Relativistic treatment of electron orbits near cyclotron resonance

RELATIVISTIC TREATMENT OF ELECTRON ORBITS NEAR CYCLOTRON RESONANCE

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Electron motion in a circularly polarized electromagnetic standing-wave, with wave vector parallel to a steady magnetic field B_0, is treated in a frame of reference rotating with the wave. Changes of the phase of the electron with respect to the wave, caused by relativistic variations of the cyclotron frequency and by the velocity parallel to B_0 are considered. The treatment admits electrostatic fields parallel to B_0 and departures from the vacuum wavelength. Analog computer solutions show that, depending on B_0 and $v(t=0)$ ejection or trapping may occur, where in the latter case the energy can be higher than in a homogeneous electric field.

1. Introduction

In recent experimental studies /1/ on the generation of plasmas in a magnetic field by electromagnetic fields with frequency $\omega \simeq \Omega = eB_0/m$, such high electron energies have been observed, that the cyclotron frequency of the particles in question is changed by a considerable amount. By consequence the synchronization of the particle motion with the field is seriously affected. This has first been pointed out by Consoli and Mourier /2/, who also gave a discussion of the mechanism of energy gain in such arrangements. The question has been further pursued at Jutphaas /3/, for homogeneous magnetic and electric fields. If that kind of treatment is applied to practical situations, the greatest error is caused by the neglect of variations of the electric field along the constant magnetic field, and particle motion in this direction.

In this paper, the effect of this parallel motion is investigated for the case of an electron in a standing, circularly polarized, electromagnetic wave, with

$\mathbf{E}_{\sim}$ — and $\mathbf{B}_{\sim}$ — fields perpendicular to the uniform constant field $\mathbf{B}_0$. The e.m. fields are assumed to vary sinusoidally in the direction parallel to $\mathbf{B}_0$, where the period may or may not be equal to the free-space wavelength. Variations of the fields in perpendicular directions are ignored; if the characteristic length L for these variations is of the order of the free-space wavelength λ_ν, this is a good approximation as long as $v_\perp/c \ll 1$. (If $L \gg \lambda_\nu$, the requirement becomes much less stringent.) The approach is taken to apply to fields in a single mode cavity (e.g. cylindrical, TE_{111}) as well as to field regions in multi-mode cavities. In the arrangements mentioned /1/, where dense hot-electron plasmas are formed, it is probable that quite strong electrostatic fields arise. As the electron mobility parallel to $\mathbf{B}_0$ highly exceeds that perpendicular to $\mathbf{B}_0$, the influence of arising space-charge potential gradients will be predominantly parallel to $\mathbf{B}_0$. A parallel electrostatic field has therefore been included in the equations. These have been written for a homogeneous medium; if they are to apply to a dense plasma, the electron density and the magnetic field must be uniform because otherwise the permittivity — and consequently the wavelength — would vary in space.

2. Equations

The electron motion is investigated in a standing electromagnetic wave of infinite extent in the transverse directions, with right-handed circular polarization, perpendicular to a constant magnetic field $\mathbf{B}_0$ which is taken to be in the z-direction (cf. Fig. 1).

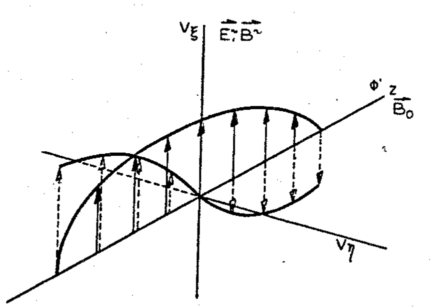

Fig. 1. Orientation of the fields.

For generality, the lower signs pertaining to left-handed polarization are retained in the derivations. The motion of positive ions could be investigated using the same formalism, if $-e$ is replaced by $+Ze$ and m_e by m_i.

Here,

$$\mathbf{E}_{\sim} = E(z)\,\{\mathbf{i}_x \cos\omega t \pm \mathbf{i}_y \sin\omega t\},$$
$$\mathbf{B}_{\sim} = \mp\frac{1}{\omega}\frac{dE}{dz}\,\{\mathbf{i}_x \cos\omega t \pm \mathbf{i}_y \sin\omega t\}. \qquad (1)$$

No assumption concerning the presence or absence of material currents is made, and $E(z) = Ef(z)$ is unspecified; further

$$\mathbf{B}_0 = \mathbf{i}_z B_0, \text{ and } \mathbf{E}_0 = -\mathbf{i}_z \frac{d\varphi}{dz}, \qquad (2), (3)$$

where $\varphi(z)$ is the electrostatic potential.

