EINDHOVEN UNIVERSITY OF TECHNOLOGY

Commentary

Citation for published version (APA):
Bruijn, de, N. G. (1978). Commentary. In E. M. J. Bertin, H. J. M. Bos, & A. W. Grootendorst (Eds.), *Two decades of mathematics in the Netherlands : 1920-1940 : a retrospection on the occasion of the bicentennial of the Wiskundig Genootschap. Part 1* (pp. 116-124). Mathematical Centre.

Document status and date:
Published: 01/01/1978

Document Version:
Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

Please check the document version of this publication:

• A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
• The final author version and the galley proof are versions of the publication after peer review.
• The final published version features the final layout of the paper including the volume, issue and page numbers.

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Download date: 04. Oct. 2023

Commentary
by
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Old and respectable as the "Wiskundig Genootschap" may be, it has never been more than a small country's mathematical society. Accordingly, it is not surprising that the society's home journal, the "Nieuw Archief voor Wiskunde", has a relatively small circulation, and, as a second order effect, that the Nieuw Archief does not get more than a small part of the more important contributions of the dutch to mathematics.

Therefore, it is particularly enjoyable to find, in the Nieuw Archief, one of the most elegant pieces of mathematics ever produced. The influence of this little paper, now 50 years old, has not yet come to an end.

In mathematics, like in art, it is true that a thing of beauty is a joy for ever. In 1947, Van der Waerden's paper was selected by A.J. Khinchin as one of the "three pearls of number theory" (Khinchin [1951], [1952]), and now, in 1978, there is no noticeabledecrease in its popularity.

It is very fortunate that we have a report, written by Van der Waerden himself ([1965], [1971]), of how the result was discovered. It was partly intended as an illustration of the author's ideas on the psychology of mathematical invention (Van der Waerden [1953]). The reading of this report is recommended to all those for whom "understanding" is not just formal verification, but rather a procedure by which intuitive ideas and experiences are linked to each other in order to build up the final mathematical structure. They know that translating this final structure into formal verification is a matter of routine, and that the level of tolerable formality is a matter of fashion.

The reading of Van der Waerden's report is also recommended to those who are interested to learn about the discussion (with Artin and Schreier) that preceded Van der Waerden's discovery, and to learn about the actual contributions of Artin and Schreier to the solution of the problem.

First a few words about Baudet. P.J.H. Baudet (born January 22, 1891) died young (December 25, 1921; see Schuh [1921] for a short obituary). He worked in group theory and number theory and wrote on various methods for introducing the real number system. In 1919 he was appointed to a professorship at the Delft Technological University. His main contact in mathematics was F. Schuh who was not only an allround mathematician but also a great puzzlist. Baudet shared Schuh's interest for puzzles and games: Baudet was an expert chess player, and a friend of the world chess champion (and mathematician) E. Lasker. In his *Brettspiele der Völker* (Berlin 1931) Lasker describes a game of "Laska" he lost from Baudet at a tournament in The Hague 1920 ("Laska" was Lasker's own invention, which he tried to promote at a time he thought that eventually all serious chess games would lead to a draw.)

Baudet's conjecture stated that if ℓ is a given natural number and if the set of natural numbers is divided over two classes, then at least one of these classes contains an arithmetic progression of length ℓ. We do not know when and in what context he stated his conjecture and what partial results he had. He of course solved the case $\ell = 3$ (if the numbers $1,\dots,9$ are divided over two classes, at least one contains an arithmetic progression of length 3). Possibly he did more. Not many mathematicians would formulate such a conjecture about arbitrarily long arithmetic progressions without checking that the corresponding conjecture for infinite arithmetic progressions is incorrect. (The following argument works: First enumerate the set of all infinite arithmetic progressions, next take integers a_1, b_1, a_2, b_2,... with $a_1 < b_2 < a_2 < b_2 < \dots$ such that a_i and b_i belong to the i-th progression. Now put all a_i's into one class, and all other positive integers, including the b_i's, into the other.)

