

# The theory of nonlinear discrete-time systems and its application to the equalization of nonlinear digital communication channels

***Citation for published version (APA):***

Etten, van, W. C. (1979). *The theory of nonlinear discrete-time systems and its application to the equalization of nonlinear digital communication channels*. (EUT report. E, Fac. of Electrical Engineering; Vol. 79-E-102). Technische Hogeschool Eindhoven.

***Document status and date:***

Published: 01/01/1979

***Document Version:***

Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

***Please check the document version of this publication:***

- A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
- The final author version and the galley proof are versions of the publication after peer review.
- The final published version features the final layout of the paper including the volume, issue and page numbers.

[Link to publication](#)

***General rights***

Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the public portal.

If the publication is distributed under the terms of Article 25fa of the Dutch Copyright Act, indicated by the "Taverne" license above, please follow below link for the End User Agreement:

[www.tue.nl/taverne](http://www.tue.nl/taverne)

***Take down policy***

If you believe that this document breaches copyright please contact us at:

[openaccess@tue.nl](mailto:openaccess@tue.nl)

providing details and we will investigate your claim.

# the

Journal of the  
IEEE  
Transactions on  
Circuits and Systems

The theory of nonlinear discrete-time systems  
and its application to the equalization of  
nonlinear digital communication channels

by

W.C. van Etten

E I N D H O V E N   U N I V E R S I T Y   O F   T E C H N O L O G Y

Department of Electrical Engineering

Eindhoven

The Netherlands

THE THEORY OF NONLINEAR DISCRETE-TIME  
SYSTEMS AND ITS APPLICATION TO THE  
EQUALIZATION OF NONLINEAR DIGITAL  
COMMUNICATION CHANNELS

by

W.C. van Etten

TH-Report 79-E-102

ISBN 90-6144-102-1

Eindhoven

November 1979

## Summary

In order to be able to deal with the equalization of nonlinear digital communication systems, one has to know something about the theory of nonlinear discrete-time systems.

As an introduction some important aspects of linear continuous systems, linear discrete-time systems and nonlinear continuous systems are repeated. Then the Volterra series description of nonlinear discrete-time systems is introduced. After the derivation of the conditions for stability, the higherdimensional z-transform representation and the analytical properties of the system functions are discussed. To facilitate the back transformation of the higherdimensional z-transform, the association of variables is introduced. Cascade connection and inversion of nonlinear discrete systems is then treated, followed by a method for the measurement of the Volterra kernels. Finally, some attention is paid to the synthesis problem.

The contemplated equalization of a nonlinear digital communication channel is reduced to an inversion problem. For the implementation of the equalizer the synthesis of nonlinear discrete-time systems is invoked. As an example the Volterra kernels of a nonlinear system have been measured and an equalizer for that system has been designed and realized. From the measured eye-pattern of the equalized system it follows that the method, as developed in this report, satisfies very well for this kind of problems.

Etten, W.C. van

THE THEORY OF NONLINEAR DISCRETE-TIME SYSTEMS AND ITS APPLICATION TO THE EQUALIZATION OF NONLINEAR DIGITAL COMMUNICATION CHANNELS.

Eindhoven University of Technology, Department of Electrical Engineering, Eindhoven, The Netherlands. November 1979.

TH-Report 79-E-102

### Address of the author:

Dr.ir. W.C. van Etten,  
Group Telecommunications,  
Department of Electrical Engineering,  
Eindhoven University of Technology,  
P.O. Box 513,  
5600 MB EINDHOVEN,  
The Netherlands

## Contents

### Summary

I.	Introduction	1
II.	Linear continuous systems	1
III.	Linear discrete systems	2
IV.	Nonlinear continuous systems	3
V.	Volterra series representation of nonlinear discrete systems	4
VI.	Conditions for stability	9
VII.	Higher dimensional z-transform description of nonlinear discrete systems	12
VIII.	Properties of the system functions	16
IX.	Association of variables	17
X.	Cascading and inversion of nonlinear systems	20
XI.	Measurement of the Volterra kernels	24
XII.	Synthesis	27
XIII.	The equalization of a nonlinear digital communication channel	30
XIV.	Concluding remarks	33
	References	34

