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Stability of non-smooth systems

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Editors: Kanat Camlibel, Nathan van de Wouw, Henk Nijmeijer
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Introduction

In this report, the results on the stability of non-smooth systems attained within WorkPackage 5 of SICONOS are presented.

Let us first motivate why the issue of stability is a crucial one within the scope of WorkPackage 5 and the project of SICONOS in general. Firstly, from the mere viewpoint of the analysis of the dynamic behaviour of non-smooth systems, knowledge on the stability of solutions of these systems is imperative for gaining understanding on the global dynamics of such systems. Secondly, from the perspective of bifurcation analysis, results on stability are of vital importance, since a bifurcation is often directly related to the (change in) stability properties of a certain solution undergoing the bifurcation. In this sense, there is a direct link with the work of WorkPackage 4. Finally, the ultimate goal of WorkPackage 5 is to develop design techniques for the synthesis of stable controllers and observers for various classes of non-smooth systems. Stability results will form the foundation on which such results will be built.

A second question that may be raised is why a considerable effort in this direction is needed given the fact that an extensive amount of literature already exists on the stability of smooth systems. The first reason is that many stability results for smooth systems are based on properties of such systems which are defined by the grace of their smoothness. For example, one may think of stability results for equilibrium points based on the linearization of smooth systems. Such linearization may not be well-defined everywhere even for systems with mild non-smoothness (e.g. systems described by differential equations with a continuous, but non-differentiable vector field). Consequently, the applicability of such results does not translate directly to non-smooth systems. A second reason may be recognized in the fact that non-smooth systems may exhibit types of limit solutions (for which the stability should be assessed), which do not exist for smooth systems; for example, equilibrium sets in Filippov-type systems.

In order to structure the results discussed in this report, we will categorize non-smooth systems with respect to their degree of non-smoothness or discontinuity:

- Non-smooth, continuous systems, such as systems described by continuous differential equations with a continuous, non-smooth vector field, which has a discontinuous Jacobian; An example can be a mechanical system with a one-sided flexible support.
- Discontinuous systems with a time-continuous state, such as systems described by differential equations with a discontinuous vector field, which can be transformed into differential inclusions with a set-valued right-hand side (often called Filippov systems, referring to the Filippov solution concept). Mechanical systems with Coulomb friction, modelled by a set-valued force law, are a well-known and important engineering example of such systems;
- Discontinuous systems with state jumps. Sometimes such systems are called impulsive systems. Mechanical systems with impacts, inducing jumps in the velocity can be formulated within this class of systems.

It is emphasized that different types of solution concepts are needed to define the solution of these different types of systems.

Moreover, the material in this report will involve stability results for different types of limit solutions of non-smooth systems:

- equilibrium points, being either on or outside switching surfaces (indicating the surfaces in state space at which the system exhibits non-smoothness and/or discontinuity);
- equilibrium sets; such limits sets can typically occur in Filippov systems;
- periodic solutions.

We like to stress that the specific results involve the formulation of conditions under which such limit solutions exhibit certain stability properties. These stability properties may vary from stability in the sense of Lyapunov, asymptotic stability to attractivity.

In every subsequent chapter the authors have chosen a mathematical framework, ranging from complementarity systems, piece-wise smooth differential equations, evolution variational inequalities and differential inclusions to measure differential equations and, which is most suitable to describe the type of system under study and the stability result obtained. In some results, the perspective of a purely mathematical system formulation, e.g. a class of complementarity systems, is chosen, whereas other contributions focus on mechanical systems with friction and/or impacts or electrical networks, in which the physical properties are used explicitly to obtain the results.

The organization of the report is as follows. It consists of six chapters each reporting the contribution of a different team. The first contribution summarizes the contribution of the French team in the frameworks of evolution variational inequalities (EVIs) and measure differential inclusions (MDIs). The authors report various stability results, mainly in the sense of Lyapunov stability, for EVIs and MDIs. Among these results, one can find extensions of absolutely stability problem and Krakovskii-LaSalle invariance principle to nonsmooth dynamical systems. The second contribution concentrates on the cone complementarity systems (CCSs). The solutions of these systems go through a succession of periods of smooth evolution separated by instantaneous events that mark transitions of one set of laws of evolution to another. Events may be externally induced or internally induced. In a framework that allows state jumps, the authors provide sufficient conditions for the (asymptotic) stability of CCSs. For somewhat restricted subclasses of CCSs, necessary and sufficient conditions for asymptotic stability are also presented. The third contribution concerns periodic orbits of certain piecewise linear hybrid systems. After presenting a new version of the generalized Bendixson's criterion, conditions in terms of linear matrix inequalities are given for the absence of limits cycles in linear relay feedback systems. In order to obtain these results, certain stability/dichotomy-like properties of solutions with respect to each other are used.

The first three contributions consider fairly general classes of nonsmooth systems. Another approach is to consider specific classes of systems. This allows to exploit further the underlying structure that is imposed by the problem under consideration. The last three contributions take this approach and look at stability issues in the contexts of power converters and mechanical systems.

The fourth contribution considers systems that are obtained from DC-to-DC buck converters by averaging techniques and investigates bifurcations of the equilibrium points, in which the stability of the equilibrium points plays a central role. It presents theoretical as well as experimental results. The fifth contribution is devoted to mechanical systems subject to unilateral constraints. The presence of unilateral constraints calls for a solution concept that allows jumps in the velocities. This contribution addresses the problem of stabilization of equilibrium points and trajectories. The sixth and the final contribution is also devoted to mechanical systems, more precisely mechanical systems with Coulomb friction. Due to the set-valued nature of the adopted friction law, the dynamics of these systems can be described by differential inclusions (of Filippov-type). In this chapter, conditions for the attractivity of equilibrium sets of such systems are formulated.

2

Some Results on the Stability and Stabilization of 1^{st} and 2^{nd} Order Nonsmooth Systems

Jean-Matthieu Bourgeot, Sophie Chareyron, Bernard Brogliato, and Pierre-Brice Wieber

INRIA Rhone-Alpes, France

1 Introduction

The works of CO1 concern several types of nonsmooth dynamical systems which are either complementarity systems, or closely related like evolution variational inequalities and measure differential inclusions. Various aspects of stability have been studied: the regulation problem (extension of the Lagrange-Dirichlet theorem, regulation of the position and contact forces), the tracking control problem (how to extend the well-known passivity-based controllers, to the case of a Lagrangian system subject to frictionless unilateral constraints), extensions of the Lyapunov second method and of the Krakovskii-LaSalle invariance principle to a class of evolution variational inequalities (relying on the co-positivity of matrices with respect to the admissible domain of the state space). These studies generally make large use of convex analysis tools and of modelling formalisms using complementarity problems and differential inclusions, like Moreau's sweeping process. The types of formalisms that have been studied, handle first order systems (with absolutely continuous solutions), and second order systems (with solutions of bounded variation in time). In a complementarity systems language, they roughly correspond to relative degree zero, one and two LCS.

2 Formalisms

2.1 Evolution variational inequalities

Let $K \subset \mathbb{R}^n$ be a nonempty closed convex set. Let $A \in \mathbb{R}^{n\times n}$ be a given matrix and $F : \mathbb{R}^n \to \mathbb{R}^n$ a nonlinear operator. For $(t_0, x_0) \in \mathbb{R} \times K$, we consider the problem $P(t_0, x_0)$: Find a function $t \to x(t)$ $(t \geqslant t_0)$ with $x \in C^0([t_0, +\infty); \mathbb{R}^n)$, $\frac{dx}{dt} \in L^\infty_{\text{loc}}(t_0, +\infty; \mathbb{R}^n)$ and such that:

$$\begin{cases} x(t) \in K,\ t \geqslant t_0 \\ \langle \frac{dx}{dt}(t) + Ax(t) + F(x(t)), v - x(t)\rangle \geqslant 0,\ \forall v \in K,\ \text{a.e. } t \geqslant t_0 \\ x(t_0) = x_0 \end{cases} \tag{1}$$

Here $\langle .,. \rangle$ denotes the Euclidean scalar product in $\mathbb{R}^n$. The corresponding norm is denoted by $\| \ . \ \|$. The system in (1) is an evolution variational inequality which we denote as LEVI(A, K) when $F \equiv 0$. It follows from standard convex analysis that (1) can be rewritten equivalently as the differential inclusion

$$\begin{cases} \frac{dx}{dt}(t) + Ax(t) + F(x(t)) \in -N_K(x(t)) \\ x(t) \in K,\ t \geqslant t_0 \end{cases} \tag{2}$$

where $N_K(x(t)) = \{s \in \mathbb{R}^n : \langle s, v - x(t)\rangle \leqslant 0,\ \forall\, v \in K\}$ is the normal cone to K at $x(t)$. In case $K = \{x \in \mathbb{R}^n : Cx + d \geqslant 0\}$ for some matrix $C \in \mathbb{R}^{m\times n}$ and vector $d \in \mathbb{R}^m$, we can rewrite (1) as

$$\begin{cases} \frac{dx}{dt}(t) + Ax(t) + F(x(t)) = C^T\lambda \\ 0 \leqslant y = Cx(t) + d \perp \lambda \geqslant 0 \end{cases} \tag{3}$$

where $\lambda \in \mathbb{R}^m$ is a Lagrange multiplier, and the second line of (3) means that both y and λ have to be non-negative, and orthogonal. System (3) belongs to the class of Linear Complementarity Systems (LCS) with a relative degree $r_{y\lambda} \geqslant 1$ between y and λ. The equivalences between various first-order formalisms has been established in [1].

2.2 Second order Complementarity Problem

Dynamical systems which may experience non-permanent contacts of perfectly rigid bodies can be described through two different formulations: one based on complementarity problem that is presented in this section 2.2; the other one based Measure Differential Equation is presented in the next sections 2.3.

First, let's see how systems with non-permanent contacts can be described through a second order complementarity problem. These systems are complementarity Lagrangian systems, with Lagrangian function $\mathcal{L} = T(\boldsymbol{q}, \dot{\boldsymbol{q}}) - U(\boldsymbol{q})$, where $T(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \frac{1}{2}\dot{\boldsymbol{q}}^T\boldsymbol{M}(q)\dot{q}$ is the kinetic energy, $U(q)$ is the differentiable potential energy. The dynamics may be written as:

$$\begin{cases} \boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{G}(\boldsymbol{q}) = \boldsymbol{u} + \nabla \boldsymbol{F}(\boldsymbol{q})\boldsymbol{\lambda}_{\boldsymbol{q}} \\ \boldsymbol{F}(\boldsymbol{q}) \geqslant 0, \quad \boldsymbol{F}(\boldsymbol{q})^T\boldsymbol{\lambda}_{\boldsymbol{q}} = 0, \quad \boldsymbol{\lambda}_{\boldsymbol{q}} \geqslant 0 \\ \text{Collision rule} \end{cases} \tag{4}$$

where $\boldsymbol{q} \in \mathbb{R}^n$ is a vector of generalized coordinates, $\boldsymbol{M}(\boldsymbol{q}) = \boldsymbol{M}^T(\boldsymbol{q}) \in \mathbb{R}^{n\times n}$ is the positive definite inertia matrix, $\boldsymbol{F}(\boldsymbol{q}) \in \mathbb{R}^m$ represent the distance to the constraints, $\boldsymbol{\lambda}_{\boldsymbol{q}} \in \mathbb{R}^m$ are the Lagrangian multipliers associated to the constraints, $u \in \mathbb{R}^n$ is the vector of generalized torque inputs, $C(\boldsymbol{q}, \dot{q})$ is the matrix of Coriolis and centripetal forces, $\boldsymbol{G}(\boldsymbol{q})$ contains conservative forces. ∇ denotes the Euclidean gradient, i.e. $\nabla \boldsymbol{F}_i(\boldsymbol{q}) = \left(\frac{\partial \boldsymbol{F}_i}{\partial q_1}, \cdots, \frac{\partial \boldsymbol{F}_i}{\partial q_n}\right)^T \in \mathbb{R}^n$ and $\nabla \boldsymbol{F}(\boldsymbol{q}) = (\nabla \boldsymbol{F}_1(\boldsymbol{q}), \cdots, \nabla \boldsymbol{F}_m(\boldsymbol{q})) \in \mathbb{R}^{n\times m}$. The impact times will be denoted generically as t_k in the following. We assume that the functions $\boldsymbol{F}_i(\cdot)$ are continuously differentiable and that $\nabla \boldsymbol{F}_i(\boldsymbol{q}(t_k)) \neq 0$ for all t_k.

A major discrepancy of complementarity systems compared to systems with switching vector fields, is that their state may be discontinuous, and that they may live on lower-dimensional spaces. This creates serious difficulties in their study [2] [8].

The Lagrangian system in (4) is *fully actuated*, i.e. $\dim(\boldsymbol{u}) = \dim(\boldsymbol{q})$. This excludes for instance lumped joint flexibilities. In case $\dim(\boldsymbol{u}) < \dim(\boldsymbol{q})$ the system is said to be *underactuated* and the control problem is much harder to solve. The first instance in the Control and Robotics literature where such a complementarity model has been used, is in [10]. One very specific feature of systems as in (4) is their intrinsic nonsmoothness, which hampers one to tangentially linearize them in the neighborhood of trajectories. Consequently linear controllers generally fail to stabilize such complementarity systems, and nonlinear feedback controllers have to be designed.

The description of a collision rule is needed to complete the description of second order complementarity problem, it is a relation based on a geometrical approach, between the post-impact velocities and the pre-impact velocities. In this work, the collision rule is chosen as in [13]:

$$\begin{aligned} \dot{\boldsymbol{q}}(t_k^+) &= -e_n\dot{\boldsymbol{q}}(t_k^-) \\ &\quad + (1+e_n)\arg\min_{z\in\mathcal{T}(\boldsymbol{q})} \tfrac{1}{2}[z - \dot{\boldsymbol{q}}(t_k^-)]^T\boldsymbol{M}(\boldsymbol{q}(t_k))[z - \dot{\boldsymbol{q}}(t_k^-)] \end{aligned} \tag{5}$$

where $\dot{\boldsymbol{q}}(t_k^+)$ is the post impact velocity, $\dot{\boldsymbol{q}}(t_k^-)$ is the pre-impact velocity, and e_n is the restitution coefficient, $e_n \in [0,1]$. We refer to the next section for the geometrical description of the constraint set $\Phi(\boldsymbol{q})$ and its tangent cone $\mathcal{T}(\boldsymbol{q})$ at $\boldsymbol{q}(t)$ (see figures 1 where the sets $\boldsymbol{q}$, $\mathcal{T}(\boldsymbol{q})$ are depicted).

2.3 Second order Measure Differential Equation

In this section, we first give a geometrical description of dynamical systems which may experience non-permanent contacts of perfectly rigid bodies and then a formulation for nonsmooth dynamical system is given based on Measure Differential Equation.

Geometrically speaking, the non-overlapping of rigid bodies can be expressed as a constraint on the position of the corresponding dynamical system, a constraint that will take the form here of a closed set $\Phi \subset \mathbb{R}^n$, assumed to be time-invariant, in which the generalized coordinates are bound to stay [13]:

$$\forall t \in \mathbb{R},\ \boldsymbol{q}(t) \in \Phi.$$

This way, contact phases correspond to phases when $\boldsymbol{q}(t)$ lies on the boundary of Φ, and non-contact phases to phases when $\boldsymbol{q}(t)$ lies in the interior of Φ. We can define then for all $\boldsymbol{q} \in \Phi$ the tangent cone [9]

$$\mathcal{T}(\boldsymbol{q}) = \left\{\boldsymbol{v} \in \mathbb{R}^n : \ \exists \tau_k \to 0,\ \tau_k > 0, \exists \boldsymbol{q}_k \to \boldsymbol{q},\ \boldsymbol{q}_k \in \Phi \text{with } \frac{\boldsymbol{q}_k - \boldsymbol{q}}{\tau_k} \to \boldsymbol{v}\right\},$$

and we can readily observe that if the velocity $\dot{\boldsymbol{q}}(t)$ has a left and right limit at an instant t, then obviously $-\dot{\boldsymbol{q}}^-(t) \in \mathcal{T}(\boldsymbol{q}(t))$ and $\dot{\boldsymbol{q}}^+(t) \in \mathcal{T}(\boldsymbol{q}(t))$.

Now, note that $\mathcal{T}(\boldsymbol{q}) = \mathbb{R}^n$ in the interior of the domain Φ, but it reduces to a half-space or even less on its boundary (Fig. 1): if the system reaches this boundary with a velocity $\dot{\boldsymbol{q}}^- \notin \mathcal{T}(\boldsymbol{q})$, it won't be able to continue its movement with a velocity $\dot{\boldsymbol{q}}^+ = \dot{\boldsymbol{q}}^-$ and still stay in Φ (Fig. 1). A discontinuity of the velocity will have to occur then, corresponding to an impact between contacting rigid bodies, the landmark of *nonsmooth* dynamical systems.

We can also define for all $\boldsymbol{q} \in \Phi$ the normal cone [9]

$$\mathcal{N}(\boldsymbol{q}) = \left\{\boldsymbol{v} \in \mathbb{R}^n : \ \forall \boldsymbol{q}' \in \Phi,\ \boldsymbol{v}^T(\boldsymbol{q}' - \boldsymbol{q}) \leqslant 0\right\},$$

and we will see in the inclusion (8) that it is directly related to the reaction forces arising from the contacts between rigid bodies.

Now, note that $\mathcal{N}(\boldsymbol{q}) = \{0\}$ in the interior of the domain Φ, and it contains at least a half-line of $\mathbb{R}^n$ on its boundary (Fig. 1): this will imply the obvious observation that non-zero contact forces may be experienced only on the boundary of the domain Φ, precisely when there is a contact. Discontinuities of the contact forces might be induced because of that, what will be discussed later.

In the end, note that with these definitions, the state $(\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t))$ appears now to stay inside the set

$$\Omega = \left\{(\boldsymbol{q}, \dot{\boldsymbol{q}}) : \ \boldsymbol{q} \in \Phi,\ \dot{\boldsymbol{q}} \in \mathcal{T}(\boldsymbol{q})\right\}.$$

Discontinuities of the velocity may have to occur in the case of Lagrangian systems experiencing non-permanent contacts between rigid bodies. A mathematically rigorous way to allow such discontinuities in the dynamics of Lagrangian system has been proposed through Measure Differential Equation [13, 14],

$$\boldsymbol{M}(\boldsymbol{q})\, d\dot{\boldsymbol{q}} + \boldsymbol{N}(\boldsymbol{q}, \dot{\boldsymbol{q}})\, \dot{\boldsymbol{q}}\, dt = \boldsymbol{f}\, dt + d\boldsymbol{r}, \tag{6}$$

with dt the Lebesgue measure and $d\boldsymbol{r}$ the reaction forces arising from the contacts between rigid bodies, an abstract measure which may not be Lebesgue-integrable. This way, the measure acceleration $d\dot{\boldsymbol{q}}$ may not be Lebesgue-integrable either so that the velocity may not be locally absolutely continuous anymore but only with locally bounded variations, $\dot{\boldsymbol{q}} \in \mathrm{lbv}([t_0, T], \mathbb{R}^n)$ [13, 14]

Functions with locally bounded variations have left and right limits at every instant, and we have for every compact subinterval $[\sigma, \tau] \subset [t_0, T]$

$$\int_{[\sigma,\tau]} d\dot{\boldsymbol{q}} = \dot{\boldsymbol{q}}^+(\tau) - \dot{\boldsymbol{q}}^-(\sigma).$$

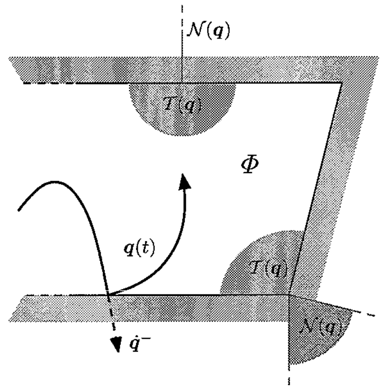

Fig. 1: Examples of tangent cones $\mathcal{T}(\boldsymbol{q})$ and normal cones $\mathcal{N}(\boldsymbol{q})$ on the boundary of the domain Φ, and example of a trajectory $\boldsymbol{q}(t) \in \Phi$ that reaches this boundary with a velocity $\dot{\boldsymbol{q}}^- \notin \mathcal{T}(\boldsymbol{q})$.

Following [13], we will consider that the non-permanent contacts that may be experienced by our Lagrangian systems are perfectly unilateral, frictionless and soft. Expressing the $\mathbb{R}^n$ valued measure $d\boldsymbol{r}$ as the product of a non-negative real measure $d\mu$ and a $\mathbb{R}^n$ valued function $\boldsymbol{r}'_\mu \in L^1_{loc}([t_0, T], d\mu; \mathbb{R}^n)$,

$$d\boldsymbol{r} = \boldsymbol{r}'_\mu \, d\mu, \tag{7}$$

The set of constraints is said frictionless and soft if the total contact impulse $d\boldsymbol{r}$ satisfies

$$\forall t \in \mathbb{R}, \qquad \dot{\boldsymbol{q}}^+(t) \in \mathcal{V}(\boldsymbol{q}),$$

$$-\boldsymbol{r}'_\mu(t) \in \mathcal{N}(\boldsymbol{q}) \tag{8}$$

$$\dot{\boldsymbol{q}}^+(t)^T \boldsymbol{r}'_\mu(t) = 0. \tag{9}$$

For a more in-depth presentation of these concepts and equations which are quite subtle, the interested reader should definitely refer to [14].

