A measure theoretical Sobolev lemma

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Abstract.

The well-known Sobolev embedding theorem is generalized in terms of geometric measure theory and Hilbert-Schmidt operators.

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Let M denote a measure space metrized by the metric d, and let μ denote a regular Borel measure on M so that bounded subsets of M have finite μ-measure. In [Fe], Theorem 2.8.18, Federer introduces conditions on the metric space (M,d) such that the following important result holds true.

(1) Theorem.

Let the function $f : M \to \mathbb{C}$ be integrable on bounded Borel sets. Then there exists a null set N_μ such that for all $r > 0$ and all $x \in M \backslash N_\mu$ the closed ball $B(x,r)$ with radius r and centre x has positive μ-measure. Moreover, the limit

$$\tilde{f}(x) = \lim_{r \downarrow 0} \mu(B(x,r))^{-1} \int_{B(x,r)} f \, d\mu$$

exists for all $x \in M \backslash N_\mu$. The function $\tilde{f} : x \mapsto \tilde{f}(x)$ is μ-measurable with $f = \tilde{f}$ almost everywhere.

Examples of such metric spaces (M,d) are the following (cf. [Fe])

- Finite dimensional vectorspaces M with $d(x,y) = v(x-y)$ where v is any norm on M.
- A Riemannian manifold (of class ≥ 2) with its usual metric (cf. [Hi]).
- M, the disjoint union of metric spaces (M_j, d_j), $j \in \mathbb{N}$ and d, the metric defined by

$$\begin{cases} d(x_\ell, y_j) = 1 & \ell \neq j \\ d(x_\ell, y_\ell) = d_\ell(x_\ell, y_\ell) . \end{cases}$$

  Here the spaces (M_j, d_j) are supposed to satisfy Federer's conditions.

Let X denote a Hilbert space and $\mathcal{R}$ a positive self-adjoint Hilbert-Schmidt operator on X. So in X there is an orthonormal basis consisting of eigenvectors of $\mathcal{R}$ with eigenvalues $\rho_k > 0$, $k \in \mathbb{N}$. The dense subspace $\mathcal{R}(X)$ of X contains all $f \in X$ satisfying

$$\sum_{k=1}^{\infty} \rho_k^{-2} \, |(f,v_k)|^2 < \infty .$$

Here $(\cdot\,,\cdot)$ denotes the inner product of X. With the sesquilinear form

$$(f,g)_\rho = (\mathcal{R}^{-1}f, \mathcal{R}^{-1}g)$$

$\mathcal{R}(X)$ becomes a Hilbert space. Now let the linear operator $\mathcal{D}$ be well-defined on $\mathcal{R}(X)$ and let it map $\mathcal{R}(X)$ into $L_2(M,\mu)$. In addition, suppose that the composition map $\mathcal{D} \circ R : X \to L_2(M,\mu)$ is Hilbert-Schmidt. This assumption ensures that the series

$$\text{(2.i)} \qquad \sum_{k=1}^{\infty} \rho_k^2 \, \|\mathcal{D}v_k\|_{L_2}^2$$

converges and hence that

$$\text{(2.ii)} \qquad \sum_{k=1}^{\infty} \rho_k^2 \, |\mathcal{D}v_k|^2 \in L_1(M,\mu) .$$

Since bounded subsets of M have finite μ-measure, every member of $L_2(M,\mu)$ is integrable on bounded sets. So we can apply Theorem (1) to each element $\mathcal{D}v_k$ of $L_2(M,\mu)$. It yields null sets $N_k^{(1)}$, $k \in \mathbb{N}$, such that the limit

$$\text{(3.i)} \qquad \varphi_k(x) = \lim_{r \downarrow 0} \ (B(x,r))^{-1} \int_{B(x,r)} (\mathcal{D}v_k) \, d\mu$$

exists for all $x \in M\setminus \mathcal{N}_k^{(1)}$ and all $k \in \mathbb{N}$. Each function φ_k extends to an everywhere defined representant of the equivalence class $\mathcal{D} v_k$.

