All binary, (n,e,r)-uniformly packed codes are known

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All binary, (n,e,r)-uniformly packed codes are known

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§ 1. Introduction

Let V be a n-dimensional vectorspace over GF(2). For $\underline{u} \in V$, the weight $w(\underline{u})$ is the number of its nonzero components. The Hamming distance $d(\underline{u},\underline{v})$ for any two vectors $\underline{u}$ and $\underline{v}$ im V is the weight of their difference, i.e. $d(\underline{u},\underline{v}) = w(\underline{u} - \underline{v})$.

A code C of length n is any subset of V, with $|C| \geq 2$; its minimum distance d(C) is the minimum value of the distance between any two distinct elements of C. A code C is called e-error-correcting iff $e = [\frac{d(C) - 1}{2}]$. The weight-enumerator of a code C is the polynomial $W_c(z)$ defined by

$$(1) \qquad W_c(z) := \sum_{i=0}^{n} A(i) z^i := \sum_{\underline{u} \in C} z^{w(\underline{u})} .$$

Clearly A(i) is the number of codewords of weight i. We need some more definitions:

$$(2) \qquad B(\underline{x},k) := |\{\underline{c} \in C \mid d(\underline{x},\underline{c}) = k\}|, \quad \underline{x} \in V, \ 0 \leq k \leq n ,$$

$$(3) \qquad p(\underline{x}) := \min\{k \mid B(\underline{x},k) \neq 0\}, \quad \underline{x} \in V ,$$

$$(4) \qquad C_e := \{\underline{x} \in V \mid p(\underline{x}) \geq e\} ,$$

$$(5) \qquad r(\underline{x}) := B(\underline{x},e) + B(\underline{x},e+1) .$$

In words: $r(\underline{x})$ is the number of code words at distance e or e + 1 from $\underline{x}$.

Let $\underline{x} \in C_e$ be fixed. By a suitable translation of the code, we may assume that $\underline{x} = \underline{0} = (0,0,\ldots,0)$.

Now $r(\underline{0})$ equals the number of codewords of weight e or e + 1. Since the mutual distance of these code words is at least 2e + 1, we have $r(\underline{0}) \leq [\frac{n+1}{e+1}]$, i.e.

$$(6) \qquad r(\underline{x}) \leq [\frac{n+1}{e+1}], \quad (\forall_{\underline{x} \in C_e}) .$$

Let r(C) be the average value of $r(\underline{x})$ for $\underline{x} \in C_e$. Since

$$(7) \qquad |C_e| = 2^n - |C| \sum_{i=0}^{e-1} \binom{n}{i}$$

and

$$(8) \qquad \sum_{\underline{x} \in C_e} r(\underline{x}) = |C|\{\binom{n}{e} + \binom{n}{e+1}\}$$

it follows that

$$(9) \qquad \frac{|C|.\{\binom{n}{e} + \binom{n}{e+1}\}}{2^n - |C| \cdot \sum_{i=0}^{e-1} \binom{n}{i}} = r(C) \leq [\frac{n+1}{e+1}] \ .$$

The inequality in (2) was originally derived in [2].

A code C is called a (n,e,r)-uniformly packed code if for all $\underline{x} \in C_e$, $r(\underline{x}) = r = r(C)$.

Clearly $r \geq 2$, since $r = 1$ implies that the code is $(e+1)$-error-correcting. We remark that this in the original definition of uniformly packed codes (see [5]).

Later this definition was generalized to other fields and the condition for r was replaced by

$$\underline{x} \in V,\ p(\underline{x}) = e \Rightarrow B(\underline{x}, e+1) = \lambda \ ,$$

$$\underline{x} \in V,\ p(x) > e \Rightarrow B(x, e+1) = \mu \ .$$

So our case reduces to $\lambda + 1 = \mu = r$ (see [1]). If $r = \frac{n+1}{e+1}$, where $e+1$ divides $n+1$, then C is called perfect. This is the case where the spheres of radius e around the codewords form a partition of V.

If $r = [\frac{n+1}{e+1}]$, where $e+1$ does not divide $n+1$, then C is called nearly perfect.

It was shown by van Lint and Tietäväinen that there are no unknown perfect codes (see [4] and [6]). Recently K. Lindström proved that there are no unknown binary, nearly perfect codes (see [3]).

It is the aim of this paper to prove:

Theorem. There are no unknown, uniformly packed binary codes.

§ 2. Lemmas

In [1] the following result is proved:

Lemma 1. If C is a (n,e,r)-uniformly packed code, $e = 1$ or 2, then either C is (nearly) perfect or we are in one of the following cases:

a) $e = 1,\ n = (2^{m-1} + 1)(2^m - 1),\ r = \begin{pmatrix} 2^{m-1} + 1 \\ 2 \end{pmatrix},\ m \geq 2;$

b) $e = 1,\ n = (2^{m-1} - 1)(2^m + 1),\ r = \begin{pmatrix} 2^{m-1} \\ 2 \end{pmatrix},\ m \geq 3;$

c) $e = 1,\ n = 2^m - 2,\ r = 2^{m-1} - 1, \qquad m \geq 3;$

d) $e = 2,\ n = 2^{2m} - 1,\ r = (2^{2m} - 1)/3, \qquad m \geq 2;$

e) $e = 2,\ n = 2^{2m+1} - 1,\ r = (2^{2m} - 1)/3, \qquad m \geq 2;$

f) $e = 2,\ n = 11,\ r = 3$.

