Estimates of optimal stopping rules for the coin tossing game

ESTIMATES OF OPTIMAL STOPPING RULES FOR THE COIN TOSSING GAME

by

J. H. van Lint
Bell Telephone Laboratories, Incorporated
Murray Hill, New Jersey

ABSTRACT

A fair coin is tossed repeatedly. Let $x_k = 1$ if the k^{th} toss results in heads and $x_k = -1$ otherwise. If we decide to stop after the n^{th} toss let $\frac{s_n}{n} = \frac{x_1+x_2+\ldots+x_n}{n}$ be the reward. An optimal stopping rule for this game (i.e. one that maximizes the expected reward) exists. Let it be: "Stop the first time $s_n > \sigma(n)$." Let $b_n(i)$ be the expected value of the game, given that $s_n = i$. This note gives an upper bound for $b_n(i)$ and a proof that

$$\overline{\lim} \frac{\sigma(n) - \frac{1}{2}n}{\sqrt{n}} > \frac{1}{5} .$$

ESTIMATES OF OPTIMAL STOPPING RULES FOR THE COIN TOSSING GAME

by

J. H. van Lint
Bell Telephone Laboratories, Incorporated
Murray Hill, New Jersey

1. INTRODUCTION

We consider the following game. A fair coin is tossed repeatedly. Let $x_k = 1$ if the k^{th} toss results in heads and $x_k = -1$ otherwise. We define $s_n = \sum_{k=1}^{n} x_k$ and let the reward be s_n/n if we decide to stop after the n^{th} toss. An unsolved problem (cf. [1]) is to find the optimal stopping rule i.e. the rule that maximizes the expected reward. This problem is apparently very difficult.

Since s_n becomes positive infinitely often with probability one, there is no sense in stopping at a point with $s_n < 0$. In [2] Chow and Robbins proved that this game has an optimal stopping rule. Let $\frac{i^+}{n} + a_n(i)$ denote the expected reward under this stopping rule, given that $s_n = i$ (by definition $a^+ = \max(a,0)$). For every $n \geq 1$ let $\sigma(n)$ denote the smallest integer with the property that $a_n(i) = 0$ for $i \geq \sigma(n)$. In [2] the following estimate was proved:

$$(1.1) \qquad \sigma(n) \leq 13n^{\frac{1}{2}} \quad \text{for } n \geq n_o.$$

In this paper we shall prove the following theorems:

Theorem 1. For $n \geq 1$ and $|i| \leq n$

$$(1.2) \qquad 0 \leq a_n(i) \leq 2^n \int_0^{\pi/4} (\sin x)^{n+|i|} (\cos x)^{n-|i|} dx.$$

Theorem 2.

$$(1.3) \qquad \overline{\lim}\, n^{-\frac{1}{2}} \sigma(n) \geq 1/5.$$

Theorem 2 shows that (1.1) cannot be improved very much and that $n^{\frac{1}{2}}$ is the right order of growth of $\sigma(n)$. Using the result (1.2) and a high speed computer better upper bounds for the numbers $a_n(i)$ were found for $n \leq 2000$. By comparing these with lower bounds also obtained with the aid of a computer the value of $\sigma(n)$ was determined for a number of values of n. The inequalities (1.1) and (1.3) and the numerical data make it plausible that $n^{-\frac{1}{2}}\sigma(n)$ tends to a limit which is less than 1.

2. STOPPING RULES

A naive stopping rule is the following: Stop the first time s_n is positive. Since

$$P(s_n > 0 \quad \text{for some } n \geq 1) = 1,$$

(cf. [3]), stopping takes place eventually with probability one, as it must for every rule. Using (cf. [3] p. 74-75)

$$P\{s_1 \leq 0,\ s_2 \leq 0, \ldots,\ s_{2n} \leq 0,\ s_{2n+1} > 0\} = \frac{1}{n+1}\binom{2n}{n}2^{-2n-1}$$

we find the expected reward under this stopping rule to be

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)(n+1)}\binom{2n}{n}2^{-2n-1} = \frac{\pi}{2} - 1.$$

Actually this is not a bad stopping rule at all. Using an iteration process described below and the estimate (1.2) on a computer we found that the expected reward under the optimal stopping rule is less than .5864 which is only 3% more than $\frac{\pi}{2} - 1$.