I. Introduction

A nonlinear digital communication system can be considered as a nonlinear discrete-time system. Speaking of systems, we consider a "black box", the contents of which is of no concern. It is assumed that the "black box" has one input and one output terminal. System theory describes, in a general way, the relation between the input and the output signal. In this approach the system itself is characterized by a system function (for a linear system) or a set of system functions (for nonlinear systems). This report starts with the system theory of nonlinear time invariant discrete-time systems, as far as it is of concern for the communication system in question. Most of this material can be found in other papers, however, as far as the knowledge of the author goes, not in such a comprehensive way as in the underlying work. As an introduction some important aspects of linear continuous systems, linear discrete-time systems and nonlinear continuous systems are repeated. For the sake of brevity we speak in the sequel of discrete systems if we mean discrete-time systems.

II. Linear continuous systems

In relating input and output signals of continuous linear time-invariant systems, Fourier and Laplace transforms appear to be elegant tools [1]. The input signal is represented by the function  $x(t)$ , whereas the corresponding output signal is denoted by  $y(t)$ . Let the impulse response of the system be given by  $h(t)$ , then the output signal  $y(t)$  is written as the convolution of the input signal  $x(t)$  and the system function  $h(t)$  as follows (see also Figure 1)

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau . \tag{1}$$

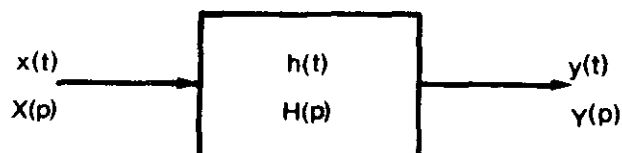


Figure 1 : "Black box" representation of a linear continuous system.

If the impulse response has the physical limitation of being causal, the integration (1) extends from zero to plus infinity. The rather complicated operation of convolution can be avoided by using Laplace or Fourier transforms. Define the transforms of the signals  $x(t)$  and  $y(t)$  by respectively  $X(p)$  and  $Y(p)$ . The system function  $H(p)$  consists of the transform of the impulse response  $h(t)$ . Now the transform of the output signal is found to be

$$Y(p) = H(p)X(p), \tag{2}$$

which means an important simplification with respect to the convolution (1). This reduction originates from the fact that the function  $e^{pt}$  is an eigenfunction of a linear time-invariant continuous system [1].

### III. Linear discrete systems

For linear discrete systems the input time sequence is given by  $\{x(m)\}$ , the set of input signal values at the characteristic instants  $t = mT$ ,  $m$  integer. The corresponding output sequence is denoted by  $\{y(m)\}$ . In this case the discrete impulse response sequence  $\{h(m)\}$  characterizes the system. For these systems the input output relation reads [2]

$$y(m) = \sum_k h(k)x(m-k). \tag{3}$$

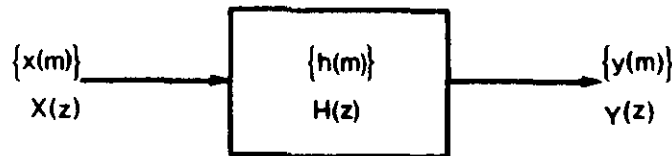


Figure 2 : "Black box" representation of a linear discrete system.

This discrete convolution is now simplified using the  $z$ -transform [2]

$$Y(z) = H(z)X(z), \tag{4}$$

where  $Y(z)$ ,  $H(z)$  and  $X(z)$  are the  $z$ -transforms of  $\{y(m)\}$ ,  $\{h(m)\}$  and  $\{x(m)\}$

respectively. Simplification (4) is related to the fact that the sequence  $\{z^{-m}\}$  is an eigenfunction of linear time-invariant discrete systems. Figure 2 gives a pictorial representation of a linear discrete system.

