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EINDHOVEN UNIVERSITY OF TECHNOLOGY
Department of Mathematics and Computing Science

RANA 93-14
September 1993

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Reports on Applied and Numerical Analysis
Department of Mathematics and Computing Science
Eindhoven University of Technology
P.O. Box 513
5600 MB Eindhoven
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ISSN: 0926-4507

On Optimally Scaled Systems for Second Order Scalar Singularly Perturbed Problems

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Abstract

The conditioning of singularly perturbed scalar Dirichlet problem is considered. It is shown how this is related to the conditioning of an appropriate associated first order system. Through this the dichotomy of the solution space (a concept that only makes sense in a vectorial setting) can be investigated. Two typical equations are studied in more detail, one with possible boundary layers on both sides of the interval and one with an internal layer (i.e. the turning point case). The results are applied to obtain estimates of global discretisation errors for difference methods. Several examples illustrate the analysis.

1 Introduction

Consider the problem

(1.1a) $\epsilon y'' + a(x)y' + b(x)y = f(x), \quad \epsilon > 0$ (but small) ,

where a, b and f are sufficiently smooth functions. Let y satisfy the Dirichlet boundary condition

(1.1b) $y(x_0) = \alpha, \; y(x_1) = \beta, \quad \alpha, \beta \in I\!R$.

An intriguing but, as it turns out, nontrivial question is how the solution of (1.1) is behaving when small perturbations of the data are taken into account, i.e. how large its conditioning constant is (cf. [1]), given the fact that ϵ is a parameter. Another question is whether this behaviour is related to dichotomy of the solution space (cf. [2]), as this is a fundamental result for first order systems (cf. [3]). We shall first try to address the conditioning question for a simple problem and then derive a meaningful corresponding first order system. In the sequel let

(1.2a) $\|y\|_\infty := \max_{x \in [x_0, x_1]} |y(x)|$,

(1.2b) $\|y\|_1 := \int_{x_0}^{x_1} |y(x)| dx$,

(which also extends to vector–functions by taking any norm for y instead of the absolute value sign; since these norms are equivalent we may as well take just the 2-norm then).
First consider the special, but instructive, example

(1.3) $\varepsilon y'' - y = f(x) \; , \quad y(0) = 0 \; , \quad y(1) = 0$.

Then, as can easily be checked, the solution is given by

(1.4a) $y(x) = \int_0^1 g(x, s) f(s) ds$,

where the kernel $g(x, s)$ is the Green's function for the scalar problem (1.3),

(1.4b) $g(x, s) := \dfrac{1}{\sqrt{\varepsilon}} \dfrac{\left(e^{\frac{1-x}{\sqrt{\epsilon}}} - e^{\frac{1+x}{\sqrt{\epsilon}}}\right)\left(e^{\frac{1-s}{\sqrt{\epsilon}}} - e^{\frac{-1+s}{\sqrt{\epsilon}}}\right)}{2\left(e^{\frac{2}{\sqrt{\epsilon}}} - 1\right)} , \quad 0 \le x \le s$;

for $s \le x \le 1$, $g(x, s)$ is given by an expression similar to (1.4b), but with x and s interchanged.

It follows from (1.4) that $\|y\|_\infty \approx \|f\|_\infty$, but $\|y\|_\infty \approx \frac{1}{\sqrt{\varepsilon}} \|f\|_1$.
This example thus shows that we cannot expect in general that $\|y\|_\infty \le C\|f\|_1$ where the constant C is independent of ε (which would amount to a notion of *well-conditioning* of the BVP (1.1)). More careful examination of this estimate (which will be done further on for the more general case) reveals that a quantity δ^{-1}, defined through

$$(1.5) \qquad \delta^{-1} := \frac{\max\limits_{x\in[x_0,x_1]} |y'(x)|}{\max\limits_{x\in[x_0,x_1]} |y(x)|} ,$$

is often showing up in the bounds.
Rather than (1.1) we may as well consider the ODE in more "standard" form

$$(1.1)' \qquad y'' + \frac{a}{\varepsilon}\, y' + \frac{b}{\varepsilon}\, y = \frac{f}{\varepsilon} =: q \ .$$

We have the following definition.

Definition 1.6. (1.1) *is called well conditioned if there exists a moderately small constant C (independent of ε) such that $\|y\|_\infty \le C\, \frac{\delta}{\varepsilon}\, \|f\|_1$.*

In the example problem (1.3) we clearly have $\delta \doteq \sqrt{\varepsilon}$. Although this means that source terms may have a large effect, we shall show in section 3 that this is not necessarily so if they arise from discretisation errors.
Next, consider the "standard" matrix–vector ODE corresponding to (1.1)

$$(1.6) \qquad \frac{d\mathbf{y}}{dt} = \mathbf{A}\mathbf{y} + \begin{pmatrix} 0 \\ q \end{pmatrix} , \qquad \text{where } \mathbf{A} := \begin{bmatrix} 0 & 1 \\ -\frac{b}{\varepsilon} & -\frac{a}{\varepsilon} \end{bmatrix} , \qquad \mathbf{y} := \begin{pmatrix} y \\ y' \end{pmatrix}$$

and a corresponding boundary condition

$$(1.7) \qquad \mathcal{B}y := \mathbf{B}_0\mathbf{y}(x_0) + \mathbf{B}_1 y(x_1) = \mathbf{c} \ ,$$

where $\mathbf{B}_0 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\mathbf{B}_1 := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $\mathbf{c} := \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ for the Dirichlet boundary conditions (1.1b). There are two problems associated with (1.6). The first one is the well-known skewness of $\mathbf{A}$. For our example problem (1.3), e.g., we see that we obtain an obvious fundamental solution

$$\mathbf{Y}(x) := \begin{bmatrix} 1 & 1 \\ \frac{1}{\sqrt{\varepsilon}} & \frac{1}{-\sqrt{\varepsilon}} \end{bmatrix} \operatorname{diag}\left(e^{\frac{1}{\sqrt{\varepsilon}}x}, e^{-\frac{1}{\sqrt{\varepsilon}}x}\right) ;$$

clearly the vectors $(1, (\sqrt{\varepsilon})^{-1})^T$, and $(1, -(\sqrt{\varepsilon})^{-1})^T$ are nearly dependent for ϵ small. Simple calculations show that they give unbounded Green's functions (as $\epsilon \downarrow 0$) (cf. also [6]); we return to this matter later.
The second problem is that the forcing term has a systematically zero first coordinate. A general sensitivity analysis of (1.6) would necessarily include perturbations of the latter as well. Below we shall see how we take care of this. Instead of (1.6) consider, more generally, a matrix vector form obtained from (1.6) by transforming the variable through

(1.8) $\mathbf{T} := \begin{pmatrix} 1 & 0 \\ 0 & \gamma \end{pmatrix}$.

This special choice of $\mathbf{T}$ is induced by the fact that we like to have the first coordinate in the system to be just y and also that we anticipate y' to be potentially large, so $\gamma y'$ is as good a choice as, say $\mu y + \gamma y'$, with μ moderate. Define

(1.9a) $\tilde{\mathbf{y}} := \mathbf{T}\mathbf{y}\ ;\ \ \tilde{\mathbf{A}} := \mathbf{T}\mathbf{A}\mathbf{T}^{-1}$.

Then

(1.9b) $\tilde{\mathbf{y}}' = \tilde{\mathbf{A}}\tilde{\mathbf{y}} + \begin{pmatrix} 0 \\ \gamma q \end{pmatrix}$.

