A brief literature survey of recursive formulations in multibody dynamics

WFW 91.057

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July 1991

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Contents

Abstract

Recursive formulation of the equations of motion for multibody systems is promising as for the reduction of computer time needed for dynamic simulations. A literature review is presented in which the state of the art of recursive techniques is discussed for systems of rigid bodies with a tree structure. Formalisms to set up the system equations of motion are reviewed and a choice for one formalism is made. Topology, kinematics and dynamics of systems with a tree structure are dealt with, and a recursive solution process is described in which the equations of motion are solved for the accelerations of bodies in the system. The case of kinematic driving constraints is also considered. Subsequently, modifications of the recursive technique, needed for the application to systems with closed loops and deformable bodies are briefly discussed. The described recursive formulation is not only applied to the kinematical part of the analysis, but is used all along the solution process of the multibody dynamics system.

Nomenclature

$\vec{a}$	vector
$\vec{e}$	unit vector
$\mathbf{A}$	second order tensor
$\mathbf{A}^c$	conjugate of tensor $\mathbf{A}$
$\mathbf{A}^{-1}$	inverse of tensor $\mathbf{A}$
$\mathrm{tr}(\mathbf{A})$	trace of tensor $\mathbf{A}$
$\vec{a} \cdot \vec{b}$	scalar product of vectors $\vec{a}$ and $\vec{b}$
$\vec{a} * \vec{b}$	vector product of vectors $\vec{a}$ and $\vec{b}$
$\mathbf{A} \cdot \vec{a}$	scalar product of tensor $\mathbf{A}$ and vector $\vec{a}$, resulting in a vector (mapping of $\vec{a}$)
$\underset{\sim}{a}$	column containing scalars
$\underset{\sim}{a}^T$	transposed column $\underset{\sim}{a}$
${}^i\underset{\sim}{a}$	representation of vector $\vec{a}$ with respect to base vectors $\underset{\sim}{\vec{e}}^i$ ($\vec{a} = (\underset{\sim}{\vec{e}}^i)^T \, {}^i\underset{\sim}{a}$)
$\underset{\sim}{A}$	matrix containing scalars
$\dot{a}$	time derivative of scalar a
δa	variation of scalar a
$\mathbf{I}$	unit tensor
$\underset{\sim}{I}$	unit matrix
$\frac{\partial \underset{\sim}{a}}{(\partial \underset{\sim}{b})^T}$	Jacobian of $\underset{\sim}{a}$ with respect to $\underset{\sim}{b}$ ($\frac{\partial \underset{\sim}{a}}{(\partial \underset{\sim}{b})^T} = \left[\frac{\partial \underset{\sim}{a}}{\partial b_1} \quad \cdots \quad \frac{\partial \underset{\sim}{a}}{\partial b_n}\right]$)

Symbols

$\vec{\alpha}$	Modal coordinate vector
$\vec{b}^{ik}$	Vector from centroid of body i to origin of k-th joint attachment frame in body i
$\mathbf{B}^{ik}$	Rotation tensor representing the mapping of i-th centroidal to k-th joint attachment frame
$\underset{\sim}{B}^{ik}$	Matrix representation of $\mathbf{B}^{ik}$ (with respect to either $\underset{\sim}{\vec{e}}^{i}$ or $\underset{\sim}{\vec{e}}^{ik}$)
β	Body connection array
$\vec{c}^{k}$	Vector from origin of k-th joint attachment frame in body i to origin of k-th joint attachment frame in body j
$\mathbf{C}^{k}$	Rotation tensor representing the mapping of k-th joint attachment frame of body i to k-th joint attachment frame of body j
$\underset{\sim}{C}^{k}$	Matrix representation of $\mathbf{C}^{k}$ (with respect to either $\underset{\sim}{\vec{e}}^{ik}$ or $\underset{\sim}{\vec{e}}^{jk}$)
$\delta\vec{\pi}^{i}$	Virtual rotation vector of body i
$\delta\vec{r}^{i}$	Virtual translation vector of body i
$\delta\underset{\sim}{\vec{u}}^{i}$	Column with virtual displacement vectors of body i
$\underset{\sim}{\vec{e}}^{i}$	Base vectors of body-fixed centroidal frame of body i
$\underset{\sim}{\vec{e}}^{ik}$	Base vectors of body-fixed joint attachment frame of joint k in body i
$\underset{\sim}{\vec{e}}^{0}$	Base vectors of inertial reference frame
$\vec{F}^{i}$	Resultant force vector acting on body i (including applied and constraint forces)
$\mathbf{J}^{i}$	Inertia tensor of body i with respect to its centroid
$\underset{\sim}{\lambda}$	Column containing Lagrange multipliers
m^{i}	Mass of body i
$\underset{\sim}{\mathbf{M}}$	Mass matrix
$\underset{\sim}{\mathbf{M}}_{r}$	Recursive mass matrix
O^{i}	Origin of body-fixed centroidal frame of body i
O^{ik}	Origin of body-fixed joint attachment frame of joint k in body i
O^{0}	Origin of global (inertial) reference frame
$\underset{\sim}{q}^{k}$	Column with relative generalized coordinates of joint k
$\underset{\sim}{\vec{Q}}$	Column with generalized force vectors
$\underset{\sim}{\vec{Q}}^{r}$	Column with recursive generalized force vectors
$\vec{r}^{i}$	Absolute position vector of centroid of body i
$\vec{r}^{ij}$	Position vector of centroid of body j relative to centroid of body i
$\mathbf{R}^{i}$	Rotation tensor representing the mapping of global reference frame to i-th centroidal frame
T	Kinetic energy
$\vec{T}^{i}$	Resultant torque vector acting on body i (including applied and constraint torques)
U	Potential energy
$\underset{\sim}{\vec{v}}^{i}$	Column with absolute linear and angular velocity vectors of body i
$\vec{\omega}^{i}$	Absolute angular velocity vector of body i
$\vec{\omega}^{ij}$	Angular velocity vector of body j relative to body i

Preface

This report is the result of a literature study on recursive formulations in multibody dynamics. The term "recursive" is encountered more and more in multibody dynamics literature, forming a reason to find out what it is all about. Furthermore, multibody dynamics software programmers tend to implement a recursive algorithm in their codes, claiming that the recursive aspect reduces computation time. At the Division of Fundamental Mechanics (WFW) in Eindhoven, an earlier contribution to this field of work has been supplied in the form of a report by Stephan van den Akker (WFW-report 1990-004). At the T.N.O. Road Vehicles Institute (TNO-IW) in Delft, the multibody code MADYMO has been developed and a new version will come along, based on a recursive algorithm.

This literature report has been realized by the supervision of Prof. dr. ir. D.H. van Campen and Prof. dr. ir. J.S.H.M. Wismans (TNO-IW). I gratefully acknowledge dr. ir. Fons Sauren and dr. ir. Willy Koppens (TNO-IW) for thoroughly correcting the first versions of this report.

Eindhoven, 1991

Paul Slaats.

Chapter 1

Introduction

In this introductory chapter, some basic terminology in multibody dynamics related to recursive formulation of equations of motion is described in order to be able to understand the following chapters better. Furthermore, literature references are categorized on the basis of their use with respect to recursive formulation. Finally, a brief description of the contents of the following chapters is given.

Multibody Systems. Multibody systems are models of mechanical systems, consisting of bodies, interconnected in such a way that (large) relative motion between the bodies can occur. Examples of multibody systems are spacecraft, mechanisms, robots, and vehicles. The systems can have a tree structure (also called "open loop" in literature) or a structure with closed chains of bodies (a kinematically "closed loop" structure). One speaks of a system with a tree structure if the path from an arbitrary body in the system to another arbitrary body in the system is unique. If this is not the case, one speaks of a kinematically closed loop system. The bodies in a system can be either rigid or deformable.

Multibody Dynamics and Recursive Formulation. The field of multibody dynamics embraces the mathematical description of the dynamic behaviour of multibody systems. The advances of computer technology and the development of numerical methods to support computational dynamics enable researchers to simulate systems with many degrees of freedom. For example, parallel processing enables even real time simulation, which is needed in robotics applications and in the support of laboratory testing. Multibody dynamics computer programs automatically generate the system equations of motion.

The system equations may have the following forms:

$$\ddot{\underset{\sim}{q}} = \underset{\sim}{g}(\underset{\sim}{q}, \dot{\underset{\sim}{q}}, t) \tag{1.1}$$

or

$$\underset{\sim}{M}(\underset{\sim}{q}, t)\ddot{\underset{\sim}{q}} = \underset{\sim}{f}(\underset{\sim}{q}, \dot{\underset{\sim}{q}}, t) \tag{1.2}$$

where

$\ddot{\underset{\sim}{q}}$	is a column with the second time derivative of the system generalized coordinates.
$\underset{\sim}{g}$	is an explicit expression of $\underset{\sim}{q}$,$\dot{\underset{\sim}{q}}$,and t.
$\underline{M}(\underset{\sim}{q}, t)$	is the mass matrix with respect to the chosen generalized coordinates.
$\underset{\sim}{f}(\underset{\sim}{q}, \dot{\underset{\sim}{q}}, t)$	is the right hand side column.
t	represents the time.

Equations of type (1.1) can be solved by numerical integration procedures having the following general equation form as a starting point (e.g., the explicit Runge Kutta method, which is conditionally stable):

$$\dot{\underset{\sim}{x}} = \underset{\sim}{c}(\underset{\sim}{x}, t) \tag{1.3}$$

Eq. (1.1) is equivalent to Eq. (1.3) if the following transformations are used:

$$\underset{\sim}{x} = \begin{bmatrix} \underset{\sim}{s} \\ \underset{\sim}{q} \end{bmatrix} \tag{1.4}$$

$$\underset{\sim}{c}(\underset{\sim}{x}, t) = \begin{bmatrix} \underset{\sim}{g} \\ \underset{\sim}{s} \end{bmatrix} \tag{1.5}$$

Equations of type (1.2) can be integrated numerically by the more common procedures that have the following general form as a starting point (e.g., the implicit Newmark-β method), where $\ddot{\underset{\sim}{q}}$ includes the (nq) second time derivatives of all generalized coordinates in the system:

$$\underline{A}\ddot{\underset{\sim}{q}} = \underset{\sim}{b} \tag{1.6}$$

with $\underline{A}$ and $\underset{\sim}{b}$ known ($\underline{A}$ not necessarily constant). For linear problems, the Newmark-β method is unconditionally stable, but if nonlinearities occur, instability may develop. If $\underline{A}$ (with dimension $nq * nq$) is not constant, then it has to be inverted at each time step in the Newmark-β method.

By using a so-called "recursive formulation", to be explained in the following chapters, the equations of motion can be written in the form of Eq. (1.1), where $\ddot{\underset{\sim}{q}}$ has a small size (containing relative coordinates of one body with respect to a preceding adjacent body in the system). In order to arrange the system equations in the form of Eq. (1.1), only small-sized matrices need to be inverted. As a result, many small-sized equations of type (1.1) have to be integrated, instead of integrating a large-sized system equation of type (1.2). Not the integration procedure itself, but the inversion of large-sized matrices takes a lot of computation time. According to literature, the recursive formulation is computationally efficient. A statement on a preferable integration method is not supplied, however, but might be interesting to explore further. The recursive formulation is the main subject of this report.

Formalisms. A large variety of formalisms for the generation of the equations of motion can be found in literature, e.g., Newton-Euler equations, d'Alembert's principle of virtual work, Kane's equations, and Lagrange's equations of the first and second kind. Each one of these formalisms leads to different forms of the equations of motion of a dynamic system and consequently different implementations can be found. Of course the equations of motion obtained with different formalisms must be equivalent, independent of the formalism. However, the efficiency of implementations of formalisms differs depending on the application. In the next chapter a survey of the five above mentioned formalisms will be presented. (Dis-)advantages of these formalisms will be discussed, and a choice for the formalism used in the following chapters will be made and motivated.

Inverse and Forward Dynamics. Distinction must be made between "inverse" and "forward" dynamics. When literature references are discussed later on in this chapter, this distinction turns out to be rather clarifying. In the problem of inverse dynamics, one assumes that the motion of the mechanical system is known as a function of time. One attempts to find the driving forces and torques necessary to realize that motion. In the problem of forward dynamics, the driving forces and torques are known as a function of time. One attempts to find the evolution of the motion of the mechanical system as a response to the applied loads and initial conditions. This report is focussed on forward dynamics.

Global and Relative Description. The kinematics of a system of bodies may be described by either global or relative quantities. In a global description, motion of all bodies in the system are represented with respect to an inertial ("global" or "absolute") coordinate frame. In a relative description, on the other hand, the motion of a particular body in the multibody system is defined with respect to an adjacent body, the motion of which has previously been defined according to the topological order of all bodies in the system. Relative description makes recursive formulation possible. A presumable drawback of the relative description is that small errors in relative quantities can cause large errors in the global kinematical quantities, especially in the case of large chains of bodies.

Recursive Formulation. Now, what is a recursive formulation? Within the context of multibody dynamics, a recursive formulation is a procedure in which elementary relationships between an arbitrary pair of contiguous bodies as part of a system of bodies can be (re-)used all along the system. Recursive relationships can be used in kinematics to generate the total system matrices, but they can as well be applied to come to a solution procedure for the multibody system equations of motion. With "solution" in this context is meant the determination of the system unknowns, such as accelerations, and not the integration of accelerations to velocities and positions. It is important to notice the dual use of the term "recursive." When consulting literature on this subject, one can easily be misled by this ambiguity.

Application of recursions in an inverse dynamics analysis is only useful when carried through as far as the kinematical part of the analysis is concerned, just to come to the system of equations. Examples of recursive formulations in inverse dynamics can be found in literature, e.g., Walker and Orin (1982), Wang and Ravani (1985), Van Woerkom (1989).

In forward dynamics, recursions are often carried through merely as far as the kinematical part of the analysis is concerned. Examples of the combination of recursive kinematics and the usual non-recursive solution procedure can be found in literature, e.g., Van Woerkom and Guelman (1987), Changizi and Shabana (1988), Wang and Ravani (1985).

However, the interest in this report is rather focussed on the use of recursions in forward dynamics extended all along the solution process. Solving the system recursively requires smaller (though more) matrices to be inverted, which is more efficient (less floating point operations) than solving a large system of linear equations for the accelerations. Examples of recursive kinematics followed by a recursive solution procedure can also be found in literature, e.g., Bae and Haug (1987; 1987-88), Kim and Haug (1988; 1989), Roberson and Schwertassek (1988), Wehage (1989).

The use of recursive techniques implies several advantages over the usual composition and solution of the system equations. Van Woerkom (1989) analyzed an inverse dynamics problem, using recursive kinematics. In this work he states that the advantages of applying recursions are threefold: easier derivation of the equations of motion, easier software coding and de-bugging, and reduced computational effort. These advantages also hold for the forward dynamics case in which recursions are used for kinematics as well as for the solution process.

Publications on recursive techniques appear often in literature but fundamentally they all come to the same findings. According to Schwertassek and Rulka (1989),recursive solution formulations have independently been developed and published at various places [Armstrong (1979), Brandl et al. (1986; 1987), Rosenthal (1987)], in part without any knowledge of earlier contributions in the field. The basic idea of recursive system solution formulations proceeds from the works of Vereshchagin (1974) and Featherstone (1983), but the different "schools" have used their own terminology to describe the derivation of the algorithms. Bae and Haug (1987) use the "variational form" of the system equations, Brandl et al. (1987) use the Newton/Euler equations as a starting point, and in Rosenthal's derivation (1987) the flavour of Kane's equations appears.

Application of Recursive Formulation. When using recursive techniques, several aspects can be considered in the formulation. To start with, recursive algorithms are often only formulated for systems with a tree structure [see, e.g., Bae and Haug (1987), Kim and Haug (1988), Van Woerkom (1987)] and worked out for some specific joints connecting the bodies. As examples, Bae and Haug (1987) and also Changizi and Shabana (1988) only treat kinematic couplings with one rotational and/or one translational degree of freedom, thus including revolute, translational, and cylindrical joints in their formulations. Featherstone (1983) presented a recursive algorithm to calculate acceleration of robot arms with revolute and translational joints only.

In literature, kinematically closed loops in multibody systems are also taken into account (e.g., Bae and Haug (1987-88), Kim and Haug (1989), Wehage (1989)). As compared with systems with a tree structure, systems with kinematically closed loops have additional joints which tie various branches in the tree together. These excess joints are also called secondary joints. The number of independent kinematically closed loops equals the number of secondary joints. When these joints are cut, all kinematic loops are opened and a tree is formed. The basic algorithms developed for kinematically open loop systems can then be applied with the introduction of constraint forces at the cut secondary joint surfaces as additional unknowns.

An important feature of the recursive formulation with respect to modern computer technology is its suitability for parallel processing implementation. Today's applications in robotics, and support of laboratory testing require real-time simulation capability. Hardware developments in the last two decades have enabled fast computations even for large systems. Computational dynamics should take advantage of the advances in hardware, such as computers with parallel processing capability. The recursive formulation presented in Chapter 3 is well suited for implementation on such computers.

Research on recursive algorithms has even overlapped the aspect of deformable bodies in multibody systems. Examples can be found in literature, e.g., see Ho (1977), Hughes (1979), Singh et al. (1985), Kim and Haug (1988), Changizi and Shabana (1988), Van Woerkom (1989), Kim and Haug (1989). In these works flexibility is restricted to small deformations of bodies experiencing large displacements.

Scope of this report. This report has been written with the intention to give a survey of the state of the art of the recursive technique in general, and to briefly mention modification of the recursive formulation to several aspects such as closed loops and deformability. In this report, these aspects are not discussed in detail.

Following chapters. Various formalisms in dynamics are surveyed in chapter 2. In chapter 3, a recursive formulation for open loop systems with rigid bodies is described. At first, a method to define the topological order of bodies in a tree structure is discussed. Then, the kinematics of an elementary system of two bodies, interconnected by a joint, are examined. An example in the form of a universal joint connecting two rigid bodies is shown. The recursive technique for setting up and solving the equations of motion is described in the dynamics section. In chapter 4, application of the recursive technique to particular fields of interest, such as closed loops, flexible bodies, and the case where relative generalized coordinates are kinematically driven are discussed. Finally, conclusions are collected in chapter 5.

Chapter 2

Formalisms in Multibody Dynamics

In literature, several ways to derive the equations of motion of mechanical systems can be distinguished. In this chapter, five formalisms are discussed:

- the Newton-Euler equations.
- d'Alembert's principle of virtual work.
- Lagrange's equations of the first kind.
- Lagrange's equations of the second kind.
- Kane's equations.

These five methods are most widely used. Other formalisms, that will not be discussed because they are less often used, are for example: Hamilton's canonical equations, the Boltzmann-Hamel equations, and the Gibbs equations.