#### IV. Nonlinear continuous systems

A well-known method for the description of nonlinear continuous time-invariant systems is given by means of the so-called Volterra series [3,4]. The output  $y(t)$  is related to the input  $x(t)$  as

$$y(t) = \int_{-\infty}^{\infty} h_1(\tau)x(t-\tau)d\tau + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h_2(\tau_1, \tau_2)x(t-\tau_1)x(t-\tau_2)d\tau_1d\tau_2 + \dots$$

$$+ \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} h_n(\tau_1, \tau_2, \dots, \tau_n) \prod_{i=1}^n x(t-\tau_i)d\tau_i + \dots, \quad (5)$$

where  $h_1(t)$  is the impulse response of the linear term as given by (1) and  $h_n(t_1, t_2, \dots, t_n)$  is called the  $n$ th order impulse response. Here multidimensional Laplace transform is helpful to evaluate (5) in a more elegant way. The  $n$ -dimensional Laplace transform of  $h_n(t_1, t_2, \dots, t_n)$  is defined as

$$H_n(p_1, \dots, p_n) \triangleq \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} h_n(t_1, \dots, t_n) \prod_{i=1}^n e^{-p_i t_i} dt_i. \quad (6)$$

For the inverse transform it is found that

$$h_n(t_1, \dots, t_n) = \frac{1}{(2\pi j)^n} \int_{\sigma_1 - j\infty}^{\sigma_1 + j\infty} \dots \int_{\sigma_n - j\infty}^{\sigma_n + j\infty} H_n(p_1, \dots, p_n) \prod_{i=1}^n e^{p_i t_i} dp_i, \quad (7)$$

where  $\sigma_1, \dots, \sigma_n$  are within the region of convergence of  $H_n(p_1, \dots, p_n)$ . By means of (6) and (7) the signal transformation  $x(t) \rightarrow y(t)$  is calculated in four steps [4].

- 1) Calculate the Laplace transform  $X(p)$  of  $x(t)$ .
- 2) Construct the set of functions

$$Y_n(p_1, \dots, p_n) = H_n(p_1, \dots, p_n) \prod_{i=1}^n X(p_i) \quad (8)$$



for all  $n$  relevant to the system in question.

3) From (8) we calculate, by means of the inverse Laplace transform,

$$y_n(t_1, \dots, t_n).$$

4) Find  $y(t)$  by setting

$$y(t) = \sum_n y_n(t, \dots, t). \quad (9)$$

Step 3) of this procedure is often rather difficult. Evaluating Equation (7) is not easy and tables of transforms for any order are not particularly practical. Moreover, step 3) results in  $y_n(t_1, \dots, t_n)$ , whereas one mostly is only interested in  $y_n(t, \dots, t)$ . This suggests that the procedure as outlined above is more complicated than necessary. Indeed, a better procedure called 'association of variables' is suggested by George [3] and in its most general form derived by Reddy and Jagan [5]. This method breaks down an  $n$ -dimensional Laplace transform  $Y_n(p_1, \dots, p_n)$  to an one-dimensional Laplace transform  $y_n(p)$ . The general formula for this process is

$$Y_n(p) = \frac{1}{(2\pi j)^{n-1}} \int_{\sigma_2 - j\infty}^{\sigma_2 + j\infty} \dots \int_{\sigma_n - j\infty}^{\sigma_n + j\infty} Y_n(p - u_1 - u_2 - \dots - u_{n-1}, u_1, u_2, \dots, u_{n-1}) \cdot du_1 du_2 \dots du_{n-1} \quad (10)$$

A repeated application of Cauchy's residue theorem makes the evaluation of this integral rather easy.

V. Volterra series representation of nonlinear discrete systems

To gain insight into the description of nonlinear discrete systems the situation as depicted in Figure 3 should be considered.

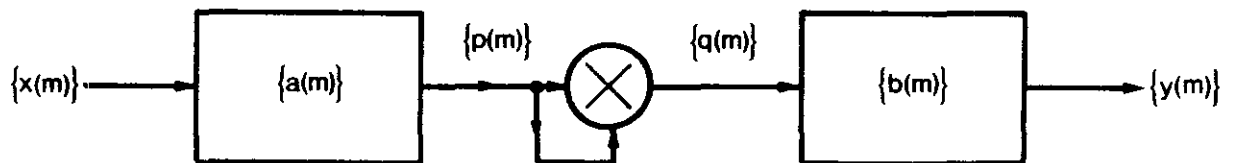


Figure 3 : Example of a second order nonlinear discrete system.