For a description of these codes see [1].

Definition. $C(n,e,r)$ denotes the set of (n,e,r)-uniformly packed codes C, where C is not perfect.

Lemma 2. If $C \in C(n,e,r)$, then $d(C) = 2e + 1$.

Proof. Assume that $d(C) = 2e + 2$. W.l.o.g. $\underline{0} \in C$ and $\underline{c} := (1,1,\ldots,1,0,0,\ldots,0)$, where $w(\underline{c}) = 2e + 2$, is in the code. Take $\underline{x} = (1,1,\ldots,1,0,\ldots,0)$, $w(\underline{x}) = e$. Then $r = r(\underline{x}) = 1$. However for $\underline{y} = (1,1,\ldots,1,0,\ldots,0)$, $w(\underline{y}) = e + 1$, we find $r = r(\underline{y}) \geq 2$. □

Lemma 3. If $C \in C(n,e,r)$, then

$$|C|\{\sum_{i=0}^{e-1} \binom{n}{i} + \frac{1}{r}(\binom{n}{e} + \binom{n}{e+1})\} = 2^n \ . \tag{10}$$

Proof. This is a reformulation of (9). □

Lemma 4. If $C(n,e,r)$ is nonempty, then the polynomial

$$Q(x) := \sum_{i=0}^{e-1} P_i^{(n)}(x) + \frac{1}{r} P_e^{(n)}(x) + \frac{1}{r} P_{e+1}^{(n)}(x) = \tag{11}$$

$$= \frac{1}{r}\{(r-1)P_{e-1}^{(n-1)}(x-1) + P_{e+1}^{(n-1)}(x-1)\} \tag{12}$$

has $e+1$ distinct integer roots $x_1, x_2, \ldots, x_{e+1}$ in $[1,n]$. Here

$$P_k^{(n)}(x) := \sum_{i=0}^{k} (-2)^i \binom{n-i}{k-i}\binom{x}{i} = \sum_{i=0}^{k} (-1)^i \binom{n-x}{k-i}\binom{x}{i} \ . \tag{13}$$

Proof. See [1]. □

Lemma 5. If $x_1 < x_2 < \ldots < x_{e+1}$ are the zeros of $Q(x)$, $e \geq 3$, then

(14) i) $$\sum_{i=1}^{e+1} x_i = \frac{(n+1)(e+1)}{2} ,$$

(15) ii) $$x_i + x_{e+1-i} = n + 1, \quad 1 \leq i \leq e+1 ,$$

(16) iii) $$\prod_{i=1}^{e+1} x_i = \frac{r(e+1)!2^{n-e-1}}{|C|} \geq \frac{(e+1)!\binom{n}{e+1}}{2^{e+1}} ,$$

(17) iv) $$2^{e+1} \prod_{i=1}^{e+1} (x_i - 1) = (n-1)(n-2)\ldots(n-e+1)\{n^2 - (2e+1)n + re(e+1)\},$$

(18) v) $$2^{e+1} \prod_{i=1}^{e+1} (x_i - 2) = (n-2)(n-3)\ldots(n-e+1)\{(r-1)(e+1)e(n-2e+1) + (n-e)(n-e-1)(n-2e-3)\} .$$

Proof. Let $C_k(p(x))$ denote the coefficients of x^k in the polynomial $p(x)$. Since

$$C_{e+1}(Q(x)) = C_{e+1}(\frac{1}{r} P_{e+1}^{(n)}(x)) = (-2)^{e+1} \frac{1}{r(e+1)!} ,$$

it follows that

(19) $$Q(x) = \frac{(-2)^{e+1}}{r(e+1)!} \prod_{i=1}^{e+1} (x - x_i) .$$

Now i) follows from (11) and the observation

$$\sum_{i=1}^{e+1} x_i = -C_e(Q(x))/C_{e+1}(Q(x)) .$$

The equality in iii) follows similarly from (11) and

$$\prod_{i=1}^{e+1} x_i = (-1)^{e+1} C_0(Q(x))/C_{e+1}(Q(x)) .$$

The inequality in iii) follows from (10) and

$$\frac{r(e+1)!2^{n-e-1}}{|C|} = \frac{(e+1)!\{\sum_{i=0}^{e-1} \binom{n}{i} + \frac{1}{r}\binom{n}{e} + \frac{1}{r}\binom{n}{e+1}\}}{2^{e+1}\,\frac{1}{r}} \geq \frac{(e+1)!\binom{n}{e+1}}{2^{e+1}} .$$