Starting with a formalism based on the Newton-Euler equation, the equations of motion for a system of nb bodies are

$$\left.\begin{array}{rcl} \vec{F}^{\,i} & = & m^i \dot{\vec{v}}^{\,i} \\ \vec{T}^{\,i} & = & \mathbf{J}^i \cdot \dot{\vec{\omega}}^{\,i} + \vec{\omega}^i * (\mathbf{J}^i \cdot \vec{\omega}^i) \end{array}\right\} (i = 1, \ldots, nb) \tag{2.1}$$

in which

$\vec{F}^{\,i}$	is the resultant of all the forces acting on body B^i.
m^i	is the mass of body B^i.
$\dot{\vec{v}}^{\,i}$	is the time derivative of the absolute velocity vector $\vec{v}$ of body B^i.
$\vec{T}^{\,i}$	is the resultant of all the torques acting on body B^i.
$\mathbf{J}^i$	is the inertia tensor of body B^i with respect to its centroid.
$\vec{\omega}^{\,i}$	is the absolute angular velocity vector of body B^i.

In order to obtain a complete, solvable system of equations in the unknown accelerations and constraint forces, some extra constraint equations need to be added to Eq. (2.1).

The above formalism is at the basis of many studies such as spacecraft and manipulator dynamics.

A second formalism is based on d'Alembert's principle of virtual work. According to this formalism, the equations of motion of a multibody system consisting of nb bodies are given by

$$\sum_{i=1}^{nb} \left[(\delta \underset{\sim}{\vec{u}}^{i})^{T} \cdot (\underline{\mathrm{M}}^{i} \cdot \dot{\underset{\sim}{\vec{v}}}^{i} - \underset{\sim}{Q}^{i}) \right] = 0 \qquad \forall \text{ kinematically admissible } \delta \underset{\sim}{\vec{u}}^{i} \tag{2.2}$$

where

$\delta \underset{\sim}{\vec{u}}^{i}$	is a column containing virtual displacement vectors (i.e. virtual translations $\delta \vec{r}^{i}$ and virtual rotations $\delta \vec{\pi}^{i}$, which will be explained in Section 3.3) of body B^{i}.
$\underline{\mathrm{M}}^{i}$	is a matrix with mass and rotational inertia terms of body B^{i}.
$\dot{\underset{\sim}{\vec{v}}}^{i}$	is the time derivative of a column containing translational and rotational velocity vectors of body B^{i}.
$\underset{\sim}{Q}^{i}$	is a column with resultant forces and torques acting on body B^{i}, and with an extra term quadratic in the absolute angular velocity of body B^{i}.

Constraint forces and constraint torques cancel each other according to d'Alembert's principle of virtual work. This formalism is also termed "generalized principle of d'Alembert" in literature.

Lagrange's equations form another formalism found quite often in literature. Lagrange's equations of the first kind are:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\underset{\sim}{q}}}\right) - \frac{\partial T}{\partial \underset{\sim}{q}} + \frac{\partial U}{\partial \underset{\sim}{q}} - \underset{\sim}{Q}^{q} = \underset{\sim}{0} \tag{2.3}$$

where

T	is the kinetic energy of the system.
$\underset{\sim}{q}$	is a set of independent system generalized coordinates.
U	is the potential energy of the system.
$\underset{\sim}{Q}^{q}$	is the set of generalized forces with respect to the chosen column $\underset{\sim}{q}$

In literature, also other terminology for this formalism is used, such as "Lagrange's modification of d'Alembert's principle", "Lagrange's form of d'Alembert's principle", or simply: "Lagrange's equations".

A generalization of the last formalism to the case where the generalized coordinates are dependent leads to the so-called "Lagrange's equations of the second kind":

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\underset{\sim}{q}}}\right) - \frac{\partial T}{\partial \underset{\sim}{q}} + \frac{\partial U}{\partial \underset{\sim}{q}} - \underset{\sim}{Q}^{q} - \underline{R}^{T} \underset{\sim}{\lambda} = \underset{\sim}{0} \tag{2.4}$$

where

$\underset{\sim}{R}$ represents the Jacobian of the (non-)holonomic constraints of the system

$\underset{\sim}{\lambda}$ denotes a column with Lagrange multipliers

Quantities in $\underset{\sim}{R}^T\underset{\sim}{\lambda}$ can be interpreted as the loads to maintain the constraints. In case of holonomic constraints, the matrix $\underset{\sim}{R}(\underset{\sim}{q},t)$ can be obtained from the partial derivative of the holonomic constraint equations $\underset{\sim}{h}$ with respect to column $\underset{\sim}{q}$ containing the generalized coordinates:

$$\underset{\sim}{h}(\underset{\sim}{q}(t),t) = \underset{\sim}{0} \tag{2.5}$$

$$\underset{\sim}{R}\dot{\underset{\sim}{q}} = -\frac{\partial \underset{\sim}{h}}{\partial t} \tag{2.6}$$

$$\underset{\sim}{R} = \underset{\sim}{R}(\underset{\sim}{q},t) = \frac{\partial \underset{\sim}{h}}{(\partial \underset{\sim}{q})^T} \tag{2.7}$$

In the case of non-holonomic constraints, the so-called "velocity equations" have to be known explicitly in order to obtain matrix $\underset{\sim}{R}$:

$$\underset{\sim}{R}(\underset{\sim}{q},t)\dot{\underset{\sim}{q}} = \underset{\sim}{c}(\underset{\sim}{q},t) \tag{2.8}$$

The formalism using Eq. (2.4) as a starting point is also termed "Euler Lagrange formalism" in literature.

The last formalism to be discussed is based on the so-called Kane's equations, met quite often in literature. The method involves two classes of quantities not employed in connection with the earlier formalisms, namely partial angular velocities and partial velocities. The dynamical equations of motion are written in terms of generalized active forces K_i and generalized inertia forces K_i^* ($i = 1,\ldots,ngc$; ngc is the number of generalized coordinates):

$$K_i + K_i^* = 0 \qquad (i = 1,\ldots,ngc) \tag{2.9}$$

with the following definitions:

$$K_i = \sum_{j=1}^{np} \vec{v}_i^{\,j} \cdot \vec{F}^{\,j} \qquad (i = 1,\ldots,ngc) \tag{2.10}$$

$$K_i^* = \sum_{j=1}^{np} \vec{v}_i^{\,j} \cdot (-m^j \vec{a}^{\,j}) \qquad (i = 1,\ldots,ngc) \tag{2.11}$$

where

np is the number of bodies in the system under consideration.

$\vec{v}_i^{\,j}$ is the i^{th} partial velocity of body j.

$\vec{F}^{\,j}$ is the resultant of all applied body and contact forces acting on the body, with respect to point j.

m^j is the mass of the body with point j

$\vec{a}^{\,j}$ is the inertial acceleration at point j in the body.

For a more extensive consideration of Kane's equations, the reader is referred to appendix A. The formalism based on these equations is used, among others, in works of Kane & Levinson (1980) and Singh et al.(1985).

After this short survey of some formalisms, one formalism has to be chosen to describe the motion of the mechanical system. Merits and demerits of the considered formalisms need to be discussed in order to reach a proper decision. D'Alembert's principle of virtual work avoids the problem of elimination of constraint forces and torques. Deformability can also be taken into account. The Newton-Euler equations are the most straightforward, but have the disadvantage that they are restricted to rigid bodies. Furthermore, the main problem with the Newton-Euler equations is the elimination of the constraint forces and torques, arising from the kinematic relations between adjacent bodies. In Lagrange's equations of the first and second kind, constraint forces and torques do not appear in the equations of motion, but this generally leads to the expense of more efforts in deriving the equations of motion. In fact, some superfluous terms in Lagrange's equations need not be elaborated since they cancel each other when they are worked out. Finally, Kane's equations also avoid the problem of elimination of constraint forces. A drawback of this last formalism is the quite untransparent way of formulating the unknown quantities of interest, yet leading to a simple result for the equations of motion (Eq. (2.9)). Experience is definitely required to select partial angular velocities and partial velocities such, that the algebraic complexity of the resulting equations of motion is minimized.

The above mentioned merits and demerits of the considered formalisms have led to the decision to use d'Alembert's principle of virtual work in the following chapters.

Chapter 3

A recursive formulation for setting up dynamic equations for systems of rigid bodies with a tree structure

3.1 Introduction

In this chapter, kinematics and dynamics of an elementary system of two bodies interconnected by an arbitrary joint are discussed. A multibody system can be thought of as being built up of a number of these elementary systems. In the dynamics section, the recursive formulation describes the contributions to the mass and load matrices of the lower numbered body from mass and load terms of the higher numbered body that it is connected to. In case of more than two bodies connected to a lower numbered body in the system, no real differences occur: effective mass and load of that lower numbered body will get a contribution from the mass and stiffness of all connected bodies then. In the following section, a bookkeeping procedure is described that takes care of correct additions of mass and load terms.

3.2 Topology

Recursive formulation of multibody dynamics is based on a procedure in which elementary relationships between an arbitrary pair of contiguous bodies as part of a system of bodies can be (re-)used all along that system. The bodies in such a pair are connected by means of a joint. Since a multibody system can be quite complex and built up of many bodies and many joints, a bookkeeping procedure of the topology of the bodies in the system is needed for the computation process. An example of such a bookkeeping procedure for multibody systems with a tree structure (i.e. with no closed loops of bodies) is the use of body connection arrays, as described by Huston (1985). The best way to clarify this procedure is by using an illustrative example.

Consider the multibody system of Fig. 3.1. The lowest numbered body (B^1), unequal to inertial space (B^0), is called the base body of the multibody system. The numbering of the bodies in the system is carried out in such a way that if body B^i is connected to body B^j, and if the number of bodies on the path from body B^i to the base body B^1 is smaller than the number of bodies from B^j to B^1 , then body B^i is assigned a lower number than body B^j.

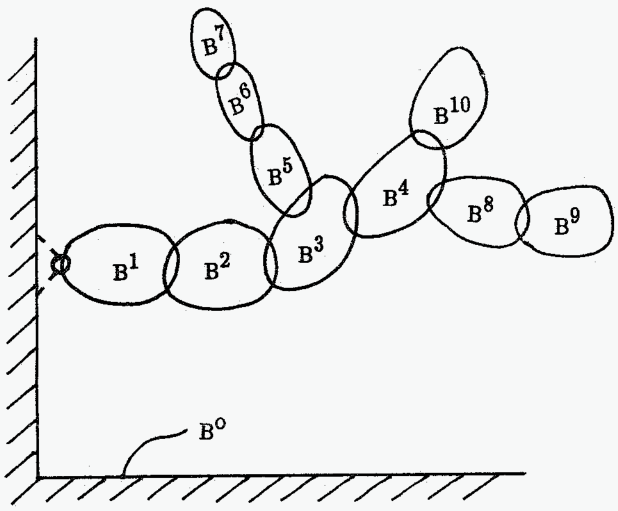

Figure 3.1: *Body numbering:* $B^i < B^j$ *if* B^i *has less links towards* B^1

Note that on this condition the numbering of bodies is not unique. For instance, the numbers of bodies B^7 and B^{10} in Fig. 3.1 might just as well be interchanged. But once a choice is made for the assignment of body numbers, each body (except inertial space B^0) is connected to a corresponding lower numbered body. The arranged set of numbers of lower numbered bodies is called the "body connection array." Let $\underset{\sim}{\beta}$ be a column containing the numbers of lower numbered bodies, then by definition $\underset{\sim}{\beta}$ is the body connection array. For the system of Fig. 3.1, the body connection array $\underset{\sim}{\beta}$ is

$$(\underset{\sim}{\beta})^T = \begin{bmatrix} 0 & 1 & 2 & 3 & 3 & 5 & 6 & 4 & 8 & 4 \end{bmatrix} \tag{3.1}$$

A formal definition of the body connection array can also be provided. Let i be the j^{th} element of column $\underset{\sim}{\beta}$, then body B^i is connected to body B^j, and body B^i is assigned a lower number than body B^j

$$i = \beta_j \quad , \quad i < j \tag{3.2}$$

3.3 Kinematics

Before describing a recursive formulation for setting up dynamic equations for systems with a tree structure, the kinematics of these systems are discussed. An elementary system of two bodies i and j, with $i = \beta_j$ (see Sect. 3.2), interconnected by an arbitrary joint k is described as generally as possible. The motion of a body is expressed in the motion of the preceding adjacent body and the relative motion between the bodies due to the joint. By repeatedly (=recursively) using this expression between a pair of bodies, equations can be obtained for the complete system. Therefore, consideration may be limited to an elementary system of two bodies interconnected by a joint. In Fig. 3.2, this elementary system is shown.

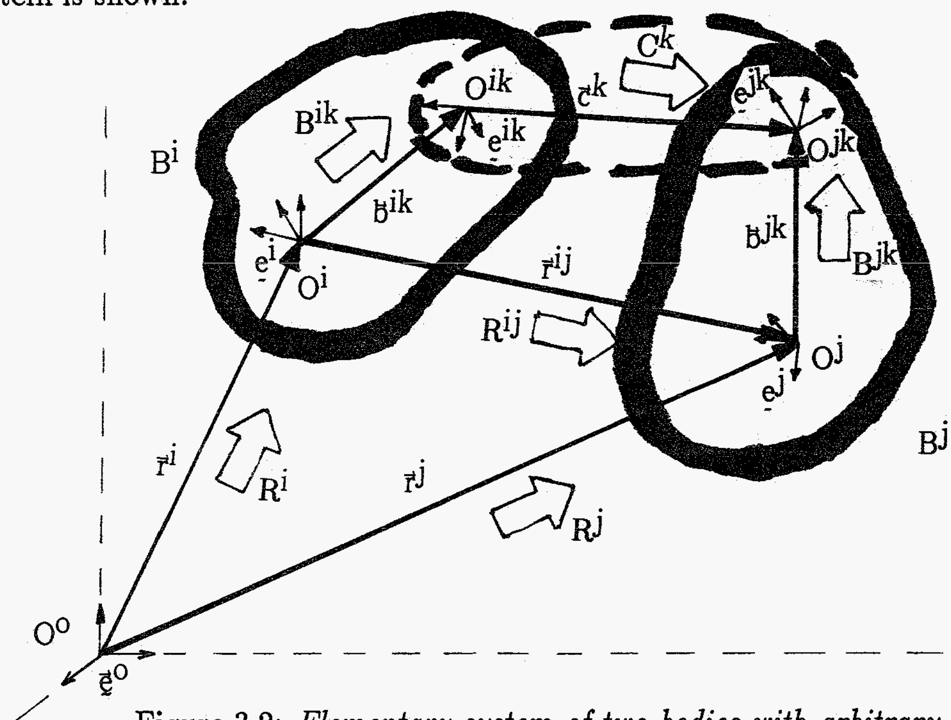

Figure 3.2: *Elementary system of two bodies with arbitrary joint*

Joint attachment frames $\underset{\sim}{\vec{e}}^{ik}$ and $\underset{\sim}{\vec{e}}^{jk}$ are introduced to describe the joint between the bodies in a modular way, such that an arbitrary joint can be substituted in the system. The joint description can be kept as simple as possible by a proper choice for the origins and orientations of the joint attachment frames. Later on in this section, a joint example with such a proper choice is discussed. Degrees of freedom in a joint are gathered in a column $\underset{\sim}{q}$ with so-called "relative coordinates". For example, a revolute joint has only one degree of freedom, so $\underset{\sim}{q}$ contains only one relative coordinate of body B^j with respect to body B^i.

In Fig. 3.2, $\mathbf{R}^i$ denotes a tensor representing the mapping of the global reference frame $\underset{\sim}{\vec{e}}^0$ to the centroidal reference frame $\underset{\sim}{\vec{e}}^i$ fixed to body B^i. So physically, $\mathbf{R}^i$ corresponds to the absolute rotation of body B^i. Mathematically, $\mathbf{R}^i$ is defined as follows:

$$(\underset{\sim}{\vec{e}}^i)^T = \mathbf{R}^i \cdot (\underset{\sim}{\vec{e}}^0)^T \tag{3.3}$$

Similarly, tensor $\mathbf{R}^j$ is defined for body B^j:

$$(\vec{\underset{\sim}{e}}^{j})^T = \mathbf{R}^j \cdot (\vec{\underset{\sim}{e}}^{0})^T \tag{3.4}$$

Tensors $\mathbf{B}^{ik}$ and $\mathbf{B}^{jk}$ in Fig. 3.2 represent the mapping of $\vec{\underset{\sim}{e}}^{i}$ to $\vec{\underset{\sim}{e}}^{ik}$ and the mapping of $\vec{\underset{\sim}{e}}^{j}$ to $\vec{\underset{\sim}{e}}^{jk}$, respectively:

$$(\vec{\underset{\sim}{e}}^{ik})^T = \mathbf{B}^{ik} \cdot (\vec{\underset{\sim}{e}}^{i})^T \tag{3.5}$$

$$(\vec{\underset{\sim}{e}}^{jk})^T = \mathbf{B}^{jk} \cdot (\vec{\underset{\sim}{e}}^{j})^T \tag{3.6}$$

For the joint in Fig. 3.2, tensor $\mathbf{C}^k$ denotes a tensor representing the mapping of the joint attachment frame $\vec{\underset{\sim}{e}}^{ik}$ fixed to body B^i, to the joint attachment frame $\vec{\underset{\sim}{e}}^{jk}$ fixed to body B^j:

$$(\vec{\underset{\sim}{e}}^{jk})^T = \mathbf{C}^k \cdot (\vec{\underset{\sim}{e}}^{ik})^T \tag{3.7}$$

The following relationship exists between the rotation tensors defined above:

$$\mathbf{R}^j = (\mathbf{B}^{jk})^c \cdot \mathbf{C}^k \cdot \mathbf{B}^{ik} \cdot \mathbf{R}^i \tag{3.8}$$

The absolute angular velocity vectors of bodies B^i and B^j, denoted by $\vec{\omega}^{\,i}$ and $\vec{\omega}^{\,j}$ resp., are defined as the axial vectors of tensors $\dot{\mathbf{R}}^i \cdot (\mathbf{R}^i)^c$ and $\dot{\mathbf{R}}^j \cdot (\mathbf{R}^j)^c$, respectively:

$$\dot{\mathbf{R}}^i \cdot (\mathbf{R}^i)^c \cdot \vec{a} = \vec{\omega}^{\,i} * \vec{a} \qquad \forall \vec{a} \tag{3.9}$$

$$\dot{\mathbf{R}}^j \cdot (\mathbf{R}^j)^c \cdot \vec{a} = \vec{\omega}^{\,j} * \vec{a} \qquad \forall \vec{a} \tag{3.10}$$

