Identification aspects of inhomogeneous materials

Identification Aspects of Inhomogeneous Materials

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Summary
This paper presents an inverse method to determine parameters in constitutive equations. The method is especially suitable to study the mechanical behavior of inhomogeneous materials. The method is based on: numerical analysis, strain distribution measurement and system identification. By means of simulations it will be shown that for a solid with a varying fiber direction it is possible to estimate stiffness parameters a well as the local fiber directions from one single test.

Introduction
Material properties in plant and animal tissues can vary with the anatomical site. Also technical materials may have inhomogeneous properties, *e.g.* reinforced composites with short fiber like particles, processed by a molding operation. These composites may be described in terms of effective mechanical behavior, *i.e.* composites are considered on a scale, several times the dimensions of the constituent materials. On this scale, a certain smoothness of the material properties is assumed. In the present paper the concept of inhomogeneity refers to a larger scale and may, for instance, be caused by different orientations of the alignment of the fibers in the material. Ideally, the inhomogeneity of the material meets the mechanical demands of the object.
Mathematical modeling of inhomogeneous materials, *e.g.* by means of a finite element model, does not lead to fundamental problems. Experimental determination of inhomogeneous properties, however, is an arduous task. A possibility to measure some of the inhomogeneous properties by means of common mechanical tests, such as uniaxial strain tests and biaxial tests, is to extract samples at different positions in the material. A disadvantage of this approach is the disruption of the structure by cutting fibers in the manufacturing of the samples. Particularly for inhomogeneous materials, an inverse approach offers better possibilities than

the common traditional testing. Despite the increasing interest for inverse methods in the realm of continuum mechanics, the identification of inhomogeneous material behavior has hardly attracted any attention. An exception is an example presented by Nappi[1] of a geotechnical problem.

The approach used in this paper is based on the combination of three elements:[2] (i) the use of digital image analysis for the measurement of non-homogeneous strain distributions on multi-axially loaded objects with arbitrary geometry , (ii) finite element modeling and (iii) application of systems identification. The third element comprises the comparison between experimental data and the outcomes of the finite element model, followed by the determination of the material parameters.

Recent publications describe the testing of this identification approach in practice, by means of experiments on an orthotropic elastic membrane.[2][3] Now the applicability of the method will be demonstrated by means of numerical simulations for inhomogeneous materials. These are carried out by computing a displacement distribution with a given constitutive model and known parameters. Subsequently, these displacements are used as fictitious 'measured data' for parameter estimation. In this way the influence of observation noise and model errors can be determined.

Identification method

In this section an outline of the identification method used is described. The reader can find further details in Hendriks[2] and Norton.[4] The method is based on the sequential minimum variance approach. The observational data are assumed to consist of a set of columns with data $\{y_k\}$, $k = 1,...,N$. The observational data of the 'experiments' described in this paper are displacement components of an inhomogeneous displacement field and are collected in a single column $\boldsymbol{y}_1$. The displacements are considered to be a nonlinear function of a set of material parameters:

$$\boldsymbol{y}_1 = \boldsymbol{h}_1(\boldsymbol{x}) + \boldsymbol{v}_1 \qquad (1)$$

where $\boldsymbol{x}$ is a column with unknown material parameters, $\boldsymbol{h}_1$ is a finite element model for the measured displacements, and $\boldsymbol{v}_1$ is a column of observation errors.

The basic estimation problem is the use of the observed displacements $\boldsymbol{y}_1$ to estimate parameter column $\boldsymbol{x}$. The estimator can be specified from the model (Eq. 1), an uncertainty model for $\boldsymbol{v}_1$ and *a priori* knowledge of $\boldsymbol{x}$. The optimal parameter column minimizes the following quadratic expression:

$$S_1 = (\boldsymbol{y}_1 - \boldsymbol{h}_1(\boldsymbol{x}))^T \boldsymbol{R}_1^{-1} \; (\boldsymbol{y}_1 - \boldsymbol{h}_1(\boldsymbol{x})) \; + \; (\hat{\boldsymbol{x}}_0 - \boldsymbol{x})^T (\boldsymbol{P}_0 + \boldsymbol{Q}_1)^{-1} (\hat{\boldsymbol{x}}_0 - \boldsymbol{x}) \qquad (2)$$