Now let $\mathbf{\Phi}$ denote the fundamental solution of (1.6), with $\mathcal{B}\mathbf{\Phi} = \mathbf{I}$; then this problem has the following Green's function $\mathbf{G}(x,s)$:

(1.10a) $\mathbf{G}(x,s) := \mathbf{\Phi}(x) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{\Phi}^{-1}(s)\ ,\quad x > s\ ,$

(1.10b) $\mathbf{G}(x,s) := -\mathbf{\Phi}(x) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \mathbf{\Phi}^{-1}(s)\ ,\quad x < s\ .$

The transformed problem (1.9b) has, correspondingly, the fundamental solution $\tilde{\mathbf{\Phi}}$, and the Green's function $\tilde{\mathbf{G}}$ of the form

(1.11a) $\tilde{\mathbf{\Phi}}(x) := \mathbf{T}\mathbf{\Phi}(x)$,

(1.11b) $\tilde{\mathbf{G}}(x,s) := \mathbf{T}\mathbf{G}(x,s)\mathbf{T}^{-1}$.

Partitioning $\mathbf{G}$ as $\begin{bmatrix} \mathbf{G}^{11} & \mathbf{G}^{12} \\ \mathbf{G}^{21} & \mathbf{G}^{22} \end{bmatrix}$, we notice that only $\tilde{\mathbf{G}}^{12}$ and $\tilde{\mathbf{G}}^{21}$ are different from $\mathbf{G}^{12}$ and $\mathbf{G}^{21}$ respectively. The function $\mathbf{G}^{12}(x,s)$, in example (1.3), appears as the kernel $g(x,s)$ in (1.4). From the estimates for $g(x,s)$ it follows that $\max_{x,s} ||\mathbf{G}^{12}|| = O(\sqrt{\varepsilon})$[1]. From some more calculations it can also be seen that $\max_{x,s} ||\mathbf{G}^{21}(x,s)|| = O((\sqrt{\varepsilon})^{-1})$, which then suggests to take $\gamma = \sqrt{\varepsilon}$ in (1.8). Actually it can be seen that with this choice of $\mathbf{T}$ the resulting Green's function $\tilde{\mathbf{G}}$ is uniformly bounded (in infinity norm). The problem (1.9b) with appropriate (though fairly general) boundary conditions may therefore be called *well-conditioned.* Recall from [3] the following

Definition 1.12. *The BVP*

(1.12a) $\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{f}$,

(1.12b) $\mathbf{B}_0\mathbf{y}(x_0) + \mathbf{B}_1\mathbf{y}(x_1) = \mathbf{c}$

[1]here and in the sequel $||\cdot||$ denotes some vector norm (and associated matrix norm)

is called well-conditioned if there exists a moderately small constant C such that

$$\|\mathbf{y}\|_\infty \leq C[\|\mathbf{f}\|_1 + \|\mathbf{c}\|]^2$$

In [3] it was shown that well–conditioning is related to dichotomy, as defined in:

Definition 1.13. *Let there exist a projection $\mathcal{P}$ and a moderately small constant κ such that*

$$\|\mathbf{\Phi}(x)\mathcal{P}\mathbf{\Phi}^{-1}(s)\| \leq \kappa\ , \quad x \geq s\ ,$$

$$\|\mathbf{\Phi}(x)(I-\mathcal{P})\mathbf{\Phi}^{-1}(s)\| \leq \kappa\ , \quad x \leq s\ ,$$

where $\mathbf{\Phi}$ is a fundamental solution of (1.12)*. Then $\mathbf{\Phi}$ is called dichotomic.*

Remark. Both well–conditioning and dichotomy should be considered in a uniform in ϵ-setting in our present context, thus giving a specific meaning to C and κ.

We have (cf. [3])

Property 1.14. *If* (1.12) *is well-conditioned, then it is dichotomic; the constant κ for the latter can be chosen as $C(1+4C)$. If* (1.12) *is dichotomic and the BC are chosen so as to make $\mathbf{B}_0\mathbf{\Phi}(x_0) + \mathbf{B}_1\mathbf{\Phi}(x_1)$ well-conditioned (assuming $\|\mathbf{\Phi}\|_\infty = 1$), then* (1.12) *is well-conditioned.*

In this paper we like to explore in more general cases how the well conditioning of the scalar second order problem is related to that of a suitable associated first order system, which we can then relate to dichotomy (or lack thereoff, see Property 1.14).

2 Optimal first order systems

In order to assess the (well) conditioning we shall use an explicit representation for the fundamental solution in terms of two basis solutions of (1.1), say y_1 and y_2. A fundamental solution, $\hat{\mathbf{\Phi}}$ say, of (1.6) is then given by

$$(2.1) \qquad \hat{\mathbf{\Phi}}(x) := \begin{bmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{bmatrix} .$$

We shall assume homogeneous Dirichlet BC throughout, so

$$(2.2) \qquad \mathbf{B}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\quad \mathbf{B}_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\quad \mathbf{c} = \mathbf{0}\ .$$

Hence

[2]The function norms (1.2) will be used for vectors as well, where $|\cdot|$ has to be replaced by $\|\cdot\|$ then.

$$
\text{(2.3)} \qquad \mathbf{G}(x,s) = \begin{cases} \hat{\mathbf{\Phi}}(x)\mathbf{Q}^{-1}\mathbf{B}_0\hat{\mathbf{\Phi}}(x_0)\hat{\mathbf{\Phi}}^{-1}(s)\,, & x > s\,, \\ -\hat{\mathbf{\Phi}}(x)\mathbf{Q}^{-1}\mathbf{B}_1\hat{\mathbf{\Phi}}(x_1)\hat{\mathbf{\Phi}}^{-1}(s)\,, & x < s\,, \end{cases}
$$

where

$$
\text{(2.4)} \qquad \mathbf{Q} := \begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y_1(x_1) & y_2(x_1) \end{bmatrix} .
$$

In order to simplify the relations somewhat we now specifically assume, that

$$
\text{(2.5a)} \qquad y_1(x_0) = 0\,, \quad y_1(x_1) = 1\,,
$$

$$
\text{(2.5b)} \qquad y_2(x_0) = 1\,, \quad y_2(x_1) = 0\,.
$$

For $x > s$ the components $\mathbf{G}^{ij}$ of $\mathbf{G}$ are then given by

$$
\text{(2.6a)} \qquad \mathbf{G}^{11}(x,s) = \frac{-y_2(x)y_1'(s)}{W(s)}\,,
$$

$$
\text{(2.6b)} \qquad \mathbf{G}^{12}(x,s) = \frac{y_2(x)y_1(s)}{W(s)}\,,
$$

$$
\text{(2.6c)} \qquad \mathbf{G}^{21}(x,s) = \frac{-y_2'(x)y_1'(s)}{W(s)}\,,
$$

$$
\text{(2.6d)} \qquad \mathbf{G}^{22}(x,s) = \frac{y_2'(x)y_1(s)}{W(s)}\,,
$$

where $W(s)$ is the *Wronskian*

$$
\text{(2.6e)} \qquad W(s) := y_1(s)y_2'(s) - y_2(s)y_1'(s)\,,
$$

which is assumed to be nonsingular (i.e. y_1, y_2 are independent solutions). $\mathbf{G}^{12}$ is the part of the Green's function that also appears if one expresses y in terms of the inhomogenity, i.e.

$$
\text{(2.7)} \qquad y(x) = \int_{x_0}^{x_1} \mathbf{G}^{12}(x,s)\, \frac{f(s)}{\epsilon}\, ds
$$

(of course, we then need a counterpart for (2.6a) with $x < s$ as well).
Now well–conditioning of the *scalar* ODE might seem to be related to a dichtomy of suitable basis solutions y_1 and y_2, say $|y_2(x)|$ growing and $|y_1(x)|$ decaying as functions of x. The following example shows that this is not the case and that we have to include the directions of the vectors $(y_i(x), y_i'(x))^T$ as well, for establishing well–conditioning (in particular we have to require these vectors to be directionally separated).