In case of rigid bodies B^i and B^j, the relative angular velocity vector $\vec{\omega}^{\,ij}$ of $\vec{\underset{\sim}{e}}^{j}$ with respect to $\vec{\underset{\sim}{e}}^{i}$ equals the axial vector of the skew-symmetric tensor $(\vec{\underset{\sim}{e}}^{ik})^T \dot{\underset{\sim}{C}}^k (\underset{\sim}{C}^k)^T \vec{\underset{\sim}{e}}^{ik}$:

$$(\vec{\underset{\sim}{e}}^{ik})^T \dot{\underset{\sim}{C}}^k (\underset{\sim}{C}^k)^T \vec{\underset{\sim}{e}}^{ik} \cdot \vec{a} = \vec{\omega}^{\,ij} * \vec{a} \qquad \forall \vec{a} \tag{3.11}$$

where $\underset{\sim}{C}^k$ in Eq. (3.11) is the matrix representation of $\mathbf{C}^k$ (with respect to either $\vec{\underset{\sim}{e}}^{ik}$ or $\vec{\underset{\sim}{e}}^{jk}$), which is not time-independent in general. Relationships of the form of Eq. (3.11) between rotation tensors, axial vectors, and angular velocities are evaluated in Appendix B. In this appendix, the general case of deformable bodies is considered first, after which restrictions to rigid bodies are specified. Since $\underset{\sim}{C}^k$ is a function of the relative generalized coordinates related to joint k and gathered in column $\underset{\sim}{q}^k$

$$\underset{\sim}{C}^k = \underset{\sim}{C}^k(\underset{\sim}{q}^k) \tag{3.12}$$

the angular velocity vector $\vec{\omega}^{\,ij}$ of body B^j relative to body B^i can be written as

$$\vec{\omega}^{\,ij} = (\vec{\underset{\sim}{\psi}}^{\,k})^T \dot{\underset{\sim}{q}}^k \tag{3.13}$$

where $\vec{\underset{\sim}{\psi}}^{\,k} = \vec{\underset{\sim}{\psi}}^{\,k}(\underset{\sim}{q}^k)$ is defined as the column with axial vectors belonging to column $(\vec{\underset{\sim}{e}}^{ik})^T \frac{\partial \underset{\sim}{C}^k}{\partial \underset{\sim}{q}^k} (\underset{\sim}{C}^k)^T \vec{\underset{\sim}{e}}^{ik}$:

$$(\underset{\sim}{\vec{e}}^{ik})^T \frac{\partial \underset{\sim}{C}^k}{\partial \underset{\sim}{q}^k} (\underset{\sim}{C}^k)^T \underset{\sim}{\vec{e}}^{ik} \cdot \vec{a} = \underset{\sim}{\vec{w}}^k * \vec{a} \qquad \forall \vec{a} \tag{3.14}$$

The angular velocities $\vec{\omega}^i$ and $\vec{\omega}^j$ are related as

$$\vec{\omega}^j = \vec{\omega}^i + (\underset{\sim}{\vec{w}}^k)^T \dot{\underset{\sim}{q}}^k \tag{3.15}$$

After describing the relationships between the angular velocities in the system of Fig. 3.2, the translational velocities are discussed. From Fig. 3.2, it follows that:

$$\vec{r}^j = \vec{r}^i + \vec{r}^{ij} \tag{3.16}$$

where

$$\vec{r}^{ij} = \vec{b}^{ik} + \vec{c}^k - \vec{b}^{jk} \tag{3.17}$$

In order to set up kinematic velocity equations for the system in Fig. 3.2, the time derivative of Eq.(3.16) combined with Eq. (3.17) is considered:

$$\dot{\vec{r}}^j = \dot{\vec{r}}^i + \dot{\vec{b}}^{ik} + \dot{\vec{c}}^k - \dot{\vec{b}}^{jk} \tag{3.18}$$

with

$$\begin{aligned} \dot{\vec{b}}^{ik} &= \frac{d}{dt}((\underset{\sim}{\vec{e}}^i)^T \, {}^i\underset{\sim}{b}^{ik}) \\ &= (\dot{\underset{\sim}{\vec{e}}}^i)^T \, {}^i\underset{\sim}{b}^{ik} + (\underset{\sim}{\vec{e}}^i)^T \, {}^i\dot{\underset{\sim}{b}}^{ik} \\ &= (\vec{\omega}^i * \underset{\sim}{\vec{e}}^i)^T \, {}^i\underset{\sim}{b}^{ik} + (\underset{\sim}{\vec{e}}^i)^T \underset{\sim}{0} \\ &= \vec{\omega}^i * \vec{b}^{ik} \end{aligned} \tag{3.19}$$

and similarly

$$\dot{\vec{b}}^{jk} = \vec{\omega}^j * \vec{b}^{jk} \tag{3.20}$$

Note that ${}^i\underset{\sim}{b}^{ik}$ and ${}^j\underset{\sim}{b}^{jk}$ are constant with respect to time. The time derivative of $\vec{c}^k$ is obtained in a similar way:

$$\begin{aligned} \dot{\vec{c}}^k &= \frac{d}{dt}((\underset{\sim}{\vec{e}}^{ik})^T \, {}^{ik}\underset{\sim}{c}^k) \\ &= \vec{\omega}^i * \vec{c}^k + (\underset{\sim}{\vec{e}}^{ik})^T \, {}^{ik}\dot{\underset{\sim}{c}}^k \end{aligned} \tag{3.21}$$

However, column ${}^{ik}\dot{\underset{\sim}{c}}^k$ is a function of the relative generalized coordinates related to joint k and gathered in column $\underset{\sim}{q}^k$:

$${}^i\underset{\sim}{c}^k = {}^i\underset{\sim}{c}^k(\underset{\sim}{q}^k) \tag{3.22}$$

With Eq. (3.22), Eq. (3.21) results in:

$$\begin{aligned}\dot{\vec{c}}^{k} &= \vec{\omega}^{i} * \vec{c}^{k} + (\underset{\sim}{\vec{e}}^{ik})^{T} \frac{\partial^{ik} \underset{\sim}{c}^{k}}{(\partial \underset{\sim}{q}^{k})^{T}} \dot{\underset{\sim}{q}}^{k} \\ &= \vec{\omega}^{i} * \vec{c}^{k} + \frac{\partial \vec{c}^{k}}{(\partial \underset{\sim}{q}^{k})^{T}} \dot{\underset{\sim}{q}}^{k} \end{aligned} \tag{3.23}$$

Substitution of Eq. (3.15) into Eq. (3.20) results in:

$$\dot{\vec{b}}^{jk} = \vec{\omega}^{i} * \vec{b}^{jk} + (\underset{\sim}{\vec{w}}^{k})^{T} \dot{\underset{\sim}{q}}^{k} * \vec{b}^{jk} \tag{3.24}$$

Note that the first term in the right hand side of Eq. (3.24) corresponds to an absolute angular velocity term, whereas the second term is expressed in terms of the time derivatives of the relative generalized coordinates of joint k. Substituting Eqs. (3.19), (3.23), and (3.24) into Eq. (3.18), the absolute velocity vector of the centroid of body B^j is:

$$\begin{aligned}\dot{\vec{r}}^{j} &= \dot{\vec{r}}^{i} + \vec{\omega}^{i} * (\vec{b}^{ik} + \vec{c}^{k} - \vec{b}^{jk}) + \\ &\quad \left(\frac{\partial \vec{c}^{k}}{(\partial \underset{\sim}{q}^{k})^{T}} - (\underset{\sim}{\vec{w}}^{k})^{T} * \vec{b}^{jk}\right) \dot{\underset{\sim}{q}}^{k} \end{aligned} \tag{3.25}$$

Again, note that the first two terms in Eq. (3.25) correspond to absolute velocity terms, whereas the last term is expressed in terms of the relative quantities gathered in column $\dot{\underset{\sim}{q}}^{k}$. Eqs. (3.15) and (3.25) may be organized in a matrix equation as follows:

$$\underset{\sim}{\vec{v}}^{j} = \underset{\sim}{\mathbf{A}}^{k} \cdot \underset{\sim}{\vec{v}}^{i} + \underset{\sim}{\vec{B}}^{k} \dot{\underset{\sim}{q}}^{k} \tag{3.26}$$

where

$$\underset{\sim}{\vec{v}}^{j} = \begin{bmatrix} \dot{\vec{r}}^{j} \\ \vec{\omega}^{j} \end{bmatrix} \tag{3.27}$$

$$\underset{\sim}{\mathbf{A}}^{k} = \begin{bmatrix} \mathbf{I} & -\mathbf{Z}^{k} \\ \mathbf{O} & \mathbf{I} \end{bmatrix} \tag{3.28}$$

$$\underset{\sim}{\vec{v}}^{i} = \begin{bmatrix} \dot{\vec{r}}^{i} \\ \vec{\omega}^{i} \end{bmatrix} \tag{3.29}$$

$$\underset{\sim}{\vec{B}}^{k} = \begin{bmatrix} \frac{\partial \vec{c}^{k}}{(\partial \underset{\sim}{q}^{k})^{T}} - (\underset{\sim}{\vec{w}}^{k})^{T} * \vec{b}^{jk} \\ (\underset{\sim}{\vec{w}}^{k})^{T} \end{bmatrix} \tag{3.30}$$

with the following definition for the skew-symmetric tensor $\mathbf{Z}^{k}$ in Eq. (3.28):

$$\mathbf{Z}^{k} \cdot \vec{a} = (\vec{b}^{ik} + \vec{c}^{k} - \vec{b}^{jk}) * \vec{a} \qquad \forall \vec{a} \tag{3.31}$$

In the right hand side of Eq. (3.26), an important separation between absolute velocity quantities (first term) and relative velocity quantities (second term) can be observed. This separation turns out to be essential in the derivation of the recursive equations later on in this chapter.

Similar to the derivations of the kinematic velocity equation (Eq. (3.26)), the virtual displacement equation (needed in the next section) can be derived:

$$\delta \underset{\sim}{\vec{u}}^{j} = \underline{\mathbf{A}}^{k} \cdot \delta \underset{\sim}{\vec{u}}^{i} + \underline{\vec{B}}^{k} \delta \underset{\sim}{q}^{k} \tag{3.32}$$

with

$$\delta \underset{\sim}{\vec{u}}^{j} = \begin{bmatrix} \delta \vec{r}^{\,j} \\ \delta \vec{\pi}^{\,j} \end{bmatrix} \tag{3.33}$$

$$\delta \underset{\sim}{\vec{u}}^{i} = \begin{bmatrix} \delta \vec{r}^{\,i} \\ \delta \vec{\pi}^{\,i} \end{bmatrix} \tag{3.34}$$

where $\delta\vec{r}$ and $\delta\vec{\pi}$ (axial vector of $\delta\mathbf{R} \cdot \mathbf{R}^{c}$) denote virtual translation and virtual rotation, respectively, and $\delta \underset{\sim}{q}$ denotes virtual relative coordinates. The matrix form of acceleration relations can be obtained by differentiation of Eq. (3.26):

$$\dot{\underset{\sim}{\vec{v}}}^{j} = \underline{\mathbf{A}}^{k} \cdot \dot{\underset{\sim}{\vec{v}}}^{i} + \underline{\vec{B}}^{k} \ddot{\underset{\sim}{q}}^{k} + \underline{\vec{D}}^{k} \tag{3.35}$$

where

$$\underline{\vec{D}}^{k} = \dot{\underline{\mathbf{A}}}^{k} \cdot \underset{\sim}{\vec{v}}^{i} + \dot{\underline{\vec{B}}}^{k} \dot{\underset{\sim}{q}}^{k} \tag{3.36}$$

$$\dot{\underline{\mathbf{A}}}^{k} = \begin{bmatrix} \mathbf{O} & -\dot{\mathbf{Z}}^{k} \\ \mathbf{O} & \mathbf{O} \end{bmatrix} \tag{3.37}$$

$$\dot{\underline{\vec{B}}}^{k} = \begin{bmatrix} \frac{d}{dt}\left(\frac{\partial \vec{c}^{\,k}}{(\partial \underset{\sim}{q}^{k})^{T}}\right) + (\vec{\omega}^{\,j} * \vec{b}^{\,jk}) * (\underset{\sim}{\vec{w}}^{k})^{T} + \vec{b}^{\,jk} * (\dot{\underset{\sim}{\vec{w}}}^{k})^{T} \\ (\dot{\underset{\sim}{\vec{w}}}^{k})^{T} \end{bmatrix} \tag{3.38}$$

with

$$(\dot{\underset{\sim}{\vec{w}}}^{k})^{T} = \left(\frac{\partial \underset{\sim}{\vec{w}}^{k}}{(\partial \underset{\sim}{q}^{k})^{T}} \dot{\underset{\sim}{q}}^{k} \right)^{T} \tag{3.39}$$

Recursive equations include a relation between absolute coordinates of one body on the one hand, and absolute coordinates of another body plus the relative coordinates due to the joint on the other hand. Eqs. (3.15), (3.16), (3.26), (3.32), and (3.35) are the essential recursive equations.

Example: universal joint

In the following, the relevant quantities needed to describe a universal joint are discussed. An extensive ilustration of the use of the kinematic equations presented so far can be found in Appendix C, where a mechanism consisting of two rigid bodies interconnected by a universal joint is described. Fig. 3.3 shows a universal joint in between the rigid bodies B^i and B^j. The joint is connected to body B^i at joint attachment point O^{ik} and connected to body B^j at joint attachment point O^{jk}. Points O^{ik} and O^{jk} are chosen to be coincident in order to keep the joint equations as simple as possible.

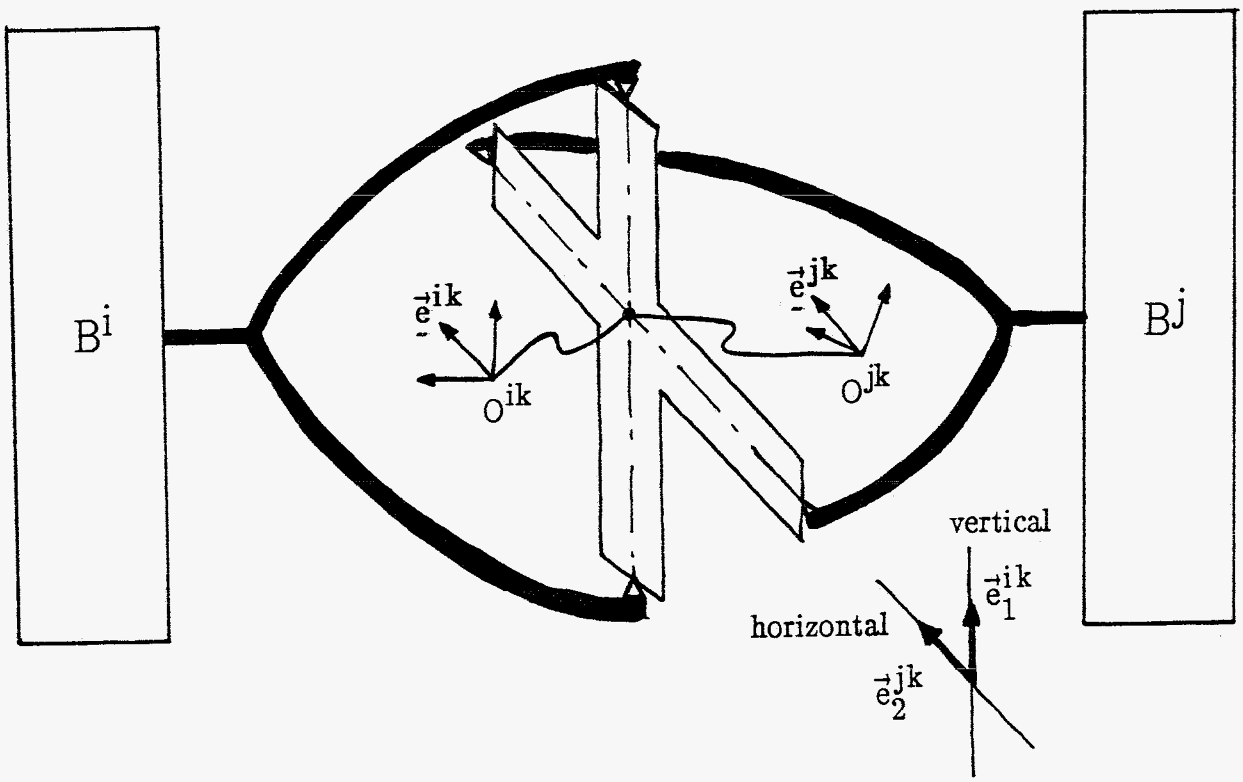

Figure 3.3: *Universal joint*

From Fig. 3.3, one can see that

$$\vec{c}^k = \vec{0} \tag{3.40}$$

The change of orientation between the joint attachment frames $\underset{\sim}{\vec{e}}^{ik}$ and $\underset{\sim}{\vec{e}}^{jk}$ is determined by tensor $\mathbf{C}^k$:

$$(\underset{\sim}{\vec{e}}^{jk})^T = \mathbf{C}^k \cdot (\underset{\sim}{\vec{e}}^{ik})^T \tag{3.41}$$

with

$$\mathbf{C}^k = (\underset{\sim}{\vec{e}}^{jk})^T \underset{\sim}{\vec{e}}^{ik} = (\underset{\sim}{\vec{e}}^{ik})^T \underset{\sim}{C}^k \underset{\sim}{\vec{e}}^{ik} \tag{3.42}$$

where $\underset{\sim}{C}^k$ can be written in terms of relative coordinates gathered in column $\underset{\sim}{q}^k$, according to Eq. (3.12). The universal joint allows two degrees of freedom between the two bodies. One could choose for two Bryant angles q_1^k and q_2^k to describe the change of orientation between the joint attachment frames $\underset{\sim}{\vec{e}}^{ik}$ and $\underset{\sim}{\vec{e}}^{jk}$. In this case, an appropriate choice for base vectors $\vec{e}_1^{ik}$ and $\vec{e}_2^{jk}$ is shown in Fig. 3.3, i.e. parallel to the "vertical" and "horizontal" rotation axes of the joint, respectively. In Fig. 3.4, the two Bryant angles q_1^k and q_2^k are depicted in detail. An intermediate frame $\underset{\sim}{\vec{e}}$ is introduced to provide a stepwise transition from $\underset{\sim}{\vec{e}}^{ik}$ to $\underset{\sim}{\vec{e}}^{jk}$.

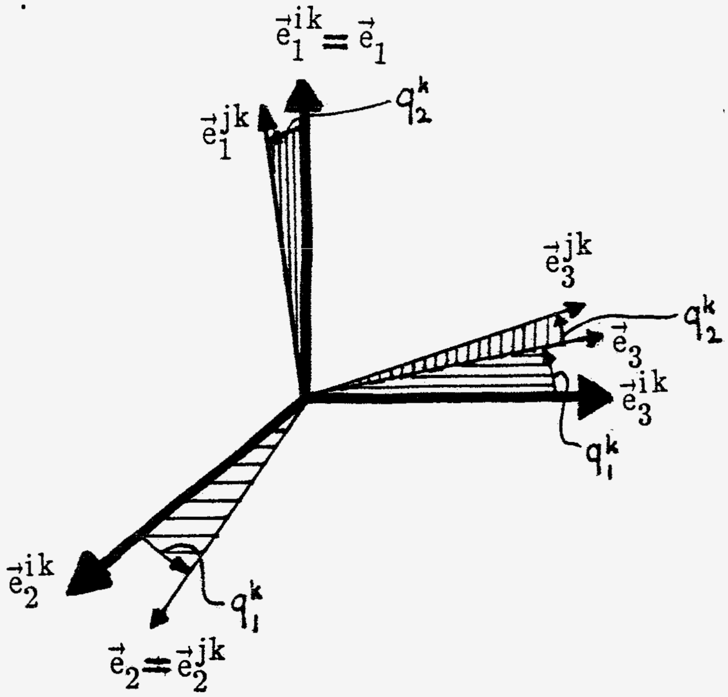

Figure 3.4: *Bryant angles with third angle equal to zero*

With this choice for the relative joint coordinates, $\underset{\sim}{C}^k$ can be expressed as (used abbreviations: $s_1 = \sin q_1^k$, $s_2 = \sin q_2^k$, $c_1 = \cos q_1^k$, and $c_2 = \cos q_2^k$):

$$\underset{\sim}{C}^k = \begin{bmatrix} c_2 & 0 & s_2 \\ s_1 s_2 & c_1 & -s_1 c_2 \\ -c_1 s_2 & s_1 & c_1 c_2 \end{bmatrix} \tag{3.43}$$

For an extensive evaluation of the universal joint in a multibody system, the reader is referred to Appendix C.