3 Stability analysis

3.1 Stability of evolution variational inequalities

The stability of EVI as in (1) has been investigated in [3, 4, 6, 7]. The work in [3] concerns an extension of the absolute stability problem in the case where the feedback branch contains maximal monotone operators (allowing for instance nonsmooth multivalued functions like the graph of complementarity relations between two variables). This stability framework has been used in [11] to design stable observers for a class of nonsmooth dynamical systems. In [7] Lyapunov's second method is investigated for linear EVIs (i.e. $F(x) = 0$ in (1)). Starting from general sufficient conditions on the Lyapunov function, various criteria are proposed which allow one to test whether or not the fixed point is stable, or unstable. Many examples are provided (among them positive real electrical circuits). In [4] the Krakovskii-LaSalle invariance principle is extended to EVIs (nonlinear vector fields $F(x)$ are included). In [6], necessary conditions for the asymptotic stability of EVIs are derived. It is known that a necessary condition for the asymptotic stability of the unique fixed point of an ODE, is that the degree of its vector field be equal to 1. This result, that is not well known in the Systems and Control community, is here extended to a class of EVIs as in (1). The proofs heavily rely on the invariance of the degree by continuous homotopy. Necessary conditions for asymptotic stabilization of controlled linear EVIs are deduced.

3.2 Stability of second order Complementarity Problem

The paper [3] also deals with Lagrangian systems as in (4). An extension of the Lagrange-Dirichlet (also known as the Lejeune-Dirichlet) theorem, which relates the potential energy to the fixed point stablity, is proposed. This is a natural extension as the nonsmoothness in (4) comes from the complementarity relations and the impact law, which are both dissipative (more exactly the complementarity conditions define a maximal multivalued monotone operator between the Lagrange multiplier and the "distance" function). The framework of the absolute stability problem (a feedback interconnection of two dissipative systems) is recovered with the feedback branch containing a maximal monotone operator representing all the nonsmooth effects.

3.3 Stability of second order Measure Differential Equation

The Lyapunov stability theory is usually presented for dynamical systems with states $x = (\boldsymbol{q}, \dot{\boldsymbol{q}})$ that vary continuously with time [12, 15]. Because of the possible discontinuities of their velocity, this might not be the case for nonsmooth Lagrangian dynamical systems. But, Lyapunov stability theory is in fact not strictly bound to continuity properties: thus we can state for example the Lyapunov Stability Theorem, see [5], that can be proved in a very similar way to what can be found in [12, 15]. This theorem, derived for nonsmooth dynamical systems, differs essentially from the smooth case by the global assumptions on the system state: a discontinuous flow allows possible jumps of the system state outside any neighborhood of the stable set, and so global conditions are needed on the system state to take into account such discontinuous behaviors.

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3

Stability of Cone Complementarity Systems

Kanat Camlibel[1], Maurice Heemels[2] Arjan van der Schaft[3], and Hans Schumacher[1]

[1] Dept. of Econometrics and Operations Research, Tilburg University, The Netherlands
[2] Dept. of Electrical Engineering, Eindhoven University of Technology, The Netherlands
[3] Fac. of Mathematical Sciences, University of Twente, The Netherlands

1 Introduction

The standard literature on dynamical systems is mostly concerned with systems that evolve in time according to a set of rules depending smoothly on the current state of the system. However, in many areas of engineering as well as in other fields, one is often confronted with systems that are most easily modelled as going through a succession of periods of smooth evolution separated by instantaneous events that mark transitions of one set of laws of evolution to another. Events may be externally induced or internally induced. To come up with a precise mathematical formulation of systems with events is a nontrivial matter, in particular because one has in general to allow for the possibility that a state jump is associated with events and so it would be too restrictive to require solutions to be continuous, let alone differentiable.

Complementarity systems (CSs) provide a modelling framework for a class of dynamical systems with externally and/or internally induced events. Our previous work [2–8, 15] was mainly focused on modelling issues, such as existence and uniqueness of solutions, and characterization of jumps at the event times. This note, however, reports the initial results on the stability of CSs that are obtained within the SICONOS project. In particular, we present sufficient conditions for the Lyapunov stability of CSs with both externally and internally induced events. For stability considerations, one of the difficulties is to cope with the externally induced jumps in the state variables. One of our contributions is to show, under a passivity condition, that the stored energy decreases instantaneously whenever a jump occurs.

The organization as follows. We begin with introducing cone complementarity systems in the next section. Afterwards, we very briefly review the notion of passivity. This will be followed the mathematical formulation of a solution concept, well-posedness results, and characterization of the jumps for cone complementarity systems. After these preparations, we attack the stability problem and present sufficient conditions. Finally, some necessary and sufficient conditions will be presented in the restricted case of bimodal planar CSs.

2 Switched cone complementarity systems

In this note we deal with the switched cone complementarity systems of the form

$$\dot{x}(t) = Ax(t) + Bz(t) \tag{1a}$$
$$w(t) = Cx(t) + Dz(t) \tag{1b}$$
$$\mathcal{C}_{\pi(t)} \ni z(t) \perp w(t) \in \mathcal{C}^{*}_{\pi(t)} \tag{1c}$$

where $(z, x, w) \in \mathbb{R}^{m+n+m}$, $\pi : \mathbb{R}_{+} \to \{-1, 0, 1\}^{m}$ is the *switching function*, and

$$\mathcal{C}_{\pi(t)} = \mathcal{K}_{\pi_1(t)} \times \mathcal{K}_{\pi_2(t)} \times \cdots \times \mathcal{K}_{\pi_m(t)} \tag{2}$$

with

$$\mathcal{K}_{-1} = \{0\} \quad \mathcal{K}_0 = \mathbb{R}_{+} \quad \mathcal{K}_1 = \mathbb{R}. \tag{3}$$

When one considers only the constant switching function $\pi(t) = 0$ for all $t \in \mathbb{R}_+$, above system is reduced to the linear complementarity system [3,4,7,8,15,16]. In case one considers the switching functions $\pi_i(t) = 0$ for all $i = 1, 2, \ldots, \ell$ and for all t, and $\pi_i : \mathbb{R}_+ \to \{-1, 1\}$ for all $i = \ell+1, \ell+2, \ldots, m$, it is reduced to the switched complementarity systems [2,6].

The systems of the type (1) were first studied in [5] with an eye towards modelling of power electronics converters.

Passivity notion will play a crucial role in our development. The next section is devoted to a quick review of the subject.

3 Passivity of a linear system

Ever since it was introduced in system theory by V. M. Popov [13,14], the notion of passivity has played an important role in various contexts such as stability issues, adaptive control, identification, etc. Particularly, the interest in stability issues led to the theory of dissipative systems [18] due to J. C. Willems.

Definition 1. [18] *A linear system $\Sigma(A, B, C, D)$ given by*

$$\dot{x}(t) = Ax(t) + Bz(t) \tag{4a}$$
$$w(t) = Cx(t) + Dz(t) \tag{4b}$$

is called passive, *or* dissipative *with respect to the supply rate $z^T w$, if there exists a nonnegative function $V : \mathbb{R}^n \to \mathbb{R}_+$ such that for all $t_0 \leqslant t_1$ and all trajectories (z, x, w) of the system (4) the following inequality holds:*

$$V(x(t_0)) + \int_{t_0}^{t_1} z^T(t)w(t)dt \geqslant V(x(t_1)).$$

If exists, the function V is called a storage function.

The following proposition is one of the classical results of systems and control theory.

Proposition 1. [18] *Consider a system $\Sigma(A, B, C, D)$ for which (A, B, C) is a minimal representation. The following statements are equivalent.*

- *$\Sigma(A, B, C, D)$ is passive.*
- *The transfer matrix $G(s) := D + C(sI - A)^{-1}B$ is* positive real, *i.e., $x^*[G(\lambda) + G^*(\lambda)]x \geqslant 0$ for all complex vectors x and all $\lambda \in \boldsymbol{C}$ such that $Re(\lambda) > 0$ and λ is not an eigenvalue of A.*
- *The matrix inequalities*

$$\begin{bmatrix} A^T K + KA & KB - C^T \\ B^T K - C & -(D + D^T) \end{bmatrix} \leqslant 0 \tag{5}$$

and $K = K^T \geqslant 0$ have a solution K.

Moreover, in case $\Sigma(A, B, C, D)$ is passive, all solutions K to the linear matrix inequalities (5) are positive definite and K is a solution to (5) if and only if $V(x) = \frac{1}{2}x^T Kx$ defines a storage function of the system $\Sigma(A, B, C, D)$.

4 Well-posedness

Our main aim is to present the available results on the stability of the switched cone complementarity systems (1). To do so, we first study well-posedness (in the sense of existence and uniqueness of solutions) of the system (1).

Let us consider only the constant switching functions for the moment, i.e. $\pi(t) = \bar{\pi} \in \{-1, 0, 1\}^m$ for all t. Define $\bar{\mathcal{C}} := \mathcal{C}_{\bar{\pi}}$. Then, we have the following cone complementarity system

$$\dot{x}(t) = Ax(t) + Bz(t) \tag{6a}$$
$$w(t) = Cx(t) + Dz(t) \tag{6b}$$
$$\bar{\mathcal{C}} \ni z(t) \perp w(t) \in \bar{\mathcal{C}}^\star. \tag{6c}$$

Some nomenclature is in order. Let $\mathcal{C} \subseteq \mathbb{R}^m$ be a cone. Given an m-vector q and an $m \times m$ matrix M, the linear cone complementarity problem LCCP$(\mathcal{C}, q, M)$ is to find an m-vector z such that

$$z \in \mathcal{C} \tag{7a}$$
$$w := q + Mz \in \mathcal{C}^\star \tag{7b}$$
$$z^T w = 0. \tag{7c}$$

If such a vector z exists, we say that z *solves* (is a *solution of*) LCCP$(\mathcal{C}, q, M)$. The following proposition is an immediate consequence of [2, Theorem II.7].

Proposition 2. *Suppose that M is nonnegative definite. Let*

$$\mathcal{Q}_M := \{z \mid z \in \mathcal{C},\ Mz \in \mathcal{C}^\star,\ \textit{and}\ z^T Mz = 0\}.$$

Then, the following statements are equivalent.

1. *LCCP$(\mathcal{C}, q, M)$ is solvable.*
2. *$q \in \mathcal{Q}_M^\star$.*

With these preparations, we are in a position to formulate a well-posedness result for systems (6).

Theorem 1. *Consider the cone complementarity system* (6). *Suppose that $\Sigma(A, B, C, D)$ is passive and the triple (A, B, C) is minimal. Let*

$$\mathcal{Q}_D := \{z \mid z \in \bar{\mathcal{C}},\ Dz \in \bar{\mathcal{C}}^\star,\ \textit{and}\ z^T Dz = 0\}.$$

Then, the following statements are equivalent.

1. *For a given initial state x_0, there exist an absolutely continuous state trajectory x with $x(0) = x_0$ and a pair $(z, w) \in \mathcal{L}_2^{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^{m+m})$ such that* (6) *are satisfied for almost all $t \geqslant 0$.*
2. *$Cx_0 \in \mathcal{Q}_D^\star$.*

Moreover, if such a triple (z, x, w) exists then the state trajectory x and $\mathrm{col}(B, D + D^T)z$ *are unique.*

Above theorem considers only a *constant* switching function. Next step is to extend this theorem to arbitrary switching functions. Arbitrary switching introduces the possibility of discontinuous state trajectories. An immediate question is how to define a jump in the state that is triggered by a switching. A natural way of introducing a jump rule can be obtained via the stored energy in the system. To formalize this idea, let

$$\mathcal{Q}_D(\mathcal{C}) := \{z \mid z \in \mathcal{C},\ Dz \in \mathcal{C}^\star,\ \text{and}\ z^T Dz = 0\}.$$

Suppose that T is an isolated switching time. Consider the minimization problem

$$\text{minimize } \tfrac{1}{2}(\bar{x} - x(T-))^T K(\bar{x} - x(T-)) \tag{8a}$$
$$\text{subject to } C\bar{x} + Fu(T+) \in \mathcal{Q}_D^\star(\mathcal{C}_{\pi(T+)}) \tag{8b}$$

where $\xi \mapsto \frac{1}{2}\xi^T K \xi$ is a storage function for the system $\Sigma(A, B, C, D)$. Passivity of the system allows us to prove that the above minimization problem has a unique solution for any $x(T-)$ and $\pi(T+)$. Note that the condition (8b) implies that there exists a solution for the initial state $\bar{x}$ after the time instant T due to Theorem 1. Our jump rule defines the solution $\bar{x}$ of the above problem as the state at $T+$. Note that $\bar{x}$ is the closest state (in the metric defined by the storage function) to the state before the jump among the states for which there exists a solution after the jump.

In the sequel of the paper, we consider switching functions $\pi : \mathbb{R}_+ \to \{-1,0,1\}^m$ that have only isolated discontinuities. Such switching functions will be called *admissible switching function.* The set Γ_f is defined as the union of the discontinuity points of a function f and zero.

Theorem 2. *Consider the switched complementarity system* (1). *Suppose that $\Sigma(A,B,C,D)$ is passive and the triple (A,B,C) is minimal. Let K be such that $\xi \mapsto \frac{1}{2}\xi^T K\xi$ is a storage function for the system $\Sigma(A,B,C,D)$. Also let an initial state x_0 and an admissible switching function π be given. Define $\Gamma = \Gamma_u \cup \Gamma_\pi$. Then, there exist a state trajectory x and a pair $(z,w) \in \mathcal{L}_2^{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^{m+m})$ such that*

1. *$x(0-) = x_0$ and x is absolutely continuous at all points $0 < t \notin \Gamma$.*
2. *(6) are satisfied for almost all $0 < t \notin \Gamma$.*
3. *For times $0 \leqslant t \in \Gamma$, $x(t+)$ is the unique minimum of*

$$\text{minimize } \tfrac{1}{2}(\bar{x} - x(t-))^T K(\bar{x} - x(t-)) \tag{9a}$$
$$\text{subject to } C\bar{x} \in \mathcal{Q}_D^\star(\mathcal{C}_{\pi(t+)}). \tag{9b}$$

Moreover, if such a triple (z,x,w) exists then the state trajectory x and $\mathrm{col}(B, D+D^T)z$ are unique.

The next lemma characterizes the jumps in the state variable.

Lemma 1. *Let $\bar{x}$ be the solution of the minimization problem* (9). *Then, there exists a unique $\bar{z} \in \mathcal{Q}(\mathcal{C}_{\pi(t+)})$ such that $\bar{x} = x(t-) + B\bar{z}$.*

As a consequence of the above lemma, one can think of the jump at time instant t as a result of an impulse $\bar{z}\delta_t$ in the z variable. Here, δ_t is the Dirac distribution that is supported at the time instant t. Alternative characterizations of the jump multiplier $\bar{z}$ can be given as follows.

Theorem 3. *Suppose that $t \in \Gamma$ and $x(t+)$ is the unique solution of the minimization problem* (9). *Let $\bar{z} \in \mathcal{Q}(\mathcal{C}_{\pi(t+)})$ be such that $\bar{x} = x(t-) + B\bar{z}$. Define $\mathcal{Q} := \mathcal{Q}_D(\mathcal{C}_{\pi(t+)})$. Then the following characterizations can be obtained for $\bar{z}$.*

1. *The jump multiplier $\bar{z}$ is the unique solution to*
$$\mathcal{Q} \ni v \perp C(x(t-) + Bv) \in \mathcal{Q}^\star \tag{10}$$
2. *The cone $\mathcal{Q}$ is equal to pos $N := \{N\lambda \mid \lambda \geqslant 0\}$ and $\mathcal{Q}^\star = \{v \mid N^T v \geqslant 0\}$ for some real matrix N. The re-initialized state $x(t+)$ is equal to $x(t-) + BN\bar{\lambda}$ and $\bar{z} = N\bar{\lambda}$ where $\bar{\lambda}$ is a solution of the following ordinary LCP.*
$$0 \leqslant \lambda \perp (N^T C x(t-) + N^T CBN\lambda) \geqslant 0. \tag{11}$$
3. *The jump multiplier $\bar{z}$ is the unique minimizer of*
$$\text{minimize } \tfrac{1}{2}(x(t-) + Bv)^T K(x(t-) + Bv) \tag{12}$$
$$\text{subject to } v \in \mathcal{Q} \tag{13}$$

5 Stability

In this section we discuss the stability of switched complementarity systems (1) under a passivity assumption. The Lyapunov stability of hybrid and switched systems in general has already received considerable attention [1,9–12,19]. From now on, we denote the unique global trajectory for a given switch function π and initial state x_0 of a switched complementarity system by $(u^{\pi,x_0}, x^{\pi,x_0}, y^{\pi,x_0})$. For the study of stability we consider the source-free case.

Definition 2 (Equilibrium point). *A state $\bar{x}$ is an equilibrium point of the switched complementarity system* (1)*, if for all admissible switching functions π $x^{\pi,\bar{x}}(t) = \bar{x}$ for almost all $t \geqslant 0$ and all π, i.e. for all solutions starting in $\bar{x}$ the state stays in $\bar{x}$.*

Note that in an equilibrium point $\dot{x} = 0$, which leads in a simple way to the following characterization of equilibria of a switched complementarity system.

Lemma 2. *A state $\bar{x}$ is an equilibrium point of the switched complementarity system* (1)*, if and only if for all $\bar{\pi} \in \{-1, 0, 1\}^m$ there exist $z^{\bar{\pi}} \in \mathbb{R}^m$ and $w^{\bar{\pi}} \in \mathbb{R}^m$ satisfying*

$$0 = A\bar{x} + Bz^{\bar{\pi}} \tag{14a}$$

$$w^{\bar{\pi}} = C\bar{x} + Dz^{\bar{\pi}} \tag{14b}$$

$$\mathcal{C}_{\bar{\pi}} \ni z^{\bar{\pi}} \perp w^{\bar{\pi}} \in \mathcal{C}^{\star}_{\bar{\pi}}. \tag{14c}$$

From this lemma it follows that $\bar{x} = 0$ is an equilibrium. Note that if A is invertible we get $\bar{x} = -A^{-1}Bz^{\bar{\pi}}$ and

$$\mathcal{C}_{\bar{\pi}} \ni z^{\bar{\pi}} \perp [-CA^{-1}B + D]z^{\bar{\pi}} \in \mathcal{C}^{\star}_{\bar{\pi}}$$

which is a homogeneous LCP over a cone.

Definition 3. *Let $\bar{x}$ be an equilibrium point of the switched complementarity system* (1) *and d denote a metric on $\mathbb{R}^n$.*

1. *$\bar{x}$ is called stable, if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $d(x^{\pi,x_0}(t), \bar{x}) < \varepsilon$ for almost all $t \geqslant 0$ whenever $d(x_0, \bar{x}) < \delta$ and π being an admissible switching function.*
2. *$\bar{x}$ is called asymptotically stable if $\bar{x}$ is stable and there exists an $\eta > 0$ such that $\lim_{t\to\infty} d(x^{\pi,x_0}(t), \bar{x}) = 0$ whenever $d(x_0, \bar{x}) < \delta$ and π being an admissible switching function. By $\lim_{t\to\infty} d(x^{\pi,x_0}(t), \bar{x}) = 0$ we mean that for every $\varepsilon > 0$ there exists a t_ε such that $d(x^{\pi,x_0}(t), \sigma), \bar{x}) < \varepsilon$ whenever $t \geqslant t_\varepsilon$.*

Theorem 4. *Consider the switched complementarity system* (1)*. Suppose that $\Sigma(A, B, C, D)$ is passive and the triple (A, B, C) is minimal. Let K be such that $\xi \mapsto \frac{1}{2}\xi^T K \xi$ is a storage function for the system $\Sigma(A, B, C, D)$. The system* (1) *has only stable equilibrium points $\bar{x}$. Moreover, if $A^T K + KA < 0$ is invertible $\bar{x} = 0$ is the only equilibrium point, which is asymptotically stable.*

6 Bimodal and planar complementarity systems

In this section, we will be dealing with the bimodal linear complementarity systems of the form

$$\dot{x} = Ax + bz \tag{15a}$$

$$w = c^T x + dz \tag{15b}$$

$$0 \leqslant z \perp w \geqslant 0 \tag{15c}$$

where $A \in \mathbb{R}^{n \times n}$, $c \in \mathbb{R}^n$, $d \in \mathbb{R}$, and $b \in \mathbb{R}^n$. Note that (15) can be obtained as a special case of the switched cone complementarity systems with $m = 1$ and $\pi(t) = 0$ for all $t \in \mathbb{R}_+$.

A solution (z, x, w) of the system is called *periodic* if all three functions are periodic.

Note that (15) can be replaced by

$$\dot{x} = \begin{cases} Ax & \text{if } c^T x \geqslant 0, \\ (A - bd^{-1}c^T)x & \text{if } c^T x \leqslant 0 \end{cases} \tag{16}$$

in case $d > 0$ and by

$$\dot{x} = \begin{cases} Ax & \text{if } (c^T x, c^T Ax, \ldots, c^T A^{n-1}x) \succcurlyeq 0, \\ PAx & \text{if } c^T x = 0 \text{ and } (c^T Ax, c^T A^2 x, \ldots, c^T A^{n-1}x) \preccurlyeq 0 \end{cases} \tag{17}$$

in case $d = 0$ and $c^T b > 0$ where $P = I - b(c^T b)^{-1} c^T$.

Consider the system (16). Suppose that A has a real eigenvalue, say ρ. Let v be an eigenvector corresponding to this eigenvalue. We can assume that $c^T v \geqslant 0$ without loss of generality. The state trajectory of (16) that starts from the initial state $x_0 = v$ is $x(t) = \exp(\rho t)v$. Depending on the sign of the eigenvalue ρ, this trajectory might be stable or unstable. This argumentation gives the following necessary condition for stability with arbitrary state space dimension n.

Lemma 3. *Suppose that $d > 0$. A necessary condition for the asymptotic stability of the system (15) is that neither A nor $A - bd^{-1}c^T$ has a real nonnegative eigenvalue.*

When the state space dimension (i.e., n) is 2, one can derive necessary and sufficient conditions as in the following theorem.