The equation of motion, written in coordinates becomes:

$$\left.\begin{aligned}
\frac{dp_x}{dt} &= -eE(z)\cos\omega t - \frac{e}{\omega} v_z \frac{dE}{dz}\sin\omega t - ev_y B_0,\\
\frac{dp_y}{dt} &= \mp eE(z)\sin\omega t \pm \frac{e}{\omega} v_z \frac{dE}{dz}\cos\omega t + ev_x B_0,\\
\frac{dp_z}{dt} &= \frac{e}{\omega}\{v_x \sin\omega t \mp v_y \cos\omega t\}\frac{dE}{dz} + e\frac{d\varphi}{dz}.
\end{aligned}\right\} \qquad (4)$$

Dividing the equations by $m\omega c$ and remembering that $d/dt = \partial/\partial t + \mathbf{v}\cdot\nabla$ yields equations in reduced or normalized quantities:

$$\left.\begin{aligned}
\frac{d}{d\tau}(P_x) &= -\frac{d}{d\tau}\{gf(k_{\parallel}z)\sin\tau\} - b\dot{Y},\\
\frac{d}{d\tau}(P_y) &= \pm\frac{d}{d\tau}\{gf(k_{\parallel}z)\cos\tau\} + b\dot{X},\\
\frac{1}{n_{\parallel}}\frac{d}{d\tau}(P_{\parallel}) &= g\frac{df}{d(k_{\parallel}z)}\{\dot{X}\sin\tau \mp \dot{Y}\cos\tau\} + \frac{d\Phi}{d(k_{\parallel}z)}.
\end{aligned}\right\} \qquad (5)$$

Here: $\tau = \omega t$;

$$\left.\begin{aligned}
&\mathbf{P} = (P_x, P_y, P_{\parallel}) \equiv (1/mc)\cdot(p_x, p_y, p_z) = \mathbf{p}/mc,\\
&\mathbf{V} = (\dot{X}, \dot{Y}, \dot{Z}) \equiv (1/c)\cdot(\dot{x}, \dot{y}, \dot{z}) = \mathbf{v}/c,\\
&\text{where in the reduced system the dots indicate differentiation with respect to } \tau\text{; further}\\
&\Phi = e\,\varphi/mc^2,\\
&(X, Y) \equiv (\omega/c)\cdot(x, y),
\end{aligned}\right\} \qquad (6)$$

while for the coordinate parallel to $\mathbf{B}_0$ the variable $k_{\parallel}z$ is introduced. Here $2\pi/k_{\parallel}$ is the length over which f and/or φ vary, if the z-dependence is sinusoidal. Finally $n_{\parallel} = k_{\parallel}c/\omega$.

The field parameters are

$$g \equiv eE/m\omega c$$

and

$$b \equiv eB_0/m\omega = \Omega_0/\omega. \qquad (7)$$

It must be noticed that, since

$$\mathbf{p} = \frac{1}{\sqrt{1 - v^2/c^2}}\cdot m\mathbf{v}, \quad \text{also} \quad \mathbf{P} = \gamma\mathbf{V}, \qquad (8)$$

where

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} = \frac{1}{\sqrt{1 - V^2}} \qquad (8)$$

is the relativistic mass or energy parameter.

The X- and Y-equations of (5) may be integrated once, to give

$$P_x=\gamma \dot{X}=-gf\sin\tau-bY+C_x,$$
$$P_y=\gamma \dot{Y}=\pm gf\cos\tau+bX+C_y, \quad (9)$$

where, as the whole problem is independent of X and Y, the origin of the coordinate system may be chosen so that

$$C_x=C_y=0, \quad (10)$$

without loss of generality. Now, the perpendicular momentum and velocity can be written in components which rotate with the electric field:

$$P_\xi\equiv\gamma V_\xi\equiv\gamma(\dot{X}\cos\tau\pm\dot{Y}\sin\tau),$$
$$P_\eta\equiv\gamma V_\eta\equiv\gamma(\mp\dot{X}\sin\tau+\dot{Y}\cos\tau), \quad (11)$$

where P_ξ, V_ξ are along the electric field, and P_η, V_η in (space) quadrature. With (10) and definitions (11), expressions (9) become

$$P_\xi=b(-Y\cos\tau\pm X\sin\tau),$$
$$\pm P_\eta=b(\pm Y\sin\tau+X\cos\tau)\pm gf. \quad (12)$$