A sequence  $\{x(m)\}$  is supplied to a linear discrete system with an impulse response sequence  $\{a(m)\}$ . The output of this subsystem is described by  $\{p(m)\}$ . Squaring the latter sequence yields the sequence  $\{q(m)\}$  which, in its turn, is supplied to a second linear discrete system, described by its impulse response sequence  $\{b(m)\}$ . As output  $\{y(m)\}$  of the system considered, we take the output of this latter linear discrete system. So the overall system, being spoken of here, consists of a cascade connection of two linear systems with a squaring device in between. The sequence  $\{p(m)\}$  is given by

$$p(m) = \sum_k a(k)x(m-k) , \quad (11)$$

whereas the output of the squaring device is readily seen to be

$$\begin{aligned} q(m) &\triangleq p^2(m) = \left[ \sum_k a_k x(m-k) \right]^2 = \\ &= \sum_k a(k)x(m-k) \sum_l a(l)x(m-l) = \\ &= \sum_k \sum_l a(k)a(l)x(m-k)x(m-l) . \end{aligned} \quad (12)$$

Output  $\{y(m)\}$  is found by applying the discrete convolution (3) to  $\{q(m)\}$  and  $\{b(m)\}$ . This yields

$$\begin{aligned} y_2(m) &= \sum_r b(r)q(m-r) = \\ &= \sum_r b(r) \sum_k \sum_l a(k)a(l)x(m-r-k)x(m-r-l) . \end{aligned} \quad (13)$$

Introducing the new variables  $i=r+k$  and  $j=r+l$ , Equation (13) is written as

$$y_2(m) = \sum_r b(r) \sum_i \sum_j a(i-r)a(j-r)x(m-i)x(m-j) \quad (14)$$

and by changing the order of summation

$$y_2(m) = \sum_i \sum_j \sum_r b(r) a(i-r) a(j-r) x(m-i) x(m-j). \quad (15)$$

As second order impulse response of the given system we define

$$h_2(i, j) \triangleq \sum_r b(r) a(i-r) a(j-r). \quad (16)$$

Substituting (16) into (15) yields

$$y_2(m) = \sum_i \sum_j h_2(i, j) x(m-i) x(m-j). \quad (17)$$

Comparing this equation with (5) we see that  $y_2(m)$  of (17) represents the second order term of the discrete Volterra series. Because of the absence of a linear term we call the system of Figure 3, a second order power system. In the general case we speak of a polynomial system.

Let us now consider the system as depicted in Figure 4, where the connection between the two linear systems, given by their impulse response sequences  $\{a(m)\}$  and  $\{b(m)\}$ , consists of two parallel signal paths: a direct connection and a squarer.

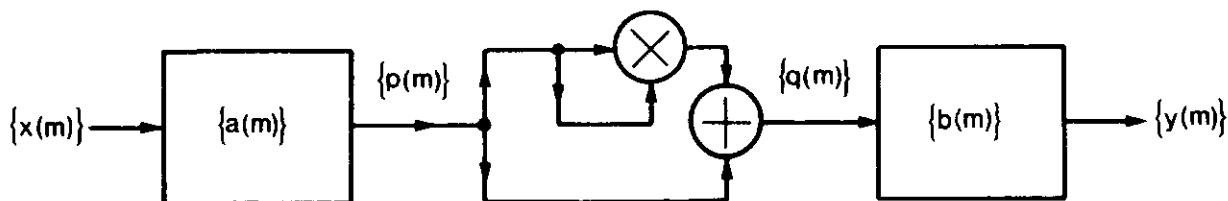


Figure 4 : Second order system with two signal paths, i.e. a second order polynomial system.

The two signals are added before supplying them to the linear system  $\{b(m)\}$ . It is obvious that the sequence  $\{p(m)\}$  is identical to that resulting from Figure 3. The sequence  $\{q(m)\}$  now reads

$$\begin{aligned}
 q(m) &= p(m) + p^2(m) = \\
 &= \sum_k a(k)x(m-k) + \sum_k \sum_l a(k)a(l)x(m-k)x(m-l) .
 \end{aligned} \tag{18}$$