Example 1. Consider the following family of ODE's

$$\epsilon y'' + \frac{2x}{(x+1)(2x+1)}\, y' - \frac{2}{(x+1)(2x+1)}\, y = f(x)\ , \quad x \in [0,T]\ , \quad T > 0\ .$$

If $\epsilon = 1$ we find the basis solutions

$$y_1(x) = \frac{1}{x+1} - \frac{x}{T(T+1)}\ , \quad y_2(x) = \frac{x}{T}\ .$$

Clearly y_1 is decaying and y_2 is growing (which can be made more dramatic by letting T increase). Even the vector solutions $(y_1(x), y_1'(x))^T$ and $(y_2(x), y_2'(x))^T$ of the associated form (1.6) seem like a dichotomic pair. However, although $\|\mathbf{y}_1(x)\| \sim \frac{1}{x}$ and $\|\mathbf{y}_2(x)\| \sim x$, it should be noted that both $\mathbf{y}_1(x)/\|\mathbf{y}_1(x)\| \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{y}_2(x)/\|\mathbf{y}_2(x)\| \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, showing that they become *directionally dependent*. In other words, they are *not* dichotomic according to definition 1.13 and therefore this problem is *not* well-conditioned (if one takes $T \to \infty$). This is also borne out by considering

$$\mathbf{G}^{12}(x,s) = \frac{(s+1)^2\, s}{(2s+1)(x+1)} \left[\frac{x(x+1) - T(T+1)}{T(T+1)}\right]\ , \quad x > s\ .$$

Note that the factor [ ] equals -1 for $T = \infty$ and x finite; for $x \approx s$ and $s \to \infty$ this expression becomes unbounded then. □

The next fundamental theorem shows how a bound for all blocks of $\mathbf{G}$ can be related to $\mathbf{G}^{12}$.

Theorem 2.8. *Let*

$$\text{(2.8a)} \qquad \delta^{-1} := \max_{j=1,2} \frac{\max_{x\in[x_0,x_1]} |y_j'(x)|}{\max_{x\in[x_0,x_1]} |y_j(x)|}\ .$$

Then, in the case of dichotomic solutions y_1 and y_2

$$\text{(2.8b)} \qquad \|\mathbf{G}^{11}\|_\infty \le \tfrac{C}{\delta}\|\mathbf{G}^{12}\|_\infty\ , \quad \|\mathbf{G}^{21}\|_\infty \le \tfrac{C}{\delta^2}\|\mathbf{G}^{12}\|_\infty\ , \quad \|\mathbf{G}^{22}\|_\infty \le \tfrac{C}{\delta}\|\mathbf{G}^{12}\|_\infty$$

where C is a moderate constant.

Proof. Since we obtain similar expressions for $\mathbf{G}^{ij}(x,s)$ as in (2.6) when $x < s$, we can restrict ourselves to the case $x > s$. The estimates then follow straightforwardly from (2.6) and explicit estimates for y_1, y_2 obtained in sections 3 and 4. □

We see that the various components of $\mathbf{G}$ have potentially quite different bounds. This suggests that using the transformation $\mathbf{T}$ (see (1.8)) with $\gamma = \delta$ will equilibrate the Green's functions; hence we have:

Corollary 2.9. *If we choose $\gamma = \delta$ in* (1.8), *where δ is defined in* (2.8a) *and assumed to be finite, then*

$$\|\tilde{\mathbf{G}}\|_\infty \le \frac{C_1}{\delta}\,\|\mathbf{G}^{12}\|_\infty \quad \text{and} \quad \frac{1}{\delta}\|\mathbf{G}^{12}\|_\infty \le C_2\|\tilde{\mathbf{G}}\|_\infty ,$$

where C_1 and C_2 are moderate constants.

We see from this corollary that the associated matrix vector system (1.9), with $\gamma = \delta$ gives unbiased estimates of the blocks. Clearly such a choice is also unique if δ is proportional to a power of ϵ (as it turns out to be in the next sections). We have then

Property 2.10. *Let $\|\mathbf{G}^{12}\|_\infty = O(\epsilon^\lambda)$, $\lambda \in \mathbb{R}_+$. Then the system (1.9) is well conditioned iff $\delta = O(\epsilon^\lambda)$ and γ is chosen asymptotically propertional to δ. As far as the choice of λ, this γ is unique.*

Proof. From Corollary 2.9 we see that $\|\tilde{\mathbf{G}}\|_\infty = O(1)$ by this choice. If either it would be chosen differently, or δ would not have the proper order, $\|\tilde{\mathbf{G}}\|_\infty$ would be unbounded.

□

3 Well conditioning; the case of no turning points

In this section and the subsequent one we shall investigate whether and how the conditioning of suitable scalar ODE can be related to the conditioning of certain optimally scaled associated first order systems. To this end we shall construct solutions y_1 and y_2 as defined in section 2 and try to estimate the quantities $\mathbf{G}^{12}$ and δ of Property 2.10.

Case(i). Consider a generalization of equation (1.3)

$$\text{(3.1)} \qquad \epsilon y'' = b(x)y + f(x)$$

with the homogeneous Dirichlet boundary conditions

$$\text{(3.2)} \qquad y(x_0) = 0\ , \quad y(x_1) = 0\ .$$

Nonhomogeneous conditions as well as the conditions of other types can be considered likewise. For now let us restrict our attention to the case when $b(x)$ does not have zeros for all $x \in [x_0, x_1]$ (i.e. no turning points). The corresponding homogeneous equation

$$\text{(3.3)} \qquad y'' = \frac{b(x)}{\epsilon}\, y$$

has two linearly independent solutions for which the asymptotic representations can easily be written out using the WKB technique (see, for example, [5]). We have to consider two possibilities: $b(x) > 0$ and $b(x) < 0$. For $b(x) > 0$ the normalized linearly independent solutions are

$$y_1(x) = \sqrt[4]{\frac{b(x_1)}{b(x)}}\ \exp\Big\{ -\int_x^{x_1} \frac{b^{\frac{1}{2}}(s)}{\sqrt{\epsilon}}\, ds\Big\}(1 + O(\sqrt{\epsilon}))\ ,$$

(3.4)
$$y_2(x) = \sqrt[4]{\frac{b(x_0)}{b(x)}} \exp\Big\{ -\int\limits_{x_0}^{x} \frac{b^{\frac{1}{2}}(s)}{\sqrt{\epsilon}}\, ds \Big\}(1 + O(\sqrt{\epsilon}))\ .$$

The asymptotic representation (3.4) is true under certain (nonrestricting) smoothness assumptions on $b(x)$: the function $b(x)$ should be three times continuously differentiable. We normalized $y_1(x)$ and $y_2(x)$ in such a way that $y_1(x_1) = 1 + O(\sqrt{\epsilon})$, $y_2(x_0) = 1 + O(\sqrt{\epsilon})$; it can also easily be seen that $y_1(x_0) = o(1)$ and $y_2(x_1) = o(1)$.
For $b(x) < 0$ the suitably scaled linearly independent solutions are