Theorem 5. *Consider the LCS (15) with $n = 2$ and (c^T, A) is an observable pair. The following statements hold.*

1. *Suppose that $d > 0$. The origin is the unique asymptotically stable equilibrium point of the LCS (15) if and only if*
   (a) *neither A nor $A - bd^{-1}c^T$ has a real nonnegative eigenvalue, and*
   (b) *if both A and $A - bd^{-1}c^T$ have nonreal eigenvalues then $\sigma_1/\omega_1 + \sigma_2/\omega_2 < 0$ where $\sigma_1 \pm i\omega_1$ ($\omega_1 > 0$) are the eigenvalues of A and $\sigma_2 \pm i\omega_2$ ($\omega_2 > 0$) are the eigenvalues of $A - bd^{-1}c^T$.*
2. *Suppose that $d > 0$. The LCS (15) has a nonconstant periodic solution if and only if both A and $A - bd^{-1}c^T$ have nonreal eigenvalues, and $\sigma_1/\omega_1 + \sigma_2/\omega_2 = 0$ where $\sigma_1 \pm i\omega_1$ ($\omega_1 > 0$) are the eigenvalues of A and $\sigma_2 \pm i\omega_2$ ($\omega_2 > 0$) are the eigenvalues of $A - bd^{-1}c^T$. Moreover, if there is one periodic solution, then all other solutions are also periodic. And, $\pi/\omega_1 + \pi/\omega_2$ is the period of any solution.*
3. *Suppose that $d = 0$. The origin is the unique asymptotically stable equilibrium point of the LCS (15) if and only if A has no real nonnegative eigenvalue and $[I - b(c^T b)^{-1} c^T]A$ has a real negative eigenvalue (note that one eigenvalue is already zero).*

Remark 1. Observe that the conditions derived in Theorem 5 item 1 are connected to the ones obtained in [17], where a stabilizing controller of the type $\max(0, Fx)$ was designed for a linear system with nonnegative control inputs. As the closed-loop actually becomes a linear complementarity system, the design of the matrix F must be such that the closed-loop system satisfies the conditions above.

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4

A Negative Bendixson-like Criterion for a Class of Hybrid Systems

Alexander Pogromsky, Henk Nijmeijer, and Koos Rooda

Dept. of Mechanical Engineering, Eindhoven University of Technology, The Netherlands

Abstract. A condition which ensures the absence of periodic orbits for nonsmooth dynamical systems is presented. The condition is a higher dimensional generalization of Bendixson's criterion applicable to differential inclusions that are useful in the description of hybrid systems. The main argument is based on contraction analysis of the d-measured volume along the system trajectories. A connection to methods for estimates of the Hausdorff dimension is emphasized. For a class of hybrid systems described by a linear system and a relay feedback the conditions are presented in the form of linear matrix inequalities. A simple but illustrative example is analyzed.

1 Introduction

Discontinuous dynamical systems and, particularly, relay systems have attracted considerable attention over the last decades. While the mathematics of smooth dynamical systems still produces new and interesting discoveries, in applied disciplines it has been realized that for many applications discontinuities should be taken into account. For example, discontinuities can be used to simplify modelling of friction in mechanical systems, to design disturbance tolerant sliding mode controllers, to deal with a switching control strategy in manufacturing systems, and so on. One of the hot topics in research in the control community are the so called hybrid dynamical systems, which combine continuous and discrete dynamics. Although the existing literature on this subject includes a growing number of monographs and papers (see e.g. [1–5], to mention a few), those systems are far from being understood.

Hybrid systems, being nonlinear dynamical systems, can have a very rich behavior and one of the main theoretical problems is to predict and to understand these without explicitly solving the equations describing the system. In this paper we study oscillatory behavior in nonlinear hybrid dynamical systems that recently receive lots of attention in the control community [6,7]. The class of systems we study is described by nonlinear differential equations with discontinuous right hand side. A particular result that is obtained here is a generalization of Bendixson's negative divergency test for that class of systems. This simple test gives a sufficient condition for the nonexistence of periodic orbits for smooth planar systems. This classical result claims that if in a simply connected domain the divergency of a vector field does not change sign, then this domain does not contain a periodic orbit. A classical proof of this statement is based on the divergence theorem and cannot be generalized to the higher dimensional case. The main purpose of this paper is to present a possible generalization of the Bendixson result in arbitrary dimension taking into account the possible discontinuity of the right hand side. There are several higher dimensional generalizations of this criterion, see, e.g. [8–12]. Muldowney and Li [9–11] used an approach based on compound matrices to prove a negative Bendixson-like criterion. In this paper we investigate this question by a method which allows to estimate the Hausdorff dimension of invariant compact sets [13–17]. In doing so, we first present a generalization for an estimate for the Hausdorff dimension formula for non smooth systems and then, based on that result we prove a negative criterion for the nonexistence of periodic orbits.

From a practical point of view a design based on global stability of a system can be too restrictive and conservative. A possible weaker criterion is that all trajectories tend to a set consisting of equilibrium points; that is the system can not exhibit oscillatory behavior. This fact indicates the importance of Bendixson-like criteria for the design and control of dynamical systems. A similar motivation can be found, for example in a recent paper [18], where for smooth systems a condition was presented which guarantees that almost all trajectories tend to an equilibrium. Another approach to simplify the stability analysis of discontinuous systems is based on the generalization of the Invariance Principle for differential inclusions, see e.g. [19].

A general idea behind the proof of the nonexistence criterion is relatively simple. If one is able to show that in a simply connected positively invariant domain the (semi)flow generated by the system contracts the area of some initial surface, it is sufficient to claim that no periodic orbits can lie inside this domain. By reversing time, the same holds true for area-expanding systems. Together with Liouville's theorem this argument gives another proof of Bendixson's criterion that can be generalized to arbitrary dimensions. To characterize the area of a surface one can use the so called Hausdorff 2-measure, so the area-contracting systems are those for which the Hausdorff 2-measure of any initial measurable set vanishes with time.

The main method employed in our study is based on stability/dichotomy-like properties of solutions with respect to each other rather than with respect to some invariant sets. The first results in this direction were developed by Demidovich [20], see also [21] and Yoshizava [22]. Methods based on similar ideas are appreciated now in the control community [23–25]. A natural way to investigate those properties is based upon linearization of the dynamical system along any given trajectory which excludes the consideration of non smooth systems. In this paper instead of linearization we investigate the behavior of some quadratic forms defined for a pair of trajectories of the system, and which allows to consider discontinuous systems. The conditions presented in this paper are formulated in terms of inequalities involving two eigenvalues of some matrix pencil.

The paper is organized as follows. In Section II we present necessary background material. Section III contains some result on estimation of the Hausdorff dimension of invariant sets. Based on these results in Section IV we present a new version of a generalized Bendixson's criterion. Particular attention is then drawn to LMI based results for linear systems with relay feedback.

2 Hausdorff dimension

Consider a compact subset K of $\mathbb{R}^n$. Given $d \geq 0$, $\varepsilon > 0$, consider a covering of K by open spheres B_i with radii $r_i \leq \varepsilon$ (see Figure 1). Denote by

$$\mu(K, d, \varepsilon) = \inf \sum_i r_i^d \tag{1}$$

the d-measured volume of covering of the set K. Here the infimum is calculated over all ε-coverings of K. There exists a limit, which may be infinite,

$$\mu_d(K) := \sup_{\varepsilon > 0} \mu(K, d, \varepsilon).$$

It can be proved that μ_d is a Borel regular measure on $\mathbb{R}^n$ (see [33]).

Definition 1 *The measure μ_d is called the* Hausdorff d-measure.

Some properties of the measure μ_d can be summarized as follows. There exists a single value $d = d_*$, such that for all $d < d_*$, $\mu_d(K) = +\infty$ and for all $d > d_*$, $\mu_d(K) = 0$, with

$$d_* = \inf\{d \ : \ \mu_d(K) = 0\} = \sup\{d \ : \ \mu_d(K) = +\infty\}.$$

(see Proposition 5.3.2 in [16]).

Definition 2 *The value d_* is called the* Hausdorff dimension *of the set K.*

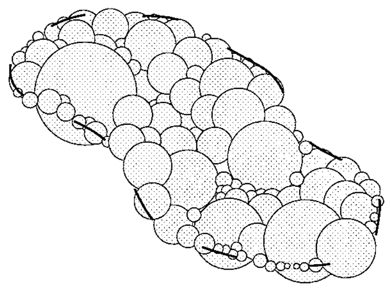

Fig. 1: The Hausdorff 2-measure is approximated by the cross-sections of little balls with the surface.

In the sequel, we will use the notation $\dim_H K$ for the Hausdorff dimension of the set K.

For the control community the notions of Hausdorff measure and Hausdorff dimension are not of common use and we like to clarify the above definitions.

Suppose we have a two-dimensional bounded surface S with area $m(S)$. We cover this surface by open spheres as required in the definition of the Hausdorff measure. Then, for $d = 1$ and $d = 3$ we have

$$\mu_1(S) = \lim_{\varepsilon \to 0} \mu(S, 1, \varepsilon) = +\infty,$$

$$\mu_3(S) = \lim_{\varepsilon \to 0} \mu(S, 3, \varepsilon) = 0,$$

while for $d = 2$ we have

$$\mu_2(S) = \frac{m(s)}{\pi}$$

This example illustrates the behavior of $\mu_d(K)$ for a given K as a function of d. Namely, for values of d less than $\dim_H K$, $\mu_d(K)$ is infinity and for all values of d greater than $\dim_H K$ $\mu_d(K)$ is zero (see Proposition 5.3.2 in [16]). This situation is schematically presented on Fig. 2.

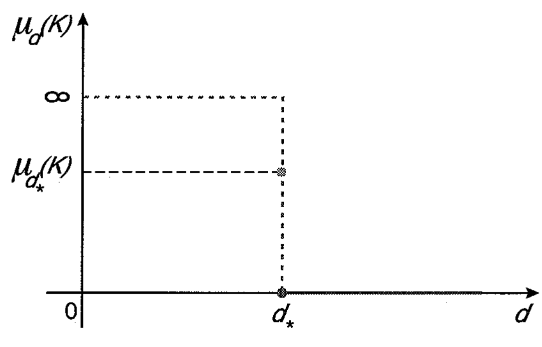

Fig. 2: Properties of the Hausdorff d-measure.

For the "good" sets such as a piece of an arc, a piece of a smooth surface, etc. the Hausdorff dimension can be used as the dimension in the normal sense (i.e. in the sense of Brouwer, or Lebesgue). This follows from the result (see, e.g. Proposition 5.3.5 in [16]) which claims that for a set K of positive n-dimensional Lebesgue measure the Hausdorff dimension of K is n. However for other sets such as the Cantor set the value of the Hausdorff dimension can be fractional. Sets of such type are often encountered as invariant sets of "chaotic" systems, that makes the Hausdorff dimension of invariant sets an important characteristic for "chaotic" systems.

3 Upper estimates for the Hausdorff dimension of invariant compact sets

The extremal property of the Hausdorff dimension suggests an idea of how to estimate it for invariant sets of dynamical systems. Namely, if one is able to prove that for a given set K its Hausdorff d-measure is zero, then it follows that d is an upper estimate of the Hausdorff dimension of K. A possible way to show that the Hausdorff d-measure is zero is to prove the following inequality

$$\mu_d(\varphi(K)) \leq \nu \mu_d(K) \tag{2}$$

where $\nu < 1$, φ is some mapping and K its invariant set, i.e. $\varphi(K) = K$ and hence $\mu_d(\varphi(K)) = \mu_d(K)$. This identity together with (2) implies that $\mu_d(K) = 0$.

When φ is a flow generated by a system of differential equations inequality (2) follows from the fact that *the d-measured volume of an open neighborhood of the invariant set K decreases along the system trajectories.* This observation suggests to employ a Lyapunov-like technique to estimate the Hausdorff dimension of invariant sets.

As has been mentioned in the Introduction, a generalization of Bendixson's criterion can be derived if one is able to show that in some simply connected region there are no invariant sets with Hausdorff dimension greater than or equal 2. This result can be obtained if one takes a 2-measured volume as a Lyapunov function candidate. However, in this section we present a more general result which holds for an arbitrary d-measure and in the next section we present a higher dimensional generalization of Bendixson's criterion.

Consider a system of differential equations

$$\dot{x} = f(x), \quad x \in \Omega \subset \mathbb{R}^n, \quad x_0 \in \Omega \tag{3}$$

where $f : \Omega \to \mathbb{R}^n$ is a (possibly) discontinuous vector field defined on some open positively invariant set Ω, and which satisfies conditions guaranteeing the existence of solutions $x(t, x_0)$ in Ω in some reasonable sense, that is, if the function f is discontinuous and satisfies some mild regularity assumptions, one can construct a set-valued function $\mathbf{f}$ bounded on any compact set according to numerous possible definitions (e.g., Filippov convex definition, Utkin's equivalent control, etc.) such that a solution of the differential inclusion

$$\dot{x} \in \mathbf{f}(x)$$

is called a solution for system (3). We require that the solution $x(t, x_0)$ is an absolutely continuous function of time. Additionally we assume that if the solution $x(t, x_0)$ is (right)-unique then continuous dependence on initial conditions in forward time is guaranteed.

The parameterized mapping $x_0 \mapsto x(t, x_0)$, $t \geq 0$, or the semi-flow will be denoted as $\varphi^t : \Omega \to \Omega$.

Consider a scalar differentiable function $V : \Omega \times \Omega \to \mathbb{R}$, $V(x, x) = 0$.

Define the time derivative of the function V along *two* solutions $x_1(t, x_{10})$, $x_2(t, x_{20})$ of (3) as follows

$$\dot{V} := \frac{\partial V(x_1, x_2)}{\partial x_1} \dot{x}_1(t, x_{10}) + \frac{\partial V(x_1, x_2)}{\partial x_2} \dot{x}_2(t, x_{20}).$$

Since V is Lipschitz continuous and the solutions $x_i(t, x_{i0})$ are absolutely continuous functions of time, the derivative

$$\dot{V}(x_1(t, x_{10}), x_2(t, x_{20}))$$

exists almost everywhere in $[0, \min_i \bar{T}_i)$, where $\bar{T}_i$ is the maximal interval of existence of the solution $x_i(t, x_{i0})$ in Ω.

For the function V we can also define its upper derivative as follows

$$\dot{V}^*(x_1,x_2) = \sup_{\xi_i \in \mathrm{f}(x_i)} \left(\frac{\partial V(x_1,x_2)}{\partial x_1}\xi_1 + \frac{\partial V(x_1,x_2)}{\partial x_2}\xi_2 \right).$$

Then for almost all $t \geq 0$ it follows that

$$\dot{V}(x_1(t,x_{10}),x_2(t,x_{20})) \leq \dot{V}^*(x_1(t,x_{10}),x_2(t,x_{20})).$$

We formulate the following hypothesis:

H1. There exists a continuously differentiable $n \times n$ symmetric matrix valued function P defined in the domain Ω, such that the function

$$V(x_1,x_2) = (x_1 - x_2)^\top P(x_1)(x_1 - x_2) \tag{4}$$

satisfies the following inequalities

$$\alpha_1 ||x_1 - x_2|| \leq V \leq \alpha_2 ||x_1 - x_2|| \tag{5}$$

and

$$\dot{V}^*(x_1,x_2) \leq (x_1 - x_2)^\top Q(x_1)(x_1 - x_2) + o(||x_1 - x_2||^2), \quad \text{as} \quad ||x_1 - x_2|| \to 0$$

for all $x_1, x_2 \in \Omega$ and some $\alpha_1, \alpha_2 > 0$ with a symmetric continuous matrix valued function Q, bounded on Ω.

H2. All solutions starting in Ω are defined for all $t \geq 0$.

We begin with the following preliminary result:

Lemma 1 *Suppose the assumptions H1 and H2 are satisfied. Then any solution $x(t,x_0)$ to (3), $x_0 \in \Omega$ is right-unique.*

Proof. We prove the statement under the weaker assumption that for all $x_1, x_2 \in \Omega$

$$\dot{V}^* \leq O((x_1 - x_2)^\top P(x_1)(x_1 - x_2)) \leq L(x_1 - x_2)^\top P(x_1)(x_1 - x_2), \quad \text{as} \quad ||x_1 - x_2|| \to 0$$

for some $L > 0$. Here we used a standard asymptotic notation $O(\cdot)$. Then

$$\frac{d}{dt}(V(x_1(t),x_2(t))e^{-Lt}) \leq 0$$

almost everywhere. The absolutely continuous function Ve^{-Lt} does not increase, and if $x_1 = x_2$ it follows that $V(x_1(t),x_2(t)) = 0$ for $t \geq 0$. Thus the right-uniqueness is proved.

The previous lemma shows that the Cauchy problem (3) is well-posed and continuous dependence on initial conditions follows.

Let $\lambda_1(x) \geq \lambda_2(x) \geq \ldots \geq \lambda_n(x)$, $x \in \Omega$ be the ordered solutions to the following generalized eigenvalue problem

$$\det(Q(x) - \lambda P(x)) = 0$$

which are real since both Q and P are symmetric.

Consider a compact set S of finite Hausdorff d-measure for some $d = d_0 + s$, $d \leq n$, where $d_0 \in \mathbb{N}$ and $s \in [0,1)$. Suppose that $S \in \Omega$, then $\varphi^t(S) \in \Omega$ for all positive t. Now we formulate the following result.

Theorem 1 *Suppose hypotheses H1 and H2 are satisfied. If for some $d = d_0 + s$, $0 < d_0 \leq n$, $0 \leq s < 1$ it follows that*

$$\sup_{x \in \Omega} (\lambda_1(x) + \ldots + \lambda_{d_0}(x) + s\lambda_{d_0+1}(x)) < 0. \tag{6}$$

Then

$$\lim_{t \to \infty} \mu_d(\varphi^t(S)) = 0.$$

The proof is based on the construction of a finite set of affine maps $\mathbb{R}^n \to \mathbb{R}^n$ which locally approximate φ^t. Then using the linear part of those maps we approximate how the d-measured volume is changed under those maps to compute the change of $\mu_d(\varphi^t(S))$. The proof of this theorem is presented in the Appendix.

The main result of this section is the following theorem.

Theorem 2 *Suppose hypotheses H1 and H2 are satisfied, and there exist positive integer d_0 and real $s \in [0, 1)$ such that*

$$\sup_{x\in\Omega} (\lambda_1(x) + \ldots + \lambda_{d_0}(x) + s\lambda_{d_0+1}(x)) < 0. \tag{7}$$

Suppose that there is an invariant compact set $K \in \Omega$. Then $\dim_H K \leq d_0 + s$.

Proof. From the previous result it follows that $\mu_d(\varphi^t(K)) \to 0$ as $t \to \infty$. Since K is invariant $\mu_d(K) = \mu_d(\varphi^t(K))$. Therefore $\mu_d(K) = 0$.

For the continuously differentiable right-hand side of system (3) the previous theorem can be formulated in infinitesimal form.

Corollary 1 *[17] Suppose that hypothesis H2 is satisfied and there is an invariant compact set $K \in \Omega$. Assume also that there is a continuously differentiable symmetric uniformly positive definite $n \times n$ matrix function P defined and bounded in Ω such that the solutions of the following generalized eigenvalue problem*

$$\det(Q(x) - \lambda P(x)) = 0$$

satisfy the inequality (7), with the matrix Q defined as

$$Q = P(x)\frac{\partial f(x)}{\partial x} + \frac{\partial f^\top(x)}{\partial x}P(x) + \dot{P}(x).$$

Then $\dim_H K \leq d_0 + s$.

In [17] this result was derived using the linearization of the flow φ^t with an approach close to that due to Douady-Oesterlé [13] and Leonov [16]. It is now seen that this result can also be obtained from a more general argument which is applicable to discontinuous systems.

We have only presented local conditions that are easier to verify analytically for particular examples. One can derive a further generalization via integral conditions of the form

$$\sup_{x_{10}} \int_0^T (\lambda_1(x(t, x_{10})) + \ldots + s\lambda_{d_0+1}(x(t, x_{10})))\, dt < 0$$

which can be useful for numerical methods.

3.1 Example: the Lorenz system

It is well known that all trajectories of the Lorenz system

$$\begin{cases} \dot{x} = \sigma(y - x) \\ \dot{y} = rx - y - xz \\ \dot{z} = -bz + xy \end{cases} \tag{8}$$

are ultimately bounded for arbitrary positive σ, r, b; that is, there is an invariant compact set K. Let us estimate its Hausdorff dimension.