Since $|\mathcal{D} v_k|^2 \in L_1(M,\mu)$, $k \in \mathbb{N}$, we get null sets $\mathcal{N}_k^{(2)}$ such that

$$(3.ii) \qquad |\varphi_k(x)|^2 = \lim_{r\downarrow 0} \mu(B(x,r))^{-1} \int_{B(x,r)} |\mathcal{D} v_k|^2 \, d\mu \quad , \quad x \in M\setminus \mathcal{N}_k^{(2)} ,$$

and because of relation (2.ii) we get a null set $\mathcal{N}^{(3)}$ such that

$$(3.iii) \qquad \sum_{k=1}^{\infty} \rho_k^2 \, |\varphi_k(x)|^2 = \lim_{r\downarrow 0} \mu(B(x,r))^{-1} \int_{B(x,r)} \left(\sum_{k=1}^{\infty} \rho_k^2 \, |\mathcal{D} v_k|^2 \right) d\mu \, , \; x \in M\setminus \mathcal{N}^{(3)} .$$

Let $\mathcal{N}_\rho$ denote the null set $\left(\bigcup_{k\in\mathbb{N}} \mathcal{N}_k^{(1)} \right) \cup \left(\bigcup_{k\in\mathbb{N}} \mathcal{N}_k^{(2)} \right) \cup \mathcal{N}^{(3)}$. For convenience sake we take $\varphi_k(x) = 0$ whenever $x \in \mathcal{N}_\rho$.

In the next lemma we put the measure theory needed for the announced main result of the paper.

(4) Lemma.

a) Let $x \in M$ and set $e_x = \sum_{k=1}^{\infty} \rho_k^2 \, \overline{\varphi_k(x)} \, v_k$. Then e_x is a member of $\mathcal{R}(X)$.

b) Let $x \in M\setminus \mathcal{N}_\rho$ and set $e_x(r) = \mu(B(x,r))^{-1} \sum_{k=1}^{\infty} \rho_k^2 \left(\int_{B(x,r)} \overline{\mathcal{D} v_k} \, d\mu \right) v_k$.

Then $e_x(r) \in \mathcal{R}(X)$ for all $r > 0$ and

$$\lim_{r\downarrow 0} \| e_x - e_x(r) \|_\rho = 0 .$$

Proof.

The proof of part a) is a consequence of the definition of the functions φ_k and of relation (3.iii).

In order to prove b) we take $x \in M \setminus \mathcal{N}_\rho$. Then for each $r > 0$ the inequality

$$\left| \int_{B(x,r)} \overline{\mathcal{D} v_k}\, d\mu \right| \leq \left(\int_{B(x,r)} d\mu \right)^{\frac{1}{2}} \left(\int_{B(x,r)} |\mathcal{D} v_k|^2 \, d\mu \right)^{\frac{1}{2}} \leq$$

$$\leq \mu(B(x,r))^{\frac{1}{2}} \, \| \mathcal{D} v_k \|_{\mathcal{L}_2(M,\mu)}$$

is valid. It yields the estimate

$$\sum_{k=1}^{\infty} \rho_k^2 \left| \int_{B(x,r)} \mathcal{D} v_k \, d\mu \right|^2 \leq \mu(B(x,r)) \left(\sum_{k=1}^{\infty} \rho_k^2 \, \| \mathcal{D} v_k \|^2_{\mathcal{L}_2(M,\mu)} \right)$$

and hence by (2.i), $\mathcal{C}_x(r) \in \mathcal{R}(X)$.