(3.5)
$$\tilde{y}_1(x) = \frac{1}{\sqrt[4]{|b(x)|}} \exp\Big\{ i \int\limits_{x_0}^{x} \frac{|b(s)|^{\frac{1}{2}} ds}{\sqrt{\epsilon}} \Big\}(1 + O(\sqrt{\epsilon}))\ ,$$
$$\tilde{y}_2(x) = \frac{1}{\sqrt[4]{|b(x)|}} \exp\Big\{ -i \int\limits_{x_0}^{x} \frac{|b(s)|^{\frac{1}{2}} ds}{\sqrt{\epsilon}} \Big\}(1 + O(\sqrt{\epsilon}))\ .$$

Note that such $\tilde{y}_1(x) = O(1)$, $\tilde{y}_2(x) = O(1)$. (We also keep the assumption on smoothness of $b(x)$.) It can easily be seen that the solutions $y_1(x)$ and $y_2(x)$ (see (3.4)) are dichtomic and even such that one of them is strictly increasing while the other is strictly decreasing for growing x (and sufficiently small ϵ). Therefore the problem (3.1), (3.2) is well conditioned for $b(x) > 0$ in the sense of Definition 1.6 with δ given by (2.8a). In our case

$$\delta^{-1} = \max_{x \in (x_0, x_1)} \Big\| \frac{M}{\sqrt{\epsilon}}(1 + O(\sqrt{\epsilon})) \Big\| = O\Big(\frac{1}{\sqrt{\epsilon}}\Big)\ ,$$

where $M = \max(b^{\frac{1}{2}}(x_0), b^{\frac{1}{2}}(x_1))$, i.e. $\gamma = O(\sqrt{\epsilon})$, and we should have from Property 2.10 (cf. Def. 1.6) that

(3.6) $$\|y\|_\infty \le C\, \frac{1}{\sqrt{\epsilon}}\, \|f\|_1\ .$$

We now estimate y and its conditioning constant also more directly and obtain a bound in terms of $\|f\|_\infty$ as well. The solution of (3.1), (3.2) can be written in the form[3]

(3.7) $$y(x, \epsilon) = \int\limits_{x_0}^{x_1} g(x, s) \frac{f(s)}{\epsilon}\, ds\ ,$$

where

(3.8) $$g(x, s) \doteq \begin{cases} -\dfrac{\sqrt{\epsilon}}{2} \dfrac{1}{\sqrt[4]{b(x)b(s)}} \exp\Big[-\displaystyle\int\limits_{x}^{s} \frac{b^{\frac{1}{2}}(\xi) d\xi}{\sqrt{\epsilon}} \Big]\ , & x_0 \le x \le s\ , \\ -\dfrac{\sqrt{\epsilon}}{2} \dfrac{1}{\sqrt[4]{b(x)b(s)}} \exp\Big[-\displaystyle\int\limits_{s}^{x} \frac{b^{\frac{1}{2}}(\xi) d\xi}{\sqrt{\epsilon}} \Big]\ , & s \le x \le x_1\ . \end{cases}$$

[3]for simplicity we shall write $g(x, s) := \mathbf{G}^{12}(x, s)$

Here and below by $\doteq$ we mean that in the right hand side we neglect higher order terms in ϵ. From (3.8) it follows immediately that

$$(3.9) \qquad \|g\|_\infty = \max_{x,s} |g(x,s)| \le \frac{\sqrt{\epsilon}}{2} \frac{1}{\min\limits_x \sqrt{b(x)}} \, .$$

Hence, by virtue of (3.7) and (3.9)

$$(3.10) \qquad \|y(\cdot,\epsilon)\|_\infty \le \|g\|_\infty \, \|f\|_1 \, \frac{1}{\epsilon} \le \frac{C}{\sqrt{\epsilon}} \, \|f\|_1 \, ,$$

as was hoped for (cf. (3.6)).
On the other hand, if we want to estimate $\|y(\cdot,\epsilon)\|_\infty$ through $\|f(\cdot)\|_\infty$, we write

$$(3.11) \qquad \|y(\cdot,\epsilon)\|_\infty \le \max_x \int_{x_0}^{x_1} \left|\frac{g(x,s)}{\epsilon}\right| ds \cdot \|f\|_\infty \, .$$

Let us estimate in (3.11) the part of the integral corresponding to integration from x_0 to x (the other part, corresponding to integration from x to x_1, can be estimated in a similar way):

$$(3.12) \qquad \max_x \int_{x_0}^{x} \left|\frac{g(x,s)}{\epsilon}\right| ds \doteq \max_x \left\{ \int_{x_0}^{x} \frac{1}{2\sqrt{\epsilon}} \cdot \frac{1}{\sqrt[4]{b(x)b(s)}} \cdot \right.$$

$$\left. \cdot \exp\left[- \int_s^x \frac{b^{\frac{1}{2}}(\xi)d\xi}{\sqrt{\epsilon}} \right] ds \right\} \le \frac{1}{2m} \max_x \left\{ \int_{x_0}^{x} \frac{1}{\sqrt{\epsilon}} \, e^{\frac{-m(x-s)}{\sqrt{\epsilon}}} \, ds \right\} \doteq$$

$$\doteq \frac{1}{2m^2} \max_x \left\{ e^{-\frac{m}{\sqrt{\epsilon}}x} \cdot \left(e^{\frac{m}{\sqrt{\epsilon}}x} - e^{\frac{m}{\sqrt{\epsilon}}x_0} \right) \right\} \doteq$$

$$\doteq \frac{1}{2m^2} \max_x \left[1 - e^{\frac{-m(x-x_0)}{\sqrt{\epsilon}}} \right] \le C \, .$$

Here $m = \min\limits_x \sqrt{b(x)}$.
From (3.11) and (3.12) we have

$$(3.13) \qquad \|y(\cdot,\epsilon)\|_\infty \le C\|f\|_\infty \, .$$

We see that, depending on the norm we use for the nonhomogeneous term, the conditioning constants will be different: the conditioning constant is of the order $O(\frac{1}{\sqrt{\epsilon}})$ in (3.10) and of the order $O(1)$ in (3.13).

