langle F \mid \text{im } G \rangle$$

holds. ■

Now, take

$$\mathcal{O} = \mathbb{R}^n, \quad \mathcal{P} = \text{im} \begin{bmatrix} I_{n_1} \\ 0 \end{bmatrix}, \quad \mathcal{Q} = \text{im} \begin{bmatrix} 0 \\ I_{n_2} \end{bmatrix}, \quad \mathcal{S} = \mathbb{R}^m \quad (77a)$$

$$F^i = (A - BK) + M^i(C - DK), \quad G^i = B + M^i D, \quad \tilde{F} = \begin{bmatrix} A_{11} & 0 \\ 0 & 0 \end{bmatrix}. \quad (77b)$$

Note that

$$\pi_{\mathcal{P}} = \begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \pi_{\mathcal{Q}} = \begin{bmatrix} 0 & 0 \\ 0 & I_{n_2} \end{bmatrix}.$$

Then, one has

$$\pi_{\mathcal{P}} F^i \pi_{\mathcal{P}} = \begin{bmatrix} A_{11} & 0 \\ 0 & 0 \end{bmatrix} \quad (78a)$$

$$\pi_{\mathcal{P}} F^i \pi_{\mathcal{Q}} = \begin{bmatrix} 0 & (L_1 + M_1^i)C_2 \\ 0 & 0 \end{bmatrix} \quad (78b)$$

$$\pi_{\mathcal{P}} G^i = \begin{bmatrix} (L_1 + M_1^i)D \\ 0 \end{bmatrix}. \quad (78c)$$

Note that the first hypothesis of Lemma 6 is satisfied due to (72). It follows from (77b) and (78a) that the second one is also satisfied. Finally, the third follows from (73). Then, Lemma 6 results in

$$\left\langle \begin{bmatrix} A_{11} & 0 \\ 0 & 0 \end{bmatrix} \mid \text{im} \begin{bmatrix} (L_1 + M_1^i)C_2 & 0 \\ (L_1 + M_1^i)D & 0 \end{bmatrix} \right\rangle \subseteq \mathcal{R}_i. \quad (79)$$

By the invertibility hypothesis, the matrix  $[C_2 \ D]$  must be of full row rank. Then, the above inclusion can be written as

$$\left\langle \begin{bmatrix} A_{11} & 0 \\ 0 & 0 \end{bmatrix} \mid \text{im} \begin{bmatrix} L_1 + M_1^i \\ 0 \end{bmatrix} \right\rangle \subseteq \mathcal{R}_i. \quad (80)$$

Summing both sides over  $i$ , one gets

$$\sum_{i=1}^r \left\langle \begin{bmatrix} A_{11} & 0 \\ 0 & 0 \end{bmatrix} \mid \text{im} \begin{bmatrix} L_1 + M_1^i \\ 0 \end{bmatrix} \right\rangle \subseteq \sum_{i=1}^r \mathcal{R}_i. \quad (81)$$

This implies that

$$\left\langle \begin{bmatrix} A_{11} & 0 \\ 0 & 0 \end{bmatrix} \mid \text{im} \begin{bmatrix} L_1 + M_1^1 & L_1 + M_1^2 & \cdots & L_1 + M_1^r \\ 0 & 0 & \cdots & 0 \end{bmatrix} \right\rangle \subseteq \sum_{i=1}^r \mathcal{R}_i. \quad (82)$$

Together with (73), the above inclusion implies that the implication  $\text{i} \Rightarrow 1$  holds. For the reverse direction, suppose that 1 holds but  $\text{i}$  does not. Then, there exist a nonzero vector  $z$  and  $\lambda \in \mathbb{C}$  such that  $z^* [\lambda I - A_{11} \ L_1 + M_1^1 \ L_1 + M_1^2 \ \cdots \ L_1 + M_1^r] = 0$ . It can be verified that the real part of  $z$ , say  $w$ , belongs to  $\mathcal{R}_i^\perp$  for all  $i$ . Thus,  $w$  belongs to  $\cap_{i=1}^r \mathcal{R}_i^\perp = (\sum_{i=1}^r \mathcal{R}_i)^\perp$ . This, however, contradicts 1.