Now let $\varepsilon > 0$. Then $k_0 \in \mathbb{N}$ can be taken so large that

$$(*) \qquad \sum_{k=k_0+1}^{\infty} \rho_k^2 \, |\varphi_k(x)|^2 < \varepsilon^2$$

and $r_0 > 0$ so small that for all r, $0 < r < r_0$, both

$$(**) \qquad \left| \varphi_k(x) - \mu(B(x,r))^{-1} \int_{B(x,r)} (\mathcal{D} v_k) \, d\mu \right| < \varepsilon$$

and

$$(***) \qquad \sum_{k=k_0+1}^{\infty} \rho_k^2 \, \mu(B(x,r))^{-1} \int_{B(x,r)} |\mathcal{D} v_k|^2 \, d\mu < 2\varepsilon^2 .$$

The inequalities (*) - (***) lead to the following estimation

$$\| e_x - e_x(r) \|_\rho^2 = \| \sum_{k=1}^{\infty} \rho_k^2 \left[\overline{\varphi_k(x)} - \mu(B(x,r))^{-1} \int_{B(x,r)} (\overline{\mathcal{D} v_k})\, d\mu \right] v_k \|_\rho^2 =$$

$$= \left(\sum_{k=1}^{k_0} + \sum_{k=k_0+1}^{\infty} \right) \rho_k^2 \left| \varphi_k(x) - \mu(B(x,r))^{-1} \int_{B(x,r)} (\mathcal{D} v_k)\, d\mu \right|^2 .$$

Now by (**)

$$\sum_{k=1}^{k_0} \rho_k^2 \left| \varphi_k(x) - \mu(B(x,r))^{-1} \int_{B(x,r)} (\mathcal{D} v_k)\, d\mu \right|^2 < \varepsilon^2 \sum_{k=1}^{\infty} \rho_k^2$$

and by (*) and (***)

$$\sum_{k=k_0+1}^{\infty} \rho_k^2 \left| \varphi_k(x) - \mu(B(x,r))^{-1} \int_{B(x,r)} (\mathcal{D} v_k)\, d\mu \right|^2 \le$$

$$\le 2 \sum_{k=k_0+1}^{\infty} \rho_k^2 \, |\varphi_k(x)|^2 + 2 \sum_{k=k_0+1}^{\infty} \rho_k^2 \left| \mu(B(x,r))^{-1} \int_{B(x,r)} (\mathcal{D} v_k)\, d\mu \right|^2 \le$$

$$\le 2\varepsilon^2 + 2 \sum_{k=k_0+1}^{\infty} \rho_k^2 \, \mu(B(x,r))^{-1} \int_{B(x,r)} |\mathcal{D} v_k|^2 \, d\mu < 6\varepsilon^2 .$$

So we have proved that

$$\| e_x - e_x(r) \|_\rho < \varepsilon \left(6 + \sum_{k=1}^{\infty} \rho_k^2 \right)^{\frac{1}{2}} , \quad 0 < r < r_0 . \qquad \square$$

We now come to the main result of this paper.

(5) Theorem (Measure theoretical Sobolev lemma).

For each $f \in \mathcal{R}(X)$ there can be chosen a representant $\tilde{\mathcal{D}}f$ in $\mathcal{D}f$ such that the following statements are valid

i) $\tilde{\mathcal{D}}f = \sum_{k=1}^{\infty} (f,v_k)\,\varphi_k$ where the series converges pointwise on M.

ii) For each $x \in M$ the linear functional $f \to \tilde{\mathcal{D}}f(x)$ is continuous on the Hilbert space $\mathcal{R}(X)$; its Riesz representant is e_x.

iii) Suppose $\sum_{k=1}^{\infty} \rho_k^2 |\varphi_k|^2$ is essentially bounded on M. Then the convergence in i) is uniform outside a set of measure zero N_0. Moreover

$$\exists_{S>0}\, \forall_{x\in M\setminus N_0} : |(\tilde{\mathcal{D}}f)(x)| < S\, \|f\|_\rho .$$

iv) For all $x \in M\setminus N_\rho$

$$(\tilde{\mathcal{D}}f)(x) = \lim_{r\downarrow 0} \mu(B(x,r))^{-1} \int_{B(x,r)} (\mathcal{D}f)\, d\mu .$$

Proof.