From this equation the output sequence becomes

$$\begin{aligned}
 y(m) &= \sum_r b(r)q(m-r) = \sum_r b(r) \sum_k a(k)x(m-r-k) + \\
 &\quad + \sum_r b(r) \sum_k \sum_l a(k)a(l)x(m-r-k)x(m-r-l) \\
 &\triangleq y_1(m) + y_2(m) ,
 \end{aligned} \tag{19}$$

where  $y_1(m)$  stands for the linear (first order) term and  $y_2(m)$  for the second order term. As can readily be seen the second order term equals Equation (13), which resulted after some simple manipulations in Equation (17). The first order term is reduced likewise

$$\begin{aligned}
 y_1(m) &= \sum_r b(r) \sum_i a(i-r)x(m-i) = \\
 &= \sum_i \sum_r b(r)a(i-r)x(m-i) = \sum_i h_1(i)x(m-i),
 \end{aligned} \tag{20}$$

where

$$h_1(i) \triangleq \sum_r b(r)a(i-r). \tag{21}$$

By means of (19), (20) and (17), the output sequence of a second order non-linear discrete system is described as a functional of the input sequence and the first and second order impulse response sequences.

Resuming we have for a second second order polynomial system

$$y(m) = \sum_i h_1(i)x(m-i) + \sum_i \sum_j h_2(i,j)x(m-i)x(m-j) . \quad (22)$$

From a generalization of this result to higher order systems, it follows that

$$y(m) = \sum_i h_1(i)x(m-i) + \sum_{i_1} \sum_{i_2} h_2(i_1, i_2)x(m-i_1)x(m-i_2) + \dots \\ + \sum_{i_1} \dots \sum_{i_n} h_n(i_1, \dots, i_n)x(m-i_1)\dots x(m-i_n) + \dots \quad (23)$$

This relation is the discrete version of Equation (5) and is called the discrete Volterra series. In many papers, for instance [6] and [7], this series has been derived from the continuous one by considering sampled-data nonlinear continuous systems. Apparently, the problem arose for the first time in that kind of systems. Considering complete discrete systems from the very beginning of the investigations, as is done in this paper, does not yield any difference in the results.

From (16) it follows that  $h_2(i, j)$  is a symmetrical function of  $i$  and  $j$ . We wonder whether it always can be assumed that the higher order impulse responses are symmetrical, because we know from literature [4] that such is the case for the higher order impulse responses of nonlinear continuous systems. This question will, under some weak conditions, be answered affirmatively as is shown below. Let us separate an arbitrary second order impulse response sequence in a symmetrical and an anti-symmetrical part

$$h_2(i, j) = h_{2s}(i, j) + h_{2a}(i, j) , \quad (24)$$

where

$$h_{2s}(i, j) \triangleq h_{2s}(j, i) \triangleq \frac{1}{2}\{h_2(i, j) + h_2(j, i)\} \quad (25)$$

and

$$h_{2a}(i, j) \triangleq -h_{2a}(j, i) \triangleq \frac{1}{2}\{h_2(i, j) - h_2(j, i)\} . \quad (26)$$

Substitute the anti-symmetrical part into (17). This yields

$$y_2(m) = \frac{1}{2} \sum_i \sum_j \{h_2(i,j) - h_2(j,i)\}x(m-i)x(m-j) \quad (27)$$

Assume that all input values  $x(m)$  are bounded, i.e.  $|x(m)| \leq C$  for all  $m$ ;  $C$  an arbitrary finite real value. Then for the first term of (27) we have

$$\frac{1}{2} \sum_i \sum_j |h_2(i,j)x(m-i)x(m-j)| \leq \frac{1}{2} C^2 \sum_i \sum_j |h_2(i,j)| \quad (28)$$

Moreover, if the series  $\sum_i \sum_j h_2(i,j)$  converges absolutely, the summations over  $i$  and  $j$  in the first term of (27), are allowed to be interchanged. Then Equation (27) reads

$$\begin{aligned} y_2(m) = & \frac{1}{2} \sum_j \sum_i h_2(i,j)x(m-i)x(m-j) + \\ & - \frac{1}{2} \sum_i \sum_j h_2(j,i)x(m-i)x(m-j). \end{aligned} \quad (29)$$

Our next step is to substitute  $i$  for  $j$  and vice versa into the first term of (29). It is then readily verified that the right-hand member of (29) vanishes. The requirement of absolute convergence of  $\sum_i \sum_j h_2(i,j)$  is a condition for stable systems, as will be shown in the next section. Thus, for stable systems with bounded input, a possible anti-symmetrical part of the second order impulse response has no influence on the output. Without loss of generality it can be assumed that  $h_2(i,j)$  is symmetrical. This result also appears to be valid for higher order impulse response sequences, as is shown by a similar reasoning as above for second order systems.