The equalities iv) and v) can easily be verified by substitution of $x = 1$ resp. $x = 2$ in (11) and (19). The definition of $P_k^{(n)}(x)$ in (13) leads to the obvious observation $P_k^{(n)}(x) = (-1)^k P_k^{(n)}(n - x)$. Using (12), one finds $Q(x) = (-1)^{e+1} Q(n + 1 - x)$. This implies ii). □

Lemma 6. Let $C \in C(n,e,r)$, $\underline{0} \in C$. Then the words of weight k in C form an $e - (n,k,\lambda(k))$ design, where $\lambda(k)$ depends on k, $\lambda(2e + 1) = r - 1$. Moreover, the words of weight k in the extended code form an $(e + 1) - (n + 1,k,\mu(k))$ design, where $\mu(k)$ depends on k, $\mu(2e + 2) = r - 1$.

Proof. See [5]. □

Lemma 7. Let $\sum_{i=0}^{n} A(i)z^i$ be the weight enumerator of a code $C \in C(n,e,r)$. Then for all $0 \le k \le n$

$$(20) \qquad \binom{n}{k} = \sum_{\delta=0}^{e+1} \alpha_\delta \sum_{i=0}^{\delta} A(k + \delta - 2i) \binom{k + \delta - 2i}{\delta - i} \binom{n - k - \delta + 2i}{i},$$

where $\alpha_0 = \alpha_1 = \ldots = \alpha_{e-1} = 1$, $\alpha_e = \alpha_{e+1} = \frac{1}{r}$.

Proof. See [5]. □

Lemma 8. If $C(n,e,r)$, $e \ge 3$, is nonempty, then $e \ge 17$ or

$e = 3$, $n \ge 90$,	$e = 8$, $n \ge 405$,	$e = 13$, $n \ge 279$,
$e = 4$, $n \ge 135$,	$e = 9$, $n \ge 262$,	$e = 14$, $n \ge 319$,
$e = 5$, $n \ge 189$,	$e = 10$, $n \ge 314$,	$e = 15$, $n \ge 361$,
$e = 6$, $n \ge 430$,	$e = 11$, $n \ge 371$,	$e = 16$, $n \ge 407$.
$e = 7$, $n \ge 324$,	$e = 12$, $n \ge 242$,	

Proof. This is done by a computer analysis. For each of the admissable parameters, we first checked whether they satisfy the necessary conditions for the existence of an $(e + 1) - (n + 1,2e + 2,r - 1)$ design (lemma 6). If so, then we applied lemma 3. This excluded all the remaining cases. The total computer time was 16 seconds on a Burroughs B6700. □

Lemma 9. If $C(n,e,r)$, $e \ge 3$, is nonempty then

i) $$n \ge \frac{(r - 1)e^2 + (3r - 2)e + (2r - 2)}{r} \quad \text{for } r \ge 4,$$

ii) $n \geq \frac{2e^2 + 8e + 4}{3}$ for $r = 3$,

iii) $n \geq \frac{e^2 + 4e + 3}{2}$ for $r = 2$.

Proof. With the aid of lemma 7, it is easy to verify that

$$A(2e + 2) = A(2e + 1)\frac{n - 2e - 1}{2(e + 1)}$$

and

$$A(2e + 3) = \frac{A(2e + 1).g(n)}{(2e + 3)(2e + 2)(r - 1)} ,$$

where $g(n) := r(n - e)(n - e - 1) - r(r - 1)e(e + 1) - (r - 1)(e + 1)(e + 3)(n - 2e - 1)$. At this point we must remark that the cases $n = 2e + 1$ and $n = 2e + 2$ never occur in $C(n,e,r)$.
Since $g(2e + 1) = r(2 - r)e(e + 1) \leq 0$, it follows that n must be greater than or equal to the largest zero of $g(x)$. Using $e^4(r - 1)^2$ as a lower bound for the discriminant of $g(n)$ for $r \geq 4$, one easily obtains i). Direct calculations for $r = 2$ and 3 lead to ii) and iii). □

Lemma 10. If $C(n,e,r)$, $e \geq 3$, is nonempty, then

$$(r - 1)(n - e + 1) \geq (e + 2)(e + 3) .$$

Proof. Since the words of weight $2e + 1$ form an e-design with $\lambda = r - 1$, one can apply the generalisation of Fisher's inequality to the parameters (see [8]). This leads to the lemma. □