Let $f \in \mathcal{R}(X)$ and put $\tilde{\mathcal{D}}f = \sum_{k=1}^{\infty} (f,v_k)\,\varphi_k$. So obviously $\tilde{\mathcal{D}}f \in \mathcal{D}f$.

i) Since $(f,e_x)_\rho = \sum_{k=1}^{\infty} (f,v_k)\,\varphi_k(x)$ and since $e_x \in \mathcal{R}(X)$ the series converges pointwise on M.

ii) Trivial, because $\tilde{\mathcal{D}}f(x) = (f,e_x)_\rho$.

iii) By assumption there is a null set N_0 such that

$$S = \sup_{x\in M\setminus N_0} \left(\sum_{k=1}^{\infty} \rho_k^2\, |\varphi_k(x)|^2 \right)^{\frac{1}{2}} < \infty .$$

Thus for each $f \in \mathcal{R}(X)$ we get the estimate

$$\left| \sum_{k=L}^{K} (f,v_k)\, \varphi_k(x) \right| \le S \left(\sum_{k=L}^{K} \rho_k^{-2}\, |(f,v_k)|^2 \right)^{\frac{1}{2}}$$

for all $K, L \in \mathbb{N}$ with $K > L$, uniformly on $M \backslash \mathcal{N}_0$.

iv) Let $x \in M \backslash \mathcal{N}_\rho$ and let $f \in \mathcal{R}(X)$. Then

$$(\tilde{\mathcal{D}} f)(x) = (f, e_x)_\rho = \lim_{r \downarrow 0} (f, e_x(r))_\rho =$$

$$= \lim_{r \downarrow 0} \mu(B(x,r))^{-1} \sum_{k=1}^{\infty} (f,v_k) \left(\int_{B(x,r)} \mathcal{D} v_k \, d\mu \right).$$

Summation and integration can be interchanged because

$$\sum_{k=1}^{\infty} \int_{B(x,r)} |(f,v_k)\, \mathcal{D} v_k| \, d\mu \le$$

$$\le \left(\sum_{k=1}^{\infty} \int_{B(x,r)} \rho_k^{-2}\, |(f,v_k)|^2 \, d\mu \right)^{\frac{1}{2}} \left(\sum_{k=1}^{\infty} \int_{B(x,r)} \rho_k^{2}\, |\mathcal{D} v_k|^2 \, d\mu \right)^{\frac{1}{2}} \le$$

$$\le \| f \|_\rho^2 \left(\sum_{k=1}^{\infty} \rho_k^2\, \| \mathcal{D} v_k \|^2_{L_2(M,\mu)} \right) \mu(B(x,r)) .$$

Thus we find

$$(\tilde{\mathcal{D}} f)(x) = \lim_{r \downarrow 0} \mu(B(x,r))^{-1} \int_{B(x,r)} \left(\sum_{k=1}^{\infty} (f,v_k)\, \mathcal{D} v_k \right) d\mu =$$

$$= \lim_{r \downarrow 0} \mu(B(x,r))^{-1} \int_{B(x,r)} (\mathcal{D} f)\, d\mu . \qquad \square$$

Illustrations (The classical Sobolev embedding theorems on $[0,2\pi]^n$).

On the n-dimensional cube $C_n = [0,2\pi]^n$ we take the usual measure $dx = dx_1 \dots dx_n$. In $L_2(C_n)$ the operator Δ,

$$\Delta = 1 - \left(\frac{\partial^2}{\partial x_1^2} + \dots + \frac{\partial^2}{\partial x_n^2}\right)$$

is well-defined and Δ has an orthonormal basis of eigenvectors

$$e_k(x) = \left(\frac{1}{2\pi}\right)^{n/2} e^{ik_1 x} \dots e^{ik_n x}$$

where $k \in \mathbb{Z}^n$, $k = (k_1,\dots,k_n)$. Obviously we have

$$\Delta e_k = (1 + k_1^2 + \dots + k_n^2)\, e_k \quad , \quad k \in \mathbb{Z}^n .$$

Theorem (5) leads to the following result.

(6) Corollary.