We now consider the truncated game, i.e. the same game with the restriction that the number of tosses may not exceed N. If we have not decided to stop before the N^{th} trial the reward will be s_N^+/N. In [2] it was shown that as $N \to \infty$ the optimal stopping rule for this truncated game converges to the optimal stopping rule for the infinite game. Let $b_n^N(i)$ denote the expected reward for the truncated game given that $s_n = i$. Then

$$(2.1)\qquad \begin{cases} b_N^N(i) = i^+/N, & \\ b_n^N(i) = \max\left(\frac{i^+}{n}\ ,\ \frac{b_{n+1}^N(i+1) + b_{n+1}^N(i-1)}{2}\right). & (n = 1,2,\ldots,\ N-1) \end{cases}$$

Define

$$(2.2) \qquad a_n^N(i) = b_n^N(i) - i^+/n.$$

Then we have (cf. [2])

$$(2.3) \quad a_n^N(i) = \begin{cases} \dfrac{a_{n+1}^N(i+1) + a_{n+1}^N(i-1)}{2} & (i \leq -1), \\[2ex] \dfrac{a_{n+1}^N(1) + a_{n+1}^N(-1)}{2} + \dfrac{1}{2(n+1)} & (i = 0), \\[2ex] \left[\dfrac{a_{n+1}^N(i+1) + a_{n+1}^N(i-1)}{2} - \dfrac{i}{n(n+1)}\right]^+ & (i \geq 1), \end{cases}$$

and

$$(2.4) \quad a_n^N(i) \text{ is increasing in } N \text{ and } \lim_{N\to\infty} a_n^N(i) = a_n(i).$$

The optimal stopping rule for the infinite game is:

$$(2.5) \qquad \text{Stop the first time } a_n(s_n) = 0.$$

So in this way we obtain lower bounds for $a_n(i)$. This can be done easily on a high speed computer. To obtain a good upper bound we used (1.2) for $n = 2000$ and observed that (2.3) also holds if the upper index N is left out. The same computer program then gives us an upper bound for $a_n(i)$ for $n \leq 2000$. If v denotes the expected reward for the game under the optimal stopping rule we found

(2.6)
$$.5854 < v < .5864 .$$

We remark that this means that the expected value of the average number of heads tossed with a fair coin under the optimal stopping rule is .793.

Some of the results of the computation will be given in Section 5.

3. Proof of Theorem 1.

The proof given here is based on the proof of [2], leema 1. We give a proof only for $i \geq 0$. The case $i < 0$ is similar. By (2.3) we have

(3.1)

$$a_n^N(i) \leq \frac{a_{n+1}^N(i+1) + a_{n+1}^N(i-1)}{2}$$

$$\leq \cdots \leq \frac{a_{n+i}^N(2i) + \binom{i}{1} a_{n+i}^N(2i-2) + \cdots + a_{n+i}^N(0)}{2^i}$$

$$\leq \frac{a_{n+i+1}^N(2i+1) + \binom{i+1}{1} a_{n+i+1}^N(2i-1) + \cdots + a_{n+i+1}^N(-1)}{2^{i+1}} + \frac{1}{2^{i+1}(n+i+1)}$$

$$\leq \cdots \leq \sum_{k=0}^{\infty} \binom{i+2k}{k} \frac{2^{-2k-i-1}}{(n+2k+i+1)}$$

since $a_N^N(i) = 0$ for all i.

Now define φ and f by

$$\varphi(y) = \sum_{k=0}^{\infty} \binom{i+2k}{k} y^k;$$

$$\frac{d}{dx}\{x^n f(x)\} = x^{n+i}\varphi(x^2) .$$

Then the sum on the right-hand side of (2.1) is $f(\frac{1}{2})$. For φ we have (cf. [4])

$$\varphi(y) = \sum_{k=0}^{\infty} \frac{(\frac{1}{2}i+\frac{1}{2})_k(\frac{1}{2}i+1)_k}{(i+1)_k}\frac{(4y)^k}{k!} = F(\tfrac{1}{2}i+\tfrac{1}{2},\tfrac{1}{2}i+1,i+1,4y)$$

$$= 2^i(1-4y)^{-\frac{1}{2}}\{1 + (1-4y)^{\frac{1}{2}}\}^{-i}$$

from which we find

$$f(\tfrac{1}{2}) = 2^{n+i}\int_0^{\frac{1}{2}} \frac{t^{n+i}}{\sqrt{1-4t^2}}\{1+\sqrt{1-4t^2}\}^{-i}dt = \frac{1}{2}\int_0^{\pi/2}\frac{(\sin x)^{n+i}}{(1+\cos x)^i}\,dx$$

$$= 2^n\int_0^{\pi/4}(\sin x)^{n+i}(\cos x)^{n-i}dx.$$

Remarks: (a) for $i = 0$ we find $a_n(0) \sim (\pi/8n)^{\frac{1}{2}}$,

(b) for "large" i we have $a_n(i) < cn^{-1}$,

(c) using the estimate (1.2) for $n = 6$ and then applying (2.3) it is easily seen that $n = 4$, $s = 2$ is a stopping point. This is the first nontrivial stopping point (it is trivial that $n = 1$, $s = 1$ is a stopping point.)