Van der Waerden denotes by $B_1(\ell)$ the statement that in every dissection of the natural numbers into two classes at least one class contains an arithmetic progression of lenght ℓ, and by $B_2(\ell,k)$ the following statement: There exists an integer $n(\ell,k)$ such that if the set of all integers $\leqslant n(\ell,k)$ is split into k classes, at least one contains an arithmetic progression of lenght ℓ. In his report of how the proof of Baudet's conjecture was found, he explains that Schreier remarked that $B_1(\ell)$ implies $B_2(\ell,2)$ (by an application of the diagonal principle; one might also phrase it as an application of König's infinity lemma). Next Artin remarked that if we have $B_2(\ell,2)$ for all ℓ, then we have $B_2(\ell,k)$ for all ℓ and k. For example, we can take

$$n(\ell,3) = n(n(\ell,2),2),$$

arguing that if we have divided $1,\ldots,n(\ell,3)$ over 3 classes, and if the first one does not contain an arithmetic progression of lenght ℓ, then the union of the second and the third class contains a progression of length $n(\ell,2)$ (we may and do assume that $n(\ell,2) > \ell$). This progression is divided over these two classes, so by $B_2(\ell,2)$ we have a sub-progression of length ℓ in one of them.

The next step, due to Artin, was to suggest that an induction proof might be arranged by assuming $B_2(\ell,k)$ for all k and then proving $B_2(\ell+1,k)$ for all k. This is connected to the idea that such a set-up will be needed if we want to make statements about repetition of blocks: if we consider "blocks" of s consecutive integers, and if the integers are divided into k classes, then there are k^s types of such blocks (if we only look at the sequence of classes inside the block). So if we want to prove $B_2(\ell+1,k)$ by means of statements on repetition of blocks, we would like to have $B_2(\ell,k^s)$, so if we have no simple bound for s, it would be nice to have $B_2(\ell,k)$ for all k.

After Schreier and Artin helped to get the problem into this form: "prove $\forall_k B_2(\ell,k)$ by induction with respect to ℓ", it was solved by young Van der Waerden.

Van der Waerden's argument is very clear from his paper, yet the notational difficulties seemed to be a bit awkward to the modern combinatorialists (Graham and Rothschild [1974]) who brought Van der Waerden's proof construction in a very condensed form:

Let [a,b] denote the set of integers with $a \leqslant x \leqslant b$. We call $(x_1,\ldots,x_m)$, $(x_1',\ldots,x_m') \in [0,\ell]^m$ ℓ-equivalent if they agree up through their last occurrences of ℓ. For any $\ell,m \geqslant 1$, consider the statement

$S(\ell,m)$ For any k, there exists $N(\ell,m,k)$ so that for any function $C: [1,N(\ell,m,k)] \to [1,k]$, there exist positive $a,d_1,\ldots,d_m$ such that $C(a + \Sigma_{i=1}^m x_i d_i)$ is constant on each ℓ-equivalence caass of $[0,\ell]^m$.

Note that $S(\ell,1)$ equals $\forall_k B_2(\ell,k)$. The authors prove $\forall_\ell \forall_m S(\ell,m)$.

The induction arrangement consists of the following two steps: (i) if $S(\ell,m)$ then $S(\ell,m+1)$, (ii) if $\forall_m S(\ell,m)$ then $S(\ell+1,1)$. It is the role of the parameter m to run from 1 to k: Van der Waerden's construction uses a kind of projected k-dimensional cube, and its definition requires induction with respect to m.

The Van der Waerden theorem has been generalized in various directions. A quite elegant generalization, due to G. Grünwald (= T. Gallai), was presented in Rado [1945]; a short proof of it in Anderson [1976]. It can be formulated by saying that instead of the set $\mathbb{N}$ of all natural numbers we take the cartesian product $\mathbb{N} \times \ldots \times \mathbb{N}$ (m factors) and instead of "arithmetic progression of length ℓ" we take "cartesian product of m arithmetic progressions, each with the same difference". Essentially the same thing was expressed by Witt in [1952].

A fairly large class of generalizations was explored by R. Rado ([1933], [1945]).

Rado also points out in [1933] that a theorem of the same type as Van der Waerden's was already obtained by I. Schur ([1916]) in a very precise form: if the integer K exceeds e.k!, and if the set {1,...,K} is partitioned into k parts then at least one part contains elements a,b,c (possibly a = b) with a+b = c. It is quite possible that this paper was known to Baudet.