**ii  $\Leftrightarrow$  2:** Note that the statement 2 is invariant under state space transformations. This means that it is enough to prove the statement for the system (23). Let  $\lambda \in \mathbb{R}$ ,  $v \in \mathbb{R}^{n_1}$ ,  $z \in \mathbb{R}^{n_2}$ , and  $w_i \in \mathbb{R}^m$  be such that

$$[v^T \ z^T \ w_i^T] \begin{bmatrix} A_{11} - \lambda I & (L_1 + M_1^i)C_2 & (L_1 + M_1^i)D \\ 0 & A_{22} + M_2^i C_2 - \lambda I & B_2 + (L_1 + M_2^i)D \\ 0 & C_2 & D \end{bmatrix} = 0.$$

This would result in

$$v^T A_{11} = \lambda v^T \quad (83)$$

$$[z^T \ w_i^T - v^T (L_1 + M_1^i)] \begin{bmatrix} A_{22} + M_2^i C_2 - \lambda I & B_2 + (L_1 + M_2^i)D \\ C_2 & D \end{bmatrix} = 0. \quad (84)$$



Note that  $\mathcal{V}^*(A_{22} + M_2^i C_2, B_2 + M_2^i D, C_2, D) = \{0\}$  for all  $i$ . Then, it follows from (84) and Proposition 3 that  $z = 0$  and  $w_i^T = v^T (L_1 + M_1^i)$ . This implies that it is equivalent to 2. ■