4. Proof of Theorem 2.

To prove theorem 2 we assume that $a_n(i) = 0$ for $i > \frac{1}{5}n^{\frac{1}{2}}$ and $n > n_1$. From this and (2.3) it follows that:

$$a_n(i) = \frac{a_{n+1}(i+1) + a_{n+1}(i-1)}{2} \qquad (i > \tfrac{1}{4}n^{\frac{1}{2}},\ n > n_2), \tag{4.1}$$

and

$$a_n(i) \geq \frac{a_{n+1}(i+1) + a_{n+1}(i-1)}{2} - \frac{i}{n(n+1)} \qquad (i \leq \tfrac{1}{4}n^{\frac{1}{2}}). \tag{4.2}$$

We now use the same iteration used in the proof of theorem 1, i.e. repeatedly applying (2.3), (4.1) and (4.2) we find for $n > n_2$

$$a_n(0) \geq \frac{1}{2}\int_0^{\pi/2} \sin^n\varphi \, d\varphi - \sum_{k=1}^{\infty} \frac{1}{2^k(n+k)(n+k+1)} \sum_i \binom{k}{i}(k-2i)$$

where the inner sum is taken over all values of i such that $i < k/2$ and $k-2i < \frac{1}{4}(n+k)^{\frac{1}{2}}$. The inner sum is less than

$\frac{1}{4}(n+k)^{\frac{1}{2}} \sum_{0 \leq i < k/2} \binom{k}{i} < 2^{k-3}(n+k)^{\frac{1}{2}}$. This leads to

$$a_n(0) \geq \frac{1}{2} \int_0^{\pi/2} \sin^n\varphi \, d\varphi - \frac{1}{8} \sum_{k=1}^{\infty} (n+k)^{-3/2}$$

$$\geq \frac{1}{2} \int_0^{\pi/2} \sin^n\varphi \, d\varphi - \frac{1}{4} n^{-\frac{1}{2}} > \frac{1}{4} n^{-\frac{1}{2}} \quad \text{for } n > n_3.$$

Trivially $b_n(i) > b_n(0)$ if $i > 0$. Therefore the inequality $a_n(0) > \frac{1}{4}n^{-\frac{1}{2}}$ for $n > n_3$ implies $a_n(i) > 0$ for $i \leq \frac{1}{4}n^{\frac{1}{2}}$ and $n > n_3$ which contradicts the assumption we made at the beginning of the proof.

Therefore $\sigma(n) > \frac{1}{5}n^{\frac{1}{2}}$ infinitely often. This completes the proof of theorem 2.

5. NUMERICAL RESULTS

In the following table we give a number of stopping points and some estimates for others. We were able to find these because the upper and lower bounds for $\sigma(n)$ coincide for some values of n. Instead of $\sigma(n)$ we list the number of heads tossed at the time of stopping i.e. $h(n) = \frac{1}{2}(n+\sigma(n))$. The largest value of n for which we know h(n) is n = 381 with h(381) = 199.

TABLE

n	h(n)	n	h(n)	LOWER BOUND	UPPER BOUND	n	h(n)	LOWER BOUND	UPPER BOUND
1	1	26	15			64	36		
2	2	27	16			81		44	45
3	3	28		16	17	100		54	55
4	3	29	17			121	65		
5	4	30	18			144	77		
6	4	31	18			169	90		
7	5	32	19			196	104		
8	5	33	19			225	119		
9	6	34	20			256	135		
10	7	35	20			289	152		
11	7	36	21			324		169	170
12	8	37	21			361		188	189
13	8	38	22			400		208	209
14	9	39	22			441		229	230
15	9	40	23			484		251	252
16	10	41	23			529		274	275
17	11	42	24			576		298	299
18	11	43	25			625		322	324
19	12	44	25			676		348	350
20	12	45	26			729		375	378
21	13	46	26			784		403	406
22	13	47	27			841		432	435
23	14	48	27			900		461	465
24	14	49	28						
25	15	50	28						

6. ACKNOWLEDGMENTS

The author is indebted to A. J. Goldstein for doing most of the computational work connected with this paper and to him and many other colleagues at Bell Telephone Laboratories, especially L. A. Shepp, for valuable discussions on the coin tossing game.

Att.
References

April 19, 1966

REFERENCES

[1] L. Breiman, Stopping-rule problems, Chapter 10 of Applied Combinational Mathematics, Wiley, New York, 1964.

[2] Y. S. Chow and H. Robbins, On optimal stopping rules for s_n/n, Ill. J. of Math. 8, (1964); 444-454.

[3] W. Feller, An introduction to probability theory and its applications, second edition, New York, Wiley 1957.

[4] Handbook of Mathematical Functions, Nat. Bureau of Standards Applied Math. Series 55, 1964.