3.4 Dynamics

After describing the kinematics of the elementary pair of bodies of Fig. 3.2, the equations of motion for a system of rigid bodies are dealt with. Consider the system with *nb* rigid bodies in Fig. 3.5.

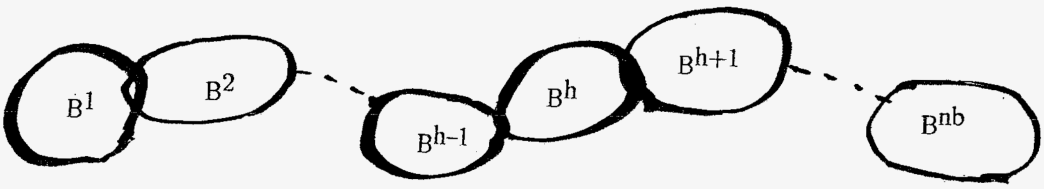

Figure 3.5: *Chain of nb rigid bodies*

According to d'Alembert's principle of virtual work, the equations of motion for a system of *nb* bodies are (see Eq. (2.2))

$$\sum_{i=1}^{nb}\left[(\delta\underset{\sim}{\vec{u}}^{\,i})^T\cdot(\underline{\mathrm{M}}^i\cdot\dot{\underset{\sim}{\vec{v}}}^{\,i}-\underset{\sim}{\vec{Q}}^{\,i})\right]=0$$

$$\forall \text{ kinematically admissible } \delta\underset{\sim}{\vec{u}}^{\,i}\ (i=1,\ldots,nb) \qquad (3.44)$$

with

$$\delta\underset{\sim}{\vec{u}}^{\,i}=\begin{bmatrix}\delta\vec{r}^{\,i}\\ \delta\vec{\pi}^{\,i}\end{bmatrix} \qquad (3.45)$$

$$\underline{\mathrm{M}}^i=\begin{bmatrix}m^i\mathbf{I} & \mathbf{O}\\ \mathbf{O} & \mathbf{J}^i\end{bmatrix} \qquad (3.46)$$

$$\dot{\underset{\sim}{\vec{v}}}^{\,i}=\begin{bmatrix}\ddot{\vec{r}}^{\,i}\\ \dot{\vec{\omega}}^{\,i}\end{bmatrix} \qquad (3.47)$$

$$\underset{\sim}{\vec{Q}}^{\,i}=\begin{bmatrix}\vec{F}^{\,i}\\ \vec{T}^{\,i}-\vec{\omega}^{\,i}*(\mathbf{J}^i\cdot\vec{\omega}^{\,i})\end{bmatrix} \qquad (3.48)$$

where $\vec{F}^{\,i}$ and $\vec{T}^{\,i}$ in Eq. (3.48) are vectors representing the resultant force and torque acting on body B^i, respectively, and where m^i and $\mathbf{J}^i$ in Eq. (3.46) are the mass of body B^i and the inertia tensor of body B^i with respect to its mass center, respectively.

It is remarked that before using d'Alembert's principle of virtual work, vectors $\vec{F}^{\,i}$ and $\vec{T}^{\,i}$ include forces and torques that are due to the connection constraints of body B^i. These constraint forces and constraint torques cancel out when d'Alembert's principle of virtual work is used to set up the equations of motion for the whole multibody system. Furthermore, the virtual displacements $\delta\underset{\sim}{\vec{u}}^i$ must be kinematically admissible, which means that they may not violate the kinematic constraints that apply to bodies B^1 to B^{nb}. In particular, $\delta\underset{\sim}{\vec{u}}^1$ of the base body (see Sect. 3.2) must satisfy not only the kinematic constraints between base body B^1 and the other directly connected higher numbered bodies, but also the kinematic constraints between body B^1 and inertial space, if body B^1 were connected to inertial space. Inertial space may be regarded as being a body (in Sect. 3.2 already numbered as B^0) with known constant position and zero velocity. The position of body B^1 relative to inertial space can be represented by kinematically admissible joint coordinates of B^1 relative to B^0 just like any other arbitrary couple of bodies B^i and B^j. In this way, introduction of Lagrange multipliers as mentioned by Bae & Haug (1987) is not needed. This provides the advantage of maintaining a system of ordinary differential equations, and avoids having to solve a system of differential/algebraic equations as in Bae & Haug (1987).

In the following, the basic recursive relations derived in Section 3.3 will be used in a procedure kind of like successive substitution. The joints in between the bodies are numbered in the following way: if joint k connects body i to body $i+1$, the joint is assigned the same number as the higher numbered body, so $k = i+1$.

Eq. (3.44) can also be written as follows:

$$\sum_{i=1}^{nb-1}\left[(\delta\underset{\sim}{\vec{u}}^i)^T\cdot(\underset{\sim}{\mathrm{M}}^i\cdot\dot{\underset{\sim}{\vec{v}}}^i-\underset{\sim}{\vec{Q}}^i)\right]+(\delta\underset{\sim}{\vec{u}}^{nb})^T\cdot(\underset{\sim}{\mathrm{M}}^{nb}\cdot\dot{\underset{\sim}{\vec{v}}}^{nb}-\underset{\sim}{\vec{Q}}^{nb})=0$$

$$\forall \text{ kinematically admissible } \delta\underset{\sim}{\vec{u}}^i \;\; (i=1,\ldots,nb) \qquad (3.49)$$

Substitution of Eqs. (3.32) and (3.35), with $i = nb-1$, $j = nb$, and $k = nb$, into Eq. (3.49) results in

$$\begin{aligned}&\sum_{i=1}^{nb-1}\left[(\delta\underset{\sim}{\vec{u}}^i)^T\cdot(\underset{\sim}{\mathrm{M}}^i\cdot\dot{\underset{\sim}{\vec{v}}}^i-\underset{\sim}{\vec{Q}}^i)\right]+\\
&\quad+(\delta\underset{\sim}{\vec{u}}^{nb-1})^T\cdot(\underset{\sim}{\mathrm{A}}^{nb})^T\cdot\left[\underset{\sim}{\mathrm{M}}^{nb}\cdot\underset{\sim}{\mathrm{A}}^{nb}\cdot\dot{\underset{\sim}{\vec{v}}}^{nb-1}+\underset{\sim}{\mathrm{M}}^{nb}\cdot\underset{\sim}{\vec{B}}^{nb}\ddot{\underset{\sim}{q}}^{nb}+\underset{\sim}{\mathrm{M}}^{nb}\cdot\underset{\sim}{\vec{D}}^{nb}-\underset{\sim}{\vec{Q}}^{nb}\right]+\\
&\quad+(\delta\underset{\sim}{q}^{nb})^T(\underset{\sim}{\vec{B}}^{nb})^T\cdot\left[\underset{\sim}{\mathrm{M}}^{nb}\cdot\underset{\sim}{\mathrm{A}}^{nb}\cdot\dot{\underset{\sim}{\vec{v}}}^{nb-1}+\underset{\sim}{\mathrm{M}}^{nb}\cdot\underset{\sim}{\vec{B}}^{nb}\ddot{\underset{\sim}{q}}^{nb}+\underset{\sim}{\mathrm{M}}^{nb}\cdot\underset{\sim}{\vec{D}}^{nb}-\underset{\sim}{\vec{Q}}^{nb}\right]=0\end{aligned}$$

$$\forall \text{ kinematically admissible } \delta\underset{\sim}{\vec{u}}^i \;\; (i=1,\ldots,nb) \text{ and } \delta\underset{\sim}{q}^{nb} \qquad (3.50)$$

Variation of column $\underset{\sim}{q}^{nb}$ containing relative coordinates must be kinematically admissible, which means that it may not violate the kinematic constraints between bodies B^{nb-1} and B^{nb}. However, since $\underset{\sim}{q}^{nb}$ contains relative joint coordinates that automatically comply with the joint constraints, $\delta\underset{\sim}{q}^{nb}$ is kinematically admissible by definition. And because $\delta\underset{\sim}{\vec{u}}^i$, $(i = 1,\ldots,nb-1)$ and $\delta\underset{\sim}{q}^{nb}$ are independent, the coefficient of $\delta\underset{\sim}{q}^{nb}$ in Eq. (3.50) must be equal to a zero column. The second time derivative $\ddot{\underset{\sim}{q}}^{nb}$ of the relative coordinates of joint nb can be written explicitly as

$$\ddot{\underset{\sim}{q}}^{nb} = -((\underset{\sim}{\vec{B}}^{nb})^T\cdot\underset{\sim}{\mathrm{M}}^{nb}\cdot\underset{\sim}{\vec{B}}^{nb})^{-1}\left[(\underset{\sim}{\vec{B}}^{nb})^T\cdot\left(\underset{\sim}{\mathrm{M}}^{nb}\cdot\underset{\sim}{\mathrm{A}}^{nb}\cdot\dot{\underset{\sim}{\vec{v}}}^{nb-1}+\right.\right.$$

$$+\underline{M}^{nb} \cdot \vec{\underset{\sim}{D}}^{nb} - \vec{\underset{\sim}{Q}}^{nb})\Big] \tag{3.51}$$

According to Bae & Haug (1987), the existence of $((\vec{\underline{B}}^{nb})^T \cdot \underline{M}^{nb} \cdot \vec{\underline{B}}^{nb})^{-1}$ can be proved. Substituting Eq. (3.51) back into Eq. (3.50) yields

$$\begin{aligned}
&\sum_{i=1}^{nb-2} \Big[(\delta\vec{\underset{\sim}{u}}^{i})^T \cdot (\underline{M}^{i} \cdot \dot{\vec{\underset{\sim}{v}}}^{i} - \vec{\underset{\sim}{Q}}^{i})\Big] \\
&+(\delta\vec{\underset{\sim}{u}}^{nb-1})^T \cdot \Big[(\underline{M}^{nb-1} + \underline{M}_r^{nb-1}) \cdot \dot{\vec{\underset{\sim}{v}}}^{nb-1} - (\vec{\underset{\sim}{Q}}^{nb-1} + \vec{\underset{\sim}{Q}}_r^{nb-1})\Big] = 0 \\
&\qquad \forall \text{ kinematically admissible } \delta\vec{\underset{\sim}{u}}^{i} \ (i = 1, \ldots, nb-1)
\end{aligned} \tag{3.52}$$

where

$$\begin{aligned}
&\underline{M}_r^{nb-1} = \\
&\quad (\underline{A}^{nb})^T \cdot \underline{M}^{nb} \cdot \underline{A}^{nb} \\
&\quad -(\underline{A}^{nb})^T \cdot \underline{M}^{nb} \cdot \vec{\underline{B}}^{nb}((\vec{\underline{B}}^{nb})^T \cdot \underline{M}^{nb} \cdot \vec{\underline{B}}^{nb})^{-1}(\vec{\underline{B}}^{nb})^T \cdot \underline{M}^{nb} \cdot \underline{A}^{nb}
\end{aligned} \tag{3.53}$$

$$\begin{aligned}
&\vec{\underset{\sim}{Q}}_r^{nb-1} = \\
&\quad (\underline{A}^{nb})^T \cdot (\underline{M}^{nb} \cdot \vec{\underset{\sim}{D}}^{nb} - \vec{\underset{\sim}{Q}}^{nb}) - (\underline{A}^{nb})^T \cdot \underline{M}^{nb} \cdot \\
&\quad \cdot \vec{\underline{B}}^{nb}((\vec{\underline{B}}^{nb})^T \cdot \underline{M}^{nb} \cdot \vec{\underline{B}}^{nb})^{-1} \Big[(\vec{\underline{B}}^{nb})^T \cdot \underline{M}^{nb} \cdot \vec{\underset{\sim}{D}}^{nb} - (\vec{\underline{B}}^{nb})^T \cdot \vec{\underset{\sim}{Q}}^{nb}\Big]
\end{aligned} \tag{3.54}$$

Note that the superscript of the summation sign has changed from $nb-1$ to $nb-2$. If this procedure is repeated down the chain to body $h+1$, it can be shown that in analogy to Eq. (3.52) the following expression results:

$$\begin{aligned}
&\sum_{i=1}^{h} \Big[(\delta\vec{\underset{\sim}{u}}^{i})^T \cdot (\underline{M}^{i} \cdot \dot{\vec{\underset{\sim}{v}}}^{i} - \vec{\underset{\sim}{Q}}^{i})\Big] = \\
&\quad +(\delta\vec{\underset{\sim}{u}}^{h+1})^T \cdot \Big[(\underline{M}^{h+1} + \underline{M}_r^{h+1}) \cdot \dot{\vec{\underset{\sim}{v}}}^{h} - (\vec{\underset{\sim}{Q}}^{h+1} + \vec{\underset{\sim}{Q}}_r^{h+1})\Big] = 0 \\
&\qquad \forall \text{ kinematically admissible } \delta\vec{\underset{\sim}{u}}^{i} \ (i = 1, \ldots, h+1)
\end{aligned} \tag{3.55}$$

By reason of better insight, the above procedure is repeated once more to show the reader that equations similar to Eq. (3.50) and (3.51) lead to an equation form similar to Eq. (3.52) or (3.55).

Substitution of Eqs. (3.32) and (3.35), with $i = h$, $j = h+1$, and $k = h+1$, into Eq. (3.55) results in

$$\begin{aligned}
&\sum_{i=1}^{h-1} (\delta\vec{\underset{\sim}{u}}^{i})^T \cdot \Big[\underline{M}^{i} \cdot \dot{\vec{\underset{\sim}{v}}}^{i} - \vec{\underset{\sim}{Q}}^{i}\Big] + \\
&\quad +(\delta\vec{\underset{\sim}{u}}^{h})^T \cdot \Big[(\underline{A}^{h+1})^T \cdot \Big[(\underline{M}^{h+1} + \underline{M}_r^{h+1}) \cdot (\underline{A}^{h+1} \cdot \dot{\vec{\underset{\sim}{v}}}^{h} + \vec{\underline{B}}^{h+1}\ddot{\underset{\sim}{q}}^{h+1} + \vec{\underset{\sim}{D}}^{h+1})
\end{aligned}$$

$$
\begin{aligned}
&-(\underset{\sim}{\vec{Q}}^{h+1} + \underset{\sim}{\vec{Q}}{}_r^{h+1})\Big] + \underline{\mathbf{M}}^h \cdot \underset{\sim}{\dot{\vec{v}}}^h - \underset{\sim}{\vec{Q}}^h\Big] \\
+(\delta \underset{\sim}{q}^{h+1})^T(\underline{\vec{B}}^{h+1})^T \cdot \Big[&(\underline{\mathbf{M}}^{h+1} + \underline{\mathbf{M}}_r^{h+1}) \cdot (\underline{\mathbf{A}}^{h+1} \cdot \underset{\sim}{\dot{\vec{v}}}^h + \underline{\vec{B}}^{h+1} \underset{\sim}{\ddot{q}}^{h+1} + \underset{\sim}{\vec{D}}^{h+1}) \\
&-(\underset{\sim}{\vec{Q}}^{h+1} + \underset{\sim}{\vec{Q}}{}_r^{h+1})\Big] = 0 \\
&\forall \text{ kinematically admissible } \delta \underset{\sim}{\vec{u}}^i \ (i = 1, \ldots, h) \text{ and } \delta \underset{\sim}{q}^{h+1}
\end{aligned}
\tag{3.56}
$$

Again, $\delta \underset{\sim}{q}^{h+1}$ is kinematically admissible by definition. And because $\delta \underset{\sim}{\vec{u}}^i$, $(i = 1, \ldots, h)$ and $\delta \underset{\sim}{q}^{h+1}$ are independent, the coefficient of $\delta \underset{\sim}{q}^{h+1}$ in Eq. (3.56) must be equal to a zero column. The second time derivative $\underset{\sim}{\ddot{q}}^{h+1}$ of the relative coordinates of joint $h+1$ can be written explicitly as

$$
\begin{aligned}
\underset{\sim}{\ddot{q}}^{h+1} = \ & -((\underline{\vec{B}}^{h+1})^T \cdot (\underline{\mathbf{M}}^{h+1} + \underline{\mathbf{M}}_r^{h+1}) \cdot \underline{\vec{B}}^{h+1})^{-1} \Big[(\underline{\vec{B}}^{h+1})^T \cdot \Big((\underline{\mathbf{M}}^{h+1} + \underline{\mathbf{M}}_r^{h+1}) \cdot \\
& \cdot \underline{\mathbf{A}}^{h+1} \cdot \underset{\sim}{\dot{\vec{v}}}^h + (\underline{\mathbf{M}}^{h+1} + \underline{\mathbf{M}}_r^{h+1}) \cdot \underset{\sim}{\vec{D}}^{h+1} - (\underset{\sim}{\vec{Q}}^{h+1} + \underset{\sim}{\vec{Q}}{}_r^{h+1})\Big)\Big]
\end{aligned}
\tag{3.57}
$$

Again, according to Bae & Haug (1987), the existence of $((\underline{\vec{B}}^{h+1})^T \cdot (\underline{\mathbf{M}}^{h+1} + \underline{\mathbf{M}}_r^{h+1}) \cdot \underline{\vec{B}}^{h+1})^{-1}$ can be proved. Substituting Eq. (3.57) back into Eq. (C.24) yields

$$
\begin{aligned}
&\sum_{i=1}^{h-1} \Big[(\delta \underset{\sim}{\vec{u}}^i)^T \cdot (\underline{\mathbf{M}}^i \cdot \underset{\sim}{\dot{\vec{v}}}^i - \underset{\sim}{\vec{Q}}^i)\Big] + \\
&\quad +(\delta \underset{\sim}{\vec{u}}^h)^T \cdot \Big[(\underline{\mathbf{M}}^h + \underline{\mathbf{M}}_r^h) \cdot \underset{\sim}{\dot{\vec{v}}}^h - (\underset{\sim}{\vec{Q}}^h + \underset{\sim}{\vec{Q}}{}_r^h)\Big] = 0 \\
&\qquad \forall \text{ kinematically admissible } \delta \underset{\sim}{\vec{u}}^i \ (i = 1, \ldots, h)
\end{aligned}
\tag{3.58}
$$

where

$$
\begin{aligned}
&\underline{\mathbf{M}}_r^h = \\
&\quad (\underline{\mathbf{A}}^{h+1})^T \cdot (\underline{\mathbf{M}}^{h+1}) \cdot \underline{\mathbf{A}}^{h+1} \\
&\quad -(\underline{\mathbf{A}}^{h+1})^T \cdot (\underline{\mathbf{M}}^{h+1} + \underline{\mathbf{M}}_r^{h+1}) \cdot \underline{\vec{B}}^{h+1}((\underline{\vec{B}}^{h+1})^T \cdot (\underline{\mathbf{M}}^{h+1} + \underline{\mathbf{M}}_r^{h+1}) \cdot \underline{\vec{B}}^{h+1})^{-1} \\
&\quad (\underline{\vec{B}}^{h+1})^T \cdot (\underline{\mathbf{M}}^{h+1} + \underline{\mathbf{M}}_r^{h+1}) \cdot \underline{\mathbf{A}}^{h+1}
\end{aligned}
\tag{3.59}
$$

$$
\begin{aligned}
&\underset{\sim}{\vec{Q}}{}_r^h = \\
&\quad (\underline{\mathbf{A}}^{h+1})^T \cdot ((\underline{\mathbf{M}}^{h+1} + \underline{\mathbf{M}}_r^{h+1}) \cdot \underset{\sim}{\vec{D}}^{h+1} - \underset{\sim}{\vec{Q}}^{h+1}) \\
&\quad -(\underline{\mathbf{A}}^{h+1})^T \cdot (\underline{\mathbf{M}}^{h+1} + \underline{\mathbf{M}}_r^{h+1}) \cdot \underline{\vec{B}}^{h+1}((\underline{\vec{B}}^{h+1})^T \cdot (\underline{\mathbf{M}}^{h+1} + \underline{\mathbf{M}}_r^{h+1}) \cdot \underline{\vec{B}}^{h+1})^{-1} \\
&\quad \Big[(\underline{\vec{B}}^{h+1})^T \cdot (\underline{\mathbf{M}}^{h+1} + \underline{\mathbf{M}}_r^{h+1}) \cdot \underset{\sim}{\vec{D}}^{h+1} - (\underline{\vec{B}}^{h+1})^T(\cdot \underset{\sim}{\vec{Q}}^{h+1} + \underset{\sim}{\vec{Q}}{}_r^{h+1})\Big]
\end{aligned}
\tag{3.60}
$$

The bottom line of the above process is reached for the base body B^1 (where the summation in Eq. (3.58) vanishes and $h = 1$):

$$
\begin{aligned}
&(\delta \underset{\sim}{\vec{u}}^1)^T \cdot \Big[(\underline{\mathbf{M}}^1 + \underline{\mathbf{M}}_r^1) \cdot \underset{\sim}{\dot{\vec{v}}}^1 - (\underset{\sim}{\vec{Q}}^1 + \underset{\sim}{\vec{Q}}{}_r^1)\Big] = 0 \\
&\qquad \forall \text{ kinematically admissible } \delta \underset{\sim}{\vec{u}}^1
\end{aligned}
\tag{3.61}
$$

where $\underset{\sim}{\mathrm{M}}_r^1$ and $\underset{\sim}{\vec{Q}}_r^1$ follow from Eqs. (3.59) and (3.60), respectively, by taking $h = 1$.