## VI. Conditions for stability

A system is stable if any bounded input signal causes an output signal that is also bounded. This means that for input signals, which satisfy  $|x(m)| \leq C$ , for all  $m$  and for  $C$  an arbitrary finite value, the corresponding output

satisfies  $|y(m)| \leq D$ , for all  $m$  and with  $D$  an arbitrary but finite value.

In the case of linear systems this means

$$|y(m)| = \left| \sum_k h(k)x(m-k) \right| \leq C \sum_k |h(k)| . \quad (30)$$

Stability follows if

$$\sum_k |h(k)| < \infty . \quad (31)$$

In a similar way as in [1] it is shown that (31) is not only a sufficient, but also a necessary condition for the stability of linear systems. To find stability conditions for nonlinear systems we take a look at Equations (19) and (22). Then it follows that if both  $y_1(m)$  and  $y_2(m)$  are bounded, we have a sufficient condition for stability. This requirement for  $y_1(m)$  leads, via the preceding argumentation directly to the conclusion

$$\sum_i |h_1(i)| < \infty . \quad (32)$$

For the second order term we have

$$\begin{aligned} |y_2(m)| &= \left| \sum_i \sum_j h_2(i,j)x(m-i)x(m-j) \right| \leq \\ &\leq \sum_i \sum_j |h_2(i,j)| \cdot |x(m-i)| \cdot |x(m-j)| \leq \\ &\leq C^2 \sum_i \sum_j |h_2(i,j)| . \end{aligned} \quad (33)$$

Stability is guaranteed if

$$\sum_i \sum_j |h_2(i,j)| < \infty . \quad (34)$$

A second order nonlinear system is stable if the conditions (32) and (34) are satisfied simultaneously. We emphasize that these conditions are sufficient but not necessary.

In general a higher order system is stable if all impulse responses  $h_n(i_1, \dots, i_n)$ , necessary to describe the system under consideration, are absolutely summable.

An example

As an example, consider the system as depicted in Figure 5. All switches are synchronously closed at the instants  $mT$  ( $m$  integer).

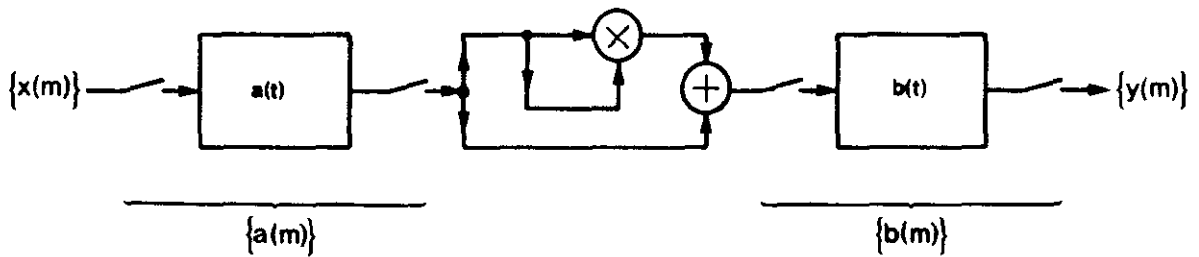


Figure 5 : A second order sampled-data nonlinear continuous system.

The continuous subsystems  $a(t)$  and  $b(t)$  are given by

$$\left. \begin{aligned} a(t) &= e^{-\alpha t} \\ b(t) &= e^{-\beta t} \end{aligned} \right\} t \geq 0 \quad \alpha, \beta > 0. \quad (35)$$

Supposing  $T = 1$ , the related discrete subsystems are given by means of

$$\left. \begin{aligned} a(m) &= e^{-\alpha m} \\ b(m) &= e^{-\beta m} \end{aligned} \right\} m \geq 0. \quad (36)$$

From this the overall system impulse responses are found, using (16) and (21)

$$\begin{aligned} h_1(i) &= \sum_{r=0}^i e^{-\beta r} e^{-\alpha(i-r)} \\ h_2(i, j) &= \sum_{r=0}^{\min(i, j)} e^{-\beta r} e^{-\alpha(i-r)} e^{-\alpha(j-r)} \end{aligned} \quad (37)$$