Lemma 11. If $C(n,e,r)$, $e \geq 3$, is nonempty, then

(21) $$n \geq \frac{2}{3}(e + 1)(e + 2) .$$

Proof. Apply lemma 9 for $r \geq 3$ and lemma 10 for $r = 2$. □

Definition. For any $m \in \mathbb{N}$, $A(m)$ is defined as the largest odd divisor of m. We define an equivalence relation on $\mathbb{N}$ by

$$m \sim n :\Leftrightarrow A(m) = A(n) .$$

Let $s(C)$, for any $C \in C(n,e,r)$, be the number of equivalence classes X_i containing at least one zero of $Q(x)$. Moreover let n_i be the number of equivalence classes containing exactly i zeros of $Q(x)$. Clearly

$$(22)\qquad \sum_{i=1}^{e+1} n_i = s(C) ,$$

$$(23)\qquad \sum_{i=1}^{e+1} i n_i = e + 1 .$$

Lemma 12. If $C(n,e,r)$, $e \geq 3$, is nonempty and $Q(x)$ has k zeros on $[0,\alpha(n+1)]$, $\alpha < \frac{1}{2}$, then

$$(24)\qquad \prod_{i=1}^{e+1} x_i \leq (4\alpha(1-\alpha))^k \left(\frac{n+1}{2}\right)^{e+1} .$$

Proof. Since $x_1 < x_2 < \ldots < x_k \leq \alpha(n+1)$ it follows from (15) that

$$x_i\, x_{e+1-i} \leq \alpha(1-\alpha)(n+1)^2 = 4\alpha(1-\alpha)\left(\frac{n+1}{2}\right)^2, \quad 1 \leq i \leq k ,$$

$$x_i\, x_{e+1-i} \leq \left(\frac{n+1}{2}\right)^2, \quad \text{for the other values of } i .$$

Together these inequalities imply the lemma. □

Lemma 13. Let $C \in C(n,e,r)$, $e \geq 3$. Then

$$(25)\qquad n + 1 \geq (e+1)^{\frac{e+1}{\log(e+1)} \frac{5 \log 2}{4} - (e+1-s(C))} \prod_{\substack{i \leq e+1-s(C) \\ i \text{ odd}}} i^2 .$$

Proof. Since

$$2^{2e} = \sum_{i=0}^{e} \binom{2e+1}{i} \leq A(|C|) \cdot \sum_{i=0}^{e} \binom{n}{i} \leq 2^{n-k} ,$$

one has $n - k - e - 1 > 0$ (here $|C| = A(|C|).2^k$). Therefore by lemma 5, iii) and by the inequality in (9)

$$(26)\qquad A\left(\prod_{i=1}^{e+1} x_i\right) = A\left(\frac{r(e+1)!\,2^{n-k-e-1}}{A(|C|)}\right) = \frac{A(r)A((e+1)!)}{A(|C|)} \leq$$

$$\leq rA((e+1)!) \leq \frac{n+1}{e+1} A((e+1)!) .$$

Tietäväinen has proved in [6] that for all $e \geq 7$

$$(27) \qquad A((e+1)!) < p(e+1)(e+1)^{[\frac{e+1}{2}]+1-\frac{e+1}{\log(e+1)}\frac{5\log 2}{4}},$$

where $p(e+1) = \prod_{\substack{i \leq e+1 \\ i \text{ odd}}} i$.

Suppose that the smallest zero x and the largest zero y in one equivalence class, satisfy $16x \leq y$. Clearly $x \leq \frac{n+1}{16}$. However (24) now implies

$$\prod_{i=1}^{e+1} x_i \leq \frac{15}{64}(\frac{n+1}{2})^{e+1} .$$

Comparing this with the inequality in (16) results in

$$\frac{15}{64} \geq \prod_{i=1}^{e+1} (1 - \frac{i}{n+1}) .$$

Since the right hand side is at least $1 - \frac{(e+1)(e+2)}{2(n+1)}$, we obtain a contradiction with lemma 11.

Therefore $n_\ell = 0$ for $\ell \geq 5$ and $n_4 \neq 0$ implies that the elements of a class x_i with four zeros look like a, 2a, 4a and 8a. Moreover, clearly $a \leq \frac{1}{8}(n+1)$. Suppose that the sum of any 2 zeros in this class is never n + 1. Let $Y := \{n+1-a,\ n+1-2a,\ n+1-4a,\ n+1-8a\}$. Now, using the arithmeticmean-geometricmean inequality, we obtain

$$\prod_{j=1}^{e+1} x_j = \prod_{x \in X_i \cup Y} x \prod_{\substack{j=1 \\ x_j \notin X_i \cup Y}}^{e+1} x \leq \frac{1}{8}\cdot\frac{7}{8}, (n+1)^2 \frac{1}{4}\frac{3}{4}(n+1)^2(\frac{n+1}{2})^4 .$$

$$\prod_{\substack{j=1 \\ x_j \notin X_i \cup Y}}^{e+1} x = \frac{21}{64}(\sum_{x \in X_i \cup Y} \frac{x}{8})^8 (\prod_{\substack{j=1 \\ x_j \notin X_i \cup Y}}^{e+1} x_j) \leq \frac{21}{64}(\sum_{x \in X_i \cup Y} \frac{x}{8})^8 (\sum_{\substack{j=1 \\ x_j \notin X_i \cup Y}}^{e+1} \frac{x_j}{e-7})^{e-7}$$

$$\leq \frac{21}{64}(\sum_{j=1}^{e+1} \frac{x_j}{e+1})^{e+1} \leq \frac{21}{64}(\frac{n+1}{2})^{e+1} .$$

This leads, as above, to a contradiction with (16) and lemma 11.