From (12), again with (9) and (11), the following differential equations for the perpendicular momentum can be deduced:

$$\frac{dP_\xi}{d\tau}=\left(1\mp\frac{b}{\gamma}\right)P_\eta-gf,$$
$$\frac{dP_\eta}{d\tau}=-\left(1\mp\frac{b}{\gamma}\right)P_\xi+g\frac{df}{d\tau}. \quad (13)$$

The set (13) can be converted into a single differential equation by introducing a new variable ψ, an artifice analogous to the one used by Kulinski /6/:

$$\psi\equiv-P_\eta+gf. \quad (14)$$

Then

$$\frac{d\psi}{d\tau}=\frac{-dP_\eta}{d\tau}+g\frac{df}{d\tau}, \quad (15)$$

entailing, with (13),

$$\frac{d\psi}{d\tau}-g\frac{df}{d\tau}=\frac{-dP_\eta}{d\tau}=\left(1\mp\frac{b}{\gamma}\right)P_\xi-g\frac{df}{d\tau},$$

so that

$$P_\xi=\frac{1}{1\mp b/\gamma}\cdot\frac{d\psi}{d\tau}. \quad (16)$$

With (15) and (16), the first equation of (13) yields

$$\frac{d}{d\tau}\left(\frac{1}{1\mp b/\gamma}\cdot\frac{d\psi}{d\tau}\right)=(1\mp b/\gamma)(gf-\psi)-gf. \quad (17)$$

Here, (14) implies that ψ is a function of τ. So is the parameter γ. The electric field gf, however, is known as a function of z. To integrate (17) therefore necessitates the knowledge of $z=z(\tau)$ and $\gamma=\gamma(\tau)$. For z, the equation of motion (5) together with definitions (11) and (14) yields

$$\frac{1}{n_{||}}\frac{dP_{||}}{d\tau}=-g\frac{df}{d(k_{||}z)}\cdot V_\eta+\frac{d\Phi}{d(k_{||}z)}=$$
$$=\frac{1}{\gamma}g\frac{df}{d(k_{||}z)}(\psi-gf)+\frac{d\Phi}{d(k_{||}z)}. \quad (18)$$

Further

$$1-\frac{1}{\gamma^2}=V^2=V_{||}^2+V_\xi^2+V_\eta^2=$$
$$=V_{||}^2+\frac{1}{\gamma^2}\left\{(\psi-gf)^2+\left(\frac{1}{1\mp b/\gamma}\frac{d\psi}{d\tau}\right)^2\right\}. \quad (19)$$

The set (17), (18) and (19), while not easily leading to an analytical solution of the problem, is well suited for treatment with an analog computer; in the main differential equation (17), time is the independent variable, while in the secondary equations for the magnetic force (18) and the energy (19), the parameters f and γ are expressed in terms of quantities already occuring in (17).

3. Results

Trajectories have been studied for $f=\cos k_{||}z$, $k_{||}=\omega/c$, $g/b=E/cB_0=0.01$, and various values of b. The initial conditions

$$\left(\frac{1}{b(1-b/\gamma)}\frac{d\psi}{dt}\right)_0=\frac{P_{\xi0}}{b}=\left(\frac{\gamma\dot{x}}{cb}\right)_0,$$
$$-\left(\frac{\psi-gf}{b}\right)_0=\frac{P_{\eta0}}{b}=\left(\frac{\gamma\dot{y}}{cb}\right)_0,$$
$$k_{||}z_0,\text{ and }\frac{P_{||0}}{b}=\left(\frac{\gamma\dot{z}}{cb}\right)_0$$

have been chosen so that the resulting curves allow significant conclusions. The information is presented in the form of $v_\perp^2/c^2$ and $v_{||}/c$ versus $k_{||}z$; hodographs in the (V_ξ, V_η) plane have also been made. Of these, only the case $v_0=0$, in the plane $z=0$ (maximum E), or homogeneous E-field, is presented in Fig. 2a. It agrees well with the analytically obtained results of reference /3/ (H. and W.), and also gives a value b_c of the magnetic field where the large, kidney-like curves, change over into the small, circle-like ones. A weakly relativistic approximation (Ref. /3/, S.) yields here $b_c-1=3(g/2)^{2/3}=9.30\times10^{-2}$, agreeing quite well with the result obtained here, $1.092<b_c<1.094$.