Proposition 1 *Suppose the parameters of the system are such that the following inequality*

$$\limsup_{t\to\infty}\left[\frac{y(t)^2}{b}+(z(t)-2r)^2\right]\le 4r^2. \tag{9}$$

is satisfied for all initial conditions. Then

$$\dim_H K \le 3-\frac{2(\sigma+b+1)}{\sigma+1+\sqrt{(\sigma-1)^2+4r\sigma}}. \tag{10}$$

Proof. The Jacobian of the Lorenz system is

$$J=\begin{pmatrix}-\sigma & \sigma & 0\\ r-z & -1 & -x\\ y & x & -b\end{pmatrix}$$

and let P be

$$P=\begin{pmatrix}r/\sigma & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

then

$$PJ+J^{\top}P=\begin{pmatrix}-2r & 2r-z & y\\ 2r-z & -2 & 0\\ y & 0 & -2b\end{pmatrix}.$$

We know that $\lambda_1+\lambda_2+\lambda_3=-2(\sigma+1+b)$, and from $-2(\sigma+1+b)<0$ it follows that $\lambda_3<0$ for all x,y,z. Hence, by Corollary 1, to find an upper estimate of s it is sufficient to find a lower estimate of $\bar{\lambda}_3$. Given this estimate, we then evaluate the upper bound on the Hausdorff dimension as

$$\dim_H K\le 3-\frac{2(\sigma+b+1)}{|\bar{\lambda}_3|}.$$

The smallest negative eigenvalue of some symmetric matrix Q is a number $\bar{\lambda}$ such that the matrix $Q-\lambda I$ is positive definite as long as $\lambda<\bar{\lambda}$. For all nonpositive λ the following matrix inequality

$$PJ+J^{\top}P-\lambda P\ge PJ+J^{\top}P-\lambda P_1$$

is satisfied, where $P_1=\operatorname{diag}\{r/\sigma,1,0\}$. Therefore, to find a lower estimate for λ_3 it is sufficient to find a lower estimate of the smallest nonpositive solution of the following equation

$$\det(PJ+J^{\top}P-\lambda P_1)=0$$

or,

$$r\left(2+\frac{\lambda}{\sigma}\right)(2+\lambda)=\frac{y^2}{b}+(2r-z)^2+\lambda\frac{y^2}{2b}.$$

Using the inequality (9) and neglecting the term $\lambda y^2/2b$ (since we are looking for a negative lower estimate of the smallest solution for λ) it is then straightforward to complete the proof.

In [26] it has been proved that for the standard values of parameters of the Lorenz system $\sigma=10, r=28, b=8/3$, the inequality (9) is satisfied. Thus in this case, $\dim_H K\le 2.4013$. A typical trajectory of the Lorenz system for those values of the parameters is presented in Fig. 3.

The inequality (10) claims that the Hausdorff dimension of the invariant set of the Lorenz system is bounded by the Lyapunov dimension of the origin. This statement is known as the Eden conjecture [27] which has been proved by Ljashko [28] for a certain set of parameters. The method proposed by Ljashko does not depend on the estimates of the ultimate bounds for the system trajectories but leads to very cumbersome calculations. Recently Leonov found a set of parameters of the Lorenz system for which the Lyapunov dimension of the Lorenz attractor equals to the local Lyapunov dimension of the origin [29]. Our result can be proved with a much simpler derivation.

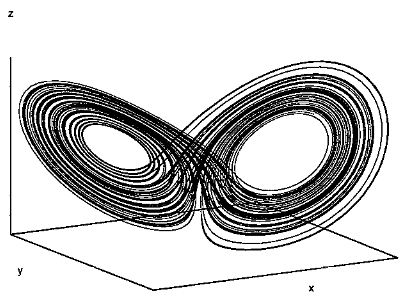

Fig. 3: A typical trajectory of the Lorenz system.

4 A higher-dimensional generalization of Bendixson's criterion

We begin with some definitions.

Definition 3 *[32] A set $S \subset \mathbb{R}^n$ is called d-dimensional rectifiable set, $d \in \mathbb{N}$ if $\mu_d(S) < \infty$ and μ_d-almost all of S is contained in the union of the images of countably many Lipschitz functions from $\mathbb{R}^d$ to $\mathbb{R}^n$.*

The rectifiable sets are generalized surfaces of geometric measure theory. Any 1-dimensional closed rectifiable contour γ bounds some two-dimensional rectifiable set, for example the cone over γ.

A set is said to be simply connected if any simple closed curve can be shrunk to a point continuously in the set.

Theorem 3 *Suppose that assumptions H1 and H2 are satisfied, let Ω be a simply connected set. Suppose that*

$$\lambda_1(x) + \lambda_2(x) < 0 \tag{11}$$

for any $x \in \Omega$. Then no periodic orbit can lie entirely in Ω.

Proof. The proof of Theorem 3 follows an idea used in the proof of the Leonov theorem ([31], see also Theorem 8.3.1 in [30]). Suppose (11) holds but there is a periodic orbit γ passing through a point x_0 $\gamma := \{x \mid \exists t \geq 0,\ x = x(t, x_0)\}$ which lies entirely in Ω.

Since as assumed the function $\mathbf{f}$ is bounded on any compact set it follows that there is a positive constant $L > 0$ such that for all $x_0 \in \gamma$

$$||x(t_1, x_0) - x(t_2, x_0)|| \leq L|t_1 - t_2|$$

and thus the set γ is an image of a Lipschitz continuous function. Therefore the set γ is a rectifiable one-dimensional set. From the theorem on existence of area-minimizing surfaces (see Theorem 5.6 in [33]) it follows that there exists a 2-dimensional rectifiable set $\bar{S} \in \mathbb{R}^n$ such that its boundary is γ and it has minimal Hausdorff 2-measure.

Let S be a rectifiable two-dimensional set $S \subset \Omega$ with boundary γ. The existence of such set follows from the fact that Ω is simply connected. As before, we denote by φ^t the flow of system (3). Let $\mu(S)$ be the Hausdorff 2-measure of a 2-dimensional surface S. Since γ is invariant under φ^t and $\varphi^t(S) \subset \Omega$ for any $t \geq 0$ (Ω is positively invariant) we have

$$\inf_{t \geq 0} \mu\left(\varphi^t(S)\right) \geq \mu(\bar{S}) > 0. \tag{12}$$

At the same time, using (11) from Theorem 1 it follows that

$$\lim_{t\to\infty} \mu\left(\varphi^t(S)\right) = 0, \tag{13}$$

which contradicts (12). Therefore, (11) ensures the absence of periodic trajectories lying in Ω.

It is worth noting that this theorem being applied to smooth systems together with its time reversed version (for smooth systems we have local right and left uniqueness) gives the classical Bendixson divergency condition.

The main idea of the proof (see [31]) is based on the existence of a surface with minimal area given its boundary. Although the mathematical problem of proving existence of a surface that has minimal area and is bounded by a prescribed curve, has long defied mathematical analysis, an experimental solution is easily obtained by a simple physical device. Plateau, a Belgian physicist, studied the problem by dipping an arbitrarily shaped wire frame into a soap solution. The resulting soap film corresponds to a relative minimum of area and thus produces a minimal surface spanned by that wire contour. A classical solution to Plateau's problem can be found, for example, in [34] with some regularity assumptions on the contour γ that can be violated if γ is a closed orbit corresponding to a periodic solution of a system of differential equations with discontinuous right hand sides. Fortunately, the argument based on geometric measure theory allows to overcome this difficulty.

It is worth mentioning that the hypothesis of the previous theorem can be relaxed. Particularly, one can relax the assumptions on positive invariance, simple connectivity of Ω and assumption H2 on the existence of the solutions for the infinite time interval. Indeed, from the proof of Theorem 1, it follows that if $S \in \Omega$ then $\pi(\varphi^t(S), d)$ decays *monotonically* provided S is of finite Hausdorff d-measure. Therefore, if condition (11) is satisfied in an open domain Ω (not necessarily simply connected and positively invariant) then Ω does not contain those invariant one-dimensional rectifiable sets γ that lie entirely in Ω and for which the corresponding 2-dimensional rectifiable sets S bounded by γ lie entirely in Ω and have minimal measure $\pi(S,2)$. The proof in this case is again by contradiction: the condition (11) implies monotonic decay of $\pi(\varphi^t(S),2)$ with time. However, γ is invariant and S has minimal measure $\pi(S,2)$ among all two-dimensional rectifiable sets bounded by γ and thus $\pi(\varphi^t(S),2)$ cannot decay. This relaxed condition can be used, for example, to disprove the existence of periodic orbits of a particular kind, as can be seen from Figure 4: if condition (11) is satisfied in torus Ω with $P(x_1) = I_n$, then the torus Ω does not contain periodic orbits of type A, while it can contain periodic orbits of type B. Note also, that this relaxed condition being applied to planar systems is tantamount to simple connectivity of Ω due to the Jordan theorem.

The final remark in this section is that the theorem can disprove not only the existence of periodic orbits, but also invariant sets of more general nature - homoclinic and/or heteroclinic orbits since they are rectifiable sets as well.

4.1 Example

Consider the following system:

$$\dot{x} = Ax + Bu, \quad u = -\mathrm{sign}(y), \quad y = Cx$$

where $x \in \mathbb{R}^3$, $u, y \in \mathbb{R}^1$ and the matrices A, B, C are given as follows

$$A = \begin{pmatrix} \alpha & 1 & 1 \\ -1 & \beta & -1 \\ -1 & 1 & -1 \end{pmatrix}, \; B = \begin{pmatrix} 0 \\ 0 \\ b \end{pmatrix}, \; C = (0\ 0\ 1)$$

with positive b. Consider the smooth function (4) in the form

$$V = (x_1 - x_2)^\top (x_1 - x_2)$$

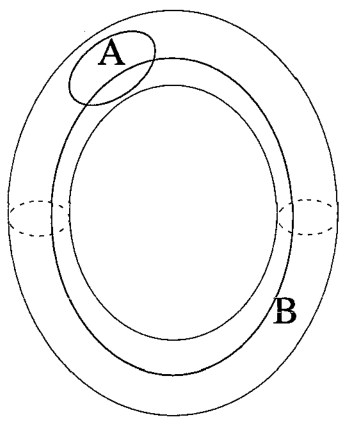

Fig. 4: If the condition (11) is satisfied in the torus Ω for $P = I_n$ then no periodic orbits of type A can lie entirely in Ω.

For this system the corresponding solution according to the Filippov convex definition coincides with the Utkin solution [1]. At the discontinuity points of the right hand side, the corresponding set valued function in the differential inclusion is obtained by the closure of the graph of the right hand side and by passing over to a convex hull. As shown in [1], p.155, these procedures do not increase the upper value of $\dot{V}^*$ and hence it is sufficient to compute the derivative of V only in the area of continuity of the right hand side. The derivative of V in this area satisfies

$$\dot{V} \le 2(x_1 - x_2)^\top \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & -1 \end{pmatrix} (x_1 - x_2)$$

The previous theorem suggests that if $\min\{\alpha, \beta\} \ge -1$, a sufficient condition for the absence of periodic solutions is

$$\alpha + \beta < 0 \tag{14}$$

To demonstrate that the violation of the condition (14) can result in oscillatory behavior we performed a computer simulation for the following parameter values: $\alpha = 1, \beta = -1/2, b = 1$. The results of the simulation are presented in Figure 5. It is seen that the system possesses orbitally stable limit cycle.

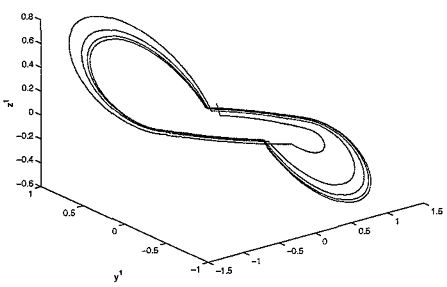

Fig. 5: Oscillatory behavior for $\alpha + \beta > 0$.

4.2 An LMI based criterion for Lur'e systems with discontinuous right hand side

In the previous example the matrix A was chosen as a sum of a diagonal and a skew-symmetric matrix that made all necessary calculations trivial. Next we present an LMI based criterion which ensures the absence of periodic solutions for the following system:

$$\dot{x} = Ax + Bu, \quad u = -b\mathrm{sign}(y), \quad y = Cx \tag{15}$$

where $x \in \mathbb{R}^n$, $n \geq 2$, $u, y \in \mathbb{R}^1$, $b > 0$ and the matrices A, B, C are of corresponding dimensions.

Theorem 4 *Suppose that there exists μ and positive definite matrix P such that the following inequality*

$$\begin{pmatrix} P(A-\mu I_n) + (A-\mu I_n)^\top P & PB - C^\top \\ B^\top P - C & 0 \end{pmatrix} \geq 0 \tag{16}$$

is satisfied. Then if

$$\mathrm{tr}A - (n-2)\mu < 0$$

the system (15) does not have periodic solutions.

Proof. According to (16) the matrix P satisfies the following equation $PB = C^\top$. Thus taking the derivative of the following function

$$V = (x_1 - x_2)^\top P(x_1 - x_2)$$

yields (as in the previous example it is sufficient to compute the derivative in the area of continuity of the right hand side)

$$\begin{aligned} \dot{V} &= (x_1 - x_2)^\top (PA + A^\top P)(x_1 - x_2) \\ &\quad -2b(Cx_1 - Cx_2)(\mathrm{sign}Cx_1 - \mathrm{sign}Cx_2) \\ &\leq (x_1 - x_2)^\top (PA + A^\top P)(x_1 - x_2) \end{aligned} \tag{17}$$

Now consider the smallest solution λ_n of the following equation

$$\det(PA + A^\top P - \lambda P) = 0 \tag{18}$$

From the hypothesis it follows that $\lambda_n \geq 2\mu$. On the other hand if λ_i, $i = 1, \ldots, n$ are the solutions of (18) then

$$\lambda_1 + \ldots + \lambda_n = 2\mathrm{tr}A$$

Since

$$\lambda_i \geq \lambda_n \tag{19}$$

it follows that

$$\lambda_1 + \lambda_2 \leq 2(\mathrm{tr}A - (n-2)\mu) < 0$$

and according to Theorem 3 the system (15) has no periodic solutions.

5 Conclusions

In this paper we presented a new *discontinuous version of a Bendixson's like criterion.* The criterion is based on a new result on estimation of the Hausdorff dimension of invariant sets for (possibly) discontinuous systems. The new criterion can be applied for the design and control of discontinuous systems when the requirement of global stability is too restrictive. Our study is based on dichotomy-like properties of solutions of dynamical systems with respect to each other rather than with respect to some invariant sets. We hope that further development of this approach will allow better understanding bifurcations in nonsmooth dynamical systems [36].

Appendix. Proof of Theorem 1

We begin with some definitions.

Let E_{x_1} be an open ellipsoid in $\mathbb{R}^n$ centered at the point x_1 and determined as

$$E_{x_1} = \{\eta \in \mathbb{R}^n \mid (x_1 - \eta)^\top P(x_1)(x_1 - \eta) < \varepsilon^2\}$$

Let $a_1(E) = a_2(E) = \ldots = a_n(E) = \varepsilon$ be the lengths of the semi-axes of the ellipsoid E computed in the metric defined by quadratic form $\eta^\top P(x_1)\eta$. Represent an arbitrary number d, $0 \le d \le n$ in the form $d = d_0 + s$, where $d_0 \in \mathbb{N}$ and $s \in [0,1)$ and introduce the following

$$\omega_d(E) = \prod_{i=1}^{d_0} a_i(E)(a_{d_0+1}(E))^s = \left(\prod_{i=1}^{d_0} a_i(E)\right)^{1-s} \left(\prod_{i=1}^{d_0+1} a_i(E)\right)^s = \varepsilon^d. \tag{20}$$

Given a compact set S. Fix a certain d and $\varepsilon > 0$ and consider all kinds of finite coverings of the compact S by ellipsoids E_{x_i}. If $d = 0$ we put $[\omega_d(E_{x_i})]^{1/d} = a_1(E_{x_i}))$. Similarly to the definition of Hausdorff d-measure we denote

$$\pi_d(S,d,\varepsilon) = \inf \sum_i \omega_d(E_{x_i}) = \inf \sum_i \varepsilon^d,$$

where the minimum is calculated over all coverings.

Since the matrix function $P(x)$ together with its inverse is bounded in an open neighborhood of S and since any ellipsoid E_{x_i} can be covered by some ball and any ball can be covered by an ellipsoid E_{x_i} for an appropriate ε, the value

$$\pi_d(S) = \sup_{\varepsilon > 0} \pi(S,d,\varepsilon)$$

can also serve as the Hausdorff measure and for definition of the Hausdorff dimension:

$$\mu\left(S,d,\frac{\varepsilon}{\sigma_{\max}}\right) \le \pi(S,d,\varepsilon) \le \mu\left(S,d,\frac{\varepsilon}{\sigma_{\min}}\right) \tag{21}$$

where $\sigma_{\min}$ is the lower bound of the smallest singular value of $P(x_1)$ over ε-neighborhood of S, $\sigma_{\max}$ is the upper bound of the greatest singular value of $P(x_1)$.

Now consider a class $\mathcal{R}$ of positive definite matrix functions defined and bounded together with its inverse in some open neighborhood of S. For $R \in \mathcal{R}$ consider ellipsoids

$$E_{x_1} = \{\eta \in \mathbb{R}^n \mid (x_1 - \eta)^\top R(x_1)(x_1 - \eta) < \varepsilon^2\}.$$

As before let $a_i(E_{x_1}),\ i = 1,\ldots,n$ be the lengths of its semiaxes computed in the metrics defined by the quadratic form $\eta^\top P(x_1)\eta$ and ordered in decreasing order. The d-measured volume of this ellipsoid computed in the same metric is then given by

$$\omega_d(E_{x_1}) = \prod_{i=1}^{d_0} a_i(a_{d_0+1})^s = \left(\prod_{i=1}^{d_0} a_i\right)^{1-s} \left(\prod_{i=1}^{d_0+1} a_i\right)^s .$$

Suppose that the ellipsoids E_{x_i} cover S and introduce the following notation

$$\tilde{\pi}_d(S,d,\varepsilon) = \inf \sum_i \omega_d(E_{x_i}),$$

where the minimum is calculated over all coverings and all $R \in \mathcal{R}$. Since $P \in \mathcal{R}$ it follows that

$$\tilde{\pi}(S,d,\varepsilon) \le \pi(S,d,\varepsilon)$$

Our next step is to consider $\pi(\varphi^t(S), d)$ as a function of time. Similarly to the proof of Lemma 1 it follows that the ellipsoid E_{x_1} is transformed by φ^t into a set that can be covered by the ellipsoid

$$\{\eta \in \mathbb{R}^n \mid (x(t,x_1)-\eta)^\top P(x(t,x_1))(x(t,x_1)-\eta) < \varepsilon^2 e^{2Lt}\}$$

From the time invariance of the system it follows that $\pi(\varphi^t(S), d)$ satisfies the following inequalities

$$\left|\pi(\varphi^t(S),d) - \pi(\varphi^{t_0}(S),d)\right| \leq \left|e^{L(t-t_0)} - 1\right| \pi(\varphi^{t_0}(S),d) \leq M|t-t_0|, \quad t \geq t_0$$

for some positive M. Therefore we have just established the following claim

Claim. $\pi(\varphi^t(S), d)$ is an absolutely continuous function of time and its derivative exists almost everywhere for $t \geq 0$ provided S is of finite Hausdorff d-measure.

According to assumption H1

$$\begin{aligned}&(x(t,x_1)-x(t,x_2))^\top P(x(t,x_1))(x(t,x_1)-x(t,x_2)) \leq (x_1-x_2)^\top P(x_1)(x_1-x_2)\\ &+t(x_1-x_2)^\top Q(x_1)(x_1-x_2) + h(t,x_1,x_2)\end{aligned}$$

for almost all $t \geq 0$ with the higher-order function h satisfying

$$\lim_{t\to 0, ||x_1-x_2||\to 0} \left(\frac{h(t,x_1,x_2)}{t(x_1-x_2)^\top P(x_1)(x_1-x_2)}\right) = 0$$

This inequality can be rewritten in the following form

$$(x(t,x_1)-x(t,x_2))^\top P(x(t,x_1))(x(t,x_1)-x(t,x_2)) \leq (x_1-x_2)^\top (P(x_1)+tQ_{t\varepsilon}(x_1))(x_1-x_2) \quad (22)$$

where

$$Q_{\tau\varepsilon}(x_1) = Q(x_1) + I_n \sup_{t,x_1}\left(\frac{h(t,x_1,x_2)}{t(x_1-x_2)^\top P(x_1)(x_1-x_2)}\right)$$

where the supremum is taken for $0 \leq t \leq \tau$ and $(x_1-x_2)^\top P(x_1)(x_1-x_2) \leq \varepsilon^2$.

Notice that $\lim_{t,\varepsilon\to 0} Q_{t\varepsilon}(x_1) = Q(x_1)$.

Given $\varepsilon > 0$ and sufficiently small $t > 0$, consider a finite covering of S by ellipsoids $E_i(\xi_i)$ centered at points ξ_i and defined by

$$E_i(\xi_i) = \{\eta \in \mathbb{R}^n \mid (\xi_i - \eta)^\top (P(\xi_i) + tQ_{t\varepsilon}(\xi_i))(\xi_i - \eta) < \varepsilon^2\}$$

Due to (22) the mapping φ^t transforms each ellipsoid of this covering into an ellipsoid $E_i'(x(t,\xi_i))$ centered at $x(t,\xi_i)$ and defined as

$$E_i'(x(t,\xi_i)) = \{\eta \in \mathbb{R}^n \mid (x(t,\xi_i)-\eta)^\top P(x(t,\xi_i))(x(t,\xi_i)-\eta) < \varepsilon^2\}$$

and therefore, if $\{E_i(\xi_i)\}$ covers S then $\{E_i'(x(t,\xi_i))\}$ covers $\varphi^t(S)$.

To estimate $\omega_d(E_i(\xi_i))$ from below notice that

$$\omega_d(E_i(x_i)) = \sigma_{i1}\sigma_{i2}\cdots\sigma_{id_0}\sigma_{id_0+1}^s \varepsilon^d \quad (23)$$

where $\sigma_{i1} \geq \sigma_{i2} \geq \ldots \geq \sigma_{in}$ are the singular values of the affine operator $\mathcal{L}_i$ that transforms the ellipsoid

$$\{\eta \in \mathbb{R}^n \mid (\eta - x_i)^\top P(x_i)(\eta - x_i) < \varepsilon^2\}$$

into the ellipsoid $E_i(x_i)$ computed in the metric defined by the quadratic form $\eta^\top P(x_i)\eta$, that is σ_{ij}^2, $j = 1, \ldots, n$ are the solutions λ_{ij}' to the following generalized eigenvalue problem

$$\det(L_i^\top P(x_i)L_i - \lambda_i' P(x_i)) = 0$$

with L_i being the matrix corresponding to the linear part of the operator $\mathcal{L}_i$.