In the above process, the second time derivatives of the relative coordinates are obtained explicitly (see Eqs. (3.51) and (3.57)), in a convenient form for numerical integration (e.g., by the Runge Kutta method). An advantage of the above derivation is the small dimension of the matrices that have to be inverted to obtain the relative coordinate accelerations as in Eqs. (3.51) and (3.57).

3.5 The recursive algorithm

In the dynamics section, a procedure has been described comparable to a successive substitution process. The solution and integration of the derived dynamic equations is discussed in this section. A brief scheme is presented which shows the characteristics of the recursive algorithm.

1. Initial conditions: relative coordinates and their time derivatives, $\underset{\sim}{q}$ and $\dot{\underset{\sim}{q}}$, are given at t_0.

2. By means of Eqs. (3.8), (3.15), (3.16), (3.17), and (3.18), positions and velocities of all rigid bodies in the system can be determined ($\vec{r}^{\,i}$, $\dot{\vec{r}}^{\,i}$, and $\vec{\omega}^{\,i}$, for $i = 1$ *to* nb). This step may be called "recursive".

3. Forces $\underset{\sim}{\vec{Q}}^{\,i}$ according to Eq. (3.48) can be worked out then, as a consequence of possibly present springs, dampers, etc. in the system (also the term $\vec{\omega}^{\,i} * (\mathbf{J}^i \cdot \vec{\omega}^{\,i})$ can be calculated), and the mass matrix $\underset{\sim}{\mathrm{M}}^i$ of each body B^i can be composed.

4. Then $\underset{\sim}{\mathrm{M}}_r^i$ and $\underset{\sim}{\vec{Q}}_r^{\,i}$ can be obtained consecutively per joint, as has been outlined in the previous section by Eqs. (3.59) and (3.60) ($\underset{\sim}{\mathrm{M}}_r^i$ and $\underset{\sim}{\vec{Q}}_r^{\,i}$, for $i = nb$ *to* 1). Note that calculations are executed from the tree end body (or bodies) to the base body in the system. This step may also be considered as a "recursive" step.

5. By means of Eq. (3.57), second time derivatives of relative coordinates $\underset{\sim}{q}^i$ can be determined ($\ddot{\underset{\sim}{q}}^{\,i}$, for $i = 1$ *to* nb). Again, this is a "recursive" step.

6. Execute a predictor/corrector operation on $\ddot{\underset{\sim}{q}}^{\,i}$ $(i = 1, \ldots, nb)$ in order to obtain $\dot{\underset{\sim}{q}}^i$ and $\underset{\sim}{q}^i$, and return to step 2. If the time is increased by this operation, and if the end time is reached, the algorithm is stopped here.

Note that in step 6 only the relative coordinates need to be determined by time integration. The time derivatives of the absolute velocities ($\dot{\underset{\sim}{\vec{v}}}^{\,i}$, $i = 1, \ldots, nb$) need not be integrated to obtain position and velocity, since position $\vec{r}^{\,i}$ and velocity $\vec{v}^{\,i}$ of one body B^i can be derived from the position $\vec{r}^{\,i-1}$ and velocity $\vec{v}^{\,i-1}$ of a preceding adjacent body B^{i-1} plus the relative coordinates $\underset{\sim}{q}^i$ and their time derivatives $\dot{\underset{\sim}{q}}^i$ between these two bodies (see Eqs. (3.16), (3.17), and (3.18)).

Chapter 4

Modification of the recursive technique to more general multibody systems

4.1 Driving constraints

In Sect. 3.4, Eq. (3.52) has been derived by elimination of the second time derivatives of the relative generalized coordinates on the assumption that there are no driving constraints, i.e. $\ddot{\underset{\sim}{q}}$ is unknown. In the case where the relative generalized coordinates of a joint are kinematically driven, the equations in Sect. 3.4 change a little. Suppose $\underset{\sim}{q}^{nb}$ in Eq. (3.50) is known, then $\delta \underset{\sim}{q}^{nb} = \underset{\sim}{0}$, and Eq. (3.50) becomes

$$
\begin{aligned}
&\sum_{i=1}^{nb-1} (\delta \vec{\underset{\sim}{v}}^{\,i})^T \cdot \left[\underline{M}^i \cdot \dot{\vec{\underset{\sim}{v}}}^{\,i} - \vec{\underset{\sim}{Q}}^{\,i}\right] + \\
&\qquad +(\delta \vec{\underset{\sim}{v}}^{\,nb-1})^T \cdot (\underline{A}^{nb})^T \cdot \left[\underline{M}^{nb} \cdot (\underline{A}^{nb} \cdot \dot{\vec{\underset{\sim}{v}}}^{\,nb-1} + \vec{\underline{B}}^{\,nb} \ddot{\underset{\sim}{q}}^{nb} + \vec{\underset{\sim}{D}}^{\,nb}) - \vec{\underset{\sim}{Q}}^{\,nb}\right] = 0 \\
&\qquad\qquad \forall \text{ kinematically admissible } \delta \vec{\underset{\sim}{v}}^{\,i} \ (i = 1, \ldots, nb-1)
\end{aligned} \tag{4.1}
$$

The term in Eq. (4.1) with the known relative accelerations $\ddot{\underset{\sim}{q}}^{nb}$ is added to column $\vec{\underset{\sim}{Q}}_r^{\,nb-1}$ of Eq. (3.54), yielding

$$
\begin{aligned}
&\sum_{i=1}^{nb-2} (\delta \vec{\underset{\sim}{v}}^{\,i})^T \cdot \left[\underline{M}^i \cdot \dot{\vec{\underset{\sim}{v}}}^{\,i} - \vec{\underset{\sim}{Q}}^{\,i}\right] + \\
&\qquad (\delta \vec{\underset{\sim}{v}}^{\,nb-1})^T \cdot \left[(\underline{M}^{nb-1} + \underline{M}_r^{nb-1}) \cdot \dot{\vec{\underset{\sim}{v}}}^{\,nb-1} - (\vec{\underset{\sim}{Q}}^{\,nb-1} + \vec{\underset{\sim}{Q}}_r^{\,nb-1})\right] = 0 \\
&\qquad\qquad \forall \text{ kinematically admissible } \delta \vec{\underset{\sim}{v}}^{\,i} \ (i = 1, \ldots, nb-1)
\end{aligned} \tag{4.2}
$$

where

$$
\underline{M}_r^{nb-1} = (\underline{A}^{nb})^T \cdot \underline{M}^{nb} \cdot \underline{A}^{nb} \tag{4.3}
$$

$$
\vec{\underset{\sim}{Q}}_r^{\,nb-1} = (\underline{A}^{nb})^T \cdot \left[\underline{M}^{nb} \cdot \left(\vec{\underline{B}}^{\,nb} \ddot{\underset{\sim}{q}}^{nb} + \vec{\underset{\sim}{D}}^{\,nb}\right) - \vec{\underset{\sim}{Q}}^{\,nb}\right] \tag{4.4}
$$

Eq. (4.2) with Eqs. (4.3) and (4.4) is derived on the assumption that all relative generalized coordinates of the joint concerned are prescribed. One might as well think of a situation in which only some of the relative generalized coordinates of the joint are kinematically driven. In this case, column $\ddot{\underset{\sim}{q}}$ could be divided into two parts: a prescribed part and an unknown part. With this division of $\ddot{\underset{\sim}{q}}$ into two parts, also the system of equations can be partitioned in a part of the form of Eq. (3.52) with Eqs. (3.53) and (3.54), and a part of the form of Eq. (4.2) with Eqs. (4.3) and (4.4), respectively. This possible partitioning is not dealt with in this report. Besides, the prescribing of driving constraints mostly applies to joints with one degree of freedom in practice.

4.2 Closed loop systems

In Chapter 3, only systems with a tree structure were described. If a mechanism contains one or more closed loops, a mathematical manipulation method described in this section enables application of the recursive technique as described in Chapter 3 to a closed loop as well.

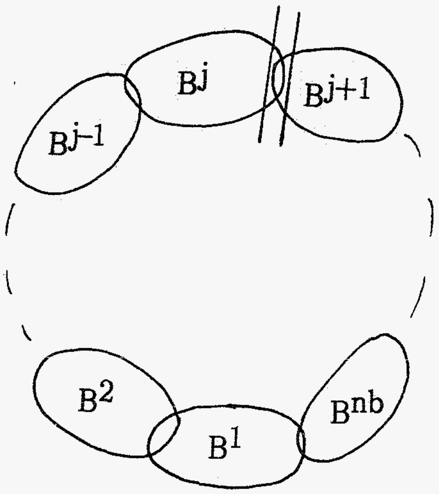

Figure 4.1: *Closed loop system*

Consider the closed loop system of Fig. 4.1. The closed loop can be opened and turned into a system with a tree structure by an imaginary cut of joint k in between bodies B^i and B^{i+1}. In the following, the joint where the loop is opened is called "cut joint." The constraint equation of joint k before "cutting" is denoted by $\underset{\sim}{h}^k$ and is a function of columns $\underset{\sim}{\vec{u}}^j$ and $\underset{\sim}{\vec{u}}^{j+1}$ with displacement vectors of bodies B^j and B^{j+1}, respectively. According to Bae & Haug (1987-88), the equations of motion for the "opened" system are

$$\sum_{i=1}^{nb}\left[(\delta\underset{\sim}{\vec{u}}^i)^T\cdot(\underset{\sim}{\mathrm{M}}^i\cdot\dot{\underset{\sim}{\vec{v}}}^i-\underset{\sim}{\vec{Q}}^i)\right]+\left[(\delta\underset{\sim}{\vec{u}}^j)^T\cdot\frac{(\partial\underset{\sim}{h}^k)^T}{\partial\underset{\sim}{\vec{u}}^j}+(\delta\underset{\sim}{\vec{u}}^{j+1})^T\cdot\frac{(\partial\underset{\sim}{h}^k)^T}{\partial\underset{\sim}{\vec{u}}^{j+1}}\right]\underset{\sim}{\lambda}=0$$

$$\forall \text{ kinematically admissible } \delta\underset{\sim}{\vec{u}}^i \text{ regardless of cut joint constraints} \tag{4.5}$$

where $\underset{\sim}{\lambda}$ denotes the introduced Lagrange multipliers.

The mathematical meaning of differentiation with respect to a vector must be defined first. A formal (coordinate-free) definition of differentiation of an arbitrary quantity $f = f(\vec{x}, t)$ (either a scalar f, a vector $\vec{f}$, or a tensor $\mathbf{F}$) with respect to vector $\vec{x}$ is the following:

$$\vec{a}\cdot\frac{\partial f}{\partial\vec{x}}=\lim_{\Delta s\to 0}\frac{f(\vec{x}+\vec{a}\Delta s,t)-f(\vec{x},t)}{\Delta s}\qquad\forall\vec{a}\tag{4.6}$$

In case of representing $\vec{x}$ with respect to an orthonormal frame of base vectors $\underset{\sim}{\vec{e}}$

$$\vec{x} = (\underset{\sim}{\vec{e}})^T \underset{\sim}{x} \tag{4.7}$$

the derivative of f with respect to $\vec{x}$ can simply be noted as

$$\frac{\partial f}{\partial \vec{x}} = (\underset{\sim}{\vec{e}})^T \frac{\partial f}{\partial \underset{\sim}{x}} \tag{4.8}$$

in which $\frac{\partial f}{\partial \underset{\sim}{x}}$ is a column with components $\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots$, etc.

The recursive technique as described in Chapter 3 can be applied from body B^j to body B^{j+1} via bodies B^{j-1}, B^{j-2}, ..., B^2, B^1, B^{nb}, B^{nb-1}, ..., B^{j+2}, or (as in Bae & Haug (1987-88)) in twofold from body B^j to base body B^1 and from body B^{j+1} to base body B^1. The number of system equations of motion according to Eq. (4.5) is smaller than the number of unknowns due to the introduction of Lagrange multipliers. In order to get a complete system with as many equations as unknowns, cut joint constraint acceleration equations must be introduced. Let the cut joint constraint equation be

$$\underset{\sim}{h}^k(\underset{\sim}{\vec{u}}^j, \underset{\sim}{\vec{u}}^{j+1}, t) = \underset{\sim}{0} \tag{4.9}$$

where $\underset{\sim}{\vec{u}}^j$ and $\underset{\sim}{\vec{u}}^{j+1}$ are columns with displacement vectors of bodies B^j and B^{j+1}, respectively. Then the first time derivative is

$$\dot{\underset{\sim}{h}}^k = \frac{\partial \underset{\sim}{h}^k}{(\partial \underset{\sim}{\vec{u}}^j)^T} \cdot \underset{\sim}{\vec{v}}^j + \frac{\partial \underset{\sim}{h}^k}{(\partial \underset{\sim}{\vec{u}}^{j+1})^T} \cdot \underset{\sim}{\vec{v}}^{j+1} + \frac{\partial \underset{\sim}{h}^k}{\partial t} = \vec{0} \tag{4.10}$$

and the second time derivative is

$$\begin{aligned}
\ddot{\underset{\sim}{h}}^k = {} & \\
& \frac{\partial}{(\partial \underset{\sim}{\vec{u}}^j)^T}\left(\frac{\partial \underset{\sim}{h}^k}{(\partial \underset{\sim}{\vec{u}}^j)^T} \cdot \underset{\sim}{\vec{v}}^j\right) \cdot \underset{\sim}{\vec{v}}^j + \frac{\partial}{\partial t}\left(\frac{\partial \underset{\sim}{h}^k}{(\partial \underset{\sim}{\vec{u}}^j)^T}\right) \cdot \underset{\sim}{\vec{v}}^j + \\
& \frac{\partial}{(\partial \underset{\sim}{\vec{u}}^{j+1})^T}\left(\frac{\partial \underset{\sim}{h}^k}{(\partial \underset{\sim}{\vec{u}}^{j+1})^T} \cdot \underset{\sim}{\vec{v}}^{j+1}\right) \cdot \underset{\sim}{\vec{v}}^{j+1} + \frac{\partial}{\partial t}\left(\frac{\partial \underset{\sim}{h}^k}{(\partial \underset{\sim}{\vec{u}}^{j+1})^T}\right) \cdot \underset{\sim}{\vec{v}}^{j+1} + \\
& \frac{\partial}{(\partial \underset{\sim}{\vec{u}}^{j+1})^T}\left(\frac{\partial \underset{\sim}{h}^k}{(\partial \underset{\sim}{\vec{u}}^j)^T} \cdot \underset{\sim}{\vec{v}}^j\right) \cdot \underset{\sim}{\vec{v}}^{j+1} + \frac{\partial}{(\partial \underset{\sim}{\vec{u}}^j)^T}\left(\frac{\partial \underset{\sim}{h}^k}{(\partial \underset{\sim}{\vec{u}}^{j+1})^T} \cdot \underset{\sim}{\vec{v}}^{j+1}\right) \cdot \underset{\sim}{\vec{v}}^j + \\
& \frac{\partial \underset{\sim}{h}^k}{(\partial \underset{\sim}{\vec{u}}^j)^T} \cdot \dot{\underset{\sim}{\vec{v}}}^j + \frac{\partial \underset{\sim}{h}^k}{(\partial \underset{\sim}{\vec{u}}^{j+1})^T} \cdot \dot{\underset{\sim}{\vec{v}}}^{j+1} + \frac{\partial^2 \underset{\sim}{h}^k}{\partial t^2} = \vec{0}
\end{aligned} \tag{4.11}$$

Note that Eq. (4.11) is of the form

$$\frac{\partial \underset{\sim}{h}^k}{(\partial \underset{\sim}{\vec{u}}^j)^T} \cdot \dot{\underset{\sim}{\vec{v}}}^j + \frac{\partial \underset{\sim}{h}^k}{(\partial \underset{\sim}{\vec{u}}^{j+1})^T} \cdot \dot{\underset{\sim}{\vec{v}}}^{j+1} = r\underset{\sim}{h}s(\underset{\sim}{\vec{u}}^j, \underset{\sim}{\vec{u}}^{j+1}, \underset{\sim}{\vec{v}}^j, \underset{\sim}{\vec{v}}^{j+1}, t) \tag{4.12}$$

where the right hand side $r\underset{\sim}{h}s(\underset{\sim}{\vec{u}}^j, \underset{\sim}{\vec{u}}^{j+1}, \underset{\sim}{\vec{v}}^j, \underset{\sim}{\vec{v}}^{j+1}, t)$ can be deduced from Eq. (4.11).

In essence, all global ingredients for a solvable system of equations are present by combining Eqs. (4.5) and (4.12). An extensive and detailed working out of the recursive technique as described in Chapter 3, applied to the above equations for closed loops is beyond the scope of this report. For literature in which the equations for closed loops have been worked out elaborately, the reader is referred to the works of Bae & Haug (1987-88) and Kim & Haug (1989).