If the sum of two zero's in X_i equals n + 1, we get in the same way, but easier, a contradiction. Hence $n_4 = 0$. Now clearly

$$A\Big(\prod_{i=1}^{e+1} x_i\Big) \geq \{1.3.5\ldots(2s(C)-1)\}.1^2.3^2\ldots(2n_3-1)^2(2n_3+1)\ldots(2n_2+2n_3-1) =$$

$$= p(2s(C)).p(2n_3).p(2(n_2+n_3)) = \tag{28}$$

$$= p(2s(C)).p(2n_3).p(2(e+1-s(C)-n_3)) \geq$$

$$\geq p(2s(C)).\{p(e+1-s(C))\}^2 \geq$$

$$\geq p(e+1)(e+1)^{s(C)-(e+1-[\frac{e+1}{2}])}.\{p(e+1-s(C))\}^2 .$$

Comparing (26) and (28) leads, with the use of (27), to the assertion of the lemmas for $e \geq 7$. For $e = 3,4,5$ and 6 the lemma follows from lemma 8. □

At this moment we have enough lower bounds on possible values of n. The next 2 lemmas will provide us with upper bounds on n.

Lemma 14. If $y_1,y_2,\ldots,y_s$ and p are positive integers such that $\frac{y_{i+1}}{y_i} \geq p$, for all $1 \leq i \leq s-1$, then

$$\prod_{i=1}^{s} y_i \leq R^{s-1}\Big(\sum_{i=1}^{s} \frac{y_i}{s}\Big)^s, \quad \text{where } R = \frac{4p}{(1+p)^2} .$$

Proof. See [7]. □

Lemma 15. If $C \in \mathcal{C}(n,e,r)$, $e \geq 3$, then

$$\Big(\frac{8}{9}\Big)^{e+1-s(C)} \geq 1 - \frac{(e+1)(e+2)}{2(n+1)} . \tag{29}$$

Proof. Let

$$Y_i := X_i \cap \{x_1,x_2,\ldots,x_{e+1}\}, \quad t(i) := |Y_i|$$

$$R_i := \Big(\prod_{x\in Y_i} x\Big)\Big/\Big(\sum_{x\in Y_i} \frac{x}{t(i)}\Big)^{t(i)} \quad \text{for } Y_i \neq \emptyset .$$

Since $x \in Y_i$, $y \in Y_i$, $y > x$ implies $y \geq 2x$, we get by lemma 14 that $R_i \leq \big(\frac{8}{9}\big)^{t(i)-1}$. Therefore, using the arithmetic-mean-geometric-mean inequality

$$\prod_{i=1}^{e+1} x_i = \prod_{i=1}^{s(C)} \Big(\prod_{x\in Y_i} x\Big) \leq \prod_{i=1}^{s(C)} \Big(\frac{8}{9}\Big)^{t(i)-1}\Big(\sum_{x\in Y_i} \frac{x}{t(i)}\Big)^{t(i)} \leq$$

$$(\frac{8}{9})^{\sum_{i=1}^{s(C)}(t(i)-1)} (\sum_{i=1}^{e+1} \frac{x_i}{e+1})^{e+1} = (\frac{8}{9})^{e+1-s(C)} (\frac{n+1}{2})^{e+1} .$$

Here we also used (22), (23) and (14).
Comparing this inequality with the inequality in (16) one obtains

$$(\frac{8}{9})^{e+1-s(C)} \geq \prod_{i=1}^{e+1} (1 - \frac{i}{n+1}) .$$

The right hand side in turn is at least $1 - \frac{(e+1)(e+2)}{2(n+1)}$. □

Lemma 16. If $C(n,e,r)$, $e \geq 3$, is nonempty, then

$$(30) \qquad (n+1)^{1-2/e} \leq (\frac{A((e+1)!)}{e+1})^{2/e} (1 + \frac{\delta_n}{2})^2 . 2(e+1)(e+2)$$

where $\delta_n := (\frac{e+1}{(n+1)A((e+1)!)})^{1/e}$.

Proof. Let us reorder the roots of $Q(x)$ in such a way that $x_i = A(x_i)2^{\alpha_i}$, $\alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_{e+1}$.

$$(31) \qquad \prod_{i=1}^{e} g.c.d.(x_i,x_{i+1}) = \prod_{i=1}^{e} g.c.d.(A(x_i),A(x_{i+1})).2^{\alpha_i} \geq \prod_{i=1}^{e} 2^{\alpha_i} =$$

$$= \frac{x_1 x_2 \ldots x_e}{A(x_1 . x_2 \ldots x_e)} .$$

As in the proof of lemma 13 we remark that $n-k-e-1 > 0$ if $|C| = A(|C|).2^k$.
Using (31) and (16) we obtain

$$(32) \qquad \prod_{i=1}^{e} \frac{|x_i - x_{i+1}|}{x_i} \geq \prod_{i=1}^{e} \frac{g.c.d.(x_i,x_{i+1})}{x_i} \geq \frac{1}{A(x_1 . x_2 \ldots x_e)} \geq \frac{1}{A(x_1 \ldots x_{e+1})}$$

$$= \frac{A(|C|)}{A(r)A((e+1)!)} \geq \frac{1}{A(r)A((e+1)!)} \geq \frac{1}{rA((e+1)!)} \geq \frac{e+1}{(n+1)A((e+1)!)} .$$

