SPIN-UP IN NON-AXISYMMETRIC CONTAINERS

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abstract

This paper reviews some general aspects of the spin-up of a fluid confined in a non-axisymmetric container, viz. the inviscid starting flow, the subsequent flow separation at the lateral walls, and the ultimate organization into a cellular flow pattern.

1. Introduction

The spin-up of a fluid in a rotating tank due to a change in the tank's rotation speed is a fundamental problem in fluid mechanics. Apart from practical engineering applications (e.g. mixing in vessels), spin-up flows are relevant to geophysical and astrophysical fluid dynamics (see e.g. Benton & Clark, 1974). In the case of an axisymmetric cylindrical container the spin-up process is mainly governed by the pumping action of the viscous Ekman layers at the horizontal boundaries of the flow domain (the rotation axis is taken vertical), whereas the sidewall plays a rather passive role. The crucial role of the Ekman layers lies in generating a secondary circulation in the flow interior. As a result of this circulation in the meridional plane, fluid elements are brought from larger to smaller radii, thereby acquiring larger azimuthal velocities in order to conserve their angular momentum. The timescale T_E for the fluid to adjust to the new rotation speed Ω is in good approximation given by $T_E = H/(\nu\Omega)^{1/2}$, with H the height of the cylinder and ν the kinematic fluid viscosity (see Greenspan & Howard, 1963). This timescale was found to apply both for the case of small changes $\Delta\Omega$ in the rotation speed (i.e. $\Delta\Omega/\Omega \ll 1$) and for the case $\Delta\Omega = \mathcal{O}(\Omega)$. Throughout the spin-up process the flow remains axisymmetric, and the sidewall plays only a minor part in the adjustment process.

The spin-up behaviour of a fluid in a non-axisymmetric container is considerably more complicated, and the lateral walls then do play a crucial role in the

J.F. Dijksman and F.T.M. Nieuwstadt (eds.), Topics in Applied Mechanics, 155–162.

dynamics. In the adjustment process leading to the ultimate state of solid-body rotation, three principal stages can be discerned, these being (1) the inviscid starting flow, characterized by zero absolute vorticity, (2) flow separation due to vorticity generation and advection at the lateral tank walls, and (3) a subsequent organization of the flow into a regular array of circulation cells. During the final stage the flow in these cells gradually decays due to the familiar spin-up/spin-down mechanism provided by the Ekman layers at the horizontal boundaries. The timescale associated with the formation of the cellular pattern is relatively short (typically less than 10 revolutions of the system), so that the overall spin-up timescale is again T_E, as in the axisymmetric case.

In order to illustrate the principal stages in the spin-up process, a number of different container shapes will be considered.

2. The starting flow

Any impulsively started flow from rest is irrotational, and this implies that the relative starting flow, arising after the tank's rotation speed is instantaneously brought from 0 to Ω, is characterized by a uniform non-zero vorticity -2Ω. Assuming that the motion in this stage is two-dimensional (free-surface deformations assumed to be small, and bottom topography being absent) the relative flow can be described by a stream function Ψ, which thus has to satisfy

$$\nabla^2\Psi = 2\Omega, \tag{1}$$

with $\Psi = 0$ at the lateral boundaries. The stream function is here defined by $\mathbf{v} = -\mathbf{k} \times \nabla\Psi$, with $\mathbf{k}$ the unit vector in the axial (i.e. vertical) direction, and $\mathbf{v}$ the (horizontal) velocity vector. Note that the position of the rotation axis is irrelevant.

- *circular tank*

  The solution of (1) on a circular domain $\{0 \le r \le 1, 0 \le \theta \le 2\pi\}$ that is bounded at $r = 0$ and satisfies $\Psi\rfloor_{r=1} = 0$ is simply

  $$\Psi(r) = \frac{1}{2}\Omega(r^2 - 1). \tag{2}$$

  The azimuthal velocity is $v_\theta = -\Omega r$, demonstrating that the relative starting flow is a rigid-body rotation in anticyclonic sense. In other words, the absolute motion is still zero at this stage.

- *semi-circular tank*

  The solution of (1) on a semi-circular domain $\{0 \le r \le 1, 0 \le \theta \le \pi\}$ can be obtained by standard Fourier techniques, yielding

  $$\Psi(r,\theta) = \Omega r^2 \sin^2\theta + \frac{8\Omega}{\pi}\sum_{n=0}^{\infty}\frac{r^{2n+1}\sin(2n+1)\theta}{(2n-1)(2n+1)(2n+3)}. \tag{3}$$

This solution has the character of a large single cell filling the domain completely. Laboratory observations (van Heijst, 1989) and numerical simulation (Andersson et al., 1992) show perfect agreement with this analytical prediction.