where $\hat{\boldsymbol{x}}_0$ is an initial guess for the parameter column $\boldsymbol{x}$. In weighted least squares estimation the matrices $\boldsymbol{R}_1$ and $\boldsymbol{P}_0$ are chosen on the basis of engineering judgement. Matrix $\boldsymbol{Q}_1$ is a nonnegative symmetric matrix. It is obvious that the introduction of $\boldsymbol{Q}_1$ makes it possible to put less weight to the *a priori* estimate $\hat{\boldsymbol{x}}_0$ (and more weight to the displacements $\boldsymbol{y}_1$). The least squares estimate does not make any use of the statistics of the observation errors. In many applications, it is not uncommon for the mean and variance of the observation error to be known. Minimum variance estimates utilize this extra information, which results in specific choices for $\boldsymbol{R}_1$ and $\boldsymbol{P}_0$. In minimum variance estimation $\boldsymbol{R}_1$ represents the covariance matrix of the observation error $\boldsymbol{v}_1$. Matrix $\boldsymbol{P}_0$ represents the covariance matrix of the estimation error in $\hat{\boldsymbol{x}}_0$. Generally: the larger $\boldsymbol{P}_0$, the smaller the influence of $\hat{\boldsymbol{x}}_0$.

Solving the nonlinear inverse problem, defined by Eqs. (1) and (2), leads to an iterative scheme, which results in an estimation $\hat{\boldsymbol{x}}_1$ for $\boldsymbol{x}$ and in a covariance matrix of the estimation error $\boldsymbol{P}_1$:

$$\hat{\boldsymbol{x}}^{(i+1)} = \hat{\boldsymbol{x}}^{(i)} + \boldsymbol{K}^{(i+1)}\,(\boldsymbol{y} - \boldsymbol{h}(\hat{\boldsymbol{x}}^{(i)})) \tag{3}$$

$$\boldsymbol{K}^{(i+1)} = (\boldsymbol{P}^{(i)} + \boldsymbol{Q})\,\boldsymbol{H}^{(i+1)^T}\,(\boldsymbol{R} + \boldsymbol{H}^{(i+1)}\,(\boldsymbol{P}^{(i)} + \boldsymbol{Q})\,\boldsymbol{H}^{(i+1)^T}\,)^{-1} \tag{4}$$

$$\boldsymbol{P}^{(i+1)} = (\boldsymbol{I} - \boldsymbol{K}^{(i+1)}\boldsymbol{H}^{(i+1)})\,(\boldsymbol{P}^{(i)} + \boldsymbol{Q})\,(\boldsymbol{I} - \boldsymbol{K}^{(i+1)}\boldsymbol{H}^{(i+1)})^T + \boldsymbol{K}^{(i+1)}\boldsymbol{R}\boldsymbol{K}^{(i+1)^T} \tag{5}$$

where the superscripts refer to the iteration number and where the subscripts are temporarily dropped. In each iteration $n + 1$ finite element calculations are executed, where n is the number of parameters. The n calculations are carried out to determine a matrix $\boldsymbol{H}_1^{(i)}$ numerically, as a linearization of $\boldsymbol{h}_1$ with respect to the most recent estimation $\hat{\boldsymbol{x}}_1^{(i-1)}$.

The sequential property of the estimator is clear when a column $\boldsymbol{y}_2$ with new observational data would become available. This can be data from another load case or from another point in time. These data can be used together with the initial conditions $\hat{\boldsymbol{x}}_1$ and $\boldsymbol{P}_1$ resulting in an improved estimation $\hat{\boldsymbol{x}}_2$ and $\boldsymbol{P}_2$. In the examples of this paper it will be shown that the data of a single load case contained in $\boldsymbol{y}_1$ is sufficient for the characterization of the material behavior. The above estimator is implemented as an extra module in the finite element code DIANA.[5]

Two approaches

Ideally, the material properties of an inhomogeneous material are determined with respect to each point of the material. In practice, however, regions surrounding a point are considered. Approaches for the identification of inhomogeneous materials can be distinguished by the size of these regions and by the inhomogeneity assumed in each region. In the example, described in the next section, two approaches are distinguished.

In the first approach a model of the entire loaded object is confronted with the 'experimental' data. The inhomogeneous properties are modeled with help of a continuous function over the finite element model. This function will be identified together with the stiffness parameters.

