

A new upper bound for the cardinality of 2-distance sets in Euclidean space

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A NEW UPPER BOUND FOR THE CARDINALITY OF
2-DISTANCE SETS IN EUCLIDEAN SPACE

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Abstract

It is proved that the cardinality of a 2-distance set S in Euclidean d -dimensional space satisfies

$$\text{card}(S) \leq \frac{1}{2}(d + 1)(d + 2) .$$

Introduction

A set S in Euclidean d -space E^d is called a 2-distance set if the distance between distinct points of S assumes only two values.

The maximum size of such a set is 5 in E^2 (Kelly), and 6 in E^3 (Croft).

Delsarte, Goethals and Seidel [1] treated the case where the points of S lie on a sphere. Their argument can be modified to obtain the bound $\text{card}(S) \leq \frac{1}{2}(d + 1)(d + 4)$ for general 2-distance sets as was established by Larman, Rogers and Seidel [2]. E. and E. Bannai [3] showed that equality doesn't occur in this case. The proof of Larman, Rogers and Seidel can be modified again to obtain $\text{card}(S) \leq \frac{1}{2}(d + 1)(d + 2)$.

Theorem.

Let S be a 2-distance set in E^d , then

$$\text{card}(S) \leq \frac{1}{2}(d + 1)(d + 2) .$$

Proof.

Let a and b the distances in S . For each point s in S and $x \in E^d$ we define

$$F_s(x) = \frac{1}{a^2 b^2} (\|x - s\|^2 - a^2)(\|x - s\|^2 - b^2) .$$

These functions form an independent set of functions since $F_s(t) = \delta_{s,t}$ for all $s, t \in S$. They are linear combinations of the following functions:

$$\|x\|^4 ; \|x\|^2 x_i ; x_i x_j ; x_i ; 1 ; \quad \text{where } 1 \leq i \leq j \leq d .$$

Hence the total number of functions F_s cannot exceed

$$1 + d + \frac{1}{2}d(d + 1) + d + 1 = \frac{1}{2}(d + 1)(d + 4) .$$

We proceed to show that in fact the set

$$\{F_s(x) , x_i , 1 \mid s \in S , 1 \leq i \leq d\}$$

is linearly independent, which implies

$$\text{card}(S) + d + 1 \leq \frac{1}{2}(d + 1)(d + 4)$$

and hence

$$\text{card}(S) \leq \frac{1}{2}(d + 1)(d + 2) .$$

Now suppose we have

$$(1) \quad \sum_{s \in S} c_s F_s(x) + \sum_{i=1}^d c_i x_i + c = 0 .$$

Inserting s in relation (1) we get

$$(2) \quad c_s + \sum_i c_i s_i + c = 0 .$$

Inserting ke_i in (1), where e_i is the i -th column of the unit matrix, we get

$$(3) \quad \frac{1}{a^2 b^2} \sum_s c_s (k^2 - 2ks_i + \|s\|^2 - a^2)(k^2 - 2ks_i + \|s\|^2 - b^2) + kc_i + c = 0 .$$

Comparing the coefficients of k^4 and of k^3 we obtain

$$(4) \quad \sum_s c_s = 0 \quad \text{and} \quad \sum_s c_s s_i = 0$$

for $i = 1, \dots, d$.

Multiply relation (2) by c_s and sum over all $s \in S$:

$$(5) \quad \sum_s c_s^2 + \sum_i c_i \sum_s c_s s_i + c \sum_s c_s = 0.$$

Now (4) and (5) yield $c_s = 0$ for all $s \in S$, whence also $c = c_i = 0$ for $i = 1, \dots, d$. This completes the proof of the theorem.

References

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