

A measure theoretical Sobolev lemma

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A MEASURE THEORETICAL SOBOLEV LEMMA

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A MEASURE THEORETICAL SOBOLEV LEMMA

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Abstract.

The well-known Sobolev embedding theorem is generalized in terms of geometric measure theory and Hilbert-Schmidt operators.

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Let M denote a measure space metrized by the metric d , and let μ denote a regular Borel measure on M so that bounded subsets of M have finite μ -measure. In [Fe], Theorem 2.8.18, Federer introduces conditions on the metric space (M, d) such that the following important result holds true.

(1) Theorem.

Let the function $f : M \rightarrow \mathbb{C}$ be integrable on bounded Borel sets. Then there exists a null set N_μ such that for all $r > 0$ and all $x \in M \setminus N_\mu$ the closed ball $B(x, r)$ with radius r and centre x has positive μ -measure. Moreover, the limit

$$\tilde{f}(x) = \lim_{r \downarrow 0} \mu(B(x, r))^{-1} \int_{B(x, r)} f \, d\mu$$

exists for all $x \in M \setminus N_\mu$. The function $\tilde{f} : x \mapsto \tilde{f}(x)$ is μ -measurable with $f = \tilde{f}$ almost everywhere.

Examples of such metric spaces (M, d) are the following (cf. [Fe])

- Finite dimensional vectorspaces M with $d(x, y) = v(x - y)$ where v is any norm on M .
- A Riemannian manifold (of class ≥ 2) with its usual metric (cf. [Hi]).
- M , the disjoint union of metric spaces (M_j, d_j) , $j \in \mathbb{N}$ and d , the metric defined by

$$\begin{cases} d(x_\ell, y_j) = 1 & \ell \neq j \\ d(x_\ell, y_\ell) = d_\ell(x_\ell, y_\ell). \end{cases}$$

Here the spaces (M_j, d_j) are supposed to satisfy Federer's conditions.

Let X denote a Hilbert space and R a positive self-adjoint Hilbert-Schmidt operator on X . So in X there is an orthonormal basis consisting of eigenvectors of R with eigenvalues $\rho_k > 0$, $k \in \mathbb{N}$. The dense subspace $R(X)$ of X contains all $f \in X$ satisfying

$$\sum_{k=1}^{\infty} \rho_k^{-2} |(f, v_k)|^2 < \infty.$$

Here (\cdot, \cdot) denotes the inner product of X . With the sesquilinear form

$$(f, g)_{\rho} = (R^{-1}f, R^{-1}g)$$

$R(X)$ becomes a Hilbert space. Now let the linear operator \mathcal{D} be well-defined on $R(X)$ and let it map $R(X)$ into $L_2(M, \mu)$. In addition, suppose that the composition map $\mathcal{D} \circ R : X \rightarrow L_2(M, \mu)$ is Hilbert-Schmidt. This assumption ensures that the series

$$(2.i) \quad \sum_{k=1}^{\infty} \rho_k^2 \|\mathcal{D}v_k\|_{L_2}^2$$

converges and hence that

$$(2.ii) \quad \sum_{k=1}^{\infty} \rho_k^2 |\mathcal{D}v_k|^2 \in L_1(M, \mu).$$

Since bounded subsets of M have finite μ -measure, every member of $L_2(M, \mu)$ is integrable on bounded sets. So we can apply Theorem (1) to each element $\mathcal{D}v_k$ of $L_2(M, \mu)$. It yields null sets $N_k^{(1)}$, $k \in \mathbb{N}$, such that the limit

$$(3.i) \quad \varphi_k(x) = \lim_{r \rightarrow 0} (B(x, r))^{-1} \int_{B(x, r)} (\mathcal{D}v_k) d\mu$$

exists for all $x \in M \setminus N_k^{(1)}$ and all $k \in \mathbb{N}$. Each function φ_k extends to an everywhere defined representant of the equivalence class $\mathcal{D}v_k$.