- *annulus with radial barrier*

  For the starting flow in an annular region $\{a \leq r \leq 1, 0 \leq \theta \leq 2\pi\}$ with a radial barrier at $\theta = 0$ (and thus at $\theta = 2\pi$), the solution is found to be (van Heijst, 1989):

$$\begin{aligned}\Psi(r,\theta) &= \Omega r^2 \sin^2\theta + \frac{32\Omega}{\pi}\sum_{n=1}^{\infty}\left[a^2\left\{\left(\frac{1}{r}\right)^{n/2} - r^{n/2}\right\} + \left(\frac{r}{a}\right)^{n/2} - \left(\frac{a}{r}\right)^{n/2}\right] \\ &\times \left[a^{-n/2} - a^{n/2}\right]^{-1} \frac{\sin\frac{1}{2}n\theta}{(n-4)n(n+4)}. \qquad (n \text{ odd}) \end{aligned} \tag{4}$$

A comparison of the streamline pattern according to (4) and the experimentally observed starting flow (visualized by streak photography) is shown in Figure 1. It is obvious that the observed motion agrees with the prediction (4) very well.

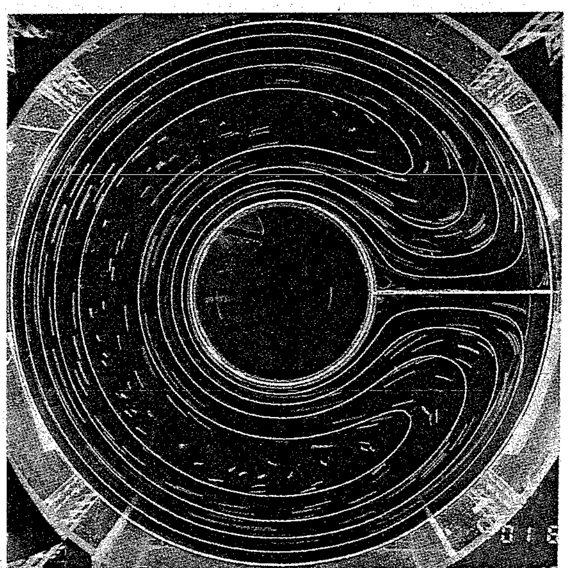

Figure 1: *The starting flow in an annular tank with a radial barrier.*

- *rectangular tank*

The solution of (1) on a rectangular domain $\{0 \leq x \leq L, 0 \leq y \leq B\}$, with x and y Cartesian coordinates, is again found by standard techniques (see van Heijst et al., 1990), yielding

$$\Psi(x,y) = (x^2 - xL)\Omega \quad + \quad \frac{8\Omega L^2}{\pi^3} \sum_{n=0}^{\infty} \frac{\sin a_n x}{(2n+1)^3} \frac{\sinh a_n(B-y) + \sinh a_n y}{\sinh a_n B}, \tag{5}$$

with $a_n \equiv (2n+1)\pi/L$. The structure of this solution is shown in Figure 2, in combination with the experimentally observed starting flow. Again, the agreement is excellent.

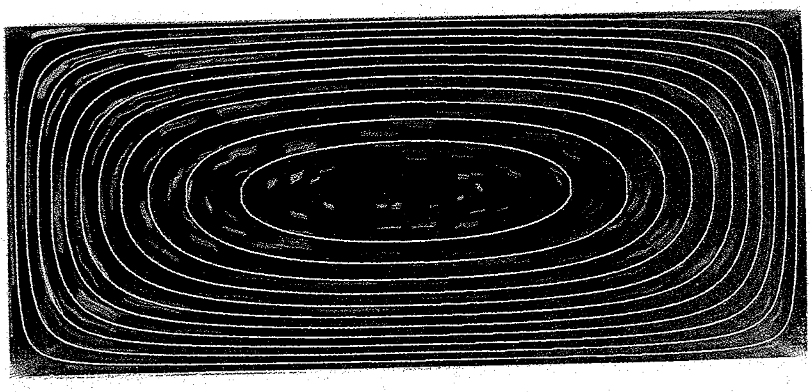

Figure 2: *The starting flow in a rectangular tank.*

In principle, solutions for the starting flow in more complex configurations can be found in the same way, although maybe not analytically. A general feature of the relative starting flow is that it has the appearance of a *single cell* with anti-cyclonic circulation (uniform vorticity -2Ω), which fills the domain completely.