Naturally, the transformation $\mathbf{T}$ of the linear system corresponding to (3.1) ($b(x) > 0$) with $\gamma = O(\delta) = O(\sqrt{\epsilon})$ will produce the optimal system (in the sence of the previous discussion). But in the case of the pulselike (δ-function) nonhomogeneous terms the conditioning constant, even for "optimal" systems, cannot be improved in comparison to (3.10). (Notice that the estimate (3.10) is exact if $f(x) = \delta(x - x^*)$, where $x^* \in (x_0, x_1)$.)
For the case $b(x) > 0$ the linearly independent solutions $\tilde{y}_1$ and $\tilde{y}_2$ (cf. (3.5)) are not dichotomic and therefore we cannot expect well conditioning (see [1]).
Let us now show that the estimate (3.10) is not really so bad if the right-hand side of the equation (3.1) with $b(x) > 0$ comes from discretization errors. Once more, consider the equation (1.3) with zero Dirichlet boundary conditions and e.g. the central (second order) difference scheme

$$(3.14) \qquad L_\epsilon y_i := \epsilon\, \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} - y_i = f(x_i) \ ,$$

where $h = \dfrac{x_1 - x_0}{N}$ for some N and $i = 1, \ldots, N-1$.
The global error $e_i := y(x_i) - y_i$ satisfies the difference equation with the local discretisation error d_i as an inhomogenity (cf. [1]):

$$(3.15) \qquad d_i := L_\epsilon y(x_i) - f(x_i) = \epsilon\Big(\frac{h^2}{12}\, y^{(4)}(x_i) + O(h^4)\Big) \ .$$

Here we used the fact that $y(x_i)$ satisfies the equation (1.3) at the point x_i. Now consider the following differential equation for the function $e(x)$, associated with (3.15),

$$(3.16) \qquad \epsilon e'' - e = d \ , \quad e(x_0) = e(x_1) = 0 \ .$$

Here $e(x)$ and $d(x)$ are continuous (e.g. interpolated) versions of e_i, d_i ($e(x_i) := e_i$, $d(x_i) := d_i$). From (3.15) we see that for a smooth solution y

$$\max\ |d_i| \le \epsilon C h^2 \quad \text{for some moderate } C \ .$$

The application of the estimate (3.10) to (3.16) gives:

$$\max_{1\le i\le N-1} |e_i| \doteq \|e(x)\|_\infty \le C\sqrt{\epsilon}(x_1 - x_0)h^2 \ .$$

Similarly, from (3.13)

$$\max_{1\le i\le N-1} |e_i| \doteq \|e(x)\|_\infty \le C\epsilon h^2.$$

Example 2. Consider the ODE

$$\epsilon y'' - y = f(x) \ , \quad x \in (0,1).$$

If f is assumed to be smooth, then y can be written as

$$y(x) = Ae^{-x/\sqrt{\epsilon}} + Be^{(x-1)/\sqrt{\epsilon}} + p(x),$$

where $|A|, |B|$ are not large (if $\epsilon \downarrow 0$) and p is smooth. If we now use some local equidistribution technique (cf. [1]), i.e. the grid is chosen such that all local errors are approximately equal, then we may assume that the modulus of the local error to be $\approx$ TOL, where TOL is some tolerance parameter. From the analysis (3.14)-(3.16) we then derive that $\|e\|_\infty \leq C$ TOL (cf. (3.13)). In Table 1 this is demonstrated for a problem with $y(0) = y(1) = 1$ and $f(x) := (-1 + 4\pi^2\epsilon)\sin 2\pi x$, whence $y(x) = e^{-x/\sqrt{\epsilon}} + e^{-(x-1)/\sqrt{\epsilon}} + \sin 2\pi x$. Note that this problem has layers and so d_i in (3.15) is not uniformly bounded in ϵ; hence equidistribution is mandatory.

ϵ	$\|e\|_\infty$
0.25D−01	0.96D−03
0.63D−02	0.37D−03
0.16D−02	0.10D−02
0.39D−03	0.10D−02
0.98D−04	0.89D−03
0.24D−04	0.86D−03
0.61D−05	0.80D−03

Table 1. Equidistribution with TOL= 10^{-3}.

On the other hand, given the nature of the local errors, the stiffness of the problem, characterised by ϵ, is alleviated if we choose $h = O(\epsilon^{\frac{1}{2}})$. Clearly the local errors (for the BC above) are largest in the boundary regious. In Table 2a we have given $\|e\|_\infty$ for various values of ϵ and $h = \sqrt{\epsilon} \cdot 2^{-m}$, $m = 1, 2, \ldots$. It can be seen from Tables 2b and 2c that both $\|d\|_\infty$ and $\|d\|_1/\sqrt{\epsilon}$ are nearly independent of ϵ. Both estimates (3.10) and (3.13) are fairly well confirmed.

m / ϵ	1	2	3	4	5	6
0.25D−01	0.15D−01	0.36D−02	0.90D−03	0.22D−03	0.55D−04	0.14D−04
0.63D−02	0.43D−02	0.11D−02	0.27D−03	0.68D−04	0.17D−04	0.43D−05
0.16D−02	0.39D−02	0.96D−03	0.24D−03	0.60D−04	0.15D−04	0.38D−05
0.39D−03	0.38D−02	0.96D−03	0.24D−03	0.60D−04	0.15D−04	0.38D−05
0.98D−04	0.38D−02	0.96D−03	0.24D−03	0.60D−04	0.15D−04	0.38D−05
0.24D−04	0.38D−02	0.96D−03	0.24D−03	0.60D−04	0.15D−04	0.38D−05

Table 2a. $\|e\|_\infty$; $h = \sqrt{\epsilon}\, 2^{-m}$.

m	1	2	3	4	5	6
ϵ						
0.25D−01	0.28D−01	0.65D−02	0.16D−02	0.40D−03	0.99D−04	0.25D−04
0.63D−02	0.13D−01	0.42D−02	0.12D−02	0.31D−03	0.79D−04	0.20D−04
0.16D−02	0.13D−01	0.41D−02	0.12D−02	0.31D−03	0.29D−04	0.20D−04
0.39D−03	0.13D−01	0.41D−02	0.12D−02	0.31D−03	0.79D−04	0.20D−04
0.98D−04	0.13D−01	0.41D−02	0.12D−03	0.31D−03	0.79D−04	0.20D−04
0.24D−04	0.13D−01	0.41D−02	0.12D−02	0.31D−03	0.79D−04	0.20D−04
0.61D−05	0.13D−01	0.41D−02	0.12D−02	0.31D−03	0.79D−04	0.20D−04

Table 2b. $\|d\|_\infty$; $h = \sqrt{\epsilon}\, 2^{-m}$.

m	1	2	3	4	5	6
ϵ						
0.25D−01	0.84D−01	0.22D−01	0.57D−02	0.14D−02	0.36D−03	0.91D−04
0.63D−02	0.30D−01	0.92D−02	0.25D−02	0.67D−03	0.17D−03	0.44D−04
0.16D−02	0.27D−01	0.84D−02	0.24D−02	0.62D−03	0.16D−03	0.41D−04
0.39D−03	0.26D−01	0.82D−02	0.23D−02	0.62D−03	0.16D−03	0.40D−04
0.98D−04	0.26D−01	0.82D−02	0.23D−02	0.61D−03	0.16D−03	0.40D−04
0.24D−04	0.26D−01	0.82D−02	0.23D−02	0.61D−03	0.16D−03	0.40D−04
0.61D−05	0.26D−01	0.82D−02	0.23D−02	0.61D−03	0.16D−03	0.40D−04

Table 2c. $\|d\|_1/\sqrt{\epsilon}$; $h = \sqrt{\epsilon}\, 2^{-m}$.