Let t be defined by

$$\frac{|x_t - x_{t+1}|}{x_t} = \max_{1 \leq i \leq e} |\frac{x_i - x_{i+1}}{x_i}| .$$

Then (32) implies

$$\frac{|x_t - x_{t+1}|}{x_t} \geq \left(\frac{e+1}{(n+1)A((e+1)!)}\right)^{1/e} = \delta_n .$$

Since the function $\frac{x}{(1+x)^2}$ is monotonically increasing on $[0,1]$ and decreasing on $[1,\infty)$, it follows that for $x_t < x_{t+1}$, i.e. $\frac{x_{t+1}}{x_t} > 1 + \delta_n$ we have

$$(33) \qquad \frac{x_t\, x_{t+1}}{\left(\frac{x_t + x_{t+1}}{2}\right)^2} = \frac{\frac{x_{t+1}}{x_t}}{\left(\frac{1 + \frac{x_{t+1}}{x_t}}{2}\right)^2} < \frac{1+\delta_n}{\left(\frac{2+\delta_n}{2}\right)^2} = 1 - \frac{\delta_n^2}{4}\left(\frac{2}{2+\delta_n}\right)^2 =: 1 - \gamma ,$$

and similarly, for $x_t > x_{t+1}$,

$$(34) \qquad \frac{x_t\, x_{t+1}}{\left(\frac{x_t + x_{t+1}}{2}\right)^2} < \frac{1-\delta_n}{\left(\frac{2-\delta_n}{2}\right)^2} = 1 - \frac{\frac{\delta_n^2}{4}}{1 - \delta_n + \frac{\delta_n^2}{4}} < 1 - \frac{\delta_n^2}{4} < 1 - \gamma ,$$

where (33) defines γ.

Using (33), (34), the arithmetic-mean geometric-mean inequality and (14), we obtain

$$\prod_{i=1}^{e+1} x_i = x_t x_{t+1} \prod_{\substack{i=1 \\ i \neq t, t+1}}^{e+1} x_i \leq (1-\gamma)\left(\frac{x_t + x_{t+1}}{2}\right)^2 \left(\sum_{\substack{i=1 \\ i \neq t, t+1}}^{e+1} \frac{x_i}{e-1}\right)^{e-1} \leq$$

$$\leq (1-\gamma)\left(\sum_{i=1}^{e+1} \frac{x_i}{e+1}\right)^{e+1} = (1-\gamma)\left(\frac{n+1}{2}\right)^{e+1} .$$

Comparing this inequality with the one in (16), yields, using again that

$$\prod_{i=1}^{e+1} \left(1 - \frac{i}{n+1}\right) \geq 1 - \frac{(e+1)(e+2)}{2(n+1)} ,$$

$$1 - \frac{\delta_n^2}{4}\left(\frac{2}{2+\delta_n}\right)^2 = 1 - \gamma > 1 - \frac{(e+1)(e+2)}{2(n+1)} , \text{ i.e.}$$

$$(n+1)\delta_n^2 < 2\left(1 + \frac{\delta_n}{2}\right)^2 (e+1)(e+2) .$$

Substitution of δ_n in the left hand side yields the lemma. □

§ 3. Proof of the theorem

Let $C \in C(n,e,r)$, $e \geq 3$. Suppose $e+1-s(C) \geq 12$. Then lemma 15 implies

$$n+1 \leq \frac{(e+1)(e+2)}{2(1-(\frac{8}{9})^{e+1-s(C)})} \leq \frac{(e+1)(e+2)}{2(1-(\frac{8}{9})^{12})} \leq \frac{2(e+1)(e+2)}{3},$$

thus violating lemma 11.

For $e+1-s(C) = 1,2,\ldots,11$, we compare lemma 13 with lemma 15. In each case we are left with a gap of admissible parameters. However all these gaps are covered by lemma 8. For instance for $e+1-s(C) = 1$, lemma 13 reads:

$$(n+1) \geq (e+1)^{\frac{e+1}{\log(e+1)} \frac{5 \log 2}{4} - 1},$$

and lemma 15 reads:

$$(n+1) \leq \frac{9}{2}(e+1)(e+2).$$

We derive a contradiction for $e \geq 9$. For $e = 3,4,5,6,7$ and 8

$$(n+1) \leq \frac{9}{2}(e+1)(e+2)$$

implies that these cases are covered by lemma 8.