If $z\neq0$ a parallel force $-e\mathbf{v}_\perp\times\mathbf{B}_\perp$ is present. The direction of the acceleration depends on the phase χ the velocity makes with $B_\sim$ (Fig. 2b). Unless $v_{\perp0}$ is large, one expects reflection, transmission or ejection

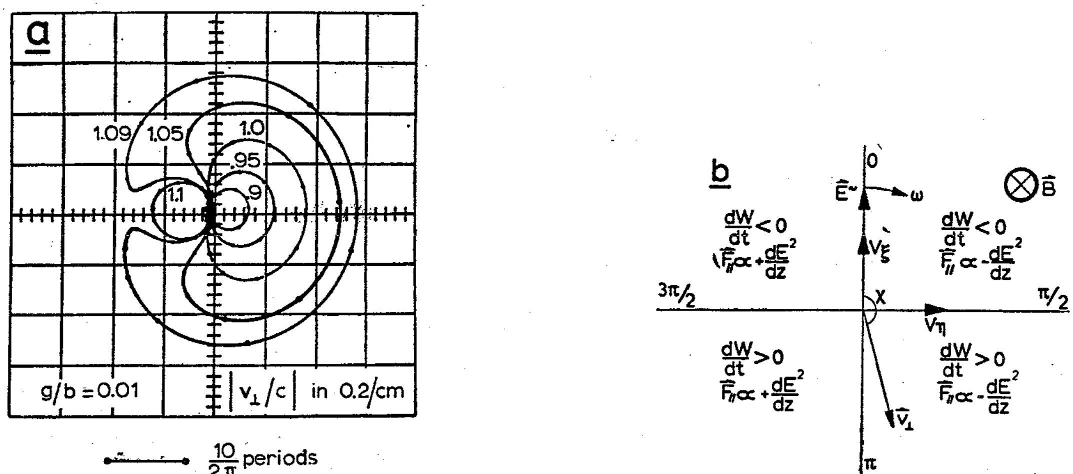

Fig. 2. Hodograph of the transverse velocity in homogeneous fields (a) and the coordinate system (b).

for $b \leqslant 1$, while for $b > b_c$ confinement in the maximum-field region will occur. The set of curves of Fig. 3 where an electron is injected into the field region, shows that for $b = 0.98$ and 1.0 the electron is either transmitted or reflected (depending on $v_{\|0}$), while obviously adiabatic reflection is an exception. The case $b = 1.05$, is remarkably insensitive to initial velocities. At $b = 1.08$ the field begins to retain the particle for some values of $v_{\|0}$, at 1.10 this is more pronounced. The next set of curves (Fig. 4). where the particle is started in the middle of the field region, more clearly shows the dependence of this trapping action on b. At $b = 1.08$, the electron is quickly ejected for all values of $v_{\|0}$, while at $b = 1.10$ many values of $v_{\|0}$ give the typical trapped orbits of $\gamma_0 v_{\|0}/cb = 0.03$. The case $\gamma_0 v_{\|0} cb = 0.40$ clearly shows that in this situation much higher energies can be obtained than in that of a homogeneous electric field.

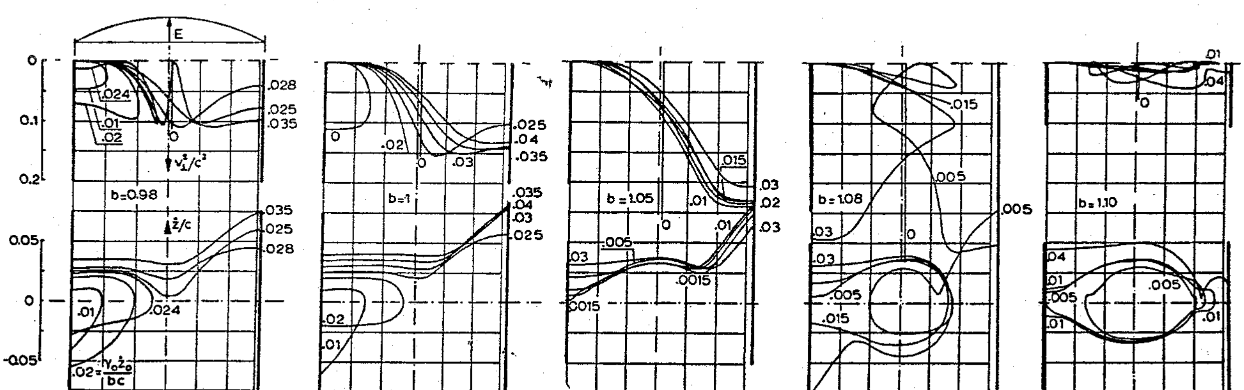

Fig. 3. Transverse energy and parallel velocity as function of $k_{\|} z$ of particles injected into the electromagnetic field (b=0.98; 1.00; 1.05; 1.08; 1.10).