Since $|\mathcal{D}v_k|^2 \in L_1(M, \mu)$, $k \in \mathbb{N}$, we get null sets $N_k^{(2)}$ such that

$$(3.ii) \quad |\varphi_k(x)|^2 = \lim_{r \rightarrow 0} \mu(B(x, r))^{-1} \int_{B(x, r)} |\mathcal{D}v_k|^2 d\mu, \quad x \in M \setminus N_k^{(2)},$$

and because of relation (2.ii) we get a null set $N^{(3)}$ such that

$$(3.iii) \quad \sum_{k=1}^{\infty} \rho_k^2 |\varphi_k(x)|^2 = \lim_{r \rightarrow 0} \mu(B(x, r))^{-1} \int_{B(x, r)} \left(\sum_{k=1}^{\infty} \rho_k^2 |\mathcal{D}v_k|^2 \right) d\mu, \quad x \in M \setminus N^{(3)}.$$

Let N_ρ denote the null set $\left(\bigcup_{k \in \mathbb{N}} N_k^{(1)} \right) \cup \left(\bigcup_{k \in \mathbb{N}} N_k^{(2)} \right) \cup N^{(3)}$. For convenience sake we take $\varphi_k(x) = 0$ whenever $x \in N_\rho$.

In the next lemma we put the measure theory needed for the announced main result of the paper.

(4) Lemma.

a) Let $x \in M$ and set $e_x = \sum_{k=1}^{\infty} \rho_k^2 \overline{\varphi_k(x)} v_k$. Then e_x is a member of $\mathcal{R}(X)$.

b) Let $x \in M \setminus N_\rho$ and set $e_x(r) = \mu(B(x, r))^{-1} \sum_{k=1}^{\infty} \rho_k^2 \left(\int_{B(x, r)} \overline{\mathcal{D}v_k} d\mu \right) v_k$.

Then $e_x(r) \in \mathcal{R}(X)$ for all $r > 0$ and

$$\lim_{r \rightarrow 0} \|e_x - e_x(r)\|_\rho = 0.$$

Proof.

The proof of part a) is a consequence of the definition of the functions φ_k and of relation (3.iii).

In order to prove b) we take $x \in M \setminus N_\rho$. Then for each $r > 0$ the inequality

$$\begin{aligned} \left| \int_{B(x,r)} \overline{\mathcal{D} v_k} d\mu \right| &\leq \left(\int_{B(x,r)} d\mu \right)^{\frac{1}{2}} \left(\int_{B(x,r)} |\mathcal{D} v_k|^2 d\mu \right)^{\frac{1}{2}} \leq \\ &\leq \mu(B(x,r))^{\frac{1}{2}} \| \mathcal{D} v_k \|_{L_2(M,\mu)} \end{aligned}$$

is valid. It yields the estimate

$$\sum_{k=1}^{\infty} \rho_k^2 \left| \int_{B(x,r)} \mathcal{D} v_k d\mu \right|^2 \leq \mu(B(x,r)) \left(\sum_{k=1}^{\infty} \rho_k^2 \| \mathcal{D} v_k \|_{L_2(M,\mu)}^2 \right)$$

and hence by (2.i), $e_x(r) \in R(X)$.

Now let $\varepsilon > 0$. Then $k_0 \in \mathbb{N}$ can be taken so large that

$$(*) \quad \sum_{k=k_0+1}^{\infty} \rho_k^2 |\varphi_k(x)|^2 < \varepsilon^2$$

and $r_0 > 0$ so small that for all r , $0 < r < r_0$, both

$$(**) \quad \left| \varphi_k(x) - \mu(B(x,r))^{-1} \int_{B(x,r)} (\mathcal{D} v_k) d\mu \right| < \varepsilon$$

and

$$(***) \quad \sum_{k=k_0+1}^{\infty} \rho_k^2 \mu(B(x,r))^{-1} \int_{B(x,r)} |\mathcal{D} v_k|^2 d\mu < 2\varepsilon^2.$$