Let us estimate the singular values of the affine operator $\mathcal{L}_i$. The linear part of this operator should satisfy

$$P(\xi_i) = L_i^\top (P(\xi_i) + tQ_{t\varepsilon}(\xi_i))L_i$$

To estimate the λ_i' it is convenient to estimate the singular values (in the same metric) of the inverse operator $\mathcal{L}_i^{-1}$, i.e. to find solutions $\lambda_{ij}'', j = 1, \dots, n$ of the following problem

$$\det(L_i^{-\top} P(x_i) L_i^{-1} - \lambda_i'' P(x_i)) = 0$$
$$L_i^{-\top} P(x_i) L_i^{-1} - (P(x_i) + tQ_{t\varepsilon}(x_i)) = 0$$

Using the Weierstrass theorem about diagonalization of a regular matrix pencil it follows that λ_{ij}'' can be represented in the form

$$\lambda_{ij}'' = 1 + t\lambda_{ij}, \quad j = 1, \dots, n$$

where λ_{ij} are the solutions of the following generalized eigenvalue problem

$$\det(Q_{t\varepsilon}(x_i) - \lambda_i P(x_i)) = 0$$

Finally, from (23) one concludes that

$$\sqrt{(1 + t\lambda_{i1})(1 + t\lambda_{i2}) \cdots (1 + t\lambda_{id_0})(1 + t\lambda_{id_0+1})^s} \omega_d(E_i(x_i)) = \varepsilon^d$$

and therefore

$$\sqrt{\sup_{i,\tau,\varepsilon}(1 + t\lambda_{i1})(1 + t\lambda_{i2}) \cdots (1 + t\lambda_{id_0})(1 + t\lambda_{id_0+1})^s} \tilde{\pi}(S, d, \varepsilon) \geq \pi(\varphi^t(S), d, \varepsilon).$$

where the supremum is calculated for $0 \leq \tau \leq t$ for sufficiently small t and $\{\eta \mid (x_i - \eta)^\top P(x_i)(x_i - \eta) < \varepsilon^2\}$. Since $\tilde{\pi}(S, d, \varepsilon) \leq \pi(S, d, \varepsilon)$ it follows that

$$\sqrt{\sup_{i,\tau,\varepsilon}(1 + t\lambda_{i1})(1 + t\lambda_{i2}) \cdots (1 + t\lambda_{id_0})(1 + t\lambda_{id_0+1})^s} \pi(S, d, \varepsilon) \geq \pi(\varphi^t(S), d, \varepsilon).$$

Taking the limit for $\varepsilon \to 0$ and $t \to 0$ and using time-invariance of the system one concludes that the right derivative of $\pi(\varphi^t(S), d)$ exists for almost all $t \geq 0$ and is bounded from above by

$$\frac{1}{2} \sup_{x \in \Omega} (\lambda_1(x) + \ldots + s\lambda_{d_0+1}(x)) \pi(\varphi^t(S), d)$$

provided S is of finite Hausdorff d-measure. Since $\pi(\varphi^t(S), d)$ is an absulutely continuous function of time it decays to zero monotonically and exponentially. The result now follows from the left inequality (21).

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5

Analysis of the Zero Average Dynamics Control Method for a Linear Converter

Fabiola Angulo[1], Mario di Bernardo[2], Enric Fossas[3], and Gerard Olivar[3]

[1] Universidad Nacional de Colombia, Colombia
[2] Dept. of Engineering Mathematics, University of Bristol, U.K.
[3] Inst. of Industrial and Control Engineering, Technical University of Catalonia, Spain

1 Introduction

Switched mode power converters can be modelled as variable structure systems because of the abrupt topological changes that the circuit, commanded by a discontinuous control action, undergoes; actually they are linear piecewise-smooth dynamical systems. As piecewise-smooth dynamical systems, sliding control mode should provide good control strategies. However, because of the discontinuities in the module of the dynamics on the switching surface, the resulting designs operate at variable switching frequency, this leading to an undesirable chattering phenomenon and hindering the design of the regulator/inverter filter elements. The Zero Average Dynamics (ZAD) control scheme, recently proposed in [4], tries to conjugate the advantages of fixed frequency implementations and the inherent robustness of sliding control modes. It is based on an appropriate design of the output that guarantees the fulfilment of the specifications and on a specific design of the Pulse Width Modulator duty cycle in such a way that the output average in each PWM-period is zero.

There are several possible PWM implementations, such as leading, trailing and centered pulse. The third, in turn, can be single or double updated. This paper deals with a buck converter controlled by a ZAD algorithm which is implemented in a single updated centered PWM. As for the output, for robustness purposes a linear combination of the error and its derivative is considered as in [2] and [3]. The error dynamics time constant appears as a bifurcation parameter. The first part of the report deals with the duty cycle equilibrium value and to steady state. Stability analysis, as the output time constant varies, is performed using a first linear approximation and Floquet and Lyapunov exponents. For the output time constant lying in certain interval, the overall system results in a stable, fixed frequency, robust controlled satisfactory performance. In the second part, average theory (perturbation theory in mathematics) is used to support the aforementioned satisfactory performance. To be precise, a centered ZAD-PWM scheme will be considered and steady-state maximum values for the error and the sliding surface in a sampling period will be computed as well.

A complete introduction to power converters can be found in [6], and a fast overview including control, in [7]. Nonlinear phenomena in power electronics, bifurcations, chaos and control of chaos are widely reported in [1]. The problem under consideration appeared when a Zero Average Dynamics control scheme was implemented to a buck converter using a centered Pulse Width Modulator. The unexpected duty cycle saturation led us to study the problem in the field of nonlinear phenomena.

2 Statement of the Problem

The dc-to-dc buck converter can be modelled as the dynamical system

$$\begin{pmatrix} \frac{dv}{d\tau} \\ \frac{di}{d\tau} \end{pmatrix} = \begin{pmatrix} -\frac{1}{RC} & \frac{1}{C} \\ -\frac{1}{L} & 0 \end{pmatrix} \begin{pmatrix} v \\ i \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{E}{L} \end{pmatrix} u \tag{1}$$

The state variables are the capacitor voltage v and the inductor current i. The control signal u takes discrete values in the set $\{-1,1\}$ depending on the switch position. The plant can also be described by the dimensionless system

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -\gamma & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u, \tag{2}$$

where $x_1 = v/E$, $x_2 = \frac{1}{E}\sqrt{\frac{L}{C}}i$, $t = \tau/\sqrt{LC}$ and $\gamma = \frac{1}{R}\sqrt{\frac{L}{C}}$. Or in a more compact form

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x} + \mathbf{b}u \tag{3}$$

where $\mathbf{x}$ and u are the state vector and the control input respectively and $\mathbf{A}$ and $\mathbf{b}$ the corresponding matrix and vector.

The sliding surface S, a proportional-derivative expression in the error, as in Bilalovic [2] and Carpita [3], is defined by

$$S(x) = (x_1 - x_{1ref}) + k_s(\dot{x}_1 - \dot{x}_{1ref}) \tag{4}$$

as the output to be regulated to zero. x_{1ref} is the reference dimensionless output voltage to be tracked and k_s a constant parameter; actually, the time constant of the first order ODE the error will fulfil. A very rich dynamics is observed as constant k_s varies. In order to obtain it, let us state some basics on PWMs and ZADs.

A centered Pulse Width Modulator with duty-cycle[4] d_k and period T operates as follows:

$$u(t) = \begin{cases} u_1 \text{ if } kT \le t < kT + \frac{d_k}{2} \\ u_2 \text{ if } kT + \frac{d_k}{2} \le t < (k+1)T - \frac{d_k}{2} \\ u_1 \text{ if } (k+1)T - \frac{d_k}{2} \le t < (k+1)T \end{cases} \tag{5}$$

Then, substituting u from Eq. (5) into Eq. (3) and solving the Ordinary Differential Equation, we obtain the T-period Poincaré map associated to the PWM, which results in

$$\mathbf{x}((k+1)T) = e^{\mathbf{A}T}\mathbf{x}(kT) + \left(e^{\mathbf{A}(T-d_k/2)} + \mathbf{I}\right)\left(e^{\mathbf{A}d_k/2} - \mathbf{I}\right)$$

$$\mathbf{A}^{-1}\mathbf{b}u_1 + e^{\mathbf{A}d_k/2}\left(e^{\mathbf{A}(T-d_k)} - \mathbf{I}\right)\mathbf{A}^{-1}\mathbf{b}u_2$$

Zero Averaged Dynamics schemes search for duty cycles d_k that force output zero average in each sampling period. That is,

$$\langle S \rangle = \int_{kT}^{(k+1)T} S(x(t))dt = 0 \tag{6}$$

Obtaining d_k from this equation demands the solution of a transcendent equation in each iteration, which is not feasible in an on line implementation. Hence, considering a linear piecewise approximation of the output S the duty cycle results in

$$\hat{d}_k = \frac{2S(0) + T\dot{S}_2}{\dot{S}_2 - \dot{S}_1} \tag{7}$$

Actually, $d_k = \mathrm{sat}(\hat{d}_k)$ because $d_k \in [0,T]$ must be fulfilled.

[4] There is an abuse of language with the word *duty-cycle*: $d_k \in [0,T]$, instead of $d_k \in [0,1]$.

3 Dynamical analysis

The system equilibrium points and equilibrium duty cycle are computed in this section as functions of the bifurcation parameter k_s. For simplicity, let us move to the regulation problem; then x_{1ref}, $x_{2ref} = \gamma x_{1ref}$ are constant. From Eq. (7) and a previous remark,

$$\hat{d}_k = \frac{2S(0) + T\dot{S}_2}{k_s(u_2 - u_1)} \tag{8}$$

Furthermore, replacing $\dot{x}_1$ and $\dot{x}_2$ from Eq. (2) with $u = u_2$, into $\dot{S} = (1 - k_s\gamma)\dot{x}_1 + k_s\dot{x}_2$ yields

$$\dot{S}_2 = (k_s\gamma^2 - \gamma - k_s)x_1 + (1 - k_s\gamma)x_2 + k_s u_2 \tag{9}$$

Assuming the equilibrium point $(x_1^*, x_2^*) = (x_{1ref}, \gamma x_{1ref})$ is reached, then $S(0) = 0$ and Eq. (9) results in

$$\dot{S}_2 = -k_s x_{1ref} + k_s u_2. \tag{10}$$

Hence, the duty cycle in equilibrium is

$$d^* = T\frac{u_2 - x_{1ref}}{u_2 - u_1} \tag{11}$$

Thus, the equilibrium duty cycle is $\frac{T}{2}(1 + x_{1ref})$ if the PWM action follows the scheme $(+1, -1, +1)$, and $\frac{T}{2}(1 - x_{1ref})$ otherwise. Additionally, since $x_{1ref} \in [-1, 1]$, both expressions are completely symmetric. The PWM scheme $(+1, -1, +1)$ is assumed throughout the paper.

Stability analysis: In several simulations, $k_s = 4.5$ was taken and no stability limit was found. However, open-loop stability analysis suggests the existence of such a limit. In order to obtain it, let us consider $x_{1ref} = 0.8$ and let the bifurcation parameter k_s vary in the interval $[0.1, 4.7]$. Then, let us

- take $d_k = d^*$ as in Eq. (11)
- linearise the system around the fixed point of the Poincaré map, and
- compute the corresponding eigenvalues.

Results are shown in Table 1. Figures were truncated at the fourth decimal.

Table 1: *Equilibrium point and eigenvalues of the linearised system.* $(x_{1ref} = 0.8)$

k_s	x_1^*	x_2^*	λ_1	λ_2
0.1	0.7999	0.2800	0.2648	-3.6551
0.6	0.7999	0.2800	0.7487	-1.2691
1.1	0.7998	0.2800	0.8528	-1.1123
1.6	0.7998	0.2800	0.8961	-1.0578
2.1	0.7997	0.2800	0.9198	-1.0303
2.6	0.7997	0.2799	0.9347	-1.0136
3.2	0.7996	0.2799	0.9466	-1.0007
3.24	0.7996	0.2799	0.9472	-1.0000
3.25	0.7996	0.2799	0.9474	-0.9998
3.7	0.7995	0.2799	0.9536	-0.9932
4.2	0.7995	0.2799	0.9590	-0.9875
4.7	0.7994	0.2799	0.9633	-0.9831

On the one hand, if parameter k_s decreases, the negative eigenvalue decreases to the point leaving the unity circle, while the other eigenvalue remains inside the unity circle. On the other hand, if parameter k_s increases, both eigenvalues remain inside the unity circle. The evolution of the stable and the unstable eigenvalues is shown in Fig. 1 and 2, respectively.

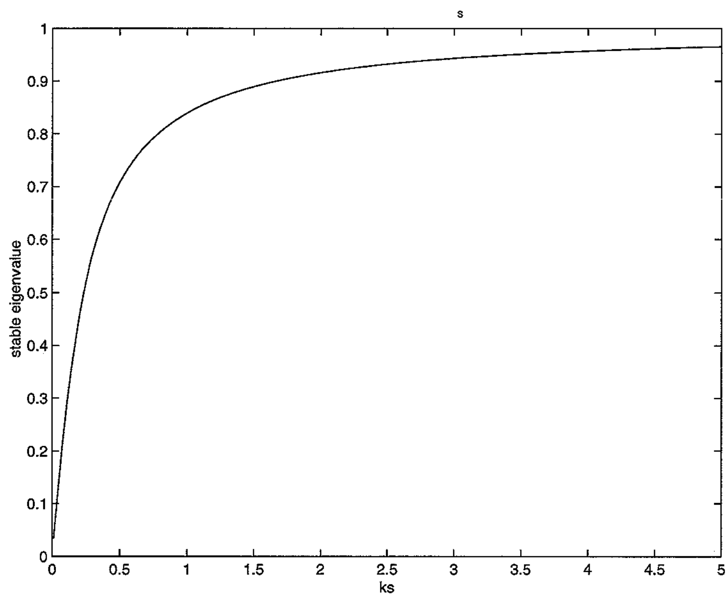

Fig. 1: Evolution of the stable eigenvalue.

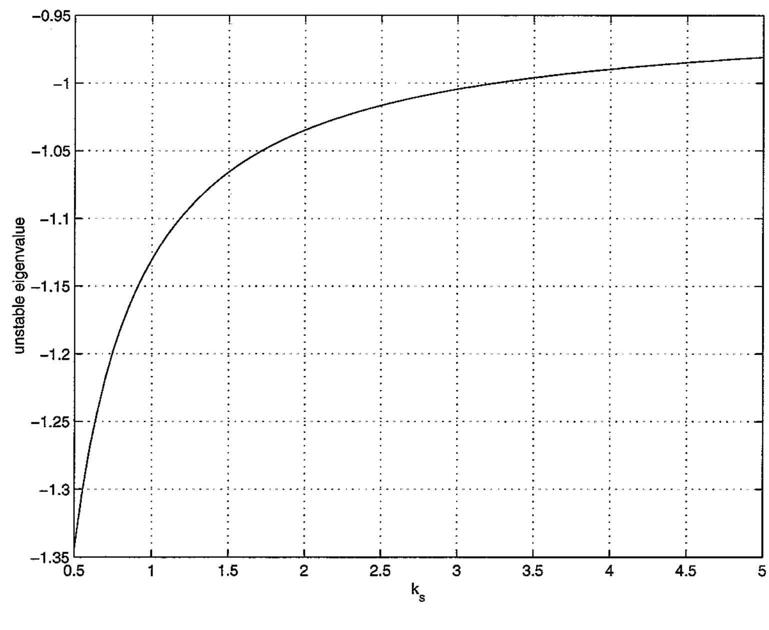

Fig. 2: Evolution of the unstable eigenvalue.

Table 2: *2T-periodic cycles.* ($x_{1ref} = 0.8$)

k_s	x_1^*	x_2^*	d_1^*
3.24375	0.79963406391683	0.27996227977072	89.99459963648458%
3.24375	0.79963406391719	0.27996228003936	89.99459978793159%
3.24374	0.79963406502222	0.27996228007770	89.99459964689876%
3.24374	0.79963406502279	0.27996228050525	89.99459988792795%
3.24	0.79966843313128	0.29771076320774	100%
3.24	0.79962048639421	0.26221558558949	79.98945599783113%
3.10	0.79961460997932	0.29769671272210	100%
3.10	0.79956665070819	0.26219199863264	79.98407921504162%

4 Bifurcations and Chaos

Previous simulations suggest different qualitative behaviors for the controlled system. Particularly interesting is the presence of unsaturated 2T-periodic orbits and a saturated duty cycle together with an unsaturated one, yielding again a 2T-periodic orbit; the latter becomes 4T-periodic and then a route to chaos follows. The first bifurcations are due to one of the eigenvalues of the Jacobian of the Poincaré map crosses the stability limit by -1, yielding flip bifurcations [5].

Since the results are very similar for all the reference values varying in the interval $[0.5, 0.8]$, from now on only a reference value of 0.8 will be considered. Some of those doubly periodic behaviors are shown in Table 2. Note the saturation value of the duty cycle for $k_s = 3.24$. More details are depicted in Figs. 3, 4 and 5, where the output voltage, the current and the duty cycle are respectively plotted versus the bifurcation parameter k_s.

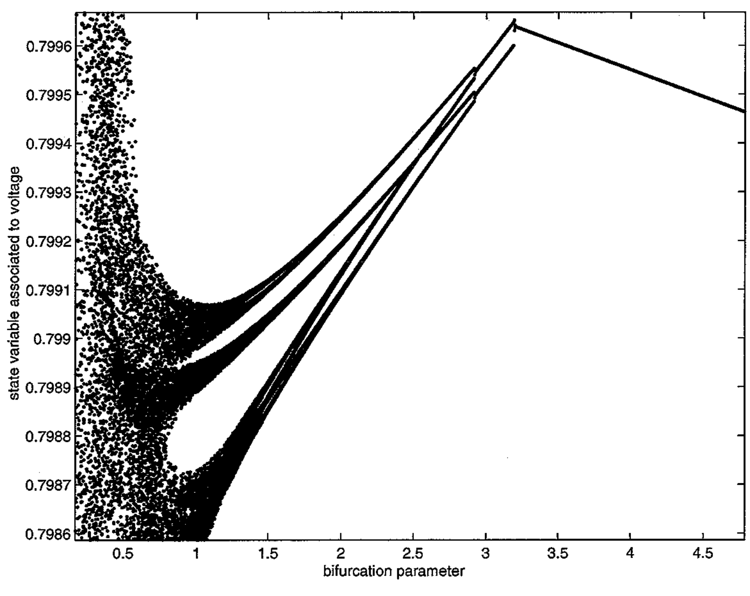

Fig. 3: Bifurcation diagram. Voltage vs k_s

As for the saturated/unsaturated duty cycle for 2T-periodic orbits, the bifurcation parameter critical value is $k_s = 3.24$, approximately. At $k_s = 3.1$, the 2T-periodic cycles yield 4T-periodic ones and these, in turn, 8T-periodic ones at $k_s = 2.8$. As k_s continues decreasing, chaos appears.

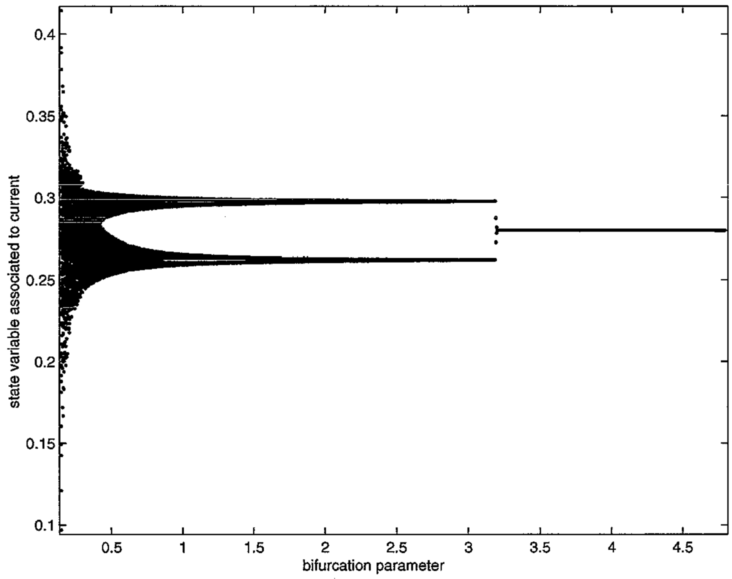

Fig. 4: Bifurcation diagram. Current vs k_s

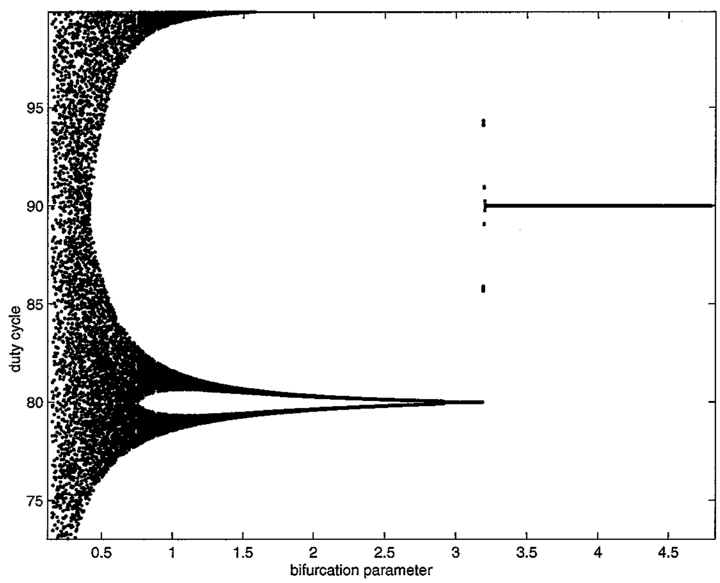

Fig. 5: Bifurcation diagram. Duty cycle vs k_s

The evolution of the Floquet exponents real part as parameter k_s varies is displayed in Fig. 6
The evolution of the Lyapunov exponents in terms of parameter k_s is displayed in Fig. 7. It is worth noting that the zero crossing point of the Floquet and Lyapunov exponents is the same. It coincides with the critical value shown in Table 1 and Fig. 2.