4.3 Systems with flexible bodies

Flexible bodies are mostly handled with modal approximations in literature. In the case of systems with flexible bodies, the equations of motion according to d'Alembert's principle of virtual work have the same form, but differ as far as the contents of columns and matrices is concerned.

$$\sum_{i=1}^{nb} \left[(\delta \underset{\sim}{\vec{u}}^{i})^{T} \cdot (\underset{\sim}{\mathrm{M}}^{i} \cdot \dot{\underset{\sim}{\vec{v}}}^{i} - \underset{\sim}{\vec{Q}}^{i}) \right] = 0 \qquad \forall \text{ kinematically admissible } \delta \underset{\sim}{\vec{u}}^{i} \tag{4.13}$$

with

$$\delta \underset{\sim}{\vec{u}}^{i} = \begin{bmatrix} \delta \vec{r}^{i} \\ \delta \vec{\pi}^{i} \\ \delta \vec{\alpha}^{i} \end{bmatrix} \tag{4.14}$$

$$\underset{\sim}{\mathrm{M}}^{i} = \underset{\sim}{\mathrm{M}}^{i}(\mathbf{P}, \vec{\alpha}^{i}) \tag{4.15}$$

$$\dot{\underset{\sim}{\vec{v}}}^{i} = \begin{bmatrix} \ddot{\vec{r}}^{i} \\ \dot{\vec{\omega}}^{i} \\ \ddot{\vec{\alpha}}^{i} \end{bmatrix} \tag{4.16}$$

$$\underset{\sim}{\vec{Q}}^{i} = \underset{\sim}{\vec{Q}}^{i}(\mathbf{P}, \dot{\vec{\alpha}}^{i}, \vec{\alpha}^{i}) \tag{4.17}$$

Extra unknowns in the form of vector $\vec{\alpha}^{i}$ have entered the system equations (Eq. (4.13)) in comparison to the system equations in previous sections. Vector $\vec{\alpha}^{i}$ contains modal coordinates of deformable body B^{i}, and $\mathbf{P}$ is a rotation tensor that depends on orientation generalized coordinates. For an elaborate development of the recursive technique applied to flexible multibody systems, the reader is referred to, e.g., Kim & Haug (1988), Kim (1988), and Kim & Haug (1989). These works are limited to small-deformation linear elastic structural theory, even when the flexible bodies undergo large translations and rotations.

Chapter 5

Conclusions

The recursive technique as described in this report has not only been applied to the kinematical part of the analysis, but it is extended to the dynamical part as well, which yields several advantages, but also a presumable disadvantage.

Inherent in the recursive formulation is the use of the relative description in which relative generalized coordinates are used. A merit of the use of relative generalized coordinates is that they automatically keep the kinematic constraints intact between bodies. This means that no extra algebraic equations need to be added to the set of ordinary differential equations of the multibody system, as is the case when the global description is used.

Furthermore, the most important advantage of the described recursive technique is the need to invert only small mass matrices, combined with backward substitution in order to obtain all quantities of interest concerning the configuration of the multibody system. This is less computation intensive than the more common inversion of the large mass matrix of the total multibody system.

An advantage not dealt with in this report is the possibility of parallel processing. In case of a computer with multiple processors, computations on several bodies in different chains in the multibody system can be executed simultaneously. The independent use of processors is allowed due to the uncoupled character of the equations of motion of the bodies in the multibody system, thanks to the use of the recursive formulation.

The recursive technique does not only have advantages though. A presumable drawback of the recursive technique, already mentioned in the introductary chapter, but not discussed in the rest of this report, is the possible accumulation of errors. A small error in the base body kinematics could lead to a large error in the kinematics of bodies further up in the chain. This might lead to a kind of snowball effect in the case of large chains of bodies. It must be remarked that this drawback is only presumed: it has not been demonstrated numerically yet.

The works of Haug c.s. have formed the basis of Chapters 3 and 4. However, some changes and additions have been made.

In Chapter 3, the use of Lagrange multipliers is avoided in the case where the base body is kinematically constrained with respect to inertial space, by considering inertial space as a normal body so that relative coordinates between base body and inertial space can be used. The avoidance of Lagrange multipliers has the advantage that no extra acceleration equations are needed to complete the solvable system. Acceleration equations are algebraic and when coupled with ordinary differential equations, numerical methods for differential/algebraic equations have to be used which correspond to more complicated integration schemes.

In Sect. 4.1, slight modifications of the system equations in case of kinematically driven joint constraints are reported. Equation forms for closed loop systems and systems with flexible bodies are considered very briefly. References in literature are provided.

Throughout most of this report, a so-called coordinate-free way of notation is used. In the works of Haug c.s., vectors are represented by columns, where the information about representations with respect to what vector bases are used, is omitted. In this report, each column representation of a vector is accompanied with the vector base with respect to which the vector is represented. This prevents confusion and avoids notation errors.

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Appendix A Kane's equations

Prior to discussing the equations of motion, some terms according to Kane's nomenclature have to be explained. Suppose a system has n generalized coordinates $q_1,\dots,q_n$ that are independent. Now, n so–called generalized speeds $u_1,\dots,u_n$ are introduced as linear combinations of $\dot{q}_1,\dots,\dot{q}_n$ by means of equations of the following form:

$$\underset{\sim}{u} = \underline{V}\,\underset{\sim}{\dot{q}} + \underset{\sim}{W} \tag{A.1}$$

where

$\underset{\sim}{u} = \underset{\sim}{u}(\underset{\sim}{q},\underset{\sim}{\dot{q}},t)$ is a column with generalized speeds.

$\underline{V} = \underline{V}(\underset{\sim}{q},t)$ is a square (nxn) coefficient matrix.

$\underset{\sim}{W} = \underset{\sim}{W}(\underset{\sim}{q},t)$ is a column with n components.

$\underline{V}$ and $\underset{\sim}{W}$ are chosen in such a way that Equation (A.1) can be solved uniquely for $\dot{q}_1,\dots,\dot{q}_n$. In this case, the linear velocity $\vec{v}^j$ of an arbitrary point P^j in a body and the angular velocity $\vec{\omega}$ of that body can always be expressed uniquely as a linear function of the generalized speeds $u_1,\dots,u_n$:

$$\vec{v}^j = \vec{v}_1^j u_1 + \dots + \vec{v}_n^j u_n \tag{A.2}$$

$$\vec{\omega} = \vec{\omega}_1 u_1 + \dots + \vec{\omega}_n u_n \tag{A.3}$$

Vectors that are the coefficients of the i^{th} generalized speed u_i in Equations (A.2) and (A.3) are called the i^{th} partial velocity $\vec{v}_i^j$ of the chosen point P^j in the body and the i^{th} partial angular velocity $\vec{\omega}_i$ of that body, respectively.

Kane's equations can be derived directly from the principle of virtual power, which is a slight modification of the principle of virtual work:

$$\int_{\Omega_0} (\vec{F}^j - \rho_0 \dot{\vec{v}}^j) \cdot \delta\vec{v}^j \, d\Omega_0 = 0 \tag{A.4}$$

$$\forall \text{ virtual velocity fields } \delta\vec{v}^j$$

in which

Ω_0 is the body volume in the reference configuration.

$\vec{F}^j$ is the resultant of all applied and constraint forces per unit of body volume, with respect to the arbitrary point P^j in the body.

ρ_0 is the mass density in the reference configuration.

$\dot{\vec{v}}^j$ is the time derivative of the linear velocity vector of point P^j in the body.

$\delta\vec{v}^j$ denotes the virtual velocity vector of point P^j in the body.

Constraint forces will cancel out due to the principle of virtual work.
Expressions for the time derivative and the variation of the velocity vector will now be generated. Suppose $\vec{r}^j$ is the position vector of an arbitrary point P^j in a body. Then its time derivative can be expressed as follows (using the regularity of matrix $\underline{V}$ in Equation (A.1)):

$$\begin{aligned} \vec{v}^j = \dot{\vec{r}}^j &= \left(\frac{\partial \vec{r}^j}{\partial \underset{\sim}{q}}\right)^T \dot{\underset{\sim}{q}} + \frac{\partial \vec{r}^j}{\partial t} \\ &= \left(\frac{\partial \vec{r}^j}{\partial \underset{\sim}{q}}\right)^T \underline{V}^{-1} \underset{\sim}{u} + \frac{\partial \vec{r}^j}{\partial t} - \left(\frac{\partial \vec{r}^j}{\partial \underset{\sim}{q}}\right)^T \underline{V}^{-1} \underset{\sim}{W} \end{aligned} \tag{A.5}$$

or concisely:

$$\vec{v}^j = (\vec{\underset{\sim}{X}}^j)^T \underset{\sim}{u} + \vec{Y}^j \tag{A.6}$$

where

$$(\vec{\underset{\sim}{X}}^j)^T = \left(\frac{\partial \vec{r}^j}{\partial \underset{\sim}{q}}\right)^T \underline{V}^{-1} \tag{A.7}$$

$$\vec{Y}^j = \frac{\partial \vec{r}^j}{\partial t} - \left(\frac{\partial \vec{r}^j}{\partial \underset{\sim}{q}}\right)^T \underline{V}^{-1} \underset{\sim}{W} \tag{A.8}$$

The time derivative and the variation of the velocity vector are as follows:

$$\dot{\vec{v}}^j = (\vec{\underset{\sim}{X}}^j)^T \dot{\underset{\sim}{u}} + \vec{Z}^j \tag{A.9}$$

and

$$\delta\vec{v}^j = (\vec{\underset{\sim}{X}}^j)^T \delta\underset{\sim}{u} \tag{A.10}$$

with

$$\vec{Z}^j = (\dot{\vec{\underset{\sim}{X}}}^j)^T \underset{\sim}{u} + \dot{\vec{Y}}^j \tag{A.11}$$

In Equation (A.10), $(\vec{\underset{\sim}{X}}^j)^T$ includes the partial velocities:

$$(\vec{\underset{\sim}{X}}^j)^T = [\ \vec{v}_1^j, \ldots, \vec{v}_n^j\ ] \tag{A.12}$$

Back to the principle of virtual power in Equation (A.4), and substituting Equation (A.10) into it:

$$\int_{\Omega_0} (\vec{F}^j - \rho_0 \dot{\vec{v}}^j) \cdot \vec{\underset{\sim}{X}}^T \delta\underset{\sim}{u}\ d\Omega_0 = 0 \tag{A.13}$$

$$\forall \text{ virtual generalized speeds } \delta\underset{\sim}{u}$$

Because of the unique relationship between the n independent time derivatives of the generalized coordinates $(\dot{q}_1, \ldots, \dot{q}_n)$ on the one hand, and the n generalized speeds $u_1, \ldots, u_n$ on the other hand (see Equation (A.1) with the regular matrix $\underline{V}$), the generalized speeds (and their variations) must also be independent. With this in mind, Equation (A.13) results in Kane's equations:

$$K_i + K_i^* = 0 \qquad\qquad (i=1,\ldots,n) \tag{A.14}$$

with "generalized active forces":

$$K_i = \int_{\Omega_0} \vec{F}^j \cdot \vec{v}_i^j\ d\Omega_0 \tag{A.15}$$

and "generalized inertia forces":

$$K_i^* = \int_{\Omega_0} (-\rho_0 \dot{\vec{v}}^j) \cdot \vec{v}_i^j \, d\Omega_0 \tag{A.16}$$

or in case of a discrete number of mass points np:

$$K_i = \sum_{j=1}^{np} \vec{F}^j \cdot \vec{v}_i^j \tag{A.17}$$

$$K_i^* = \sum_{j=1}^{np} (-m^j \dot{\vec{v}}^j) \cdot \vec{v}_i^j \tag{A.18}$$

In the case of a rigid body in the system, Equation (A.17) can be modified as follows. If a set of contact and/or body forces acting on a rigid body of the system is replaced with a combination of torque $\vec{T}$ together with a force $\vec{S}$ applied at a given point of the body, then the contribution of this set of forces to K_i is given by $\vec{\omega}_i \cdot \vec{T} + \vec{v}_i \cdot \vec{S}$, where $\vec{\omega}_i$ is now the i^{th} partial angular velocity of the rigid body, and $\vec{v}_i$ is the i^{th} partial velocity of the point of application of $\vec{S}$.

Appendix B

Relations between rotation tensors, axial vectors and angular velocities in the flexible and the rigid case

Consider the system in Fig. B.1 with deformable bodies B^i and B^j, interconnected by an arbitrary joint.

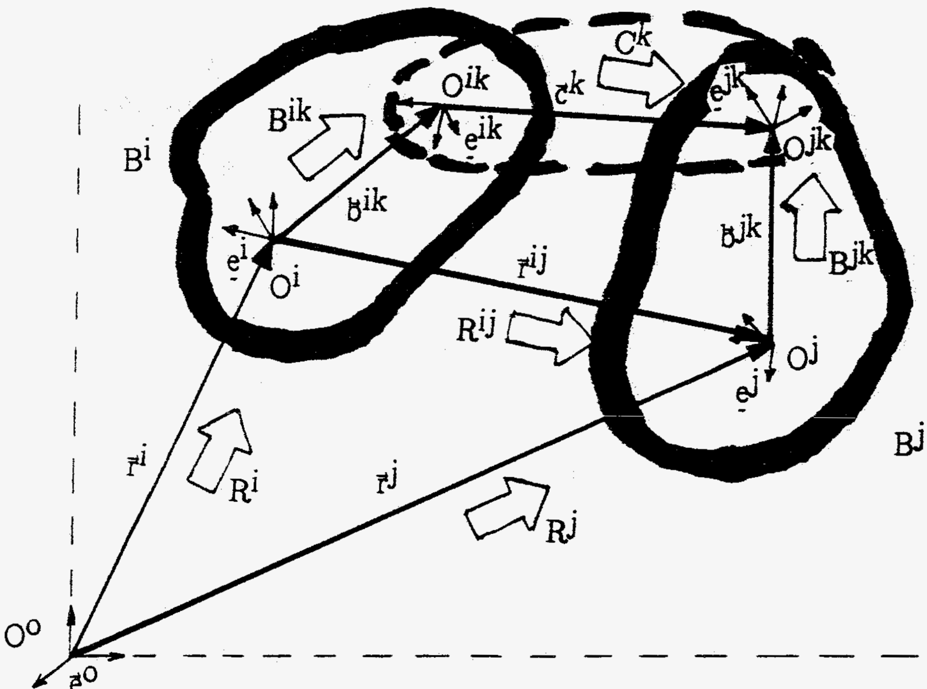

Figure B.1: *Arbitrary joint between two deformable bodies*

Let $\mathbf{R}^i$, $\mathbf{B}^{ik}$, $\mathbf{C}^k$, $\mathbf{B}^{jk}$, $\mathbf{R}^j$, and their corresponding matrix representations $\underset{\sim}{R}^i$, $\underset{\sim}{B}^{ik}$, $\underset{\sim}{C}^k$, $\underset{\sim}{B}^{jk}$, and $\underset{\sim}{R}^j$ be defined as follows

$$(\underset{\sim}{\vec{e}}^{\,i})^T = \mathbf{R}^i \cdot (\underset{\sim}{\vec{e}}^{\,0})^T = (\underset{\sim}{\vec{e}}^{\,0})^T \underset{\sim}{R}^i \tag{B.1}$$

$$(\underset{\sim}{\vec{e}}^{ik})^T = \mathbf{B}^{ik} \cdot (\underset{\sim}{\vec{e}}^{i})^T = (\underset{\sim}{\vec{e}}^{i})^T \underline{B}^{ik} \tag{B.2}$$

$$(\underset{\sim}{\vec{e}}^{i})^T = \mathbf{C}^{k} \cdot (\underset{\sim}{\vec{e}}^{ik})^T = (\underset{\sim}{\vec{e}}^{ik})^T \underline{C}^{k} \tag{B.3}$$

$$(\underset{\sim}{\vec{e}}^{jk})^T = \mathbf{B}^{jk} \cdot (\underset{\sim}{\vec{e}}^{j})^T = (\underset{\sim}{\vec{e}}^{j})^T \underline{B}^{jk} \tag{B.4}$$

$$(\underset{\sim}{\vec{e}}^{i})^T = \mathbf{R}^{j} \cdot (\underset{\sim}{\vec{e}}^{0})^T = (\underset{\sim}{\vec{e}}^{0})^T \underline{R}^{j} \tag{B.5}$$

and let $\vec{\omega}^{\,i}$ and $\vec{\omega}^{\,j}$ be defined as

$$\dot{\mathbf{R}}^{i} \cdot (\mathbf{R}^{i})^c \cdot \vec{a} = \vec{\omega}^{\,i} * \vec{a} \qquad \forall \vec{a} \tag{B.6}$$

$$\dot{\mathbf{R}}^{j} \cdot (\mathbf{R}^{j})^c \cdot \vec{a} = \vec{\omega}^{\,j} * \vec{a} \qquad \forall \vec{a} \tag{B.7}$$

With the definition in Eq. (B.6), the time derivative of $\underset{\sim}{\vec{e}}^{i}$ can be written as follows:

$$\begin{aligned}
(\dot{\underset{\sim}{\vec{e}}}^{i})^T &= \dot{\mathbf{R}}^{i} \cdot (\underset{\sim}{\vec{e}}^{0})^T \\
&= \dot{\mathbf{R}}^{i} \cdot (\mathbf{R}^{i})^c \cdot \mathbf{R}^{i} \cdot (\underset{\sim}{\vec{e}}^{0})^T \\
&= \vec{\omega}^{\,i} * (\underset{\sim}{\vec{e}}^{i})^T
\end{aligned} \tag{B.8}$$

For the time derivative of $\underset{\sim}{\vec{e}}^{ik}$, according to Eq. (B.2), we may write:

$$\begin{aligned}
(\dot{\underset{\sim}{\vec{e}}}^{ik})^T &= \frac{d}{dt}(\mathbf{B}^{ik} \cdot \mathbf{R}^{i} \cdot (\underset{\sim}{\vec{e}}^{0})^T) \\
&= (\dot{\mathbf{B}}^{ik} \cdot \mathbf{R}^{i} + \mathbf{B}^{ik} \cdot \dot{\mathbf{R}}^{i}) \cdot (\underset{\sim}{\vec{e}}^{0})^T \\
&= (\dot{\mathbf{B}}^{ik} \cdot (\mathbf{B}^{ik})^c \cdot \mathbf{B}^{ik} \cdot \mathbf{R}^{i} + \\
&\quad + \mathbf{B}^{ik} \cdot \dot{\mathbf{R}}^{i} \cdot (\mathbf{R}^{i})^c \cdot (\mathbf{B}^{ik})^c \cdot \mathbf{B}^{ik} \cdot \mathbf{R}^{i}) \cdot (\underset{\sim}{\vec{e}}^{0})^T \\
&= (\dot{\mathbf{B}}^{ik} \cdot (\mathbf{B}^{ik})^c + \mathbf{B}^{ik} \cdot \dot{\mathbf{R}}^{i} \cdot (\mathbf{R}^{i})^c \cdot (\mathbf{B}^{ik})^c) \cdot (\underset{\sim}{\vec{e}}^{ik})^T \\
&= (\dot{\mathbf{R}}^{i} \cdot (\mathbf{R}^{i})^c + \dot{\mathbf{B}}^{ik} \cdot (\mathbf{B}^{ik})^c + \\
&\quad + \mathbf{B}^{ik} \cdot \dot{\mathbf{R}}^{i} \cdot (\mathbf{R}^{i})^c \cdot (\mathbf{B}^{ik})^c - \dot{\mathbf{R}}^{i} \cdot (\mathbf{R}^{i})^c) \cdot (\underset{\sim}{\vec{e}}^{ik})^T
\end{aligned} \tag{B.9}$$