The inequalities (*) - (***) lead to the following estimation

$$\begin{aligned} \|e_x - e_x(r)\|_\rho^2 &= \left\| \sum_{k=1}^{\infty} \rho_k^2 \left[\overline{\varphi_k(x)} - \mu(B(x,r))^{-1} \int_{B(x,r)} (\mathcal{D} v_k) d\mu \right] v_k \right\|_\rho^2 = \\ &= \left(\sum_{k=1}^{k_0} + \sum_{k=k_0+1}^{\infty} \right) \rho_k^2 \left| \varphi_k(x) - \mu(B(x,r))^{-1} \int_{B(x,r)} (\mathcal{D} v_k) d\mu \right|^2. \end{aligned}$$

Now by (**)

$$\sum_{k=1}^{k_0} \rho_k^2 \left| \varphi_k(x) - \mu(B(x,r))^{-1} \int_{B(x,r)} (\mathcal{D} v_k) d\mu \right|^2 < \varepsilon^2 \sum_{k=1}^{\infty} \rho_k^2$$

and by (*) and (***)

$$\begin{aligned} &\sum_{k=k_0+1}^{\infty} \rho_k^2 \left| \varphi_k(x) - \mu(B(x,r))^{-1} \int_{B(x,r)} (\mathcal{D} v_k) d\mu \right|^2 \leq \\ &\leq 2 \sum_{k=k_0+1}^{\infty} \rho_k^2 |\varphi_k(x)|^2 + 2 \sum_{k=k_0+1}^{\infty} \rho_k^2 \left| \mu(B(x,r))^{-1} \int_{B(x,r)} (\mathcal{D} v_k) d\mu \right|^2 \leq \\ &\leq 2\varepsilon^2 + 2 \sum_{k=k_0+1}^{\infty} \rho_k^2 \mu(B(x,r))^{-1} \int_{B(x,r)} |\mathcal{D} v_k|^2 d\mu < 6\varepsilon^2. \end{aligned}$$

So we have proved that

$$\|e_x - e_x(r)\|_\rho < \varepsilon \left(6 + \sum_{k=1}^{\infty} \rho_k^2 \right)^{\frac{1}{2}}, \quad 0 < r < r_0. \quad \square$$

We now come to the main result of this paper.

(5) Theorem (Measure theoretical Sobolev lemma).

For each $f \in R(X)$ there can be chosen a representant $\tilde{\mathcal{D}}f$ in $\mathcal{D}f$ such that the following statements are valid

- i) $\tilde{\mathcal{D}}f = \sum_{k=1}^{\infty} (f, v_k) \varphi_k$ where the series converges pointwise on M .
- ii) For each $x \in M$ the linear functional $f \rightarrow \tilde{\mathcal{D}}f(x)$ is continuous on the Hilbert space $R(X)$; its Riesz representant is e_x .
- iii) Suppose $\sum_{k=1}^{\infty} \rho_k^2 |\varphi_k|^2$ is essentially bounded on M . Then the convergence in i) is uniform outside a set of measure zero N_0 . Moreover

$$\exists_{S>0} \forall_{x \in M \setminus N_0} : |(\tilde{\mathcal{D}}f)(x)| < S \|f\|_{\rho}.$$

- iv) For all $x \in M \setminus N_{\rho}$

$$(\tilde{\mathcal{D}}f)(x) = \lim_{r \downarrow 0} \mu(B(x, r))^{-1} \int_{B(x, r)} (\mathcal{D}f) d\mu.$$

Proof.

Let $f \in R(X)$ and put $\tilde{\mathcal{D}}f = \sum_{k=1}^{\infty} (f, v_k) \varphi_k$. So obviously $\tilde{\mathcal{D}}f \in \mathcal{D}f$.