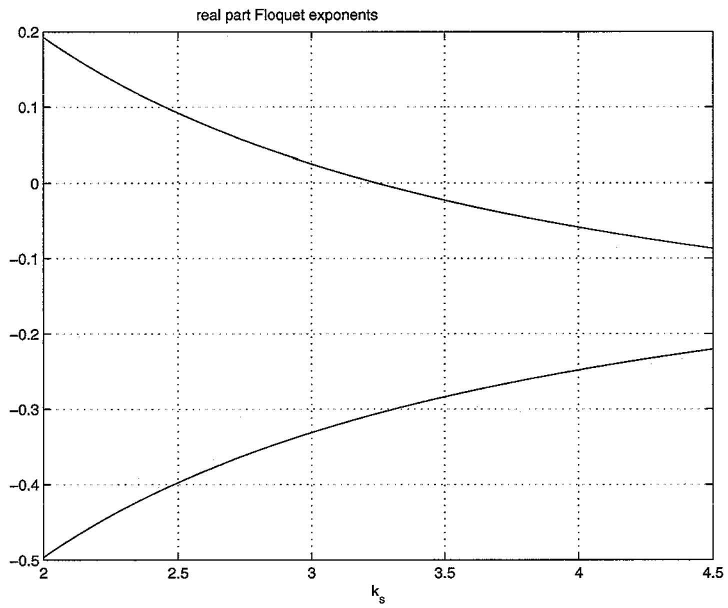

Fig. 6: Floquet exponents.

5 Applied Perturbation Theory to Power Converters Regulation

Let us rewrite system dynamics in the more suitable variables $e = x_1 - v_{ref}$ and $s = e + k_s\dot{e}$, being thereafter $t = \tau/\varepsilon$ the independent variable and $\varepsilon = T$,

$$\begin{pmatrix} \dot{e} \\ \dot{s} \end{pmatrix} = \varepsilon \begin{pmatrix} -\frac{1}{k_s} & \frac{1}{k_s} \\ \gamma - \frac{1}{k_s} - k_s & \frac{1}{k_s} - \gamma \end{pmatrix} \begin{pmatrix} e \\ s \end{pmatrix} + k_s \varepsilon \begin{pmatrix} 0 \\ u - v_{ref} \end{pmatrix} \tag{12}$$

This system reads as $\dot{x} = \varepsilon f(x,t)$ as usual in averaging (perturbation) methods. u is defined as

$$u = \begin{cases} 1 \text{ if } k \leqslant t \leqslant k + d/2 \\ -1 \text{ if } k + d/2 < t < k + 1 - d/2 \\ 1 \text{ if } k + (1 - d/2) \leqslant t \leqslant k + 1 \end{cases} \tag{13}$$

In this case $d = D/T = D/\varepsilon \simeq \frac{1+v_{ref}}{2}$. Taking $\mathbf{x} = (e, s)$ equation (12) can be written in a compact form as $\dot{\mathbf{x}} = \varepsilon\,\mathbf{A}\mathbf{x} + k_s\,\varepsilon\,\hat{\mathbf{u}}$ where

$$\hat{\mathbf{u}} = \begin{pmatrix} 0 \\ u - v_{ref} \end{pmatrix} \quad \text{and} \quad \mathbf{A} = \begin{pmatrix} -\frac{1}{k_s} & \frac{1}{k_s} \\ \gamma - \frac{1}{k_s} - k_s & \frac{1}{k_s} - \gamma \end{pmatrix}$$

being u defined by equation (13) or similar, depending on the pulse generation scheme (PWMC or PWML). The control technique selected according to equation (6) guarantees $\langle x_2 \rangle = 0$ and $\langle \hat{\mathbf{u}}_2 \rangle = 0$.

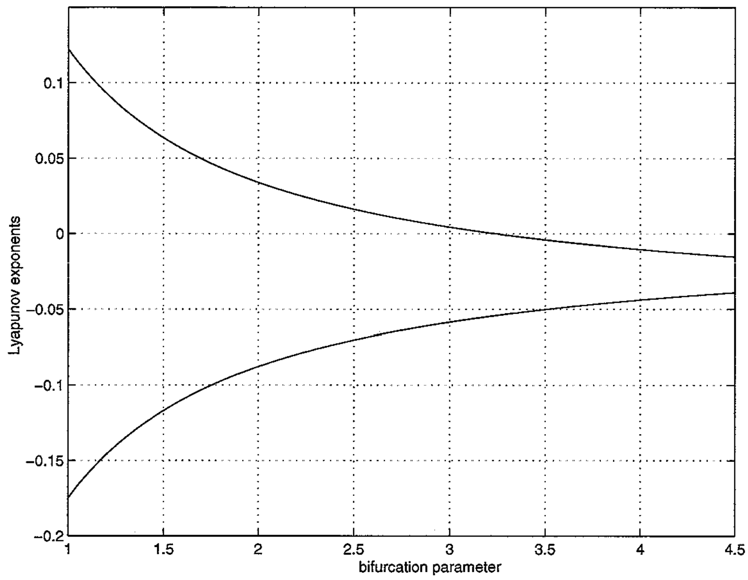

Fig. 7: Lyapunov exponents.

In order to average the system, let us define the change of variables $\mathbf{y} = \mathbf{x} - \varepsilon\overline{\mathbf{u}}^1$, where $\overline{\mathbf{u}}^1 = k_s\int_0^t \hat{\mathbf{u}} d\tau$. Since $\langle\hat{\mathbf{u}}\rangle = 0$, $\overline{\mathbf{u}}^1$ is also periodic. It is straightforward to obtain,

$$\dot{\mathbf{y}} = \varepsilon\mathbf{A}\mathbf{y} + \varepsilon^2\mathbf{A}\overline{\mathbf{u}}^1 \tag{14}$$

For the next change of variables, let us note that the function $\int_0^t A\overline{u}^1$ can not be presumed periodic. To solve this problem let us define $\hat{\overline{u}}^1 = A\overline{u}^1$, $a = \int_0^1 \hat{\overline{u}}^1 dt$ as the mean of $\hat{\overline{u}}^1 = A\overline{u}^1$ and $\overline{u}^2 = \int_0^t (\hat{\overline{u}}^1 - a) d\tau$, which is periodic.

Now, let us define a new change of variables, namely

$$z = y - \varepsilon^2\overline{u}^2. \tag{15}$$

Then,

$$\dot{\mathbf{z}} = \dot{\mathbf{y}} - \varepsilon^2\dot{\overline{\mathbf{u}}}^2$$

This equation is also well defined in the switching instant because $\overline{u}^2$ is $\mathcal{C}^1$. Hence

$$\dot{\mathbf{z}} = \dot{\mathbf{y}} - \varepsilon^2\left(\mathbf{A}\overline{\mathbf{u}}^1 - \mathbf{a}\right) \tag{16}$$

Replacing (14) and (15) in (16) yields

$$\dot{\mathbf{z}} = \varepsilon\mathbf{A}\mathbf{z} + \varepsilon^2\mathbf{a} + \varepsilon^3\mathbf{A}\overline{\mathbf{u}}^2 \tag{17}$$

Let $\mathbf{z}^* = -\varepsilon\mathbf{A}^{-1}\mathbf{a}$ be the equilibrium point of equation (17) for $\overline{\mathbf{u}}^2 = \mathbf{0}$ and $\mathbf{w} = \mathbf{z} - \mathbf{z}^*$. Then,

$$\dot{\mathbf{w}} = \varepsilon\mathbf{A}\mathbf{w} + \varepsilon^3\hat{\overline{\mathbf{u}}}^2$$

where $\hat{\overline{\mathbf{u}}}^2 = \mathbf{A}\overline{\mathbf{u}}^2$. The general solution for this equation is

$$\mathbf{w}(t) = e^{\varepsilon\mathbf{A}t}\mathbf{w}(0) + \varepsilon^3\int_0^t e^{\varepsilon\mathbf{A}(t-\sigma)}\hat{\overline{\mathbf{u}}}^2(\sigma)\,d\sigma \tag{18}$$

A steady-state solution will be periodic; thus $\mathbf{w}(1) = \mathbf{w}(0)$. Hence,

$$\mathbf{w}(0) = \left(\mathbf{I} - e^{\varepsilon \mathbf{A}}\right)^{-1} \varepsilon^3 \int_0^1 e^{\varepsilon \mathbf{A}(1-\sigma)} \hat{\overline{\mathbf{u}}}^2(\sigma)\, d\sigma \tag{19}$$

Finally, the general solution in the original variables $\mathbf{x}$ of the system is

$$\mathbf{x}(t) = \varepsilon \overline{\mathbf{u}}^1(t) + \varepsilon^2 \overline{\mathbf{u}}^2(t) + \mathbf{w}(t) - \varepsilon \mathbf{A}^{-1}\mathbf{a} \tag{20}$$

where $\mathbf{w}(t)$ is given by equations (18) and (19).

As $\overline{\mathbf{u}}^1$ and $\overline{\mathbf{u}}^2$ only depends on the input control signal, their integrals are well known. Then, in order to bound $\mathbf{x}(t)$, we will proceed in obtaining bounds for each component of the variable $\mathbf{w}(t)$, namely the state transition matrix, $\left(\mathbf{I} - e^{\varepsilon \mathbf{A}}\right)^{-1}$ and $\int_0^1 e^{\varepsilon \mathbf{A}(1-\sigma)} \hat{\overline{\mathbf{u}}}^2(\sigma)\, d\sigma$.

To conclude, using the estimated bound of $\mathbf{w}(t)$, is easy to find from

$$\mathbf{x}(t) = \varepsilon \overline{\mathbf{u}}^1(t) + \varepsilon^2 \overline{\mathbf{u}}^2(t) + \mathbf{w}(t)$$

the maximum values for the error $e = x_1$ and for the sliding surface $s = x_2$.

All of these calculus are particularised for $d = 0.9$ and the parameter values $k_s = 4.5$, $v_{ref} = 0.8$, $\varepsilon = T = 0.1767$, $E = 40$V and $\gamma = 0.35$. The later corresponds to $R = 20\Omega$, $C = 40\mu$F, $L = 2$mH in physical parameters.

Therefore $\det(I - e^{\varepsilon \mathbf{A}}) \geq 0.0285$, $|e_{11}| \leq 1.0083$, $|e_{12}| \leq 0.0393$, $|e_{21}| \leq 0.7726$, $|e_{22}| \leq 1.0083$, $I_1 \leq 0.0043$, $I_2 \leq 0.0379$, $|m_{11}| \leq 0.0392$, $|m_{12}| \leq 0.0440$, $|m_{21}| \leq 0.8640$ and $|m_{22}| \leq 0.0559$.

Hence

$$\begin{array}{l} w_1(0) \leq 0.00036 \; w_2(0) \leq 0.0011 \\ w_1(t) \leq 0.00043 \; w_2(t) \leq 0.0016 \end{array}$$

Output error estimation As in this case $\mathbf{a} = \mathbf{0}$ and the first component of the input vector $\overline{\mathbf{u}}^1$ is zero, error dynamics is defined by the first component of $\mathbf{w}(t)$ and $\varepsilon^2 \overline{\mathbf{u}}^2$, thus

$$e(t) = w(t)_1 + \varepsilon^2 \overline{u}_1^2$$

The maximum of $|\overline{u}_1^2| = |\overline{u}_{21}^2|$ holds at $t = 0.5$ and is 0.0225, then

$$\max |e(t)| \leq 0.0011$$

This is equivalent to a maximum error value of 0.14% in steady state when the reference reaches 0.8 value. In Figure 8 the error behavior obtained from a numerical simulation is depicted. Notice that real error value is lower than the estimated. However, the bound is really close.

Sliding surface error estimation In this case the expression is adjusted to analyze the second component as

$$s(t) = \varepsilon \overline{u}_2^1 + \varepsilon^2 \overline{u}_2^2 + w_2(t)$$

The maximum associated to $|\overline{u}_2^1| = k_s |\overline{u}_{21}^1|$ holds at $t = 0.45$ and is 0.4050; while the maximum of $|\overline{u}_2^2| = (1 - k_s\gamma)|\overline{u}_{21}^2|$, holds at $t = 0.5$ and is 0.0129, then

$$\max |s(t)| \leqslant \varepsilon |\overline{u}_2^1| + \varepsilon^2 |\overline{u}_2^2| + w_2(t) = 0.0728$$

Notice the agreement between the maximum of the piecewise sliding surface approximation and our result. The leading term in the later inequality, $\varepsilon \overline{u}_2^1$ and is 0.0716. Simulation results are depicted in Figure 9.

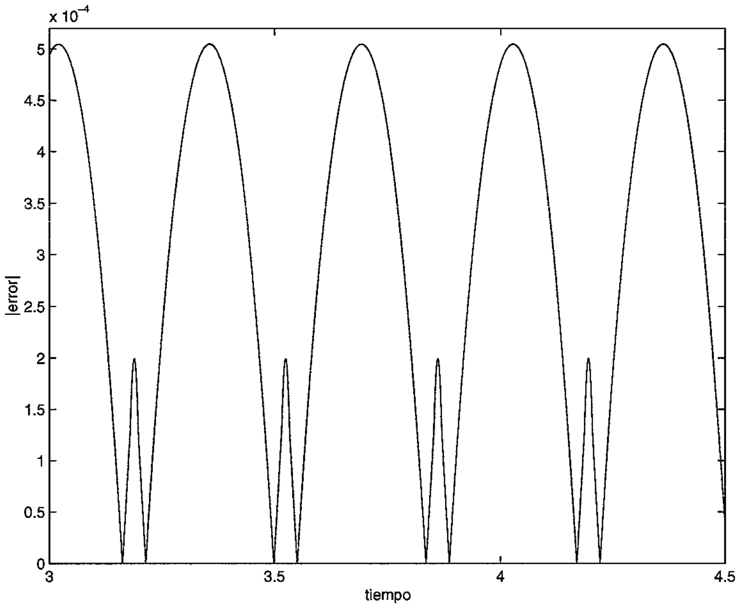

Fig. 8: Behavior of error in sampling interval for PWMC scheme

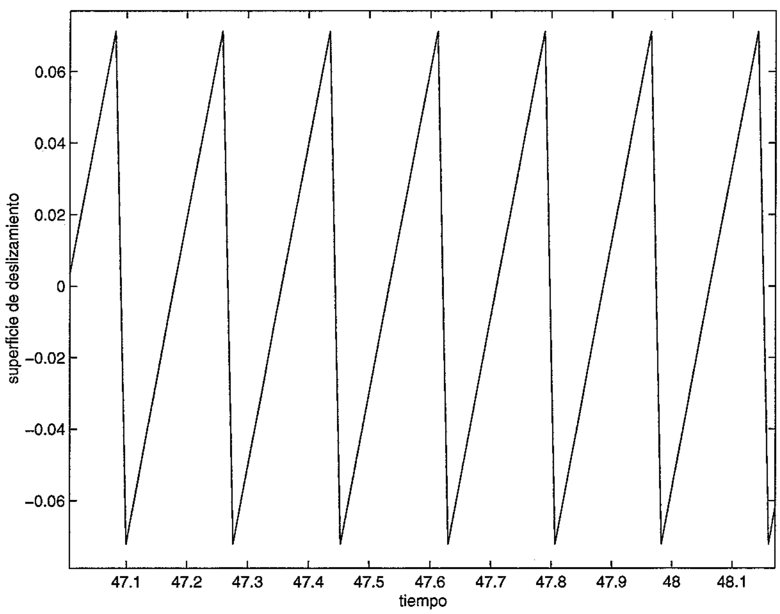

Fig. 9: Behavior of sliding surface in a sampling interval for PWMC scheme

These results were generalised to second order systems modelled by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u$$
$$y = \mathbf{c}\mathbf{x}$$

where $\mathbf{x} \in \mathcal{R}^2$, $u \in \mathcal{R}$, $y \in \mathcal{R}$ and matrix $\mathbf{A}$ and vectors $\mathbf{b}$ and $\mathbf{c}$ have appropriate dimensions. We assume y has relative degree 2, which imply a transfer function given by:

$$Y(s) = \frac{K}{s^2 + \alpha_2 s + \alpha_1} U(s)$$

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6

Some Special Features of Stability for Mechanical Systems Subject to Unilateral Constraints

L. Menini and A. Tornambè

Dip. Informatica, Sistemi e Produzione, Università di Roma Tor Vergata, Italy

1 Introduction

Mechanical systems subject to non-smooth impacts arising from the presence of unilateral constraints have been considered in the scientific literature starting from the very early works of Newton and Hertz. For an extensive review of the existing results, the interested reader is referred to [2]. Nowadays, the renewed interest for such systems is due not only to their technological applications (e.g., walking, hopping and juggling robots, gear or cam based mechanisms, hammering tasks, manipulation problems referred to artificial satellites) but particularly to the interesting control problems involved. As a matter of fact, many of the available control algorithms, e.g., those based on classical Lyapunov stability theory, require uniqueness and continuity with respect to the initial conditions of the solution of the dynamic system to be controlled, which can be lost due to the unilateral constraints, even in the simple case when both the equations of motion and the constraints are linear (e.g., multiple impacts or finite accumulation points of the impact times can cause the loss of such properties). In the last years, a lot of research has been devoted to the control of hybrid dynamic systems (see, e.g., [1, 8, 9]), of which mechanical systems subject to non-smooth impacts are a subclass. However, the Hamiltonian structure of the unconstrained mechanical systems and the special type of discontinuities that are generated by the impacts, make such a subclass easier to deal with than the whole class of hybrid dynamic systems, so that specific can be obtained.

We consider finite-dimensional mechanical systems which can be described by a vector $\mathbf{q}(t) \in \mathbb{R}^n$ of generalized coordinates. If the vector $\mathbf{q}(t)$ is constrained to belong to an admissible region:

$$\mathcal{A} := \{\mathbf{q} \in \mathbb{R}^n : \quad f_i(\mathbf{q}) \le 0, \quad i = 1, 2, ..., m\}, \tag{1}$$

there can be times t_c at which $\mathbf{q}(t)$ is not differentiable, i.e., impact times. If $f_i(\mathbf{q}(t)) = 0$ for some $i \in \{1, 2, ..., m\}$ and for some $t \in \mathbb{R}$, then some parts of the mechanical system are, at such a time, in contact with themselves or with the external environment. An impact can occur at a certain time $t_c \in \mathbb{R}$ only if, at such a time, one has $f_i(\mathbf{q}(t_c)) = 0$ for some $i \in \{1, 2, ..., m\}$. If $f_i(\mathbf{q}(t)) = 0$ for more than one index $i \in \{1, 2, ..., m\}$ and for the same time t, then there is a multiple contact at time t.

By assuming that the impacts cause no instantaneous loss of energy, the method of the Valentine variables allows one to model mechanical systems subject to inequality constraints [7] by means of the Hamilton principle. Such a method consists in a double transformation of the inequality constraints, which are, first, transformed into equality constraints and, finally, converted into differential constraints for convenience.

If $T(\mathbf{q}(t),\, \dot{\mathbf{q}}(t)) = \frac{1}{2}\, \dot{\mathbf{q}}^T(t)\, \mathbf{B}(\mathbf{q}(t))\, \dot{\mathbf{q}}(t)$ is the kinetic energy and $U_t(\mathbf{q}(t)) = U(\mathbf{q}(t)) - \mathbf{q}^T(t)\, \mathbf{E}\, \mathbf{u}(t)$ is the total potential energy of the mechanical system (taking into account the action of the vector $\mathbf{u}(t) \in \mathbb{R}^p$ of the control generalized forces), denote by $L(\mathbf{q}(t),\, \dot{\mathbf{q}}(t)) := T(\mathbf{q}(t),\, \dot{\mathbf{q}}(t)) - U(\mathbf{q}(t))$ the

Lagrangian function. Then, the equations describing the motion of the system are the following

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\mathbf{q}}} - \frac{\partial L}{\partial \mathbf{q}} + \sum_{i=1}^{m} \dot{\lambda}_i \, \mathbf{J}_i(\mathbf{q}) = \mathbf{E}\,\mathbf{u}, \tag{2a}$$

$$2\,\gamma_i \dot{\lambda}_i = 0, \quad i = 1, \ldots, m, \tag{2b}$$

$$\mathbf{J}_i^T(\mathbf{q})\dot{\mathbf{q}} + 2\,\gamma_i \dot{\gamma}_i = 0, \quad i = 1, \ldots, m, \tag{2c}$$

where $\dot{\lambda}_i$ are the derivatives (defined only in the distributional sense) of the Lagrange multipliers used to take into account the differential constraints, and $\mathbf{J}_i(\mathbf{q})$ denotes the (column) Jacobian vector of $f_i(\mathbf{q})$. In order to compute the system behaviour from given initial conditions, the equation above is integrated in the intervals between impact times, whereas, at the impact times, the following Erdmann-Weierstrass corner conditions need to be satisfied:

$$\frac{1}{2}\,\dot{\mathbf{q}}^T(t_c^-)\,\mathbf{B}(\mathbf{q}(t_c))\,\dot{\mathbf{q}}(t_c^-) = \frac{1}{2}\,\dot{\mathbf{q}}^T(t_c^+)\,\mathbf{B}(\mathbf{q}(t_c))\,\dot{\mathbf{q}}(t_c^+), \tag{3a}$$

$$\mathbf{B}(\mathbf{q}(t_c))\,\dot{\mathbf{q}}(t_c^-) + \sum_{i=1}^{m} \lambda_i(t_c^-)\,\mathbf{J}_i(\mathbf{q}(t_c)) = \mathbf{B}(\mathbf{q}(t_c))\,\dot{\mathbf{q}}(t_c^+) + \sum_{i=1}^{m} \lambda_i(t_c^+)\,\mathbf{J}_i(\mathbf{q}(t_c)), \tag{3b}$$

$$2\,\gamma_i(t_c)\dot{\lambda}_i(t_c^-) = 2\,\gamma_i(t_c)\dot{\lambda}_i(t_c^+), \quad i = 1, \ldots, m, \tag{3c}$$

The Erdmann-Weierstrass corner condition (3a) shows that the kinetic energies immediately before and immediately after an impact must be equal, whereas equation (3b) relates the jump $\dot{\mathbf{q}}(t_c^+) - \dot{\mathbf{q}}(t_c^-)$ of the generalized velocities with the jumps $\lambda_i(t_c^+) - \lambda_i(t_c^-)$ of the Lagrange multipliers. In the case of single impacts, equations (3a)-(3b) can be solved uniquely by requiring that the mechanical system does not leave the admissible region; hence, such equations allow the post-impact velocities to be determined as functions of the pre-impact velocities and of the system configuration. Therefore, the system described by (2a)-(3) is a special case of the hybrid system (50) in [9]; moreover, as well known, it can also be transformed into the complementarity formulation used for hybrid systems in [3, 8].