If, on the other hand, the layers are absent, as in the case for $y(0) = y(1) = 0$, we may expect the estimate (3.10) to be a qualitatively gross overestimation. This is confirmed by Tables 3a, 3b and 3c where we have given $\|e\|_\infty$, $\|d\|_\infty$ and $\|d\|_1/\sqrt{\epsilon}$ respectively for this problem.

h	0.25D+00	0.12D+00	0.62D−01	0.31D−01	0.16D−01	0.78D−02
ϵ						
0.25D−01	0.10D+00	0.26D−01	0.64D−02	0.16D−02	0.40D−03	0.10D−03
0.63D−02	0.39D−01	0.10D−01	0.25D−02	0.64D−03	0.16D−03	0.40D−04
0.16D−02	0.11D−01	0.29D−02	0.74D−03	0.19D−03	0.47D−04	0.12D−04
0.39D−03	0.29D−02	0.77D−03	0.19D−03	0.49D−04	0.12D−04	0.30D−05
0.98D−04	0.73D−03	0.19D−03	0.49D−04	0.12D−04	0.31D−05	0.77D−06
0.24D−04	0.18D−03	0.48D−04	0.12D−04	0.31D−05	0.77D−06	0.19D−06
0.61D−05	0.46D−04	0.12D−04	0.31D−05	0.77D−06	0.19D−06	0.48D−07

Table 3a. $\|e\|_\infty$.

h / ϵ	0.25D+00	0.12D+00	0.62D-01	0.31D-01	0.16D-01	0.78D-02
0.25D-01	0.19D+00	0.50D-01	0.13D-01	0.32D-02	0.79D-03	0.20D-02
0.63D-02	0.47D-01	0.12D-01	0.32D-02	0.79D-03	0.20D-03	0.50D-04
0.16D-02	0.12D-01	0.31D-02	0.79D-03	0.20D-03	0.50D-04	0.12D-04
0.39D-03	0.29D-02	0.78D-03	0.20D-03	0.49D-04	0.12D-04	0.31D-05
0.98D-04	0.73D-03	0.19D-03	0.49D-04	0.12D-04	0.31D-05	0.77D-06
0.24D-04	0.18D-03	0.49D-04	0.12D-04	0.31D-05	0.77D-06	0.19D-06
0.61D-05	0.46D-04	0.12D-04	0.31D-05	0.77D-06	0.19D-06	0.48D-07

Table 3b. $\|d\|_\infty$.

h / ϵ	0.25D+00	0.12D+00	0.62D-01	0.31D-01	0.16D-01	0.78D-02
0.25D-01	0.30D+00	0.16D+00	0.48D-01	0.13D-01	0.32D-02	0.80D-03
0.63D-02	0.15D+00	0.81D-01	0.24D-01	0.63D-02	0.16D-02	0.40D-03
0.16D-02	0.74D-01	0.40D-01	0.12D-01	0.31D-02	0.80D-03	0.20D-03
0.39D-03	0.37D-01	0.20D-01	0.60D-02	0.16D-02	0.40D-03	0.10D-03
0.98D-04	0.18D-01	0.10D-01	0.30D-02	0.79D-03	0.20D-03	0.50D-04
0.24D-04	0.92D-02	0.51D-02	0.15D-02	0.39D-03	0.99D-04	0.25D-04
0.61D-05	0.46D-02	0.25D-02	0.75D-03	0.20D-03	0.50D-04	0.12D-04

Table 3c. $\|d\|_1/\sqrt{\epsilon}$.

Case (ii). Consider the equation

$$\text{(3.17a)} \quad \epsilon y'' + a(x)y' + b(x)y = f(x) \ ,$$

with the homogeneous Dirichlet boundary conditions

$$\text{(3.17b)} \quad y(x_0) = 0 \ , \quad y(x_1) = 0 \ .$$

We will discuss only the case when $a(x) \neq 0$ for $x \in [x_0, x_1]$. Without loss of generality, let us take $a(x) > 0$ (the case $a(x) < 0$ can be treated in a similar way).
The two linearly independent solutions y_1 and y_2 of the homogeneous equation corresponding to (3.17) and satisfying the conditions $y_1(x_0) = 0, y_2(x_1) = 0$ and $y_1(x_1) = 1, y_2(x_0) = 1$, can be found to be (see Appendix for details):

$$\text{(3.18)} \quad y_1(x,\epsilon) \doteq \frac{\tilde{y}_1(x,\epsilon) - y_2(x,\epsilon)}{\tilde{y}_1(x_1,\epsilon)} \ ,$$

$$\text{(3.19)} \quad y_2(x,\epsilon) \doteq \frac{a(x_0)}{a(x)} \exp\Big[-\int_{x_0}^{x} \frac{a(s)ds}{\epsilon}\Big] \exp\Big[+\int_{x_0}^{x} \frac{b(s)}{a(s)}\, ds\Big] \ ,$$

where

$$(3.20)\qquad \tilde{y}_1(x,\epsilon) \doteq \exp\Big[-\int\limits_{x_0}^{x} \frac{b(s)}{a(s)}\,ds\Big]$$

is a solution of the homogeneous equation (independent of y_2). For the solutions y_1 and y_2 to be dichotomic we require $\tilde{y}_1(x,\epsilon)$ to be nondecreasing (it can easily be seen that $\tilde{y}_1 > 0$ for sufficiently small ϵ). Therefore we have the following condition on $a(x)$ and $b(x)$ (which is related to dichotomy of y_1 and y_2)

$$(3.21)\qquad \int\limits_{x_0}^{x} \frac{b(s)}{a(s)}\,ds \le 0\ .$$

Using $y_1(x,\epsilon)$ and $y_2(x,\epsilon)$ the Green's function for (3.17), (3.18) can be constructed in the form

$$(3.22)\qquad g(x,s) = \begin{cases} \dfrac{y_1(x,\epsilon)y_2(s,\epsilon)}{W(s)}\ , & x_0 \le x \le s\ , \\[2ex] \dfrac{y_2(x,\epsilon)y_1(s,\epsilon)}{W(s)}\ , & s \le x \le x_1\ . \end{cases}$$

Here $W(s) = y_1(s)y_2'(s) - y_2(s)y_1'(s)$. It can be shown from (3.18), (3.19) and (3.20) that

$$(3.23)\qquad \frac{1}{W(s)} \doteq -\frac{\epsilon}{a(x_0)}\tilde{y}_1(x_1,\epsilon)\exp\Big[+\int\limits_{x_0}^{s} \frac{a(s)ds}{\epsilon}\Big]\ .$$

Let us now find estimates for $\|y\|_\infty$ (here y is the solution of the problem (3.17)) in terms of $\|f\|_\infty$ and $\|f\|_1$. It can easily be seen that in our case

$$\delta^{-1} = \max_{j=1,2} \frac{\max\limits_{x\in[x_0,x_1]} |y_j'(x)|}{\max\limits_{x\in[x_0,x_1]} |y_j(x)|} = O(\epsilon^{-1})\ .$$

By virtue of (3.18), (3.19), (3.20) and (3.23) it follows from (3.22), that

$$(3.24)\qquad \|g\|_\infty = \max_{x,s\in[x_0,x_1]} |g(x,s)| \le C\epsilon\ , \quad C > 0\ .$$

Hence we conclude from Property 2.10 that $\|y\|_\infty \le C_1\|f\|_1$, $C_1 > 0$. In a similar way it can be seen that $\|y\|_\infty \le C_2\|f\|_\infty$, $C_2 > 0$.

4 The turning point case

Here we will discuss some examples involving equations with turning points. We start with the equation similar to (3.1).