In the first term $\dot{\mathbf{R}}^{i} \cdot (\mathbf{R}^{i})^c$, the absolute angular velocity vector $\vec{\omega}^{\,i}$ of body B^i is recognized. Physically, the remaining terms represent the angular velocity of $\underset{\sim}{\vec{e}}^{ik}$ relative to $\underset{\sim}{\vec{e}}^{i}$. It can be shown that these remaining terms can be formulated in a more compact form:

$$\dot{\mathbf{B}}^{ik} \cdot (\mathbf{B}^{ik})^c + \mathbf{B}^{ik} \cdot \dot{\mathbf{R}}^{i} \cdot (\mathbf{R}^{i})^c \cdot (\mathbf{B}^{ik})^c - \dot{\mathbf{R}}^{i} \cdot (\mathbf{R}^{i})^c = (\underset{\sim}{\vec{e}}^{i})^T \dot{\underline{B}}^{ik} (\underline{B}^{ik})^T \underset{\sim}{\vec{e}}^{i} \tag{B.10}$$

Proof of Eq. (B.10):

$$\begin{aligned}
\dot{\mathbf{B}}^{ik} &= (\dot{\underset{\sim}{\vec{e}}}^{i})^T \underline{B}^{ik} \underset{\sim}{\vec{e}}^{i} + (\underset{\sim}{\vec{e}}^{i})^T \dot{\underline{B}}^{ik} \underset{\sim}{\vec{e}}^{i} + (\underset{\sim}{\vec{e}}^{i})^T \underline{B}^{ik} \dot{\underset{\sim}{\vec{e}}}^{i} \\
&= (\underset{\sim}{\vec{e}}^{i})^T (\underline{R}^{i})^T \dot{\underline{R}}^{i} \underline{B}^{ik} \underset{\sim}{\vec{e}}^{i} + (\underset{\sim}{\vec{e}}^{i})^T \dot{\underline{B}}^{ik} \underset{\sim}{\vec{e}}^{i} + (\underset{\sim}{\vec{e}}^{i})^T \underline{B}^{ik} (\dot{\underline{R}}^{i})^T \underline{R}^{i} \underset{\sim}{\vec{e}}^{i} \\
&= (\underset{\sim}{\vec{e}}^{i})^T \left((\underline{R}^{i})^T \dot{\underline{R}}^{i} \underline{B}^{ik} + \dot{\underline{B}}^{ik} + \underline{B}^{ik} (\dot{\underline{R}}^{i})^T \underline{R}^{i} \right) \underset{\sim}{\vec{e}}^{i}
\end{aligned} \tag{B.11}$$

$$(\mathbf{B}^{ik})^c = (\vec{\underset{\sim}{e}}^{\,i})^T (\underline{B}^{ik})^T \vec{\underset{\sim}{e}}^{\,i} \tag{B.12}$$

$$\dot{\mathbf{R}}^i = (\vec{\underset{\sim}{e}}^{\,i})^T \left((\underline{R}^i)^T \dot{\underline{R}}^i \underline{R}^i + \dot{\underline{R}}^i + \underline{R}^i (\dot{\underline{R}}^i)^T \underline{R}^i \right) \vec{\underset{\sim}{e}}^{\,i} \tag{B.13}$$

Substitution of Eqs. (B.11) through (B.13) into the left hand side of Eq. (B.10) and omitting multiplication by $(\vec{\underset{\sim}{e}}^{\,i})^T$ and $\vec{\underset{\sim}{e}}^{\,i}$ for convenience:

$$\begin{aligned}
&\left((\underline{R}^i)^T \dot{\underline{R}}^i \underline{B}^{ik} + \dot{\underline{B}}^{ik} + \underline{B}^{ik} (\dot{\underline{R}}^i)^T \underline{R}^i \right) (\underline{B}^{ik})^T + \\
&\underline{B}^{ik} \left((\underline{R}^i)^T \dot{\underline{R}}^i \underline{R}^i + \dot{\underline{R}}^i + \underline{R}^i (\dot{\underline{R}}^i)^T \underline{R}^i \right) (\underline{R}^i)^T (\underline{B}^{ik})^T - \\
&\left((\underline{R}^i)^T \dot{\underline{R}}^i \underline{R}^i + \dot{\underline{R}}^i + \underline{R}^i (\dot{\underline{R}}^i)^T \underline{R}^i \right) (\underline{R}^i)^T = \\
&(\underline{R}^i)^T \dot{\underline{R}}^i + \dot{\underline{B}}^{ik} (\underline{B}^{ik})^T + \\
&\underline{B}^{ik} \left[(\dot{\underline{R}}^i)^T \underline{R}^i + (\underline{R}^i)^T \dot{\underline{R}}^i + \dot{\underline{R}}^i (\underline{R}^i)^T + \underline{R}^i (\dot{\underline{R}}^i)^T \right] (\underline{B}^{ik})^T + \\
&-(\underline{R}^i)^T \dot{\underline{R}}^i - \left[(\dot{\underline{R}}^i \underline{R}^i)^T + \underline{R}^i \dot{\underline{R}}^i)^T \right] = \\
&\dot{\underline{B}}^{ik} (\underline{B}^{ik})^T
\end{aligned} \tag{B.14}$$

Q.E.D.

With this result, the angular velocity vector $\vec{\omega}^{\,ik}$ of $\vec{\underset{\sim}{e}}^{\,ik}$ relative to $\vec{\underset{\sim}{e}}^{\,i}$ is defined as:

$$(\vec{\underset{\sim}{e}}^{\,i})^T \dot{\underline{B}}^{ik} (\underline{B}^{ik})^T \vec{\underset{\sim}{e}}^{\,i} \cdot \vec{a} = \vec{\omega}^{\,ik} * \vec{a} \qquad \forall \vec{a} \tag{B.15}$$

which turns Eq. (B.9) into the following compact equation form:

$$(\dot{\vec{\underset{\sim}{e}}}^{\,ik})^T = (\vec{\omega}^{\,i} + \vec{\omega}^{\,ik}) * (\vec{\underset{\sim}{e}}^{\,ik})^T \tag{B.16}$$

Similar to Eq. (B.15), the angular velocity vectors $\vec{\omega}^{\,k}$ (of $\vec{\underset{\sim}{e}}^{\,jk}$ relative to $\vec{\underset{\sim}{e}}^{\,ik}$) and $\vec{\omega}^{\,jk}$ (of $\vec{\underset{\sim}{e}}^{\,jk}$ relative to $\vec{\underset{\sim}{e}}^{\,j}$) are defined as follows:

$$(\vec{\underset{\sim}{e}}^{\,ik})^T \dot{\underline{C}}^k (\underline{C}^k)^T \vec{\underset{\sim}{e}}^{\,ik} \cdot \vec{a} = \vec{\omega}^{\,k} * \vec{a} \qquad \forall \vec{a} \tag{B.17}$$

$$(\vec{\underset{\sim}{e}}^{\,j})^T \dot{\underline{B}}^{jk} (\underline{B}^{jk})^T \vec{\underset{\sim}{e}}^{\,j} \cdot \vec{a} = \vec{\omega}^{\,jk} * \vec{a} \qquad \forall \vec{a} \tag{B.18}$$

With these definitions, an expression for the angular velocity vector $\vec{\omega}^{\,ij}$ of $\vec{\underset{\sim}{e}}^{\,j}$ relative to $\vec{\underset{\sim}{e}}^{\,i}$, defined as

$$\vec{\omega}^{\,ij} = \vec{\omega}^{\,j} - \vec{\omega}^{\,i} \tag{B.19}$$

can be found rather easily by considering the time derivatives of the following two alternative descriptions of $\vec{\underset{\sim}{e}}^{\,jk}$:

$$(\vec{\underset{\sim}{e}}^{\,jk})^T = (\vec{\underset{\sim}{e}}^{\,ik})^T \underline{C}^k \tag{B.20}$$

$$(\underset{\sim}{\vec{e}}^{jk})^T = (\underset{\sim}{\vec{e}}^{j})^T \underline{B}^{jk} \tag{B.21}$$

Looking at Eq. (B.20), the time derivative of $\underset{\sim}{\vec{e}}^{jk}$ results as follows:

$$\begin{aligned} (\dot{\underset{\sim}{\vec{e}}}^{jk})^T &= (\dot{\underset{\sim}{\vec{e}}}^{ik})^T \underline{C}^k + (\underset{\sim}{\vec{e}}^{ik})^T \dot{\underline{C}}^k \\ &= (\vec{\omega}^{\,i} + \vec{\omega}^{\,ik}) * (\underset{\sim}{\vec{e}}^{ik})^T \underline{C}^k + (\underset{\sim}{\vec{e}}^{ik})^T \dot{\underline{C}}^k (\underline{C}^k)^T \underset{\sim}{\vec{e}}^{ik} \cdot (\underset{\sim}{\vec{e}}^{ik})^T \underline{C}^k \\ &= (\vec{\omega}^{\,i} + \vec{\omega}^{\,ik} + \vec{\omega}^{\,k}) * (\underset{\sim}{\vec{e}}^{jk})^T \end{aligned} \tag{B.22}$$

And by Eq. (B.21), the following result is obtained:

$$\begin{aligned} (\dot{\underset{\sim}{\vec{e}}}^{jk})^T &= (\dot{\underset{\sim}{\vec{e}}}^{j})^T \underline{B}^{jk} + (\underset{\sim}{\vec{e}}^{j})^T \dot{\underline{B}}^{jk} \\ &= \vec{\omega}^{\,j} * (\underset{\sim}{\vec{e}}^{j})^T \underline{B}^{jk} + (\underset{\sim}{\vec{e}}^{j})^T \dot{\underline{B}}^{jk} (\underline{B}^{jk})^T \underset{\sim}{\vec{e}}^{j} \cdot (\underset{\sim}{\vec{e}}^{j})^T \underline{B}^{jk} \\ &= (\vec{\omega}^{\,j} + \vec{\omega}^{\,jk}) * (\underset{\sim}{\vec{e}}^{jk})^T \end{aligned} \tag{B.23}$$

Combining Eqs. (B.22) and (B.23), and substitution in Eq. (B.19) results in the following expression for $\vec{\omega}^{\,ij}$:

$$\vec{\omega}^{\,ij} = \vec{\omega}^{\,ik} + \vec{\omega}^{\,k} - \vec{\omega}^{\,jk} \tag{B.24}$$

In the case that bodies B^i and B^j are rigid, the matrix representation of $\mathbf{B^{ik}}$ (with respect to either $\underset{\sim}{\vec{e}}^{i}$ or $\underset{\sim}{\vec{e}}^{ik}$) and the matrix representation of $\mathbf{B^{jk}}$ (with respect to either $\underset{\sim}{\vec{e}}^{j}$ or $\underset{\sim}{\vec{e}}^{jk}$) are time-independent

$$\dot{\underline{B}}^{ik} = \underline{O} \tag{B.25}$$

$$\dot{\underline{B}}^{jk} = \underline{O} \tag{B.26}$$

and according to Eqs. (B.15) and (B.18) the angular velocity vectors $\vec{\omega}^{\,ik}$ and $\vec{\omega}^{\,jk}$ are zero, as already expected physically:

$$\vec{\omega}^{\,ik} = \vec{0} \tag{B.27}$$

$$\vec{\omega}^{\,jk} = \vec{0} \tag{B.28}$$

And thus, in case of two rigid bodies B^i and B^j, the angular velocity vector $\vec{\omega}^{\,ij}$ of $\underset{\sim}{\vec{e}}^{j}$ relative to $\underset{\sim}{\vec{e}}^{i}$ results as follows:

$$\vec{\omega}^{\,ij} = \vec{\omega}^{\,k} \tag{B.29}$$

Appendix C

Kinematics example: two rigid bodies interconnected by a universal joint

In order to illustrate use of the equations presented in Section 3.3, a universal joint in the two body mechanism of Fig. C.1 is used. The equations derived in Section 3.3 with the indices i,j, and k will be used in the following with the values 1,2, and 1, respectively. Each joint reference frame in a body stays parallel to the centroidal reference frame in that body, so tensors $\mathbf{B}^{11}$ and $\mathbf{B}^{21}$ are identities ($\mathbf{I}$). Joint attachment points are unit distances from the centroidal reference frames for each body, to simplify vectors $\vec{b}^{11}$ and $\vec{b}^{21}$. Initially, all frames are parallel, for simplicity.

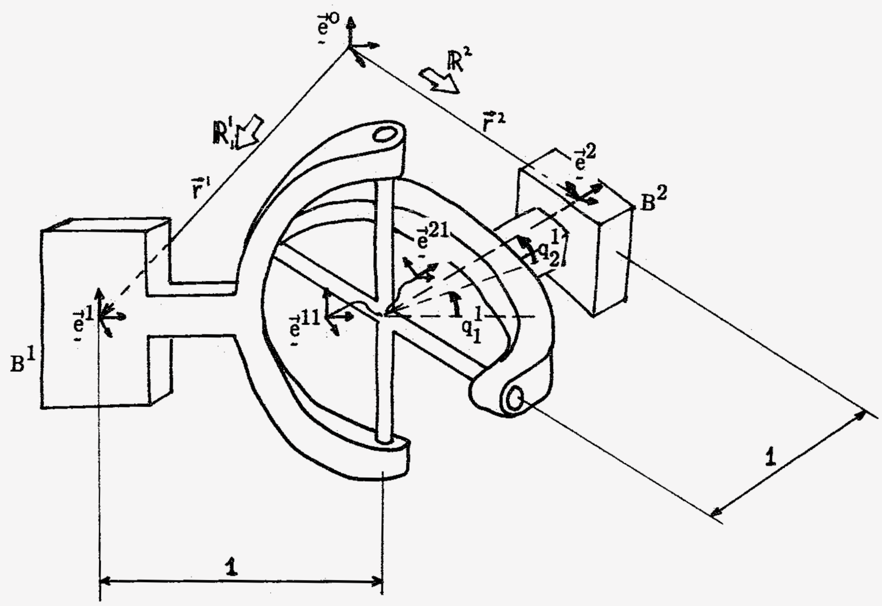

Figure C.1: *Universal joint*

The position and orientation of body B^2 can be expressed in terms of the position and orientation of body B^1 and relative coordinates q_1^1 and q_2^1, gathered in column $\underset{\sim}{q}^1$. The position of the centroid of body B^1 is represented by the absolute position vector $\tilde{\vec{r}}^1$. The orientation of body B^1 is determined by rotation tensor $\mathbf{R}^1$:

$$(\underset{\sim}{\vec{e}}^1)^T = \mathbf{R}^1 \cdot (\underset{\sim}{\vec{e}}^0)^T \tag{C.1}$$

with

$$\mathbf{R}^1 = (\underset{\sim}{\vec{e}}^1)^T \underset{\sim}{\vec{e}}^0 = (\underset{\sim}{\vec{e}}^0)^T \underset{\sim}{R}^1 \underset{\sim}{\vec{e}}^0 \tag{C.2}$$

where $\underset{\sim}{R}^1$ can be written in terms of Bryant angles α_1^1, α_2^1, and α_3^1, respectively (used abbreviations: $s\alpha_1 = \sin\alpha_1^1$, $s\alpha_2 = \sin\alpha_2^1$, $s\alpha_3 = \sin\alpha_3^1$, $c\alpha_1 = \cos\alpha_1^1$, $c\alpha_2 = \cos\alpha_2^1$, and $c\alpha_3 = \cos\alpha_3^1$):

$$\underset{\sim}{R}^1 = \begin{bmatrix} c\alpha_2 c\alpha_3 & -c\alpha_2 s\alpha_3 & s\alpha_2 \\ c\alpha_1 s\alpha_3 + s\alpha_1 s\alpha_2 c\alpha_3 & c\alpha_1 c\alpha_3 - s\alpha_1 s\alpha_2 s\alpha_3 & -s\alpha_1 c\alpha_2 \\ s\alpha_1 s\alpha_3 - c\alpha_1 s\alpha_2 c\alpha_3 & s\alpha_1 c\alpha_3 + c\alpha_1 s\alpha_2 s\alpha_3 & c\alpha_1 c\alpha_2 \end{bmatrix} \tag{C.3}$$

It is noted that singularity problems, inherently encountered when using Tait-Bryan angles (i.e. when $\alpha_2 = \frac{\pi}{2} \pm n\pi$ $(n = 0, 1, \ldots)$), are not taken into consideration in this example.