All these theorems have the form: "if a certain set is split arbitrarily into classes, then a certain phenomenon happens in at least one of these classes". A famous set-theoretical theorem with this structure is the one of Ramsey ([1930]). Let p,k,m be given positive integers, $p \leqslant m$. Then N = N(p,k,m) exists such that the following is true. Let S be any set with at least N elements. Let the set of all p-subsets of S be partitioned into k parts. Then S has an m-subset M such that all p-subsets of M fall into the same part.

At first sight this has little to do with Van der Waerden's theorem, but in Graham and Rothschild [1971] we find a theorem with a large number of corollaries among which are both Van der Waerden's and Ramsey's theorem. For further extensions, successfully using the language of categories, see Graham, Leeb and Rothschild [1972].

Quite a different line of generalizations was opened by Erdös and Turan in [1936]: If k is fixed and n is large, and if we split the set {1,...,n} into k classes, then not all of these classes can be very thin, and the idea is that "not being very thin" is in itself sufficient for a set in order to contain an arithmetic progression of length ℓ. To be most precise, we define $r_\ell(n)$ to be the largest integer m for which $\{1,\ldots,n\}$ has a subset of m elements which does not contain an arithmetic progression of length ℓ. Erdös and Turan conjectured that for all ℓ we have $\lim_{n\to\infty} r(n)/n = 0$. This is stronger than Van der Waerden's theorem, since $r_\ell(n) < n/k$ implies $B_2(\ell,k)$. Van der Waerden's theorem does not say which one of the k classes contains an arithmetic progression, whereas the Erdös-Turan conjecture provides a very easy criterion: Just look whether the classes are big enough.

The Erdös-Turan conjecture was proved for $k = 3$ by K.F. Roth [1951], who actually proved that $r_3(n) < cn/\log\log n$ for some constant c. The case $\ell = 4$ was settled in Szemerédi [1969], and the case of general ℓ finally in Szemerédi [1975]. This proof is very complex. A proof on entirely different principles (ergodic theory) was recently produced by H. Furstenberg (see Kolata [1977]).

After the Erdös-Turan conjecture was settled by Szemerédi, Erdös issued the stronger conjecture that if a sequence $n_1, n_2, \ldots$ of positive integers satisfies $\Sigma_{i=1}^{\infty} n_i^{-1} = \infty$, then it contains arbitrarily long arithmetic progressions.

A lower bound for $r_3(n)$ was given by Behrend in [1946]: $r_3(n) > n \exp(-c(\log n)^{\frac{1}{2}})$ with some positive constant c.

All these theorems express the existence of just one arithmetic progression, but it is natural to expect that there are many. Varnavides in [1959] proved a best possible answer to this question for $\ell = 3$. For every $\delta > 0$ there exist constants c_1 and c_2 such that, for all $n > c_1$, every subset of $\{1,\ldots,n\}$ with more than δn elements contains at least $c_2 n^2$ arithmetic progressions of length 3. Note that the number of such progressions in the full $\{1,\ldots,n\}$ is of the same order. A slightly stronger result was obtained for general ℓ by Choi in [1975], as a relatively simple application of Szemerédi's theorem.

Coming back to Van der Waerden's theorem: there has not been much success in finding reasonable upper bounds for the n(ℓ,k). For lower bounds see Berlekamp [1968].

Compared to the work of Szemerédi [1975], Van der Waerden's proof is not very hard. Yet one may ask for a simpler proof. Hahn [1971] asks whether one cannot even find a simpler proof for the case that it is not the set of integers but the set of reals that is split into k classes, or other sets which are essentially bigger than the set of integers.

On the other hand, Van der Waerden's argument is so nice that one might secretly hope that a simpler proof does not exist!

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References

ANDERSON, P.G. [1976], "A generalization of Baudet's conjecture (Van der Waerden's theorem)", *American Math. Monthly*, 83 (1976), pp. 359-361.

