A new upper bound for the cardinality of 2-distance sets in Euclidean space

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Abstract

It is proved that the cardinality of a 2-distance set S in Euclidean d-dimensional space satisfies

$$\mathrm{card}(S) \le \tfrac{1}{2}(d+1)(d+2)\ .$$

Introduction

A set S in Euclidean d-space E^d is called a 2-distance set if the distance between distinct points of S assumes only two values.
The maximum size of such a set is 5 in E^2 (Kelly), and 6 in E^3 (Croft). Delsarte, Goethals and Seidel [1] treated the case where the points of S lie on a sphere. Their argument can be modified to obtain the bound $\mathrm{card}(S) \le \frac{1}{2}(d+1)(d+4)$ for general 2-distance sets as was established by Larman, Rogers and Seidel [2] . E. and E. Bannai [3] showed that equality doesn't occur in this case. The proof of Larman, Rogers and Seidel can be modified again to obtain $\mathrm{card}(S) \le \frac{1}{2}(d+1)\ d+2)$.

Theorem.

Let S be a 2-distance set in E^d, then

$$\mathrm{card}(S) \le \tfrac{1}{2}(d+1)(d+2)\ .$$

Proof.

Let a and b the distances in S. For each point s in S and $x \in E^d$ we define

$$F_s(x) = \frac{1}{a^2b^2}\left(\|x-s\|^2 - a^2\right)\left(\|x-s\|^2 - b^2\right)\ .$$

These functions form an independent set of functions since $F_s(t) = \delta_{s,t}$ for all $s,t \in S$. They are linear combinations of the following functions:

$$\|x\|^4 \;;\; \|x\|^2 x_i \;;\; x_i x_j \;;\; x_i \;;\; 1 \;; \qquad \text{where } 1 \le i \le j \le d \;.$$

Hence the total number of functions F_s cannot exceed

$$1 + d + \tfrac{1}{2}d(d+1) + d + 1 = \tfrac{1}{2}(d+1)(d+4) \;.$$

We proceed to show that in fact the set

$$\{F_s(x) \;,\; x_i \;,\; 1 \;\mid\; s \in S \;,\; 1 \le i \le d\}$$

is linearly independent, which implies

$$\text{card}(S) + d + 1 \le \tfrac{1}{2}(d+1)(d+4)$$

and hence

$$\text{card}(S) \le \tfrac{1}{2}(d+1)(d+2) \;.$$

Now suppose we have

$$(1) \qquad \sum_{s\in S} c_s F_s(x) + \sum_{i=1}^{d} c_i x_i + c = 0 \;.$$

Inserting s in relation (1) we get

$$(2) \qquad c_s + \sum_i c_i s_i + c = 0 \;.$$

Inserting ke_i in (1), where e_i is the i-th column of the unit matrix, we get

$$(3) \qquad \frac{1}{a^2 b^2} \sum_s c_s (k^2 - 2ks_i + \|s\|^2 - a^2)(k^2 - 2ks_i + \|s\|^2 - b^2) + \\ + kc_i + c = 0 \;.$$

Comparing the coefficients of k^4 and of k^3 we obtain

(4) $$\sum_s c_s = 0 \qquad \text{and} \qquad \sum_s c_s s_i = 0$$

for $i = 1,\dots,d$.

Multiply relation (2) by c_s and sum over all $s \in S$:

(5) $$\sum_s c_s^2 + \sum_i c_i \sum_s c_s s_i + c \sum_s c_s = 0 .$$

Now (4) and (5) yield $c_s = 0$ for all $s \in S$, whence also $c = c_i = 0$ for $i = 1,\dots,d$. This completes the proof of the theorem.

References

[1] Ph. Delsarte, J.M. Goethals and J.J. Seidel;
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[2] D.G. Larman, C.A. Rogers and J.J. Seidel;
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[3] E. and E. Bannai;
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