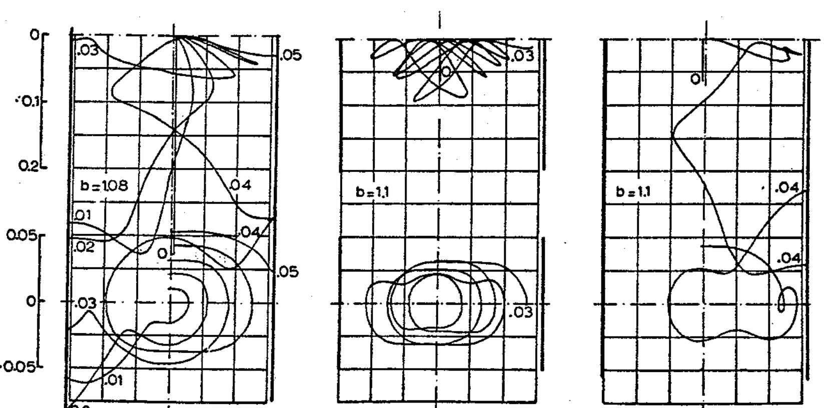

Fig. 4. Transverse energy and parallel velocity as function of $k_{\|} z$ of confined and ejected particles (b=1.08; 1.10).

This result observed here in a specific case, should be expected to be of a more general nature; a resonance interaction can be assumed with one of the two travelling-waves, which compose a standing-wave. For these, the result of the parallel motion is a doppler shift of the wave frequency from ω to $\omega \pm k_{||} v_{||}$. When $|\omega - \Omega|$ is sufficiently large and $v_{||}$ satisfies $\omega + k_{||} v_{||} = \Omega$ for one travelling-wave, one is inclined to neglect the non-resonant wave, and to calculate only the resonant interaction. That this is only observed for $b \gtrsim b_c$ indicates that for other values of b the interaction length does not suffice. Nevertheless it would be interesting to compare the results given here, with those of a travelling-wave treatment, such as the one by Roberts and Buchsbaum /4/, or an extension of the formalism developed above to the travelling-wave case. The latter will allow a better comparison as the treatment of reference /4/ does not allow for electrostatic fields. This indicates that the "synchronous" case (where the electron is trapped by the wave), given some emphasis in /4/, and before by Davydovskii /5/, may

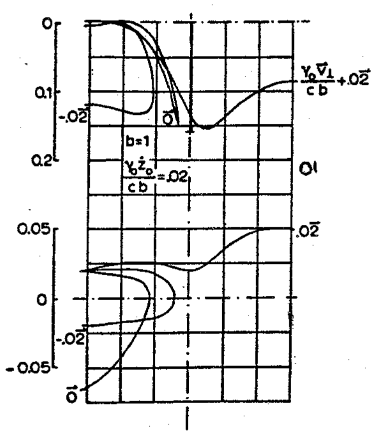

Fig. 5. Dependence on the transverse initial velocity.

not correspond to physical reality in a laboratory plasma. In general, space charge actions in such a plasma prevent the electrons from following the wave freely, and this is required for the synchronous case.

Figures 3 and 4 have been obtained with $v_{\perp 0} = 0$. Figure 5 clearly shows the marked dependence on $v_{\perp 0}$ of the acceleration parallel to B_0.

The accuracy and stability of th machine allow the study of phenomena of longer duration than e.g. the one of Fig. 4 ($b = 1.10$, $v_{||0}/c = 0.03$); for times appreciably beyond this the figures still reproduce very well.

Acknowledgement

We would like to express our gratitude to Prof. C.M. Braams for his valuable criticism.

This work was performed as part of the research program of the association agreement of Euratom and the "Stichting voor Fundamenteel Onderzoek der Materie" (FOM) with financial support from the "Nederlandse Organisatie voor Zuiver Wetenschappelijk Onderzoek" (ZWO) and Euratom.

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