Case (iii). Consider the problem

(4.1a) $\epsilon y'' = b(x)y + f(x),$

(4.1b) $y(x_0) = 0\ ,\quad y(x_1) = 0\ ,$

Here (4.1a) is defined in the interval (x_0, x_1), containing the point $x = 0$. Suppose, that $b(x)$ has a simple zero at $x = 0$: $b(0) = 0$. Without loss of generality we may assume that $\frac{b(x)}{x} > 0$ and is a twice continuously differentiable function. Then it is well-known (cf. [5]) that the homogeneous equation corresponding to (4.1) has twice continuously differentiable linearly independent solutions $y_1(x, \epsilon)$ and $y_2(x, \epsilon)$ such that

(4.2a)
$$y_1(x,\epsilon) = \sqrt[4]{\frac{\zeta}{b(x)}}\left\{Ai\left(\frac{\zeta}{\epsilon^{1/3}}\right) + R_1\right\},$$
$$y_2(x,\epsilon) = \sqrt[4]{\frac{\zeta}{b(x)}}\left\{Bi\left(\frac{\zeta}{\epsilon^{1/3}}\right) + R_2\right\}.$$

Here ζ is a new variable related to x by the formulae

(4.2b)
$$\tfrac{2}{3}\zeta^{3/2} = \int_0^x b^{\frac{1}{2}}(s)ds \quad \text{for } x \geq 0\ ,$$
$$\tfrac{2}{3}(-\zeta)^{3/2} = \int_x^0 (-b(s))^{\frac{1}{2}}ds \quad \text{for } x \leq 0\ .$$

Under natural conditions on $b(x)$ the remainders R_1 and R_2 in (4.2) are of the order $O(\sqrt{\epsilon})$ (see [5, pp. 397-400] for details). The solutions (4.2a) (as well as any lineair combination) are not dichotomic; this follows from the properties of the *Airy functions Ai* and *Bi*; both are oscillatory for $\zeta < 0$ $(x < 0)$.

Case (iv). Consider the problem

(4.3)
$$\epsilon y'' + xy' = f(x),$$
$$y(-1) = y(1) = 0\ .$$

Two linearly independent solutions of the corresponding homogeneous equation can immediately be written down (we normalize them in such a way that $y_1(-1) = 0$, $y_1(1) = 1$ and $y_2(-1) = 1$, $y_2(1) = 0$):

(4.4)
$$y_1(x,\epsilon) = E\int_{-1}^{x} e^{-\frac{\xi^2}{2\epsilon}}\, d\xi\ ,$$
$$y_2(x,\epsilon) = E\int_{x}^{1} e^{-\frac{\xi^2}{2\epsilon}}\, d\xi\ ,$$

where

$$E = \left(\int_{-1}^{1} e^{-\frac{\xi^2}{2\epsilon}} \, d\xi \right)^{-1} .$$

It can easily be seen that these solutions are dichotomic (y_1 is monotonically increasing and y_2 is monotonically decreasing).
We obtain

$$(4.5) \qquad \delta^{-1} = \max_{j=1,2} \frac{\max_x \ |y_j'(x)|}{\max_x \ |y_j(x)|} = E = O\Big(\frac{1}{\sqrt{\epsilon}}\Big) .$$

The Green's function for problem (4.3) can be constructed in the form, similar to (3.22):

$$(4.6) \qquad g(x,s) = \begin{cases} \dfrac{y_1(x,\epsilon)y_2(s,\epsilon)}{W(s,\epsilon)} =: g_1 \ , & -1 \le x \le s \ , \\ \dfrac{y_2(x,\epsilon)y_1(s,\epsilon)}{W(s,\epsilon)} =: g_2 \ , & s \le x \le 1 \ ; \end{cases}$$

here $W(s,\epsilon) = -E \cdot e^{-\frac{s^2}{2\epsilon}}$. Hence, the solution of (4.3) can be expressed as:

$$(4.7) \qquad y(x,\epsilon) = \int_{-1}^{1} g(x,s) \frac{f(s)}{\epsilon} \, ds .$$

From (4.7) we have the estimate

$$(4.8) \qquad \|y\|_\infty \le \tfrac{1}{\epsilon} \ \|g\|_\infty \ \|f\|_1 \ .$$

From (4.6) we see

$$(4.9) \qquad \|g\|_\infty \le \|g_1\|_\infty + \|g_2\|_\infty \ .$$

Let us estimate the first term in the sum above (the second term can be estimated in a similar manner). From (4.4) and expression for $W(s)$ we have for $s \ge 0$ (a similar expression holds for $s \le 0$)

$$(4.10) \qquad \|g_1\|_\infty = \max_{x,s \in [0,1]} \ E \ e^{\frac{s^2}{2\epsilon}} \int_{s}^{1} e^{-\frac{\xi^2}{2\epsilon}} \, d\xi \int_{-1}^{x} e^{-\frac{\xi^2}{2\epsilon}} \, d\xi =$$

$$= \max_{s \in [0,1]} \ e^{\frac{s^2}{2\epsilon}} \int_{s}^{1} e^{-\frac{\xi^2}{2\epsilon}} \, d\xi =$$

$$= \max_{s\in[0,1]} \epsilon \int_0^{\sqrt{(1-s^2)/2\epsilon}} \frac{2\eta e^{-\eta^2} d\eta}{\sqrt{s^2+2\epsilon\eta^2}} \leq C\epsilon \int_0^{\frac{1}{2\epsilon}} \frac{e^{-\eta^2} d\eta}{\sqrt{\epsilon}} \leq C\sqrt{\epsilon}\,, \quad C>0\,.$$

Here we made a change of variable $\eta^2 = \frac{\xi^2 - s^2}{2\epsilon}$.
Taking into account the analogous estimate for $\|g_2\|_\infty$, we can substitute (4.9) into (4.8) to get

$$(4.11) \qquad \|y\|_\infty \leq \frac{C\cdot\sqrt{\epsilon}}{\epsilon} \|f\|_1 = \frac{C}{\sqrt{\epsilon}} \|f\|_1 \,.$$

The estimate above is attainable, for example, for $f(x) = \delta(x)$ ($\delta(x)$ is a Dirac δ-function). Normally, the estimate for $\|y\|_\infty$ through $\|f\|_\infty$ involves a conditioning constant that is smaller in comparision to the one appearing in the estimate through $\|f\|_1$. In the case of problem (4.3) we have

$$(4.12) \qquad \|y\|_\infty \leq \tfrac{1}{\epsilon} \max_{x\in[-1,1]} \int_{-1}^{1} |g(x,s)| ds \cdot \|f\|_\infty \,.$$

The upper bound for a conditioning constant in (4.12) can be easily obtained using (4.10):

$$(4.13) \qquad \max_{x\in[-1,1]} \int_{-1}^{1} |g(x,s)| ds \leq \int_{-1}^{1} \|g\|_\infty \, ds \leq C\sqrt{\epsilon}\,,$$

and therefore

$$(4.14) \qquad \|y\|_\infty \leq \frac{C}{\sqrt{\epsilon}} \|f\|_\infty \,.$$

It is interesting to see how the estimate (4.11) pertains to discretisation errors. Given the nature of the problem it is only realistic to use non equispaced grids when (standard) finite differences are being used. Let, defining $h_i := x_{i+1} - x_i$,

$$(4.15a) \quad y''(x_i) \doteq \Big[\frac{y(x_{i+1}) - y(x_i)}{h_i} - \frac{y(x_i) - y(x_{i-1})}{h_{i-1}}\Big] \cdot \frac{2}{h_i + h_{i-1}}$$

$$(4.15b) \quad y'(x_i) \doteq \frac{y(x_{i+1}) - y(x_{i-1})}{h_i + h_{i-1}} \,.$$

Then we have a local discretization error

$$(4.16) \quad d_i \doteq \frac{\epsilon}{3}(h_i - h_{i-1})\Big[\frac{\epsilon}{3}\, y'''(x_i) + \frac{x_i}{2}\, y''(x_i)\Big] + (h_i^2 - h_i h_{i-1} + h_{i-1}^2)\Big[\frac{\epsilon}{12}\, y''''(x_i) + \frac{x_i}{6}\, y'''(x_i)\Big] \,.$$

Example 3. Consider the BVP

$$\epsilon y'' + xy' = \frac{1}{\sqrt{\pi\epsilon}}\, e^{-x^2/\epsilon} ,$$

$$y(-1) = y(1) = 0 .$$

For small ϵ, the right hand–side of the equation approximates a δ function, centered at 0. If we use a local error distributed mesh, with errors of ≈ 0.05, we may expect results accurate in approximately 2 digits. (N.B. this equidistribution was obtained through using $e^{-x^2/\epsilon}$ as a distribution function). The results for various ϵ are given in Table 4.