- i) Since $(f, e_x)_{\rho} = \sum_{k=1}^{\infty} (f, v_k) \varphi_k(x)$ and since $e_x \in R(X)$ the series converges pointwise on M .

- ii) Trivial, because $\tilde{\mathcal{D}}f(x) = (f, e_x)_{\rho}$.

- iii) By assumption there is a null set N_0 such that

$$S = \sup_{x \in M \setminus N_0} \left(\sum_{k=1}^{\infty} \rho_k^2 |\varphi_k(x)|^2 \right)^{\frac{1}{2}} < \infty.$$

Thus for each $f \in R(X)$ we get the estimate

$$\left| \sum_{k=L}^K (f, v_k) \varphi_k(x) \right| \leq S \left(\sum_{k=L}^K \rho_k^{-2} |(f, v_k)|^2 \right)^{\frac{1}{2}}$$

for all $K, L \in \mathbb{N}$ with $K > L$, uniformly on $M \setminus N_0$.

iv) Let $x \in M \setminus N_\rho$ and let $f \in R(X)$. Then

$$\begin{aligned} (\tilde{\mathcal{D}}f)(x) &= (f, e_x)_\rho = \lim_{r \rightarrow 0} (f, e_x(r))_\rho = \\ &= \lim_{r \rightarrow 0} \mu(B(x, r))^{-1} \sum_{k=1}^{\infty} (f, v_k) \left(\int_{B(x, r)} \mathcal{D} v_k d\mu \right). \end{aligned}$$

Summation and integration can be interchanged because

$$\begin{aligned} &\sum_{k=1}^{\infty} \int_{B(x, r)} |(f, v_k) \mathcal{D} v_k| d\mu \leq \\ &\leq \left(\sum_{k=1}^{\infty} \int_{B(x, r)} \rho_k^{-2} |(f, v_k)|^2 d\mu \right)^{\frac{1}{2}} \left(\sum_{k=1}^{\infty} \int_{B(x, r)} \rho_k^2 |\mathcal{D} v_k|^2 d\mu \right)^{\frac{1}{2}} \leq \\ &\leq \|f\|_\rho^2 \left(\sum_{k=1}^{\infty} \rho_k^2 \|\mathcal{D} v_k\|_{L_2(M, \mu)}^2 \right) \mu(B(x, r)). \end{aligned}$$

Thus we find

$$\begin{aligned} (\tilde{\mathcal{D}}f)(x) &= \lim_{r \rightarrow 0} \mu(B(x, r))^{-1} \int_{B(x, r)} \left(\sum_{k=1}^{\infty} (f, v_k) \mathcal{D} v_k \right) d\mu = \\ &= \lim_{r \rightarrow 0} \mu(B(x, r))^{-1} \int_{B(x, r)} (\mathcal{D}f) d\mu. \end{aligned} \quad \square$$

Illustrations (The classical Sobolev embedding theorems on $[0, 2\pi]^n$).

On the n -dimensional cube $C_n = [0, 2\pi]^n$ we take the usual measure $dx = dx_1 \dots dx_n$. In $L_2(C_n)$ the operator Δ ,

$$\Delta = 1 - \left(\frac{\partial^2}{\partial x_1^2} + \dots + \frac{\partial^2}{\partial x_n^2} \right)$$

is well-defined and Δ has an orthonormal basis of eigenvectors

$$e_k(x) = \left(\frac{1}{2\pi} \right)^{n/2} e^{ik_1 x} \dots e^{ik_n x}$$

where $k \in \mathbb{Z}^n$, $k = (k_1, \dots, k_n)$. Obviously we have

$$\Delta e_k = (1 + k_1^2 + \dots + k_n^2) e_k, \quad k \in \mathbb{Z}^n.$$

Theorem (5) leads to the following result.

(6) Corollary.