BEHREND, F.A. [1946], "On sets of integers which contain no three terms in arithmetic progression", *Proc. Nat. Acad. Sci. USA*, 32 (1946), pp. 331-332.

BERLEKAMP, E.R. [1968], "A construction for partitions which avoid long arithmetic progressions", *Canadian Math. Bull.*, 11 (1968), pp. 409-414.

CHOI, S.G.L. [1975], "On arithmetic progressions in sequence", *J. London Math. Soc.* (2), 10 (1975), pp. 427-430.

ERDÖS,P. and TURÁN, P. [1936], "On some sequences of integers", *J. London Math. Soc.*, 11 (1936), pp. 261-264.

GRAHAM, R.L. and ROTHSCHILD, B.L. [1971], "Ramsey's theorem for n-parameter sets", *Transactions Amer. Math. Soc.*, 30 (1971), pp. 1-36.

GRAHAM, R.L., LEEB, K. and ROTHSCHILD, B.L. [1972], "Ramsey's theorem for a class of categories", *Advances in Mathematics*, 8 (1972), pp. 417-433.

GRAHAM, R.L. and ROTHSCHILD, B.L. [1974], "A short proof of Van der Waerden's theorem on arithmetic progressions", *Proc. American Math. Soc.*, 42 (1974), pp. 385-386.

HAHN, L.S. [1971], "Advanced Problem nr. 5808", *American Math. Monthly*, 78 (1971), p. 798.

KHINCHIN, A.J. [1951], *Drei Perlen der Zahlentheorie*, Akademie-Verlag, Berlin 1951.

——— [1952], *Three Pearls of Number Theory*, Graylock Press, Rochester N.Y., 1952.

KOLATA, G.B. [1977], "Structure in large sets: two proofs where there were none", *Science*, 195 (no. 4280) (1977), pp. 767-768.

RADO, R. [1933], "Studien zur Kombinatorik", *Math. Zeitschrift*, 36 (1933), pp. 424-480.

——— [1945], "Note on combinatorial analysis", *Proc. London Math. Soc.*, 48 (1945), pp. 122-160.

RAMSEY, F.P. [1930], "On a problem of formal logic", *Proc. London Math. Soc.* (2), 30 (1930), pp. 262-286.

ROTH, K.F. [1953], "On certain sets of integers", *J. London Math. Soc.*, 28 (1953), pp. 104-109.

SCHUH, F. [1921], "In Memoriam Pierre Joseph Henry Baudet", *Christiaan Huygens*, 1 (1921-1922), p. 145.

SCHUR, I. [1916], "Ueber die Kongruenz $x^m + y^m \equiv z^m \pmod p$", *Jahresber. Deutsch. Math. Verein*, 25 (1916), pp. 114-117.

SZEMEREDI, E. [1969], "On sets of integers containing no four elements in arithmetic progression", *Acta Math. Acad. Sci. Hungar.*, 20 (1969), pp. 89-104.

——— [1975], "On sets of integers containing no k elements in arithmetic progression", *Acta Arith.*, 27 (1975), pp. 199-245.

VARNAVIDES, P. [1959], "On certain sets of positive density", *J. London Math. Soc.*, 34 (1959), pp. 358-360.

WAERDEN, B.L. van der [1927], "Beweis einer Baudet'schen Vermutung", *Nieuw Arch. Wisk.* (2), 15 (1927), pp. 212-216.

———————— [1953], "Einfall und Ueberlegung in der Mathematik", *Elemente der Math.*, 8 (1953), pp. 121-129; 9 (1954), pp. 1-9; 9 (1954), pp. 49-56.

———————— [1965], "Wie der Beweis der Vermutung von Baudet gefunden wurde", *Abh. Math. Sem. Univ. Hamburg*, 28 (1965), pp. 6-15.

———————— [1971], "How the proof of Baudet's conjecture was found", *Studies in Pure Mathematics* (papers presented to Richard Rado on the occasion of his sixty-fifth birthday, ed. by MIRSKY, L., Academic Press, London-New York,1971), pp. 252-260.

WITT, E. [1952], "Ein kombinatorischer Satz der Elementargeometrie", *Mathematische Nachrichten*, 6 (1952), pp. 261-262.