If we want to answer the question of stability, as far as the first order term goes, we have to look at the series



$$\begin{aligned} \sum_{i=0}^{\infty} \left| e^{-i\alpha} \sum_{r=0}^i \{e^{(\alpha-\beta)}\}^r \right| &= \sum_{i=0}^{\infty} \left| e^{-i\alpha} \frac{1 - e^{i(\alpha-\beta)}}{1 - e^{\alpha-\beta}} \right| = \\ &= \left| \frac{1}{1 - e^{\alpha-\beta}} \right| \sum_{i=0}^{\infty} |e^{-i\alpha} - e^{-i\beta}| \leq \frac{1}{|1 - e^{\alpha-\beta}|} \left\{ \sum_{i=0}^{\infty} e^{-i\alpha} + \sum_{i=0}^{\infty} e^{-i\beta} \right\}, \\ &\alpha \neq \beta. \end{aligned} \quad (38)$$

By means of d'Alembert's ratio test [8] it is easy to see that both series of (38) converge and thus the first order impulse response is absolutely summable. Let us now consider the second order term

$$\begin{aligned} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \left| \sum_{r=0}^{\min(i,j)} e^{-\beta r} e^{-\alpha(i-r)} e^{-\alpha(j-r)} \right| &= \\ &= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \left| e^{-i\alpha} e^{-j\alpha} \frac{1 - \{e^{2\alpha-\beta}\}^{\min(i,j)}}{1 - e^{2\alpha-\beta}} \right| = \\ &= \frac{1}{|1 - e^{2\alpha-\beta}|} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} |e^{-(i+j)\alpha} - e^{-(i+j)\alpha + \min(i,j)(2\alpha-\beta)}| \leq \\ &\leq \frac{1}{|1 - e^{2\alpha-\beta}|} \left\{ \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} e^{-(i+j)\alpha} + \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} e^{-(i+j)\alpha + \min(i,j)2\alpha} e^{-\min(i,j)\beta} \right\} \\ &\leq \frac{1}{|1 - e^{2\alpha-\beta}|} \left\{ \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} e^{-(i+j)\alpha} + \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} e^{-\min(i,j)\beta} \right\}, \\ &\alpha \neq \frac{\beta}{2}. \end{aligned} \quad (39)$$

The general term of both series of (39) tends to zero if  $i$  and  $j$  go to infinity. This is Stolz's necessary and sufficient condition for the convergence of double series [8]. Now it is concluded that the system as given in Figure 5 is stable because it is shown that both conditions (31) and (34) are satisfied.

## VII. Higher dimensional z-transform description of nonlinear discrete systems

Define the two-dimensional z-transform as

$$X_2(z_1, z_2) \triangleq \sum_{m_1} \sum_{m_2} x_2(m_1, m_2) z_1^{-m_1} z_2^{-m_2}, \quad (40)$$

then the inverse transform is given by [2]

$$x_2(m_1, m_2) = \frac{1}{(2\pi j)^2} \oint_{C_1} \oint_{C_2} X_2(z_1, z_2) z_1^{m_1-1} z_2^{m_2-1} dz_1 dz_2, \quad (41)$$

where  $C_1$  and  $C_2$  are contours within the region of convergence of  $X(z_1, z_2)$ .  
Recalling (3) and (4) an artificial variable is introduced in (15) such that this equation is rewritten as [6]

$$\begin{aligned} y_2(m_1, m_2) &= \sum_i \sum_j \sum_r b(r) a(i-r) a(j-r) x(m_1-i) x(m_2-j) = \\ &= \sum_r b(r) \sum_i a(i-r) x(m_1-i) \sum_j a(j-r) x(m_2-j). \end{aligned} \quad (42)$$

Taking the two-dimensional  $z$ -transform of (42) yields

$$\begin{aligned} Y_2(z_1, z_2) &= \sum_{m_1} \sum_{m_2} \sum_r b(r) \sum_i a(i-r) x(m_1-i) z_1^{-m_1} \\ &\cdot \sum_j a(j-r) x(m_2-j) z_2^{-m_2}. \end{aligned} \quad (43)$$