The influence of the model errors, depends on the suitability of this function to describe the true inhomogeneity.
In the second approach only a part of the loaded object is modeled and confronted with the 'measured' displacements of that region. The properties of this region are assumed to be homogeneous, which leads to model errors, depending on the size of the region and the level of inhomogeneity. The second approach has a very important consequence from finite element viewpoint. The boundaries of the region are no longer the actual boundaries of the loaded object. Practically, now only prescribed displacements can be used as boundary conditions and no forces. It is obvious that, with such a model, no stiffness parameters can be determined. The next section, however, will show that it is still possible to estimate the ratios between the different stiffness parameters.

Example: curvilinear orthotropy

Curvilinear orthotropy is the term used to describe a material, in which the orientation of the orthotropic symmetry coordinate system is different from point to point.[6]

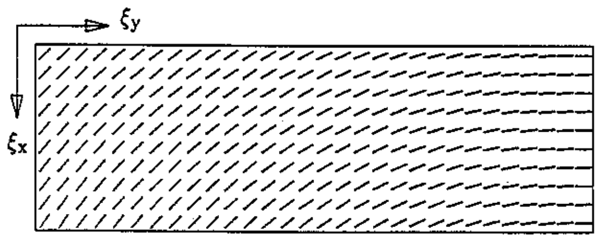

Fig. 1 Sample shape and orientation of local planes of symmetry.

Figure 1 shows a flat membrane (dimensions: $1 \times 3 \times 0.02$) with curvilinear orthotropic behavior. An orthotropic material has three mutually perpendicular planes of symmetry with respect to each point of the material. In the present example, it is assumed that one plane of symmetry coincides with the plane of the sample. The normal of one of the other planes of symmetry is indicated in the figure with a short line. These lines may be interpreted as the orientation of fiber like particles in a reinforced composite. The axes shown in figure 1 are tangent to concentric circles, where ($\xi_x = 3.0$, $\xi_y = 3.0$) denotes the center. This type of circumferential orthotropy is typical for wood, where one axis is tangent to the growth rings.
In each point of the sample the stiffness properties with respect to the local symmetry axes are the same. The material parameters are chosen arbitrarily: $E_1 = 1.0, E_2 = 0.2, \nu_{12} = 0.3$, $G_{12} = 0.2$, where E_1 is the stiffness in material 1-direction as indicated in figure 1, E_2 is the stiffness in perpendicular direction and ν_{12} and G_{12} denote the Poisson's ratio and shear modulus respectively.

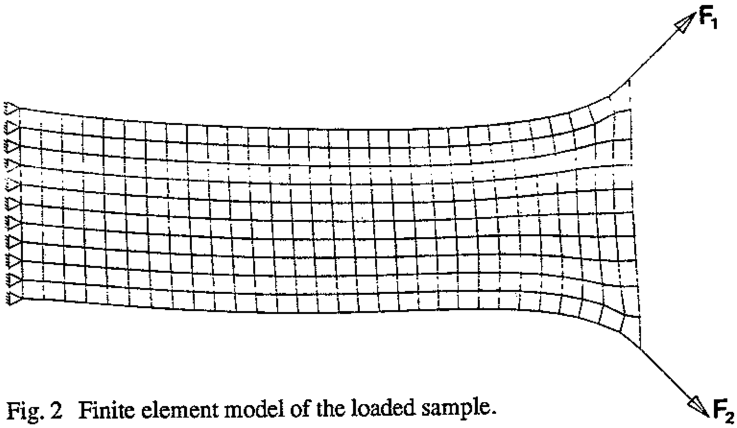

Fig. 2 Finite element model of the loaded sample.

Figure 2 shows the finite element model of the sample, used for the artificial generation of the displacement data. The model consists of 4-noded plane stress elements. The membrane is symmetrically loaded with two equal forces working in the plane of the sample. It will be clear that the deformation is not symmetrical, which is caused by the varying fiber direction.

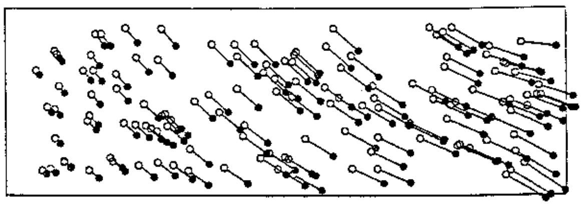

Fig. 3 Measured displacements for approach 1.

Two sets of measured displacements will be distinguished. The first set consists of the displacement components of 128 material points, as shown in figure 3. The initial positions of these points are a realization of a 2-dimensional uniform random distribution. The second set of measured displacements is demonstrated in figure 4.