3. Organization into cells

In practical situations the stage of the starting flow lasts only for a short period of time (typically less than $\frac{1}{4}T$, with $T = 2\pi/\Omega$ the rotation period), and the boundary layers at the bottom and at the sidewalls soon modify the flow pattern completely. The lateral boundary layers contain cyclonic relative vorticity, which is advected by the main flow along the sidewalls in anticyclonic direction. It is then quickly observed that cyclonic vorticity becomes concentrated in cyclonic circulation cells located in the "corners" of the domain; in other words: *flow* *separation* occurs at the lateral walls. Subsequently, the flow becomes irregular, with possibly three-dimensional motion. The two-dimensional (2D) nature of the flow is rapidly restored, however, because of the background rotation. A characteristic feature of 2D turbulent flows is the so-called "inverse energy cascade", i.e. the spectral flux of kinetic energy to the larger scales of motion. This effect can be observed from the growth of larger vortices (see e.g. McWilliams, 1984), which give the flow an ordered appearance. This phenomenon is also referred to as the *self-organization* of 2D flow.

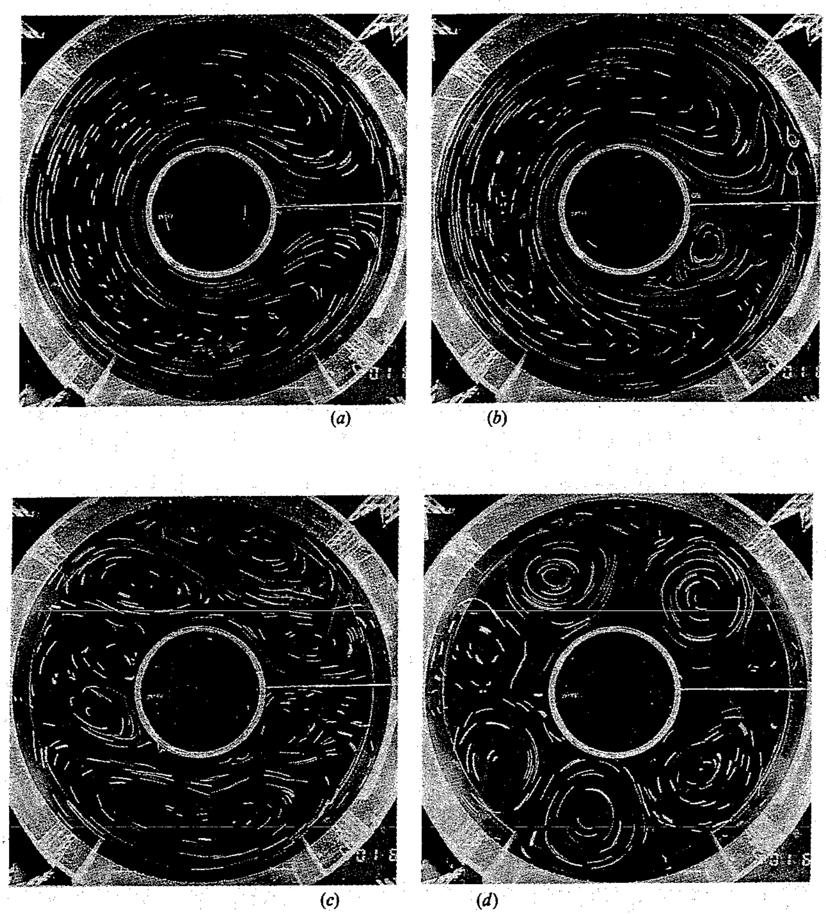

Figure 3: *The flow evolution during spin-up in an annular container with a radial barrier.*

The various stages in the flow evolution can be clearly observed in the sequence of photographs presented in Figure 3. Immediately after the starting stage (Fig. 3a), flow separation occurs near the barrier (Fig. 3b). Because the point of separation at the inner cylinder wall shifts in clockwise direction, the flow soon becomes highly irregular throughout the domain (Fig. 3c). After a few revolutions, however, one observes an organization in cells of alternating circulation sense (Fig. 3d). The cellular pattern is highly persistent, and the motion in the cells gradually decays due to the spin-up/spin-down mechanism provided by the Ekman layer at the bottom of each cell.

Although the inverse energy cascade allows a continuous growth of the cells, their maximum size is determined by the smallest length scale imposed by the geometry, here being the annulus width. Experiments with different widths indeed revealed different numbers of cells. As a rule of thumb, their number (N) is given by $N = env\{(D_0 + D_i)\pi/(D_0 - D_i)\}$, with D_0 and D_i the diameters of the outer and inner cylinders, respectively.