So from now on we may assume $e+1-s(C) = 0$. Let $m(e)$ be the right hand side of (25) after substitution of $e+1-s(C) = 0$.

Since $\delta_n \leq \delta_{m(e)}$ we may replace δ_n by $\delta_{m(e)}$ in (30). Then (30) yields an upperbound for $n+1$ which contradicts (25) for $e \geq 11$. Hence $3 \leq e \leq 10$. At this moment we are left with a finite (but still large) set of admissible parameters. We could let the computer do the rest for us.

The rest of this article is devoted to avoiding the use of a computer for this part of the proof.

Since $e+1-s(C) = 0$, it follows from (26) that

$$(35) \qquad \prod_{i=1}^{e+1} (2i-1) \leq A\left(\prod_{i=1}^{e+1} x_i\right) \leq \frac{n+1}{e+1} A((e+1)!).$$

This gives a lower bound $a(e)$ for $n+1$.

Since $\delta_n \leq \delta_{a(e)}$, we find, after replacing δ_n by $\delta_{a(e)}$ in (30), that lemma 16 contradicts (35) for $e \geq 7$. For instance: $e = 7$;

(35) implies $n+1 \geq 51480 = a(7)$. Replacing δ_n by $\delta_{a(7)}$ in (30) yields $n+1 \leq 5418$ a clear contradiction.

The cases e = 3,4,5,6 will now be treated separately.

e = 6. (35) yields $n+1 \geq 3003 = a(6)$.
After replacement of δ_n by $\delta_{a(6)}$ in (35), it follows that $n+1 \leq 9735$.
Suppose that Q(x) has a zero on $[0,0.45(n+1)]$. Then it is not difficult to verify that lemma 12 contradicts the inequality in (16) for $n+1 \geq 3003$. Hence the roots x_i of Q(x) are all in $[0.45(n+1),0.55(n+1)]$. Hence by the two bounds on $(n+1)$, we know that

$$1352 \leq x_i \leq 5354, \quad i = 1,\ldots,7 . \tag{36}$$

Suppose that all zeros of Q(x) have an odd part ≥ 3, then the left inequality in (35) can be sharpened by

$$3.5.7.9.11.13.15 \leq A\Big(\prod_{i=1}^{7} x_i\Big) .$$

Now (35) contradicts $n+1 \leq 9735$. So one zero, let us say x_1, has odd part 1. In the same way one zero, let us say x_2, has odd part 3. The only possibilities for x_1 by (36) are 2^{11} and 2^{12}, and for x_2 3.2^9 and 3.2^{10}.
However $x_i \in [0.45(n+1),0.55(n+1)]$ implies for x_1

$$n+1 \in [3723,4551] \text{ or } n+1 \in [7447,9102]$$

and for x_2

$$n+1 \in [2792,3413] \text{ or } n+1 \in [5585,6826] .$$

A contradiction.

e = 5. We repeat the argument of the case e = 6 and get $1386 \leq n+1 \leq 7944$. Each zero of Q(x) is in $[0.42(n+1),0.58(n+1)]$. So each zero is in $[582,4607]$. Again we find that one zero x_1 has odd part 1. So $x_1 = 2^{10}, 2^{11}$ or 2^{12} and we find

$$n+1 \in [1765,2438], [3531,4876] \text{ or } [7062,9752] .$$

The assumption that some zero x_i of Q(x) has odd part 5 leads to $x_i = 5.2^7$, 5.2^8 or 5.2^9.
The corresponding admissable intervals of $n+1$ have an empty intersection with the ones before. So we have a contradiction. Now (35) can be sharpened to

$$1.3.7.9.11.13 \leq \frac{n+1}{6} A(6!), \text{ i.e. } \quad n+1 \geq 3603 .$$

Now we start all over again. However we can now deduce that all zeros of $Q(x)$ are in $[0.45(n+1), 0.58(n+1)]$. Knowing that $Q(x)$ has no zero with odd part 5, implies that it has a zero, let us say x_2, with $A(x_2) = 3$. Now $x_1 = 2^{11}$ or 2^{12} implies

$$n + 1 \in [3723, 4551] \text{ or } n + 1 \in [7447, 9102] ,$$

and $x_2 = 3.2^{10}$ (the only possibility) implies $n + 1 \in [5585, 6826]$. A contradiction.

e = 4. Repeating the initial arguments of the case e = 6 yields

$$n + 1 \in [315, 15255] ,$$

and each zero is at least $0.35(n+1)$, so at least 111.
Let $x_1 < x_2 < x_3 < x_4 < x_5$ be the zeros of $Q(x)$. Lemma 5, ii) implies $x_3 = \frac{n+1}{2}$.
Let $n + 1 = A(n+1).2^a$. Then (35) reads

$$1 \cdot 3 \cdot \frac{n+1}{2^{a+1}} \cdot 5 \cdot 7 = 1.3.A(x_3).5.7 \leq \frac{n+1}{5} A(5!) \text{ i.e. } 5.7 \leq 2^{a+1} .$$