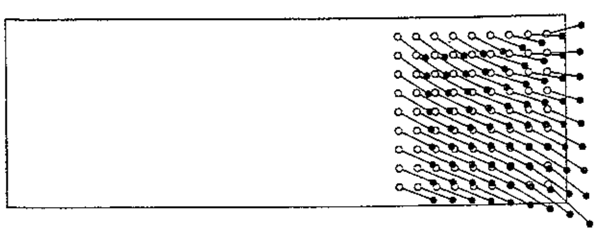

Fig. 4 Measured displacements for approach 2.

Approach 1

For the fiber direction two models will be distinguished. Model 1 is given by:

$$\alpha(\xi) = \begin{cases} -arctan\left(\dfrac{\xi_x - c_1}{\xi_y - c_2}\right) & \text{for } \xi_y \neq c_2 \\ \dfrac{1}{2}\pi & \text{for } \xi_y = c_2 \end{cases} \tag{6}$$

In this equation α denotes the positive rotation of the material 1-direction from the model ξ_x-axis. This rotation is a function of the position coordinates ξ_x and ξ_y. The parameters c_1 and c_2 in Eq. (6) can be interpreted as the coordinates of the centroid of the concentric circles. Model 2 is given by:

$$\alpha(\xi) = b_0 + b_1\xi_x + b_2\xi_y \tag{7}$$

This bilinear function with unknown parameters b_0, b_1 and b_2 is used to investigate the influence of model errors. Clearly this function cannot pinpoint the actual inhomogeneity, as shown in figure 1, with any set of parameters b_i.

Para-meter	Exact value	Initial guess	Estimations		
			No noise	$\sigma = 0.001$	$\sigma = 0.01$
E_1	1.000	0.666	1.000	0.993	0.931
E_2	0.200	0.133	0.200	0.200	0.198
ν_{12}	0.300	0.200	0.300	0.301	0.305
G_{12}	0.200	0.133	0.200	0.201	0.211
c_1	3.000	2.000	2.998	3.004	3.055
c_2	3.000	2.000	2.999	3.010	3.106

Table 1: Estimation results after 10 iterations with the tangential function (6).

Table 1 shows the estimation results with Eq. (6) Here 6 parameters are estimated using the experimental data shown in figure 3. The fourth column of the table shows the estimates of these parameters without observation errors. The fifth and sixth column show the estimation results when the displacement data are disturbed with a zero mean normal distribution. The average displacement of the sample is 0.1 It can be observed that identification approach works well, even with a noise signal rate of 10% ($\sigma = 0.01$) Similar results are presented in table 2, but now 7 parameters are estimated using Eq. (7) The table shows that in this case the obvious model errors scarcely effect the estimations for E_1, E_2, ν_{12} and G_{12}. Some discussion on the exact values is worthwhile in this case The model errors make the use of the

Para-meter	Exact value	Initial guess	Estimations		
			No noise	$\sigma = 0.001$	$\sigma = 0.01$
E_1	1.000	0.666	1.031	1.022	0.944
E_2	0.200	0.133	0.199	0.199	0.200
ν_{12}	0.300	0.200	0.294	0.296	0.304
G_{12}	0.200	0.133	0.193	0.194	0.209
b_0	*does*	-0.784	-0.755	-0.753	-0.735
b_1	*not*	0.262	0.200	0.199	0.193
b_2	*apply*	-0.262	-0.274	-0.273	-0.263

Table 2: Estimation results after 10 iterations with the bilinear function (7).

term "exact" misleading. The exact values of column 2 are no longer necessarily the optimal parameters, in the sense that they minimize expression (2). Exact values for the parameters b_i can not be given. Nevertheless, a comparison of the fiber directions calculated with Eq. (6) and Eq. (7) show hardly any difference[2]. This is visualized in figure 5 where the bilinear inhomogeneity is drawn based on the estimated b_i parameters, whereas figure 1 shows the actual circumferential orthotropy. Evidently, in this case, the estimation results are neither very sensitive to this type of model errors, nor are they sensitive for the combination of model errors and random observation errors.

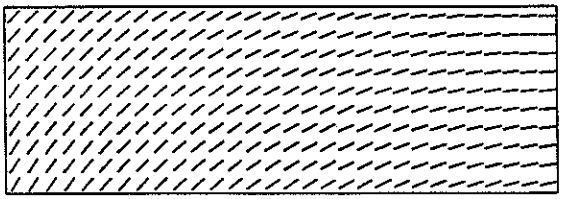

Fig. 5 Estimated inhomogeneity using a bilinear function.