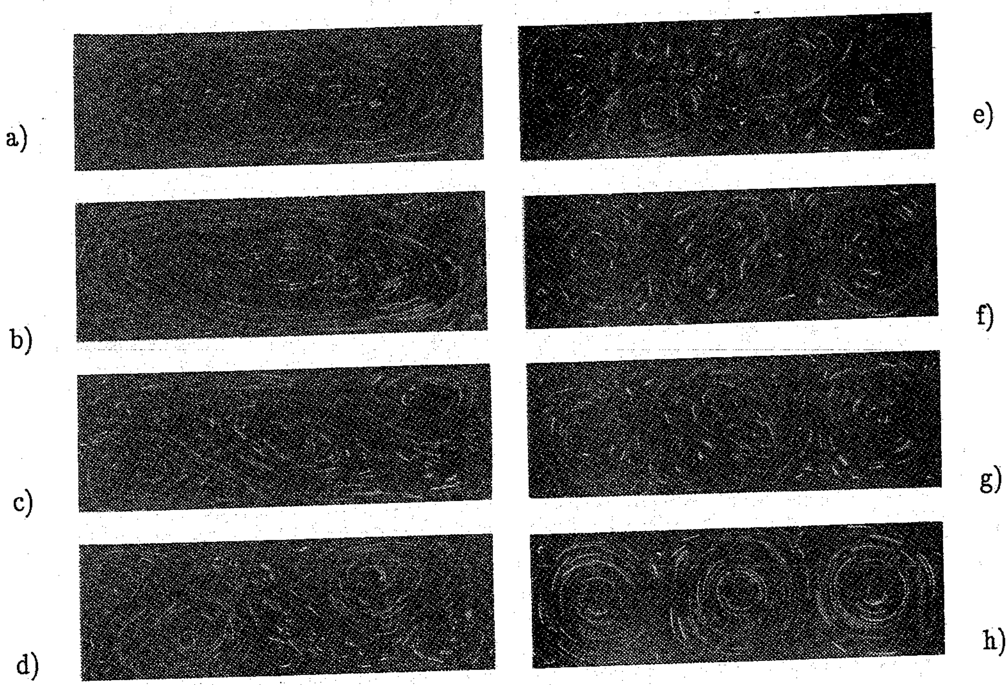

Figure 4: *The flow evolution during spin-up in a rectangular container with aspect ratio* $\delta = 3$.

The same sequence of events can be observed in the rectangular container, although the way in which the organization of the flow is achieved is slightly different, owing to the rectangular configuration. The flow evolution during spin-up is illustrated by the sequence of photographs in Figure 4 for a rectangular tank of aspect ratio $\delta \equiv L/B = 3$. Immediately after the starting stage (Fig. 4a), cyclonic vorticity cells are seen to develop in the corners downstream of the longer sidewalls (Fig. 4b,c). While gradually increasing in size, these cells move radially inwards to the centre, thus pinching the original anticyclonic central cell (Fig. 4d,e). Once they get close enough, these cyclonic cells merge (Fig. 4f,g), thus leading to the formation of a single central cell. In the ultimate stage, the central cyclonic cell is flanked by two cells with anticyclonic circulation. The number of cells ($N = 3$) in this organized state corresponds with the aspect ratio ($\delta = 3$) of the tank. Detailed observations have revealed (see van Heijst et al., 1990) that the inward motion of the cyclonic cells initially formed in the tank corners, and their subsequent merging leading to the central cyclonic cell is caused by the parabolic curvature of the free surface: cyclonic vortices tend to drift to shallower parts of the tank, whereas anticyclonic vortices tend to move into deeper parts. In free-surface experiments with the tank centre positioned on the rotation axis, one can thus expect to observe an odd number of cells, irrespective of the aspect ratio δ. This feature is illustrated in Figure 5, where the organized flow in the tank with $\delta = 4$ consists of 5 cells of alternating circulation sense.

Figure 5: *The organized cellular flow pattern in a rectangular tank with aspect ratio* $\delta = 4$.

The general effect of topography variations (associated either with free-surface deformation or with bottom topography) has recently been investigated in a rectangular tank with a linearly sloping bottom (van Heijst *et al.*, 1992). It was found that no organized cellular flow pattern is established when the topography is steeper than some critical angle, and that instead the relative flow remains irregular and unsteady while decaying, thus enhancing the mixing efficiency of the flow system.

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