In this work, we briefly report some results on stabilization of equilibrium points and trajectories for the considered class of systems, with the goal of pointing out some relevant differences with the traditional stability definitions and related techniques.

2 Stabilization of equilibrium points

For simplicity we consider the case in which the system is "fully actuated", i.e., $\mathbf{E} = \mathbf{I}$, and all the state variables $\mathbf{q}(t)$ and $\dot{\mathbf{q}}(t)$ are measured. Consider the following "proportional-derivative" control law (whose efficacy for unconstrained mechanical systems is standard):

$$\mathbf{u}(t) = -\mathbf{K}_v\,\dot{\mathbf{q}}(t) - \mathbf{K}_p\,(\mathbf{q}(t) - \mathbf{q}_R), \tag{4}$$

where $\mathbf{K}_v$ is a positive definite matrix of dimensions $p \times p$ (which guarantees energy dissipation), $\mathbf{q}_R$ is an equilibrium point of the system and $\mathbf{K}_p$ is suitably chosen in order to "dominate" the potential forces of the system, if needed.

By using the total energy of the system (including a term due to the proportional action in the control law (4)) as Lyapunov function and making use of a suitable extension of the LaSalle's theorem, under mild assumptions on the solutions of the system and on the shape of the constraints, it can be proven (see [7] for details) that the equilibrium point of the closed-loop system $\mathbf{q} = \mathbf{q}_R$, $\dot{\mathbf{q}} = \mathbf{0}$, $\dot{\lambda}_i = \dot{\lambda}_{i,R}$ (where $\dot{\lambda}_{i,R}$ are suitable positive constants) is asymptotically stable for practical purposes. Actually, such an equilibrium point has a sort of stability property limited to the generalized coordinates $\mathbf{q}$ and to the generalized velocities $\dot{\mathbf{q}}$, and the classical attractivity property with respect to $\mathbf{q}$, $\dot{\mathbf{q}}$ and to the generalized reaction forces $\dot{\lambda}_i$ (which, on the contrary, are necessarily excluded from the stability requirement). In particular, it can be proven that:

1. for each real number $\varepsilon > 0$, there exists a real number $\delta > 0$ such that for every initial condition $\mathbf{q}(0)$, $\dot{\mathbf{q}}(0^-)$, $\lambda_i(0^-)$, $\gamma_i(0) = \sqrt{-f_i(0)}$, $i = 1, 2, ..., m$, satisfying $\mathbf{q}(0) \in \mathcal{A}$, with $\|\mathbf{q}(0) - \mathbf{q}_R\| < \delta$, and $\dot{\mathbf{q}}(0^-) \in \mathbb{R}^n$, with $\|\dot{\mathbf{q}}(0^-)\| < \delta$, the corresponding solution $\mathbf{q}(t)$, $\dot{\mathbf{q}}(t)$, $\lambda_i(t)$, $\gamma_i(t) = \sqrt{-f_i(\mathbf{q}(t))}$ of (2) under conditions (3) is such that $\|\mathbf{q}(t) - \mathbf{q}_R\| < \varepsilon$ and $\|\dot{\mathbf{q}}(t)\| < \varepsilon$ for all times $t \geq 0$;
2. for each real number $\delta > 0$, with δ being arbitrarily large, and for any initial condition $\mathbf{q}(0)$, $\dot{\mathbf{q}}(0^-)$, $\lambda_i(0^-)$, $\gamma_i(0) = \sqrt{-f_i(0)}$, $i = 1, 2, ..., m$, satisfying $\mathbf{q}(0) \in \mathcal{A}$, with $\|\mathbf{q}(0) - \mathbf{q}_R\| < \delta$, and $\dot{\mathbf{q}}(0^-) \in \mathbb{R}^n$, with $\|\dot{\mathbf{q}}(0^-)\| < \delta$, the corresponding solution $\mathbf{q}(t)$, $\dot{\mathbf{q}}(t)$, $\lambda_i(t)$, $\gamma_i(t) = \sqrt{-f_i(\mathbf{q}(t))}$ of (2) under conditions (3) is such that

$$\lim_{t \to +\infty} \|\mathbf{q}(t) - \mathbf{q}_R\| = 0, \tag{5a}$$

$$\lim_{t \to +\infty} \|\dot{\mathbf{q}}(t)\| = 0, \tag{5b}$$

$$\lim_{t \to +\infty} \|\dot{\lambda}_i(t) - \dot{\lambda}_{i,R}\| = 0, \quad i = 1, 2, ..., m. \tag{5c}$$

By making suitable assumptions, mainly on the potential energy of the system, and, possibly, on the dissipation terms due to viscous friction that we have neglected here for simplicity, the hypotheses that the system is fully actuated can be relaxed [7].

3 Asymptotic tracking of trajectories

For any real number g, let $\lfloor g \rfloor$ denote the largest integer smaller than or equal to g, and let $[g]$ denote the closest integer to g, *i.e.*, $[g] := \lfloor g \rfloor$ if $g - \lfloor g \rfloor \leqslant 1/2$, and $[g] := \lfloor g \rfloor + 1$ if $g - \lfloor g \rfloor > 1/2$.

We assume that a desired trajectory, $\mathbf{q}_d(t)$ is defined with the property that the only impacts occur at each integer value of time t, *i.e.*, for each t such that $t = \lfloor t \rfloor$ there is an impact time, with jumps in the velocity variables that do not converge to zero as time increases. As an example, in [5], the problem is studied of tracking trajectories of a body moving in a billiard. Defining the tracking error at time t as $\mathbf{e}(t) := \mathbf{q}(t) - \mathbf{q}_d(t)$, the following problem can be considered.

Problem 1 *Find a piece-wise continuous control law (where $\dot{\mathbf{q}}(t)$ is to be understood as $\dot{\mathbf{q}}(t^-)$ at the impact times):*

$$\mathbf{u}(t) = \varphi(\mathbf{q}(t), \dot{\mathbf{q}}(t), t), \tag{6}$$

such that the following properties hold for the closed-loop system:

(a) for each $\varepsilon > 0$, for each $t_0 \in \mathbb{R}$ and for each $\gamma \in (0, 1/2)$, there exists $\delta_{\varepsilon, t_0, \gamma} > 0$ such that if $\left[\mathbf{q}^T(t_0)\ \dot{\mathbf{q}}^T(t_0^+)\right]^T \in \hat{\mathcal{A}}$, $\|\mathbf{e}(t_0)\| < \delta_{\varepsilon, t_0, \gamma}$ and $\|\dot{\mathbf{e}}(t_0^+)\| < \delta_{\varepsilon, t_0, \gamma}$, then

$$\|\mathbf{e}(t)\| < \varepsilon, \quad \forall t \in \mathbb{R},\ t \geqslant t_0, \tag{7a}$$

$$\|\dot{\mathbf{e}}(t^-)\| < \varepsilon, \quad \forall t \in \mathbb{R},\ t > t_0,\ |t - [t]| > \gamma, \tag{7b}$$

$$\|\dot{\mathbf{e}}(t^+)\| < \varepsilon, \quad \forall t \in \mathbb{R},\ t > t_0,\ |t - [t]| > \gamma; \tag{7c}$$

(b) for each $t_0 \in \mathbb{R}$, there exists a neighborhood Θ_{t_0} of $\left[\mathbf{q}_d^T(t_0)\ \dot{\mathbf{q}}_d^T(t_0^+)\right]^T$ such that the following relationships hold for each $\left[\mathbf{q}^T(t_0)\ \dot{\mathbf{q}}^T(t_0^+)\right]^T \in \Theta_{t_0} \cap \hat{\mathcal{A}}$:

$$\lim_{t \to +\infty} \|\mathbf{e}(t)\| = 0, \tag{8a}$$

$$\lim_{k \to +\infty} \|\dot{\mathbf{e}}((k + \tau)^-)\| = 0, \quad \forall \tau \in (0, 1), \tag{8b}$$

$$\lim_{k \to +\infty} \|\dot{\mathbf{e}}((k + \tau)^+)\| = 0, \quad \forall \tau \in (0, 1), \tag{8c}$$

where the limits in equations (8b) *and* (8c) *are taken with k being integer, whereas the limit in equation* (8a) *is taken with t being real.*

Remark 1 Mechanical systems subject to non-smooth impacts are a special subclass of the set of impulsive differential systems: see equations (2.9.1) in [4]. For impulsive differential systems, it is already pointed out in [4] that the usual notions of stability require suitable modifications; in particular, the notion of *quasi stability* is proposed in Definition 2.9.1, at page 103 of the same reference. This definition relaxes the usual requirements of stability and attractivity about the times at which the differential system is subject to impulses; in the same book, criteria for quasi stability are also given. However, these criteria cannot be used as such to prove our requirements, which are stronger. As a matter of fact, the class of impulsive differential systems is larger than the class of mechanical systems subject to inequality constraints, since the latter class allows jumps only in the velocity variables, and not in the position variables, whereas the former can allow jumps in all the state variables. For this reason our stability definition, which is very close to the definition of quasi stability, is more strict than quasi stability. Indeed, our stability requirement (i.e., requirement (a) of Problem 1) coincides with the requirement $(S_{1\eta})$ of quasi stability when restricted to the velocity variables and with the classical stability requirement when restricted to the position variables. Our attractivity requirement (i.e., requirement (b) of Problem 1) is in spirit very similar to the requirement $(S_{3\eta})$ of quasi stability when restricted to the velocity variables and coincides with the classical attractivity requirement when restricted to the position variables. □

In the example considered in [5], due to the fact that the desired trajectory was an admissible trajectory for the unforced system (though not an orbitally stable one), a simple PD control law was proven to solve Problem 1. On the other hand, in [6], where the problem was considered of tracking "rolling trajectories" for a rocking block, it was shown that the performance of the control law can be improved by switching the control off in small intervals around the desired impact times.

4 Conclusions

In this work we have reported some results on the stabilization of equilibrium points and trajectories for mechanical systems subject to non-smooth impacts.

In both cases, the more relevant issue is that, when dealing with such a class of systems, it seems appropriate to re-define the concepts of stability (of the desired equilibrium or of the desired trajectory) in order to obtain reasonable control problems: in the case of stability of an equilibrium point it seems natural to exclude from the stability requirement the contact forces (since they necessarily become infinite at the impact times), whereas in the case of stability of a trajectory it is convenient to define the stability and the attractivity requirement with special care for the velocity variables, if it is desired that they are subject to non vanishing jumps at arbitrarily high impact times.

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7

Attractivity of Equilibrium Sets of Systems with Dry Friction

Nathan van de Wouw[1] and Remco Leine[2]

[1] Dept. of Mechanical Engineering, Eindhoven University of Technology, The Netherlands
[2] IMES–Center of Mechanics, ETH Zentrum, Switzerland

1 Introduction

The presence of dry friction can influence the behaviour and performance of mechanical systems as it can induce several phenomena, such as friction-induced limit-cycling, damping of vibrations and stiction. Dry friction in mechanical systems is often modelled using set-valued constitutive models [4], such as the set-valued Coulomb's law. Set-valued friction models have the advantage to properly model stiction, since the friction force is allowed to be non-zero at zero relative velocity. The dynamics of mechanical systems with set-valued friction laws are described by differential inclusions. We limit ourselves to set-valued friction laws which lead to Filippov-type systems [3]. Filippov systems, describing systems with friction, can exhibit equilibrium sets, which correspond to the stiction behaviour of those systems.

The overall dynamics of mechanical systems is largely affected by the stability and attractivity properties of the equilibrium sets. For example, the loss of stability of the equilibrium set can, in certain applications, cause limit-cycling. Moreover, the stability and attractivity properties of the equilibrium set can also seriously affect the performance of control systems. In [1,2,8], stability and attractivity properties of (sets of) equilibria in differential inclusions are studied. More specifically, in [1,8] the attractivity of the equilibrium set of a passive, one-degree-of-freedom friction oscillator with one switching boundary (i.e. one dry friction element) is discussed. Moreover, in [2,8] the Lyapunov stability of an equilibrium point in the equilibrium set is shown. However, most papers are limited to either one-degree-of-freedom systems or to systems exhibiting only one switching boundary.

We will provide conditions under which the equilibrium set is attractive for multi-degree-of-freedom mechanical systems with an arbitrary number of Coulomb friction elements using Lyapunov-type stability analysis and a generalisation of LaSalle's invariance principle for non-smooth systems. Moreover, passive as well as non-passive systems will be considered. The non-passive systems that will be studied are linear mechanical systems with a non-positive definite damping matrix with additional dry friction elements. The non-positive-definiteness of the damping matrix of linearised systems can be caused by fluid, aeroelastic, control and gyroscopical forces, which can cause instabilities. It will be demonstrated in this paper that the presence of dry friction in such an unstable linear system can (conditionally) ensure the local attractivity of the equilibrium set of the resulting system with dry friction. Moreover, an estimate of the region of attraction for the equilibrium set will be given. A rigid multibody approach is used for the description of mechanical systems with friction, which allows for a natural physical interpretation of the conditions for attractivity.

In section 2, the equations of motion for linear mechanical systems with frictional elements are formulated and the equilibrium set is defined. In section 3, the attractivity properties of the equilibrium set are studied by means of a generalisation of LaSalle's invariance principle. In section 4, an example is studied in order to illustrate the theoretical results and to investigate the correspondence between the estimated and actual region of attraction. Finally, a discussion of the obtained results and concluding remarks are given in section 5.

2 Modelling of Mechanical Systems with Coulomb Friction

In this section, we will formulate the equations of motion for linear mechanical systems with m frictional translational joints. These translational joints restrict the motion of the system to a manifold determined by the bilateral holonomic constraint equations imposed by these joints (sliders). Coulomb's friction law is assumed to hold in the tangential direction of the manifold.

Let us formulate the equations of motions for such systems by:

$$\boldsymbol{M}\ddot{\boldsymbol{q}} + \boldsymbol{C}\dot{\boldsymbol{q}} + \boldsymbol{K}\boldsymbol{q} - \boldsymbol{W}_T\boldsymbol{\lambda}_T = \mathbf{0}, \tag{1}$$

in which $\boldsymbol{q}$ is a column of independent generalised coordinates, $\boldsymbol{M}$, $\boldsymbol{C}$ and $\boldsymbol{K}$ represent the mass-matrix, damping-matrix and stiffness-matrix, respectively, and $\boldsymbol{\lambda}_T$ is a column of friction forces in the translational joints. These friction forces obey the following set-valued force law:

$$\boldsymbol{\lambda}_T \in -\boldsymbol{\Lambda}\mathrm{Sign}(\dot{\boldsymbol{g}}_T), \quad \text{with} \quad \boldsymbol{\Lambda} = \mathrm{diag}\left(\left[\mu_1|\lambda_{N_1}| \ldots \mu_m|\lambda_{N_m}|\right]\right). \tag{2}$$

Herein, λ_{N_i} and μ_i, $i = 1, \ldots, m$, are the normal contact force and the friction coefficient in translational joint i. Moreover, $\boldsymbol{W}_T^{\mathrm{T}} = \frac{\partial \dot{\boldsymbol{g}}_T}{\partial \dot{\boldsymbol{q}}}$ is a matrix reflecting the generalised force directions of the friction forces. Herein, $\dot{\boldsymbol{g}}_T$ is a column of relative sliding velocities in the translational joints. Equation (1) forms, together with a set-valued friction law (2), a differential inclusion. Differential inclusions of this type are called Filippov systems which obey Filippov's solution concept (Filippov's convex method). Consequently, the existence of solutions of system (1) is guaranteed. Moreover, due to the fact that $\mu_i \geqslant 0$, $i = 1, \ldots, m$, which excludes the possibility of repulsive sliding modes along the switching boundaries, also uniqueness of solutions in forward time is guaranteed [6].

Due to the set-valued nature of the friction law (2), the system exhibits an equilibrium set. Since we assume that $\dot{\boldsymbol{g}}_T = \boldsymbol{W}_T^T\dot{\boldsymbol{q}}$, $\dot{\boldsymbol{q}} = \mathbf{0}$ implies $\dot{\boldsymbol{g}}_T = \mathbf{0}$. This means that every equilibrium implies sticking in all contact points and obeys the equilibrium inclusion:

$$\boldsymbol{K}\boldsymbol{q} + \boldsymbol{W}_T\boldsymbol{\Lambda}\mathrm{Sign}(\mathbf{0}) \ni \mathbf{0}. \tag{3}$$

The equilibrium set is therefore given by

$$\mathcal{E} = \left\{(\boldsymbol{q}, \dot{\boldsymbol{q}}) \in \mathbb{R}^{2n} | \, (\dot{\boldsymbol{q}} = \mathbf{0}) \wedge \boldsymbol{q} \in -\boldsymbol{K}^{-1}\boldsymbol{W}_T\boldsymbol{\Lambda}\mathrm{Sign}(\mathbf{0})\right\} \tag{4}$$

and is positively invariant due to the uniqueness of the solutions in forward time.

3 Attractivity Analysis of the Equilibrium Set

Let us now study the attractivity properties of this equilibrium set $\mathcal{E}$. Hereto, we will use LaSalle's principle [5], but applied to Filippov systems with uniqueness of solutions in forward time [9].

Let us consider the stability of linear systems with friction and positive definite matrices $\boldsymbol{M}$, $\boldsymbol{K}$ and a non-positive damping matrix $\boldsymbol{C}$. Note that this implies that the equilibrium point of the linear system without friction is either stable or unstable (not asymptotically stable). In the following theorem we state the condition under which (part of) the equilibrium set of the system with friction is locally attractive.

Theorem 1 (Local attractivity of a subset of the equilibrium set). *Consider system (1) with friction law (2). If the matrices $\boldsymbol{M}$, $\boldsymbol{K}$ are positive definite and the matrix $\boldsymbol{C}$ is not positive definite but symmetric, then a convex subset of the equilibrium set (4) is locally attractive under the following condition: $\boldsymbol{U}_{c_i} \in span\{\boldsymbol{W}_T\}$ for $i = 1, \ldots, n_q$, where $\boldsymbol{U}_c = \{\boldsymbol{U}_{c_i}\}$ is a matrix containing the n_q eigencolumns corresponding to the eigenvalues of $\boldsymbol{C}$, which lie in the closed left-half complex plane.*

Proof. We consider a positive definite function

$$V = \frac{1}{2}\dot{\boldsymbol{q}}^{\mathrm{T}}\boldsymbol{M}\dot{\boldsymbol{q}} + \frac{1}{2}\boldsymbol{q}^{\mathrm{T}}\boldsymbol{K}\boldsymbol{q}. \tag{5}$$

Using friction law (2) and the fact that $\dot{\boldsymbol{g}}_T = \boldsymbol{W}_T^{\mathrm{T}}\dot{\boldsymbol{q}}$, the time-derivative of V is

$$\begin{aligned} \dot{V} &= \dot{\boldsymbol{q}}^{\mathrm{T}}\left(-\boldsymbol{C}\dot{\boldsymbol{q}} - \boldsymbol{K}\boldsymbol{q} + \boldsymbol{W}_T\boldsymbol{\lambda}_T\right) + \dot{\boldsymbol{q}}^{\mathrm{T}}\boldsymbol{K}\boldsymbol{q} \\ &= -\dot{\boldsymbol{q}}^{\mathrm{T}}\boldsymbol{C}\dot{\boldsymbol{q}} - \dot{\boldsymbol{g}}_T^{\mathrm{T}}\boldsymbol{\Lambda}\mathrm{Sign}(\dot{\boldsymbol{g}}_T) = -\dot{\boldsymbol{q}}^{\mathrm{T}}\boldsymbol{C}\dot{\boldsymbol{q}} - \boldsymbol{p}^{\mathrm{T}}|\dot{\boldsymbol{g}}_T| = -\dot{\boldsymbol{q}}^{\mathrm{T}}\boldsymbol{C}\dot{\boldsymbol{q}} - \boldsymbol{p}^{\mathrm{T}}|\boldsymbol{W}_T^{\mathrm{T}}\dot{\boldsymbol{q}}| \end{aligned} \tag{6}$$

where the columns $\boldsymbol{p}$ and $|\dot{\boldsymbol{g}}_T|$ are defined by $\boldsymbol{p} = \{\Lambda_{ii}\}$, $|\dot{\boldsymbol{g}}_T| = \{|\dot{g}_{T_i}|\}$, for $i = 1, \dots, m$. (6) implies that $\dot{V}$ is a continuous single-valued function (of $\boldsymbol{q}$ and $\dot{\boldsymbol{q}}$). It holds that $\boldsymbol{p} \geqslant \boldsymbol{0}$ and that if $\dot{\boldsymbol{q}} = \boldsymbol{0}$ then $\dot{\boldsymbol{g}}_T = \boldsymbol{0}$.