## References

1. H. Aling and J.M. Schumacher. A nine-fold canonical decomposition for linear systems. *Int. J. Control*, 39(4):779–805, 1984.
2. A. Bemporad, G. Ferrari-Trecate, and M. Morari. Observability and controllability of piecewise affine and hybrid systems. *IEEE Transactions on Automatic Control*, 45(10):1864–1876, 2000.
3. V.D. Blondel and J.N. Tsitsiklis. Complexity of stability and controllability of elementary hybrid systems. *Automatica*, 35(3):479–490, 1999.
4. R.F. Brammer. Controllability in linear autonomous systems with positive controllers. *SIAM J. Control*, 10(2):329–353, 1972.
5. B. Brogliato. Some results on the controllability of planar variational inequalities. *Systems and Control Letters*, 2005. to appear.
6. M.K. Çamlıbel, W.P.M.H. Heemels, and J.M. Schumacher. On the controllability of bimodal piecewise linear systems. In R. Alur and G.J. Pappas, editors, *Hybrid Systems: Computation and Control*, LNCS 2993, pages 250–264. Springer, Berlin, 2004.
7. M.K. Çamlıbel, W.P.M.H. Heemels, and J.M. Schumacher. On linear passive complementarity systems. *European Journal of Control*, 8(3), 2002.
8. M.K. Çamlıbel, W.P.M.H. Heemels, and J.M. Schumacher. Stability and controllability of planar bimodal complementarity systems. In *Proc. of the 42th IEEE Conference on Decision and Control*, Hawaii (USA), 2003.
9. M.K. Çamlıbel, J.S. Pang, and J. Shen. Observability of conewise linear systems. In *Proc. of the 44th IEEE Conference on Decision and Control*, Sevilla (Spain), 2005. (accepted for publication).
10. M.K. Çamlıbel, W.P.M.H. Heemels, A.J. van der Schaft, and J.M. Schumacher. Switched networks and complementarity. *IEEE Transactions on Circuits and Systems I*, 50(8):1036–1046, 2003.
11. M.K. Çamlıbel, L. Iannelli, and F. Vasca. Modelling switching power converters as complementarity systems. In *Proc. of the 43th IEEE Conference on Decision and Control*, Paradise Islands (Bahamas), 2004.
12. R.W. Cottle, J.-S. Pang, and R.E. Stone. *The Linear Complementarity Problem*. Academic Press, Boston, 1992.
13. B.D. Craven. *Mathematical Programming and Control Theory*. Chapman and Hall, London, 1978.
14. R. Haber and H. Unbehauen. Structure identification of nonlinear dynamic systems - a survey on input/output approaches. *Automatica*, 9:311–349, 1990.
15. L.C.G.J.M. Habets and J.H. van Schuppen. A controllability result for piecewise-linear hybrid systems. In *Proc. of the European Control Conference*, Porto (Portugal), 2001.
16. W.P.M.H. Heemels and B. Brogliato. The complementarity class of hybrid dynamical systems. *European Journal of Control*, 26(4):651–677, 2003.
17. W.P.M.H. Heemels, M.K. Çamlıbel, and J.M. Schumacher. On the dynamic analysis of piecewise-linear networks. *IEEE Trans. on Circuits and Systems-I: Fundamental Theory and Applications*, 49(3):315–327, March 2002.
18. W.P.M.H. Heemels, J.M. Schumacher, and S. Weiland. Linear complementarity systems. *SIAM J. Appl. Math.*, 60(4):1234–1269, 2000.
19. L. Hörmander. *The Analysis of Linear Partial Differential Operators I*. Springer-Verlag, Berlin, 1983.
20. T. Kailath. *Linear Systems*. Prentice-Hall, Englewood Cliffs, NJ, 1980.
21. R.E. Kalman. On the general theory of control systems. In *Proceedings of the 1st World Congress of the International Federation of Automatic Control*, pages 481–493, 1960.
22. G. Lafferriere, G.J. Pappas, and S. Yovine. Reachability computation for linear hybrid systems. In *Proc. of the 14th IFAC World Congress*, volume E, pages 7–12, Beijing (China), 1999.
23. E.B. Lee and L. Markus. *Foundations of Optimal Control Theory*. John Wiley & Sons, New York, 1967.
24. K.K. Lee and A. Arapostathis. On the controllability of piece-wise linear hypersurface systems. *Systems and Control Letters*, 9:89–96, 1987.
25. A.G.J. MacFarlane and N. Karcanias. Poles and zeros of linear multivariable systems: a survey of algebraic, geometric and complex-variable theory. *Int. J. Contr.*, 24(1):33–74, 1976.
26. D. Nescic. Controllability for a class of simple wiener-hammerstein systems. *Systems and Control Letters*, 36(1):51–59, 1999.

27. S.H. Saperstone and J.A. Yorke. Controllability of linear oscillatory systems using positive controls. *SIAM J. Control*, 9(2):253–262, 1971.
28. A.J. van der Schaft and J.M. Schumacher. The complementary-slackness class of hybrid systems. *Mathematics of Control, Signals and Systems*, 9:266–301, 1996.
29. A.J. van der Schaft and J.M. Schumacher. Complementarity modelling of hybrid systems. *IEEE Transactions on Automatic Control*, 43(4):483–490, 1998.
30. A.J. van der Schaft and J.M. Schumacher. *An Introduction to Hybrid Dynamical Systems*. Springer-Verlag, London, 2000.
31. J.M. Schumacher. Complementarity systems in optimization. *Mathematical Programming Series B*, 101:263–295, 2004.
32. G.V. Smirnov. *Introduction to the Theory of Differential Inclusions*. American Mathematical Society, Rhode Island, 2000.
33. E.D. Sontag. *Mathematical Control Theory: Deterministic Finite Dimensional Systems*. Springer, New York, 2nd edition, 1998.
34. H.L. Trentelman, A.A. Stoorvogel, and M.L.J. Hautus. *Control Theory for Linear Systems*. Springer, London, 2001.