Let $m \in \mathbb{N}$ with $m > n/2$, and let $0 \leq \ell < m - n/2$, $\ell \in \mathbb{Z}$. Then there is a null set $\mathcal{N}_n^{(\ell)}$ such that for each $u \in \Delta^{-m/2}(L_2(C_n))$ there exists a representant $\tilde{u}$ of u with the following property: For all $\alpha \in (\mathbb{N} \cup \{0\})^n$, $|\alpha| \leq \ell$, there exists γ_α such that

$$\forall_{x \in C_n \setminus \mathcal{N}_n^{(\ell)}} : |(\mathcal{D}^\alpha u)(x)| \leq \gamma_\alpha \, \|u\|_m .$$

Here $\mathcal{D}^\alpha$ denotes the differential operator $\mathcal{D}^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \dots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}$, and $\|\cdot\|_m$ the Hilbert space norm of $\Delta^{-m/2}(L_2(C_n))$.

Proof.

Note first that $\Delta^{-m/2}$ is a bounded operator on $L_2(C_n)$ satisfying $\Delta^{-m/2} e_k = (1 + k_1^2 + \ldots + k_n^2)^{-m/2} e_k$, and further that $\mathcal{D}^\alpha e_k = i^{|\alpha|} k_1^{\alpha_1} \ldots k_n^{\alpha_n} e_k$. So the operator $\mathcal{D}^\alpha \Delta^{-m/2}$ is Hilbert-Schmidt if the series

$$\sum_{k \in \mathbb{Z}^n} \frac{k_1^{2\alpha_1} \ldots k_n^{2\alpha_n}}{(1 - k_1^2 + \ldots + k_n^2)^m}$$

converges. Comparison with the integral

$$\int_{\mathbb{R}^n} \frac{x_1^{2\alpha_1} \ldots x_n^{2\alpha_n}}{(1 + x_1^2 + \ldots + x_n^2)^m} \, dx_1 \ldots dx$$

shows that for $2m - 2|\alpha| - (n-1) > 1$ this indeed is the case. Hence we find that for $|\alpha| \le \ell < m - n/2$ the operator $\mathcal{D}^\alpha \Delta^{-m/2}$ is Hilbert-Schmidt. Since $|e_k(x)| = 1$, $x \in [0,2\pi]^n$, it also follows that the function $x \mapsto \sum_{k \in \mathbb{Z}^n} |(\mathcal{D}^\alpha \Delta^{-m/2} e_k)(x)|$ is bounded on C_n.
So Theorem (5) and the previous observations yield the desired result.
(Cf. [Yo].) □

Epilogue.

One of the authors (De Graaf) has set up a new theory of generalized functions, [G]. This theory is based on holomorphic semigroups. Each nonnegative self-adjoint operator A in a Hilbert space X gives rise to a space of generalized functions $T_{X,A}$. In [G] each generalized function F is an initial condition u(0) = F of the equation

$$\frac{du}{dt} = -Au \, .$$

The corresponding solution u_F has to satisfy $u_F(t) \in X$, $t > 0$, and $u_F(t+\tau) = e^{-\tau A} u_F(t)$, $t,\tau > 0$. (Heuristically, $u_F(t) = e^{-tA} u_F(0)$.) E.g. for each $w \in X$ and $m > 0$, $A^m w$ is a member of $T_{X,A}$.

Based on this theory of generalized functions, a theory of generalized eigenfunctions has been developed, see [EG], where a central role is played by Theorem (5) and by the so called commutative multiplicity theory (cf. [Br]). The main result in [EG] can be stated as follows:

Let A be a self-adjoint operator in X such that the operators e^{-tA}, $t > 0$, are Hilbert-Schmidt. Then any self-adjoint operator T extendable to a closed operator in $T_{X,A}$ has a complete set of generalized eigenfunctions in $T_{X,A}$. Moreover to almost each point λ in the spectrum of T with multiplicity m_λ there correspond precisely m_λ generalized eigenfunctions out of this complete set.

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