We now apply a spectral decomposition of $\boldsymbol{C} = \boldsymbol{U}_c^{-\mathrm{T}}\boldsymbol{\Omega}_c\boldsymbol{U}_c^{-1}$, where $\boldsymbol{U}_c$ is the matrix containing all eigencolumns and $\boldsymbol{\Omega}_c$ is the diagonal matrix containing all eigenvalues of $\boldsymbol{C}$, which are real. Moreover, we introduce coordinates $\boldsymbol{\eta}$ such that $\boldsymbol{q} = \boldsymbol{U}_c\boldsymbol{\eta}$. Consequently, $\dot{V}$ satisfies

$$\dot{V} = -\dot{\boldsymbol{q}}^{\mathrm{T}}\boldsymbol{U}_c^{-\mathrm{T}}\boldsymbol{\Omega}_c\boldsymbol{U}_c^{-1}\dot{\boldsymbol{q}} - \boldsymbol{p}^{\mathrm{T}}|\boldsymbol{W}_T^{\mathrm{T}}\dot{\boldsymbol{q}}| = -\dot{\boldsymbol{\eta}}^{\mathrm{T}}\boldsymbol{\Omega}_c\dot{\boldsymbol{\eta}} - \boldsymbol{p}^{\mathrm{T}}|\boldsymbol{W}_T^{\mathrm{T}}\boldsymbol{U}_c\dot{\boldsymbol{\eta}}|. \tag{7}$$

The matrix $\boldsymbol{C}$ has n_q eigenvalues in the closed left-half complex plane; all other eigenvalues lie in the open right-half complex plane. Consequently, $\dot{V}$ obeys the inequality

$$\dot{V} \leqslant -\sum_{i=1}^{n_q}\lambda_i\dot{\eta}_i^2 - \boldsymbol{p}^{\mathrm{T}}|\boldsymbol{W}_T^{\mathrm{T}}\boldsymbol{U}_c\dot{\boldsymbol{\eta}}| \quad \forall\,\dot{\boldsymbol{\eta}}, \tag{8}$$

where we assumed that the eigenvalues (and eigencolumns) of $\boldsymbol{C}$ are ordered in such a manner that λ_i, $i = 1, \dots, n_q$, correspond to the eigenvalues of $\boldsymbol{C}$ in the closed left-half complex plane. Assume that $\exists\,\alpha > 0$ such that

$$\sum_{i=1}^{n_q}|\boldsymbol{e}_i^{\mathrm{T}}\dot{\boldsymbol{\eta}}| \leqslant \alpha\boldsymbol{p}^{\mathrm{T}}|\boldsymbol{W}_T^{\mathrm{T}}\boldsymbol{U}_c\dot{\boldsymbol{\eta}}| \quad \forall\,\dot{\boldsymbol{\eta}}. \tag{9}$$

Herein, $\boldsymbol{e}_i$ is a unit-column with a non-zero element on the i-th position. Assuming that such an α can be found, (8) results in

$$\dot{V} \leqslant -\sum_{i=1}^{n_q}\lambda_i\dot{\eta}_i^2 - \beta\sum_{i=1}^{n_q}|\dot{\eta}_i| \leqslant 0, \quad \forall\,\dot{\boldsymbol{\eta}} \in \left\{\dot{\boldsymbol{\eta}} \mid \frac{\beta}{\lambda_i} \leqslant \dot{\eta}_i \leqslant -\frac{\beta}{\lambda_i},\ i = 1, \dots, n_q\right\}, \tag{10}$$

with $\beta = \frac{1}{\alpha}$ and $\dot{\eta}_i = \boldsymbol{e}_i^{\mathrm{T}}\dot{\boldsymbol{\eta}}$. Let us now investigate when $\exists\,\alpha > 0$ such that (9) is satisfied. Note, hereto, that if

$$\boldsymbol{e}_i \in \mathrm{span}\left\{\boldsymbol{U}_c^{\mathrm{T}}\boldsymbol{W}_T\right\}, \quad \forall\, i \in [1, \dots, n_q],$$

then $\exists\,\boldsymbol{\gamma}^{\mathrm{T}}$ such that $\boldsymbol{e}_i^{\mathrm{T}} = \boldsymbol{\gamma}^{\mathrm{T}}\boldsymbol{W}_T^{\mathrm{T}}\boldsymbol{U}_c$. It therefore holds that $|\boldsymbol{e}_i^{\mathrm{T}}\dot{\boldsymbol{\eta}}| = |\boldsymbol{\gamma}^{\mathrm{T}}\boldsymbol{W}_T^{\mathrm{T}}\boldsymbol{U}_c\dot{\boldsymbol{\eta}}|$ and $|\boldsymbol{e}_i^{\mathrm{T}}\dot{\boldsymbol{\eta}}| \leqslant |\boldsymbol{\gamma}^{\mathrm{T}}||\boldsymbol{W}_T^{\mathrm{T}}\boldsymbol{U}_c\dot{\boldsymbol{\eta}}|$. Choose the smallest $\tilde{\alpha}_i$ such that $|\boldsymbol{\gamma}^{\mathrm{T}}| \leqslant \tilde{\alpha}_i\boldsymbol{p}^{\mathrm{T}}$, where the sign $\leqslant$ has to be understood component-wise. Then it holds that $|\boldsymbol{e}_i^{\mathrm{T}}\dot{\boldsymbol{\eta}}| \leqslant \tilde{\alpha}_i\boldsymbol{p}^{\mathrm{T}}|\boldsymbol{W}_T^{\mathrm{T}}\boldsymbol{U}_c\dot{\boldsymbol{\eta}}| \quad \forall\,\dot{\boldsymbol{\eta}},\ \forall\, i \in [1, \dots, n_q]$. Note that α in (9) can be taken as $\alpha = \sum_{i=1}^{n_q}\tilde{\alpha}_i$. Finally, one should realise that if and only if

$$\boldsymbol{U}_c\boldsymbol{e}_i \in \mathrm{span}\left\{\boldsymbol{U}_c\boldsymbol{U}_c^{\mathrm{T}}\boldsymbol{W}_T\right\}, \tag{11}$$

or, in other words, if the i-th column $\boldsymbol{U}_{c_i}$ of $\boldsymbol{U}_c$ satisfies $\boldsymbol{U}_{c_i} \in \mathrm{span}\{\boldsymbol{W}_T\}$ (note in this respect that $\boldsymbol{U}_c$ is real and symmetric), then it holds that $\boldsymbol{e}_i \in \mathrm{span}\left\{\boldsymbol{U}_c^{\mathrm{T}}\boldsymbol{W}_T\right\}$. Therefore, a sufficient condition for the validity of (10) can be given by

$$\boldsymbol{U}_{c_i} \in \mathrm{span}\{\boldsymbol{W}_T\}, \quad \forall\, i \in [1, \dots, n_q]. \tag{12}$$

Now, we apply LaSalle's Invariance Principle. Let us, hereto, define a set $\mathcal{C}$ by

$$\mathcal{C} = \left\{ (\boldsymbol{q}, \dot{\boldsymbol{q}}) \mid |\left(\boldsymbol{U}_c^{-1}\dot{\boldsymbol{q}}\right)_i| \leqslant -\frac{\beta}{\lambda_i},\ i = 1, \ldots, n_q \right\}, \tag{13}$$

where $\left(\boldsymbol{U}_c^{-1}\dot{\boldsymbol{q}}\right)_i$ denotes the i-th element of the column $\boldsymbol{U}_c^{-1}\dot{\boldsymbol{q}}$. Moreover, let us define a set $\mathcal{I}_\rho$ such that $\mathcal{I}_\rho = \{(\boldsymbol{q}, \dot{\boldsymbol{q}}) \mid V(\boldsymbol{q}, \dot{\boldsymbol{q}}) \leqslant \rho\}$ and choose the constant ρ such that $\mathcal{I}_\rho \subset \mathcal{C}$. Moreover, we define a set $\mathcal{S} \subset \mathcal{I}_\rho$ by $\mathcal{S} = \{(\boldsymbol{q}, \dot{\boldsymbol{q}}) \in \mathcal{I}_\rho : \dot{\boldsymbol{q}} = \boldsymbol{0}\}$. Furthermore, the largest invariant set in $\mathcal{S}$ is a subset $\tilde{\mathcal{E}}$ of the equilibrium set $\mathcal{E}$, where $\tilde{\mathcal{E}} = \mathcal{E} \bigcap \operatorname{int}(\mathcal{I}_{\rho^*})$ and

$$\rho^* = \max_{\{\rho : \mathcal{I}_\rho \subset \mathcal{C}\}} \rho. \tag{14}$$

Note that $\dot{V} = 0$ if and only if $(\boldsymbol{q}, \dot{\boldsymbol{q}}) \in \mathcal{S}$ and $\dot{V} < 0$ otherwise. Application of LaSalle's invariance principle concludes the proof of the local attractivity of $\tilde{\mathcal{E}}$ under condition (12).

At this point several remarks should be made:

1. It should be noted that the proof of Theorem 1 provides us with a conservative estimate of the region of attraction $\mathcal{A}$ of the locally attractive equilibrium set $\mathcal{E}$. The estimate $\mathcal{B}$ can be formulated in terms of the generalised displacements and velocities: $\mathcal{B} = \mathcal{I}_{\rho^*}$, where ρ^* satisfies (14), the set $\mathcal{C}$ is given by (13) and V is given by (5); In [9], an explicit expression for ρ^* is provided which allows to estimate the region of attraction of the equilibrium set:

$$\rho^* = \min_{i=1,\ldots,n_q} \rho_i, \quad \text{with } \rho_i = \frac{\beta^2}{2\lambda_i^2} \frac{1}{\|e_{n+i}^T \boldsymbol{S}^{-1}\|^2}, \tag{15}$$

   where $\boldsymbol{S}$ is the square root of $\boldsymbol{P}$ ($\boldsymbol{P} = \boldsymbol{S}^T\boldsymbol{S}$) and $\boldsymbol{P}$ is given by

$$\boldsymbol{P} = \begin{bmatrix} \boldsymbol{U}_c^T \boldsymbol{K} \boldsymbol{U}_c & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{U}_c^T \boldsymbol{M} \boldsymbol{U}_c \end{bmatrix}. \tag{16}$$

2. the proof of Theorem 1 also shows that boundedness of solutions (starting in $\mathcal{B}$) is ensured and that the equilibrium *point* $(\boldsymbol{q}, \dot{\boldsymbol{q}}) = (\boldsymbol{0}, \boldsymbol{0})$ is Lyapunov stable;
3. it can be shown that if it holds that $\boldsymbol{\Lambda}^{\mathrm{T}} \boldsymbol{W}_T^{\mathrm{T}} \boldsymbol{K}^{-\mathrm{T}} \boldsymbol{W}_T \boldsymbol{\Lambda} < 2\rho^*$, then $\mathcal{E} \subset \mathcal{I}_{\rho^*}$. In that case the entire equilibrium set $\mathcal{E}$ is locally attractive.
4. an important consequence of Theorem 1 is that when the damping-matrix $\boldsymbol{C}$ is positive definite, global attractivity of the equilibrium set is assured. Note, hereto, that in the proof of Theorem 1, (12) is automatically satisfied and ρ can be taken arbitrarily large in that case.

4 Illustrating example

In this section, we will illustrate the results of the previous section by means of an example concerning a 2DOF mass-spring-damper system, see Figure 1. The equation of motion of this system can be written in the form (1), with $\boldsymbol{q}^{\mathrm{T}} = [x_1\ x_2]$ and the generalised friction forces $\boldsymbol{\lambda}_T$ given by the Coulomb friction law (2). Herein the matrices $\boldsymbol{M}$, $\boldsymbol{C}$, $\boldsymbol{K}$, $\boldsymbol{W}_T$ and $\boldsymbol{\Lambda}$ are given by

$$\begin{aligned} \boldsymbol{M} &= \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}, \quad \boldsymbol{C} = \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_1 + c_2 \end{bmatrix}, \quad \boldsymbol{K} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_1 + k_2 \end{bmatrix} \\ \boldsymbol{W}_T &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \boldsymbol{\Lambda} = \begin{bmatrix} \mu_1 m_1 g & 0 \\ 0 & \mu_2 m_2 g \end{bmatrix}, \end{aligned} \tag{17}$$

with $m_1, m_2, k_1, k_2 > 0$ and $\mu_1, \mu_2 \geqslant 0$. Moreover, the tangential velocity $\dot{\boldsymbol{g}}_T$ in the frictional contacts is given by $\dot{\boldsymbol{g}}_T = [\dot{x}_1\ \dot{x}_2]^{\mathrm{T}}$. Let us first compute the spectral decomposition of the damping-matrix, $\boldsymbol{C} = \boldsymbol{U}_c^{-\mathrm{T}} \boldsymbol{\Omega}_c \boldsymbol{U}_c^{-1}$, with (for non-singular $\boldsymbol{C}$):

$$\boldsymbol{U}_c = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \quad \boldsymbol{\Omega}_c = \begin{bmatrix} c_1 & 0 \\ 0 & c_1 + 2c_2 \end{bmatrix}. \tag{18}$$

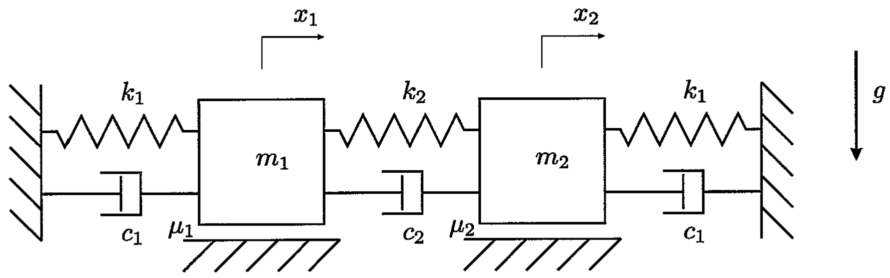

Fig. 1: 2DOF mass-spring-damper system with Coulomb friction.

The equilibrium set $\mathcal{E}$, as defined by (4), is given by

$$\begin{aligned}\mathcal{E} = \{(x_1, x_2, \dot{x}_1, \dot{x}_2) \mid |x_1| &\leqslant \frac{(k_1 + k_2)\mu_1 m_1 g + k_2\mu_2 m_2 g}{k_1^2 + 2k_1k_2} \wedge \\ |x_2| &\leqslant \frac{(k_1 + k_2)\mu_2 m_2 g + k_2\mu_1 m_1 g}{k_1^2 + 2k_1k_2} \wedge \dot{x}_1 = 0 \wedge \dot{x}_2 = 0\}\,.\end{aligned} \tag{19}$$

Let us now consider two different cases for the damping parameters c_1 and c_2: Firstly, we consider the case that $\mathbf{c_1 > 0}$ **and** $\mathbf{c_2 > -c_1/2}$. Note that $\boldsymbol{C} > 0$ if and only if $c_1 > 0$ and $c_2 > -c_1/2$. Consequently, the global attractivity of the equilibrium set $\mathcal{E}$ is assured. It should be noted that this is also the case when one or both of the friction coefficients μ_1 and μ_2 vanish.

Secondly, we consider the case that $\mathbf{c_1 > 0}$ **and** $\mathbf{c_2 < -c_1/2}$. Clearly, the damping matrix is not positive definite in this case. As a consequence, the equilibrium point of the system without friction is unstable. Still the equilibrium set of the system with friction can be locally attractive. Therefore, Theorem 1 can be used to investigate the attractivity properties of (a subset of) the equilibrium set. For the friction situation depicted in Figure 1, condition (12) is satisfied if $\mu_1 > 0$ and $\mu_2 > 0$. Namely, $\boldsymbol{W}_T$ spans the two-dimensional space and, consequently, the eigencolumn of the damping matrix corresponding to the unstable eigenvalue $c_1 + 2c_2$, namely $\begin{bmatrix}-1 & 1\end{bmatrix}^{\mathrm{T}}$, lies in the space spanned by the columns of $\boldsymbol{W}_T$.

Since the attractivity is only local, it is desirable to provide an estimate $\mathcal{B}$ of the region of attraction $\mathcal{A}$ of (a subset of) the equilibrium set Here, we present a comparison between the actual region of attraction (obtained by numerical simulation) and the estimate $\mathcal{B}$ for the following parameter set: $m_1 = m_2 = 1\,\mathrm{kg}$, $k_1 = k_2 = 1\,\mathrm{N/m}$, $c_1 = 0.5\,\mathrm{Ns/m}$, $c_2 = -0.375\,\mathrm{Ns/m}$, $\mu_1 = \mu_2 = 0.1$ and $g = 10\mathrm{m/s}^2$. The numerical simulations are performed using an event-driven integration method as described in [7]. The event-driven integration method is a hybrid integration technique that uses a standard ODE solver for the integration of smooth phases of the system dynamics and a LCP (Linear Complementarity Problem) formulation to determine the next hybrid mode at the switching boundaries. For these parameter settings, $\mathcal{E} \subset \mathrm{int}(\mathcal{I}_{\rho^*})$ and the local attractivity of the entire equilibrium set $\mathcal{E}$ is ensured. In Figure 2, we show a cross-section of $\mathcal{A}$ with the plane $\dot{x}_1 = 0$ and $\dot{x}_2 = 0$, denoted by $\hat{\mathcal{A}}$, which was obtained numerically. Hereto, a grid of initial conditions in the plane $\dot{x}_1 = \dot{x}_2 = 0$ was defined, for which the solutions were obtained by numerically integrating the system over a given time span T. Subsequently, a check was performed to inspect whether the state of the system at time T was in the equilibrium set $\mathcal{E}$. Initial conditions corresponding to attractive solutions are depicted with a light colour (set $\hat{\mathcal{A}}$) and initial conditions corresponding to non-attractive solutions are depicted with a dark grey colour (set $\hat{\mathcal{D}}$). Moreover, $\hat{\mathcal{E}}$ and $\hat{\mathcal{B}}$ are also shown in the figure, where the ^ indicates that we are referring to cross-sections of the sets. It should be noted that $\hat{\mathcal{E}} \subset \hat{\mathcal{B}}$. As expected the set $\mathcal{B}$ is a conservative estimate for the region of attraction $\mathcal{A}$. In [9], more examples are discussed in which the crucial condition for local attractivity (12) is not satisfied.

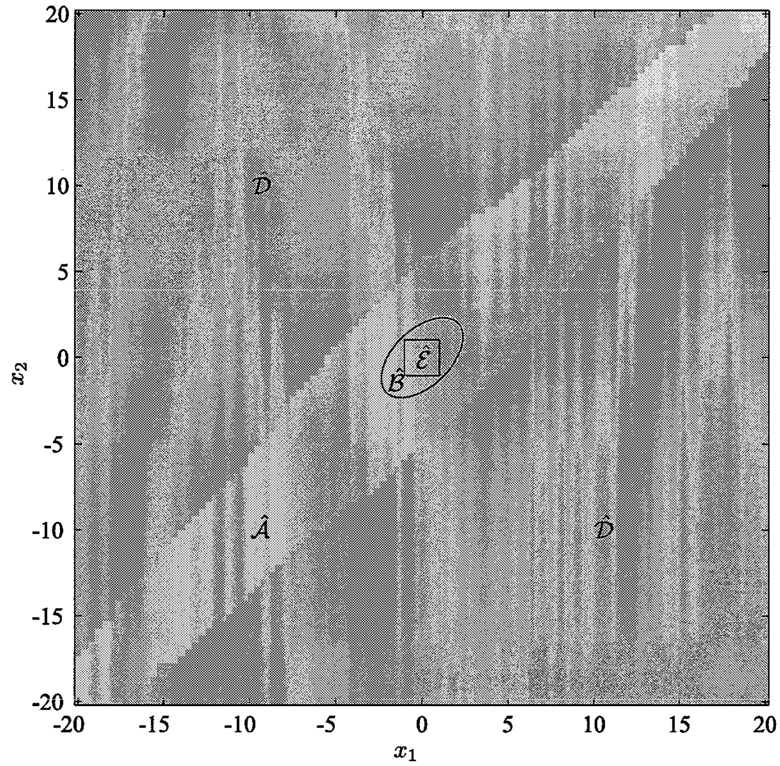

Fig. 2: Cross-section of the region of attraction $\mathcal{A}$ with the plane defined by $\dot{x}_1 = 0$ and $\dot{x}_2 = 0$.

5 Conclusions

Conditions for the (local) attractivity of (subsets of) equilibrium sets of mechanical systems with friction are derived. The systems are allowed to have multiple degrees-of-freedom and multiple switching boundaries (friction elements). It is shown that the equilibrium set $\mathcal{E}$ of a linear mechanical system, which without friction exhibits a stable equilibrium point E, will always be attractive when Coulomb friction elements are added. Moreover, it has been shown that even if the system without friction has an unstable equilibrium point E, then (a subset of) the equilibrium set $\mathcal{E}$ of the system with friction can under certain conditions be locally attractive and the equilibrium point $E \subset \mathcal{E}$ is stable. The crucial condition can be interpreted as follows: the space spanned by the eigendirections of the damping matrix, related to negative eigenvalues, lies in the space spanned by the generalised force directions of the dry friction elements.

Lyapunov stability of the equilibrium set of non-passive systems is not addressed, however, the combination of the attractivity property of the equilibrium set and the boundedness of solutions within $\mathcal{B}$ can be a valuable characteristic when the equilibrium set is a desired steady state of the system. Moreover, an estimate of the region of attraction of the equilibrium set is provided.

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