ϵ	$\|\|y\|\|_1$	$\|\|y\|\|_\infty$	$\|\|y\|\|_\infty/\sqrt{\epsilon}$
0.50D–01	0.71D+00	0.13D+01	0.29D+00
0.25D–01	0.72D+00	0.18D+01	0.29D+00
0.13D–01	0.76D+00	0.25D+01	0.28D+00
0.63D–02	0.71D+00	0.36D+01	0.28D+00
0.31D–02	0.77D+00	0.51D+01	0.28D+00
0.16D–02	0.79D+00	0.71D+01	0.28D+00
0.78D–03	0.78D+00	0.10D+02	0.28D+00

Table 4.

The order of conditioning constant in (4.13) is the same as the one in (4.11). The estimate (4.14) is very rough, though. It can be shown that the more accurate estimate is

(4.17) $\qquad ||y||_\infty \le C|\ln\epsilon|\;||f||_\infty .$

This estimate is attainable, for example, for $f(x) = -1$, so it cannot be improved.
It follows from (4.17) that the conditioning constant in the estimate of $||y||_\infty$ through $||f||_\infty$ is large for small ϵ.

Example 4. If we take the BVP

$$\epsilon y'' + xy' = -1 ,$$

$$x(-1) = x(1) = 0 ,$$

we obtain a confirmation of the bound (4.17), see Table 5 (where we have use a locally equidistributed error of ≈ 0.05 in order to have at least two significant digits).

ϵ	$\|y\|_\infty$	$\|y\|_\infty/\|\ln\ \epsilon\|$
0.10D+00	0.10D+01	0.43D+00
0.50D-01	0.12D+01	0.39D+00
0.25D–01	0.14D+01	0.38D+00
0.13D–01	0.17D+01	0.38D+00
0.63D–02	0.19D+01	0.38D+00
0.31D–02	0.22D+01	0.39D+00
0.16D–02	0.26D+01	0.39D+00
0.78D–03	0.29D+01	0.40D+00

Table 5.

Appendix 1

Here we show how to obtain the formulae (3.18), (3.19), (3.20). The homogeneous equation corresponding to (3.17) will be

$$(A.1)\qquad y'' + \frac{a(x)}{\epsilon}\, y' + \frac{b(x)}{\epsilon}\, y = 0.$$

We seek the two linearly independent solutions of (A.1) in the form (see [5]:

$$(A.2)\qquad \tilde{y}(x) = \exp[-\frac{1}{2\epsilon}\int_{x_0}^{x} a(s)ds]w(x).$$

After the substitution into (A.1) we obtain the following equation for w:

$$(A.3)\qquad w'' - \frac{1}{\epsilon}[\frac{a^2(x)}{4\epsilon} + \frac{a'(x)}{2} - b(x)]w = 0.$$

We assume $a(x) =\neq 0$, $|a'(x)| < \infty$, $b(x) < \infty$. For small ϵ the dominant term in the coefficient by w is $a^2(x)/4\epsilon^2$. This dominant term is always positive if $a(x) \neq 0$ and therefore the whole coefficient by w is positive for sufficiently small ϵ. For the following we can fix the sign of $a(x)$ without loss of generality. Suppose $a(x) > 0$. We now can rewrite (A.3) in a form similar to (3.3):

$$(A.4)\qquad w'' = \frac{h(x,\epsilon)}{\epsilon^2}\, w,$$

where

$$h(x,\epsilon) = \frac{a^2(x)}{4} + \epsilon\, \frac{a'(x)}{2} - \epsilon b(x) > 0$$

for small ϵ. But this means that we can write two linearly independent solutions w_1, w_2 of (A.4) using the formulae, similar to (3.4). Substituting the w_1 and w_2 into (A.2) and taking into account the asymptotics for $h^{\frac{1}{2}}$

$$h^{\frac{1}{2}}(x,\epsilon) = \frac{a(x)}{2}[1 + \frac{1}{2}\,\epsilon\Big(\frac{2a'(x)}{a^2(x)} - \frac{4b(x)}{a^2(x)}\Big) + O(\epsilon^2)] =$$

$$\frac{a(x)}{2} + \epsilon[\frac{1}{2}\,\frac{a'(x)}{a(x)} - \frac{b(x)}{a(x)}] + O(\epsilon^2),$$

we can write the formulae for $\tilde{y}_1$ and $\tilde{y}_2$ (in normalized form):

$$\tilde{y}_1(x,\epsilon) = \sqrt{\frac{a(x_0)}{a(x)}}\,\exp\Big[\int_{x_0}^{x}(\frac{1}{2}\,\frac{a'(s)}{a(s)} - \frac{b(s)}{a(s)})ds\Big] + \ldots =$$

$$\exp\Big[-\int_{x_0}^{x}\frac{b(s)}{a(s)}\,ds\Big](1+\ldots)\,,$$

$$\tilde{y}_2(x,\epsilon) = \frac{a(x_0)}{a(x)}\,\exp\Big[-\int_{x_0}^{x}\frac{a(s)}{\epsilon}\,ds\Big]\,\exp\Big[\int_{x_0}^{x}\frac{b(s)}{a(s)}\,ds\Big](1+\ldots).$$

We need to find the linearly independent solutions y_1 and y_2 of (A.1) satisfying the conditions $y_1(x_0) = 0$, $y_1(x_1) = 1$ and $y_2(x_0) = 1$, $y_2(x_1) = 0$. Naturally, $y_2 = \tilde{y}_2 + \ldots$ and y_1 is given by the linear combination (3.18) of $\tilde{y}_1$ and $\tilde{y}_2$.

References

1. U. Ascher, R. Mattheij, and R. Russell, *Numerical Solution of Boundary Value Problems for Ordinary Differential Equations*, Prentice–Hall, Englewood Cliffs, 1988.

2. W. Coppel, *Dichotomies in Stability Theory*, Springer Verlag, Berlin, 1978.

3. F. de Hoog and R. Mattheij, *On dichotomy and well- conditioning in BVP*, SIAM J. Numer. Anal., 24 (1987), pp. 98-105.

4. J. Kevorkian and J.P. Cole, *Perturbation Methods in Applied Mathematics*, Springer Verlag, Berlin 1981.

5. F. Olver, *Asymptotics and Special Function*, Academic Press, New York–London, 1974.

6. G. Söderlind and R. Mattheij, *Stability and asymptotic estimates in nonautonomous linear differential systems*, SIAM J. Math. Anal., 16 (1985), pp. 69-92.

Acknowledgement

The second author would like to thank the department of Applied Mathematics and Prof. O'Malley in particular, for the hospitality he enjoyed during his stay at of the University of Washington at Seattle.