The change of orientation between the joint attachment frames in bodies B^1 and B^2 is determined by tensor $\mathbf{C}^1$:

$$(\underset{\sim}{\vec{e}}^{21})^T = \mathbf{C}^1 \cdot (\underset{\sim}{\vec{e}}^{11})^T \tag{C.4}$$

with

$$\mathbf{C}^1 = (\underset{\sim}{\vec{e}}^{21})^T \underset{\sim}{\vec{e}}^{11} = (\underset{\sim}{\vec{e}}^{11})^T \underset{\sim}{C}^1 \underset{\sim}{\vec{e}}^{11} \tag{C.5}$$

where $\underset{\sim}{C}^1$ can be written in terms of the rotational generalized coordinates between bodies B^1 and B^2, q_1^1 and q_2^1 (used abbreviations: $s_1 = \sin q_1^1$, $s_2 = \sin q_2^1$, $c_1 = \cos q_1^1$, and $c_2 = \cos q_2^1$):

$$\underset{\sim}{C}^1 = \begin{bmatrix} c_2 & 0 & s_2 \\ s_1 s_2 & c_1 & -s_1 c_2 \\ -c_1 s_2 & s_1 & c_1 c_2 \end{bmatrix} \tag{C.6}$$

The absolute angular velocity vector $\vec{\omega}^1$ of body B^1 and the angular velocity vector $\vec{\omega}^{12}$ of body B^2 relative to body B^1 are defined as the axial vectors of tensors $\dot{\mathbf{R}}^1 \cdot (\mathbf{R}^1)^c$ and $(\underset{\sim}{\vec{e}}^{11})^T \dot{\underset{\sim}{C}}^1 (\underset{\sim}{C}^1)^T \underset{\sim}{\vec{e}}^{11}$, respectively:

$$\dot{\mathbf{R}}^1 \cdot (\mathbf{R}^1)^c \cdot \vec{a} = \vec{\omega}^1 * \vec{a} \qquad \forall \vec{a} \tag{C.7}$$

$$(\underset{\sim}{\vec{e}}^{11})^T \dot{\underset{\sim}{C}}^1 (\underset{\sim}{C}^1)^T \underset{\sim}{\vec{e}}^{11} \cdot \vec{a} = \vec{\omega}^{12} * \vec{a} \qquad \forall \vec{a} \tag{C.8}$$

where

$$
\begin{aligned}
\dot{\mathbf{R}}^1 \cdot (\mathbf{R}^1)^c &= (\vec{\underset{\sim}{e}}^0)^T \dot{\underset{\sim}{R}}^1 \vec{\underset{\sim}{e}}^0 \cdot (\vec{\underset{\sim}{e}}^0)^T (\underset{\sim}{R}^1)^T \vec{\underset{\sim}{e}}^0 \\
&= (\vec{\underset{\sim}{e}}^0)^T \dot{\underset{\sim}{R}}^1 (\underset{\sim}{R}^1)^T \vec{\underset{\sim}{e}}^0
\end{aligned} \tag{C.9}
$$

with

$$
\dot{\underset{\sim}{R}}^1 (\underset{\sim}{R}^1)^T = \begin{bmatrix} 0 & -\dot{\alpha}_2^1 s\alpha_1 - \dot{\alpha}_3^1 c\alpha_1 c\alpha_2 & \dot{\alpha}_2^1 c\alpha_1 - \dot{\alpha}_3^1 s\alpha_1 c\alpha_2 \\ \dot{\alpha}_2^1 s\alpha_1 + \dot{\alpha}_3^1 c\alpha_1 c\alpha_2 & 0 & -\dot{\alpha}_1^1 - \dot{\alpha}_3^1 s\alpha_2 \\ -\dot{\alpha}_2^1 c\alpha_1 + \dot{\alpha}_3^1 s\alpha_1 c\alpha_2 & \dot{\alpha}_1^1 + \dot{\alpha}_3^1 s\alpha_2 & 0 \end{bmatrix} \tag{C.10}
$$

and where

$$
(\vec{\underset{\sim}{e}}^{11})^T \dot{\underset{\sim}{C}}^1 (\underset{\sim}{C}^1)^T \vec{\underset{\sim}{e}}^{11} = (\vec{\underset{\sim}{e}}^{11})^T \begin{bmatrix} 0 & -\dot{q}_2^1 s_1 & \dot{q}_2^1 c_1 \\ \dot{q}_2^1 s_1 & 0 & -\dot{q}_1^1 \\ -\dot{q}_2^1 c_1 & \dot{q}_1^1 & 0 \end{bmatrix} \vec{\underset{\sim}{e}}^{11} \tag{C.11}
$$

The components in column ${}^0\underset{\sim}{u}$ of an arbitrary axial vector $\vec{u}$ with respect to the base vectors $\vec{\underset{\sim}{e}}^0$ can be recognized in the matrix representation of the corresponding skew-symmetrical tensor $\mathbf{U}$ with respect to the base vectors $\vec{\underset{\sim}{e}}^0$ in the following way (see e.g. Roberson & Schwertassek, 1988, page 49):

$$
\begin{aligned}
\vec{u} &= (\vec{\underset{\sim}{e}}^0)^T \, {}^0\underset{\sim}{u} = (\vec{\underset{\sim}{e}}^0)^T \begin{bmatrix} {}^0u_1 \\ {}^0u_2 \\ {}^0u_3 \end{bmatrix} \Rightarrow \\
\mathbf{U} &= (\vec{\underset{\sim}{e}}^0)^T \begin{bmatrix} 0 & -{}^0u_3 & {}^0u_2 \\ {}^0u_3 & 0 & -{}^0u_1 \\ -{}^0u_2 & {}^0u_1 & 0 \end{bmatrix} \vec{\underset{\sim}{e}}^0
\end{aligned} \tag{C.12}
$$

Therefore, the angular velocity vectors $\vec{\omega}^{\,1}$ and $\vec{\omega}^{\,12}$ can be written as follows:

$$
\vec{\omega}^{\,1} = (\vec{\underset{\sim}{e}}^0)^T \, {}^0\underset{\sim}{\omega}^1 = (\vec{\underset{\sim}{e}}^0)^T \begin{bmatrix} \dot{\alpha}_1^1 + \dot{\alpha}_3^1 s\alpha_2 \\ \dot{\alpha}_2^1 c\alpha_1 - \dot{\alpha}_3^1 s\alpha_1 c\alpha_2 \\ \dot{\alpha}_2^1 s\alpha_1 + \dot{\alpha}_3^1 c\alpha_1 c\alpha_2 \end{bmatrix} \tag{C.13}
$$

$$
\begin{aligned}
\vec{\omega}^{\,12} &= (\vec{\underset{\sim}{e}}^{11})^T \, {}^{11}\underset{\sim}{\omega}^{12} = (\vec{\underset{\sim}{e}}^{11})^T \begin{bmatrix} \dot{q}_1^1 \\ \dot{q}_2^1 c_1 \\ \dot{q}_2^1 s_1 \end{bmatrix} \\
&= (\vec{\underset{\sim}{e}}^0)^T \underset{\sim}{R}^1 \begin{bmatrix} 1 & 0 \\ 0 & c_1 \\ 0 & s_1 \end{bmatrix} \begin{bmatrix} \dot{q}_1^1 \\ \dot{q}_2^1 \end{bmatrix} = (\vec{\underset{\sim}{w}}^{\,1})^T \dot{\underset{\sim}{q}}^1
\end{aligned} \tag{C.14}
$$

with

$$
\vec{\underset{\sim}{w}}^{\,1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_1 & s_1 \end{bmatrix} (\underset{\sim}{R}^1)^T \vec{\underset{\sim}{e}}^0 \tag{C.15}
$$

And:

$$\vec{\omega}^{2} = \vec{\omega}^{1} + (\underset{\sim}{\vec{w}}^{1})^{T} \dot{\underset{\sim}{q}}^{1} \tag{C.16}$$

In order to look at translational velocities, the position vectors are considered first:

$$\vec{r}^{2} = \vec{r}^{1} + \vec{r}^{12} \tag{C.17}$$

where

$$\vec{r}^{12} = \vec{b}^{11} + \vec{c}^{1} - \vec{b}^{12} \tag{C.18}$$

Since the origins of the vector bases $\underset{\sim}{\vec{e}}^{11}$ and $\underset{\sim}{\vec{e}}^{21}$ coincide, vector $\vec{c}^{1}$ equals the zero vector:

$$\vec{c}^{1} = \vec{0} \tag{C.19}$$

Combining Eqs. (C.16) through (C.18), and taking the time derivative results in the following velocity equation:

$$\begin{aligned} \dot{\vec{r}}^{2} &= \dot{\vec{r}}^{1} + \dot{\vec{b}}^{11} - \dot{\vec{b}}^{21} \\ &= \dot{\vec{r}}^{1} + \vec{\omega}^{1} * (\vec{b}^{11} - \vec{b}^{21}) - (\underset{\sim}{\vec{w}}^{1})^{T} \dot{\underset{\sim}{q}}^{1} * \vec{b}^{21} \\ &= \dot{\vec{r}}^{1} - (\vec{b}^{11} - \vec{b}^{21}) * \vec{\omega}^{1} - (\underset{\sim}{\vec{w}}^{1})^{T} * \vec{b}^{21} \dot{\underset{\sim}{q}}^{1} \end{aligned} \tag{C.20}$$

Eqs. (C.16) and (C.20) may be organized in a matrix equation:

$$\underset{\sim}{\vec{v}}^{2} = \underline{\mathbf{A}}^{1} \cdot \underset{\sim}{\vec{v}}^{1} + \underline{\vec{B}}^{1} \dot{\underset{\sim}{q}}^{1} \tag{C.21}$$

where

$$\underset{\sim}{\vec{v}}^{i} = \begin{bmatrix} \dot{\vec{r}}^{i} \\ \vec{\omega}^{i} \end{bmatrix} \qquad (i = 1, 2) \tag{C.22}$$

$$\underline{\mathbf{A}}^{1} = \begin{bmatrix} \mathbf{I} & -\mathbf{Z}^{1} \\ \mathbf{O} & \mathbf{I} \end{bmatrix} \tag{C.23}$$

$$\underline{\vec{B}}^{1} = \begin{bmatrix} -(\underset{\sim}{\vec{w}}^{1})^{T} * \vec{b}^{21} \\ (\underset{\sim}{\vec{w}}^{1})^{T} \end{bmatrix} \tag{C.24}$$

$$\dot{\underset{\sim}{q}}^{1} = \begin{bmatrix} \dot{q}_{1}^{1} \\ \dot{q}_{2}^{1} \end{bmatrix} \tag{C.25}$$

Tensor $\mathbf{Z}^1$ in matrix $\underline{\mathbf{A}}^1$ in Eq. (C.23) can be derived from the second term in the right hand side of Eq. (C.20). But first, vectors $\vec{b}^{11}$ and $\vec{b}^{21}$ are evaluated. From now on, all vectors and tensors in this example will be worked out in matrix representation with respect to the inertial frame $\underset{\sim}{\vec{e}}^{0}$. In Fig. C.1, vector bases $\underset{\sim}{\vec{e}}^{1}$ and $\underset{\sim}{\vec{e}}^{2}$ are chosen in such a way that their third base vectors are parallel to $\vec{b}^{11}$ and $\vec{b}^{21}$, respectively:

$$\vec{b}^{11} = \vec{e}_3^{\,1} = (\underset{\sim}{\vec{e}}^{1})^T \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = (\underset{\sim}{\vec{e}}^{0})^T \underline{R}^1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = (\underset{\sim}{\vec{e}}^{0})^T \, {}^0\underset{\sim}{b}^{11} \tag{C.26}$$

$$\vec{b}^{21} = -\vec{e}_3^{\,2} = (\underset{\sim}{\vec{e}}^{2})^T \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} = (\underset{\sim}{\vec{e}}^{0})^T \underline{R}^1 \underline{C}^1 \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} = (\underset{\sim}{\vec{e}}^{0})^T \, {}^0\underset{\sim}{b}^{21} \tag{C.27}$$

These relations are substituted in the second term in the right hand side of Eq. (C.20) in order to obtain tensor $\mathbf{Z}^1$:

$$\begin{aligned}
(\vec{b}^{11} - \vec{b}^{21}) * \vec{\omega}^{\,1} &= (({}^0\underset{\sim}{b}^{11})^T - ({}^0\underset{\sim}{b}^{21})^T)\underset{\sim}{\vec{e}}^{0} * \vec{\omega}^{\,1} \\
&= (\begin{bmatrix} 0 & 0 & 1 \end{bmatrix} (\underline{R}^1)^T - \begin{bmatrix} 0 & 0 & -1 \end{bmatrix} (\underline{C}^1)^T (\underline{R}^1)^T)\underset{\sim}{\vec{e}}^{0} * \vec{\omega}^{\,1} \\
&= \underset{\sim}{z}^T \underset{\sim}{\vec{e}}^{0} * (\underset{\sim}{\vec{e}}^{0})^T \, {}^0\underset{\sim}{\omega}^1 \\
&= \underset{\sim}{z}^T \begin{bmatrix} \vec{0} & \vec{e}_3^{\,0} & -\vec{e}_2^{\,0} \\ -\vec{e}_3^{\,0} & \vec{0} & \vec{e}_1^{\,0} \\ \vec{e}_2^{\,0} & -\vec{e}_1^{\,0} & \vec{0} \end{bmatrix} {}^0\underset{\sim}{\omega}^1 \\
&= (\underset{\sim}{\vec{e}}^{0})^T \begin{bmatrix} 0 & -z_3 & z_2 \\ z_3 & 0 & -z_1 \\ -z_2 & z_1 & 0 \end{bmatrix} {}^0\underset{\sim}{\omega}^1 \\
&= (\underset{\sim}{\vec{e}}^{0})^T \begin{bmatrix} 0 & -z_3 & z_2 \\ z_3 & 0 & -z_1 \\ -z_2 & z_1 & 0 \end{bmatrix} \underset{\sim}{\vec{e}}^{0} \cdot (\underset{\sim}{\vec{e}}^{0})^T \, {}^0\underset{\sim}{\omega}^1 \\
&= \mathbf{Z}^1 \cdot \vec{\omega}^{\,1}
\end{aligned} \tag{C.28}$$

where

$$z_1 = s\alpha_2 + c\alpha_2 c\alpha_3 s_2 + c\alpha_2 s\alpha_3 s_1 c_2 + s\alpha_2 c_1 c_2 \tag{C.29}$$

$$z_2 = -s\alpha_1 c\alpha_2 + (c\alpha_1 s\alpha_3 + s\alpha_1 s\alpha_2 c\alpha_3) s_2 - (c\alpha_1 c\alpha_3 - s\alpha_1 s\alpha_2 s\alpha_3) s_1 c_2 - s\alpha_1 c\alpha_2 c_1 c_2 \tag{C.30}$$

$$z_3 = c\alpha_1 c\alpha_2 + (s\alpha_1 s\alpha_3 - c\alpha_1 s\alpha_2 c\alpha_3) s_2 - (s\alpha_1 c\alpha_3 + c\alpha_1 s\alpha_2 s\alpha_3) s_1 c_2 + c\alpha_1 c\alpha_2 c_1 c_2 \tag{C.31}$$

Matrix $\underline{\vec{B}}^{\,1}$ can also be worked out:

$$\underline{\vec{B}}^{\,1} = \begin{bmatrix} \vec{b}^{21} * (\underset{\sim}{\vec{w}}^{\,1})^T \\ (\underset{\sim}{\vec{w}}^{\,1})^T \end{bmatrix} \tag{C.32}$$

where in the lower part of $\underline{\vec{B}}^{\,1}$ (by using Eq. (C.15)):

$$\underset{\sim}{\vec{w}}^{\,1} = {}^0\underline{W}^1 \underset{\sim}{\vec{e}}^{0} \tag{C.33}$$

with

$$({}^0\underset{\sim}{W}^1)^T = \begin{bmatrix} c\alpha_2 c\alpha_3 & -c\alpha_2 s\alpha_3 c_1 + s\alpha_2 s_1 \\ c\alpha_1 s\alpha_3 + s\alpha_1 s\alpha_2 c\alpha_3 & (c\alpha_1 c\alpha_3 - s\alpha_1 s\alpha_2 s\alpha_3)c_1 - s\alpha_1 c\alpha_2 s_1 \\ s\alpha_1 s\alpha_3 - c\alpha_1 s\alpha_2 c\alpha_3 & (s\alpha_1 c\alpha_3 + c\alpha_1 s\alpha_2 s\alpha_3)c_1 + c\alpha_1 c\alpha_2 s_1 \end{bmatrix} \tag{C.34}$$

and in the upper part of $\vec{\underset{\sim}{B}}^1$:

$$\begin{aligned} \vec{b}^{21} * (\vec{\underset{\sim}{w}}^1)^T &= ({}^0\underset{\sim}{b}^{21})^T \vec{\underset{\sim}{e}}^0 * (\vec{\underset{\sim}{e}}^0)^T ({}^0\underset{\sim}{W}^1)^T = \\ &= (\vec{\underset{\sim}{e}}^0)^T \begin{bmatrix} 0 & -{}^0b_3^{21} & {}^0b_2^{21} \\ {}^0b_3^{21} & 0 & -{}^0b_1^{21} \\ -{}^0b_2^{21} & {}^0b_1^{21} & 0 \end{bmatrix} ({}^0\underset{\sim}{W}^1)^T \end{aligned} \tag{C.35}$$

with

$${}^0b_1^{21} = -c\alpha_2 c\alpha_3 s_2 - c\alpha_2 s\alpha_3 s_1 c_2 - s\alpha_2 c_1 c_2 \tag{C.36}$$

$${}^0b_2^{21} = -(c\alpha_1 s\alpha_3 + s\alpha_1 s\alpha_2 c\alpha_3)s_2 + (c\alpha_1 c\alpha_3 - s\alpha_1 s\alpha_2 s\alpha_3)s_1 c_2 + s\alpha_1 c\alpha_2 c_1 c_2 \tag{C.37}$$

$${}^0b_3^{21} = -(s\alpha_1 s\alpha_3 - c\alpha_1 s\alpha_2 c\alpha_3)s_2 + (s\alpha_1 c\alpha_3 + c\alpha_1 s\alpha_2 s\alpha_3)s_1 c_2 - c\alpha_1 c\alpha_2 c_1 c_2 \tag{C.38}$$

Virtual displacement relations follow from Eq. (3.32), as

$$\delta\vec{\underset{\sim}{v}}^2 = \underset{\sim}{\mathbf{A}}^1 \cdot \delta\vec{\underset{\sim}{v}}^1 + \vec{\underset{\sim}{B}}^1 \delta\underset{\sim}{q}^1 \tag{C.39}$$

with

$$\delta\vec{\underset{\sim}{v}}^i = \begin{bmatrix} \delta\vec{r}^i \\ \delta\vec{\pi}^i \end{bmatrix} \qquad (i = 1, 2) \tag{C.40}$$

$$\delta\underset{\sim}{q}^1 = \begin{bmatrix} \delta q_1^1 \\ \delta q_2^1 \end{bmatrix} \tag{C.41}$$

Accelerations of body B^2 can be obtained by taking the time derivative of Eq. (C.21):

$$\dot{\vec{\underset{\sim}{v}}}^2 = \underset{\sim}{\mathbf{A}}^1 \cdot \dot{\vec{\underset{\sim}{v}}}^1 + \vec{\underset{\sim}{B}}^1 \ddot{\underset{\sim}{q}}^1 + \vec{\underset{\sim}{D}}^1 \tag{C.42}$$

where

$$\vec{\underset{\sim}{D}}^1 = \dot{\underset{\sim}{\mathbf{A}}}^1 \cdot \vec{\underset{\sim}{v}}^1 + \dot{\vec{\underset{\sim}{B}}}^1 \dot{\underset{\sim}{q}}^1 \tag{C.43}$$

$$\dot{\underset{\sim}{\mathbf{A}}}^1 = \begin{bmatrix} \mathbf{O} & -\dot{\mathbf{Z}}^1 \\ \mathbf{O} & \mathbf{O} \end{bmatrix} \tag{C.44}$$

$$\dot{\mathbf{Z}}^1 = (\vec{\underset{\sim}{e}}^0)^T \begin{bmatrix} 0 & -\dot{z}_3 & \dot{z}_2 \\ \dot{z}_3 & 0 & -\dot{z}_1 \\ -\dot{z}_2 & \dot{z}_1 & 0 \end{bmatrix} \vec{\underset{\sim}{e}}^0 \tag{C.45}$$

in which $\dot{z}_1$, $\dot{z}_2$, and $\dot{z}_3$ can be obtained from Eqs. (C.29) through (C.31), and finally

$$\dot{\vec{\underset{\sim}{B}}}^1 = \begin{bmatrix} ({}^0\dot{\underset{\sim}{b}}^{21})^T \vec{\underset{\sim}{e}}^0 * (\vec{\underset{\sim}{e}}^0)^T ({}^0\underset{\sim}{W}^1)^T + ({}^0\underset{\sim}{b}^{21})^T \vec{\underset{\sim}{e}}^0 * (\vec{\underset{\sim}{e}}^0)^T ({}^0\dot{\underset{\sim}{W}}^1)^T \\ (\vec{\underset{\sim}{e}}^0)^T ({}^0\dot{\underset{\sim}{W}}^1)^T \end{bmatrix} \tag{C.46}$$

where ${}^0\dot{\underset{\sim}{b}}^{21}$ and ${}^0\dot{\underset{\sim}{W}}^1$ are obtained by taking the time derivatives of the right hand sides of Eqs. (C.36), (C.37), (C.38), and (C.34).