Hence $n + 1 = A(n+1).2^a$, $a \geq 5$. Let us now suppose that one zero x_i is odd. Clearly $i \neq 3$. Since also $n + 1 - x_i$ is odd in this case. Hence

$$A(x_i \cdot (n + 1 - x_i)) = x_i(n + 1 - x_i) \geq 111.(315 - 111) .$$

Substitution of this in (35) leads to an immediate contradiction. Hence all zeros are even. Let us now write down (17).

$$2^5 . \prod_{i=1}^{5} (x_i - 1) = (n-1)(n-2)(n-3)(n^2 - 9n + 20r), \text{ i.e.}$$

$$2^5 . \prod_{i=1}^{5} (x_i - 1) = ((n+1)-2)((n+1)-3)((n+1)-4)((n+1)^2 -$$

$$+ 11(n+1) + 10 + 20r) .$$

Since all zeros x_i are even, it follows that the left hand side is divisible by 2^5. The right hand side has as highest power of two $2^1.2^0.2^2.2^1 = 2^4$, since $2^5 \mid (n+1)$. This is a contradiction.

e = 3. The hardest case. Using (35) and subsequently lemma 16 yields

$$140 \leq n + 1 \leq 65.886 .$$

Using lemma 12 as before we observe that all zeros of $Q(x)$ are at least $\frac{1}{15}(n+1)$. Suppose that some zero x_i of $Q(x)$ is odd. Then (35) implies

$$1 . 3 . 5 . \frac{n+1}{15} \leq 1.3.5.x_i = 1.3.5.A(x_i) \leq \frac{n+1}{4} A(4!) = \frac{3}{4}(n+1).$$

i.e. $n + 1 \leq \frac{3}{4}(n+1)$. A clear contradiction.
Let $x_1 < x_2 < x_3 < x_4$ be the zeros of $Q(x)$. Let $x_i = A(x_i)2^{\alpha_i}$. Since

$$x_3 \geq \frac{n+1}{2}, \quad A(x_3) = \frac{x_3}{2^{\alpha_3}} \geq \frac{n+1}{2^{\alpha_e+1}}.$$

Substitution of this in (35) learns that $\alpha_3 \geq 4$. Similarly $\alpha_4 \geq 4$. Using lemma 12 as before, it follows that $x_2 \geq 0.403\ (n+1)$, hence

$$A(x_2) = \frac{x_2}{2^{\alpha_2}} \geq \frac{0.403(n+1)}{2^{\alpha_2}}.$$

Substitution of this in (35) also learns that $\alpha_2 \geq 4$. Hence $n + 1 = x_2 + x_3$ by (15) is divisible by $2^4 = 16$. We again write down (17)

$$2^4 \prod_{i=1}^{4} (x_i - 1) = (n-1)(n-2)\{n^2 - 7n + 12r\} =$$

$$= ((n+1)-2)((n+1)-3)\{(n+1)^2 - 9(n+1) + 8 + 12r\}.$$

Since all x_i's are even and $n+1$ is divisible by 16, it follows that $r \equiv 0 \pmod 4$.
For $e = 3$ it is not difficult to find the zeros of $Q(x)$. They are

$$x_{1234} = \frac{n+1 \pm \sqrt{3n - 6r - 1 \pm \sqrt{6n^2 - 6n - 24rn + 36r^2 + 4}}}{2}.$$

Let us define s, ℓ and m by

$$6n^2 - 6n - 24rn + 36r^2 + 4 = s^2 \tag{37}$$

$$3n - 6r - 1 + s = \ell^2 \tag{38}$$

$$3n - 6r - 1 - s = m^2. \tag{39}$$

Let us denote $n + 1 = A(n+1)2^a$, $\ell = A(\ell)2^b$, $m = A(m)2^c$, $s = A(s)2^u$, $r = A(r).2^z$ and $|C| = A(|C|)2^k$.
Then (37), (38), and (39) can be rewritten

$$3A^2(n+1)2^{2a+1} - 9A(n+1)2^{a+1} - 3A(r)A(n+1)2^{z+a+3} + 9A^2(r)2^{2z+2} + \tag{40}$$

$$+ 3A(r)2^{z+3} + 2^4 = A^2(s)2^{2u}.$$

(41) $$3A(n+1)2^a - 3A(r)2^{z+1} - 2^2 + A(s)2^u = A^2(\ell)2^{2b}$$

(42) $$3A(n+1)2^a - 3A(r)2^{z+1} - 2^2 - A(s)2^u = A^2(m)2^{2c} .$$

Considering the powers of 2 in each term we deduce from (40) that, since $a \geq 4$ and $z \geq 2$, u equals 2. Now (41) implies $b \geq 2$ and (42) implies $c \geq 2$. However since exactly one of $A(s) + 1$ and $A(s) - 1$ is congruent to 2 mod 4 and the other congruent to 0 mod 4, one of these equations will imply that $z = 2$ and the other $z \geq 3$. A contradiction. □

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