Let $m \in \mathbb{N}$ with $m > n/2$, and let $0 \leq \ell < m - n/2$, $\ell \in \mathbb{Z}$. Then there is a null set $N_n^{(\ell)}$ such that for each $u \in \Delta^{-m/2}(L_2(C_n))$ there exists a representant \tilde{u} of u with the following property: For all $\alpha \in (\mathbb{N} \cup \{0\})^n$, $|\alpha| \leq \ell$, there exists γ_α such that

$$\forall_{x \in C_n \setminus N_n^{(\ell)}} : |(\mathcal{D}^\alpha u)(x)| \leq \gamma_\alpha \|u\|_m.$$

Here \mathcal{D}^α denotes the differential operator $\mathcal{D}^\alpha = \left(\frac{\partial}{\partial x_1} \right)^{\alpha_1} \dots \left(\frac{\partial}{\partial x_n} \right)^{\alpha_n}$, and $\|\cdot\|_m$ the Hilbert space norm of $\Delta^{-m/2}(L_2(C_n))$.

Proof.

Note first that $\Delta^{-m/2}$ is a bounded operator on $L_2(C_n)$ satisfying

$$\Delta^{-m/2} e_k = (1 + k_1^2 + \dots + k_n^2)^{-m/2} e_k, \text{ and further that } \mathcal{D}^\alpha e_k = i^{|\alpha|} k_1^{\alpha_1} \dots k_n^{\alpha_n} e_k.$$

So the operator $\mathcal{D}^\alpha \Delta^{-m/2}$ is Hilbert-Schmidt if the series

$$\sum_{k \in \mathbb{Z}^n} \frac{k_1^{2\alpha_1} \dots k_n^{2\alpha_n}}{(1 + k_1^2 + \dots + k_n^2)^m}$$

converges. Comparison with the integral

$$\int_{\mathbb{R}^n} \frac{x_1^{2\alpha_1} \dots x_n^{2\alpha_n}}{(1 + x_1^2 + \dots + x_n^2)^m} dx_1 \dots dx_n$$

shows that for $2m - 2|\alpha| - (n-1) > 1$ this indeed is the case. Hence we find

that for $|\alpha| \leq \ell < m - n/2$ the operator $\mathcal{D}^\alpha \Delta^{-m/2}$ is Hilbert-Schmidt. Since

$|e_k(x)| = 1, x \in [0, 2\pi]^n$, it also follows that the function

$$x \mapsto \sum_{k \in \mathbb{Z}^n} |(\mathcal{D}^\alpha \Delta^{-m/2} e_k)(x)| \text{ is bounded on } C_n.$$

So Theorem (5) and the previous observations yield the desired result.

(Cf. [Yo].)

□

Epilogue.

One of the authors (De Graaf) has set up a new theory of generalized functions, [G]. This theory is based on holomorphic semigroups. Each nonnegative self-adjoint operator A in a Hilbert space X gives rise to a space of generalized functions $T_{X,A}$. In [G] each generalized function F is an initial condition $u(0) = F$ of the equation

$$\frac{du}{dt} = -Au.$$

The corresponding solution u_F has to satisfy $u_F(t) \in X$, $t > 0$, and $u_F(t+\tau) = e^{-\tau A} u_F(t)$, $t, \tau > 0$. (Heuristically, $u_F(t) = e^{-tA} u_F(0)$.) E.g. for each $w \in X$ and $m > 0$, $A^m w$ is a member of $T_{X,A}$.

Based on this theory of generalized functions, a theory of generalized eigenfunctions has been developed, see [EG], where a central role is played by Theorem (5) and by the so called commutative multiplicity theory (cf. [Br]). The main result in [EG] can be stated as follows:

Let A be a self-adjoint operator in X such that the operators e^{-tA} , $t > 0$, are Hilbert-Schmidt. Then any self-adjoint operator T extendable to a closed operator in $T_{X,A}$ has a complete set of generalized eigenfunctions in $T_{X,A}$. Moreover to almost each point λ in the spectrum of T with multiplicity m_λ there correspond precisely m_λ generalized eigenfunctions out of this complete set.

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