Approach 2

Now the finite element model is based on the measured displacements, shown in figure 4. The figure shows that the material points are positioned in a square. For this square a finite element model is derived (figure 6). The prescribed displacements of the four edges are derived from the displacements of the outer material points. The displacements of the inner material points are considered as measured data. It is assumed that the material properties, the material orientation included, are homogeneous over the sample part.

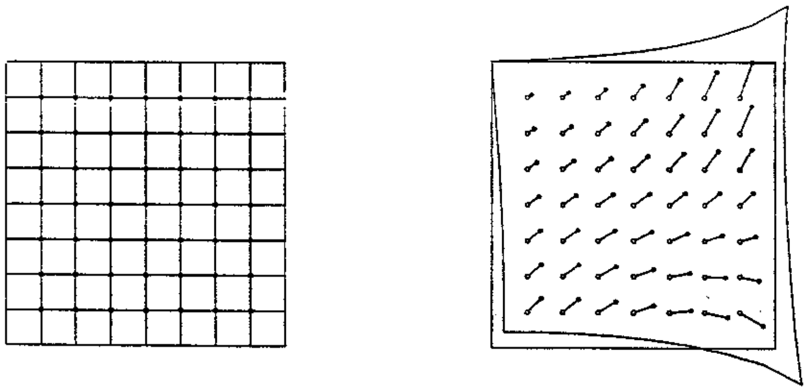

Fig. 6 Finite element model of a square part of the sample (left panel); Kinematic boundary conditions and measured data (right panel).

It is obvious that stiffness parameters can not be identified with the model presented here. In the present simulation we will investigate whether or not the combination of model and measured data does contain information about the ratios between the stiffness parameters. In the present example the following dimensionless parameters will be identified:

$$x^T = (\frac{E_2}{E_1}, \nu_{12}, \frac{G_{12}}{E_1}, cotan(\alpha)) \tag{8}$$

where α denotes the positive rotation of the material axes system.

Para-meter	Exact value	Initial guess	Estimations	
			No noise	$\sigma = 0.001$
$\frac{E_2}{E_1}$	0.200	0.133	0.216	0.221
ν_{12}	0.300	0.200	0.322	0.330
$\frac{G_{12}}{E_1}$	0.200	0.133	0.202	0.220
$cotan(\alpha)$	[-0.444,-0.052]	-0.100	-0.093	-0.096

Table 3: Estimation results after 10 iterations using a homogeneous model for a part of the sample.

Table 3 shows the estimation results for the dimensionless parameters. The true values are given in the second column. For the cotangential value of the rotation of the material axes, a range is given representing the true occurring values. In the case of perfect observations, it can be observed that there is a good agreement between the estimation results and the true parameters, although the comparison is less favorable as in the first approach. However, also here a discussion on the specification of "true" or "exact" values is in place. In the model it is assumed that the sample part has homogeneous properties. This model error makes the term "exact" misleading. The biased parameters of the fourth column may give better results in the homogeneous model than the true parameters[2].

Returning to the results of table 3, it can be observed that the results for the cases with disturbed data is less favorable. If the standard deviation of the noise is 1% of the average displacement of the sample ($\sigma = 0.001$), the estimation results differ slightly from the results in the perfect observation case. However, if the standard deviation increases to 10% the identification fails after two iterations, since the thermodynamical constraints on the stiffness parameters are violated. Apparently, this approach is more sensitive to measuring errors than the first approach. A possible explanation is that in the second approach measurement errors on the displacements enter as model errors via the specification of the kinematic boundary conditions. Hence the originally random observation errors cause systematic errors in the model.

Concluding remarks

For the identification of inhomogeneous materials a mixed numerical-experimental approach is favorable. Via two approaches, but using the same identification idea, it is shown that a nondestructive characterization is possible.

The advantages of the first approach, where the entire sample is identified, are:

- The procedure leads to a complete quantification of the entire sample.
- The identification is not sensitive to observation errors.

The advantages of the second approach are:

- The *a priori* specification of a function representing the inhomogeneity can be omitted.
- The finite element models are smaller.
- In general the models contain less parameters.
- The method meets to practical problems of determining the exact geometry and boundary conditions.

Note that the two approaches are in fact two extreme cases of a whole range of possible approaches The boundary conditions may be partly kinematic and partly dynamic. In addition, also in the second approach an inhomogeneous model for the sample part can be considered. Experimental investigations have to learn whether these kind of model errors will disturb the identification process.

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