Substituting into this equation the new variables

$$\begin{aligned} p &\triangleq i - r \\ q &\triangleq j - r, \end{aligned} \quad (44)$$

gives

$$\begin{aligned} Y_2(z_1, z_2) &= \sum_{m_1} \sum_{m_2} \sum_r b(r) \sum_p a(p) x(m_1-r-p) z_1^{-m_1} \\ &\cdot \sum_q a(q) x(m_2-r-q) z_2^{-m_2}. \end{aligned} \quad (45)$$

Introduce the variables

$$\begin{aligned} n_1 &\triangleq m_1 - r \\ n_2 &\triangleq m_2 - r. \end{aligned} \quad (46)$$

This changes (45) as follows

$$\begin{aligned}
 Y_2(z_1, z_2) &= \sum_r b(r) \sum_{n_1} \sum_p a(p) x(n_1 - p) z_1^{-n_1 - r} \cdot \\
 &\cdot \sum_{n_2} \sum_q a(q) x(n_2 - q) z_2^{-n_2 - r} = \\
 &= \sum_r b(r) (z_1 z_2)^{-r} \sum_{n_1} \sum_p a(p) x(n_1 - p) z_1^{-n_1} \cdot \\
 &\cdot \sum_{n_2} \sum_q a(q) x(n_2 - q) z_2^{-n_2} = \\
 &= B(z_1 z_2) A(z_1) A(z_2) X(z_1) X(z_2) \cdot
 \end{aligned} \tag{47}$$

We have now developed an input output relation in the two-dimensional  $z$ -domain, of a second order power system. This relation is in its general form written as

$$Y_2(z_1, z_2) = H_2(z_1, z_2) X(z_1) X(z_2) \cdot \tag{48}$$

In the given example of Figure 5 the second order system function reads

$$H_2(z_1, z_2) = B(z_1 z_2) A(z_1) A(z_2) \cdot \tag{49}$$

The first order system function of this example is readily found by applying (4) to (37). This yields

$$H_1(z) = A(z) B(z) \cdot \tag{50}$$

For  $A(z)$  and  $B(z)$  we have

$$\begin{aligned}
 A(z) &= \sum_{m=0}^{\infty} e^{-\alpha m} z^{-m} = \frac{z}{z - e^{-\alpha}} \\
 B(z) &= \sum_{m=0}^{\infty} e^{-\beta m} z^{-m} = \frac{z}{z - e^{-\beta}} \cdot
 \end{aligned} \tag{51}$$

Substituting these expressions into (49) and (50) gives

$$H_1(z) = \frac{z^2}{(z-e^{-\alpha})(z-e^{-\beta})} ,$$

$$H_2(z_1, z_2) = \frac{z_1 z_2}{z_1 z_2 - e^{-\beta}} \frac{z_1}{z_1 - e^{-\alpha}} \frac{z_2}{z_2 - e^{-\alpha}} . \quad (52)$$

For arbitrary higher order systems we define the  $n$ -dimensional  $z$ -transform (see [6],[7] and [9])

$$X_n(z_1, \dots, z_n) \triangleq \sum_{m_1} \dots \sum_{m_n} x_n(m_1, \dots, m_n) z_1^{-m_1} \dots z_n^{-m_n} , \quad (53)$$

with the inverse transform

$$x_n(m_1, \dots, m_n) = \frac{1}{(2\pi j)^n} \oint_{C_1} \dots \oint_{C_n} X_n(z_1, \dots, z_n) z_1^{m_1-1} \dots z_n^{m_n-1} dz_1 \dots dz_n , \quad (54)$$

with  $C_1, \dots, C_n$  contours within the region of convergence of  $X(z_1, \dots, z_n)$ .

An  $n$ th order system is defined by a set of system functions

$\{H_1(z), H_2(z_1, z_2), \dots, H_n(z_1, \dots, z_n)\}$ . From the definitions it will be clear that the higher order system functions  $H_i(z_1, \dots, z_i)$  are the  $i$ -dimensional  $z$ -transforms of the higher order impulse responses  $h_i(m_1, \dots, m_i)$  and the other way around, i.e.

$$H_i(z_1, \dots, z_i) = \sum_{m_1} \dots \sum_{m_i} h_i(m_1, \dots, m_i) z_1^{-m_1} \dots z_i^{-m_i} \quad (55)$$

and

$$h_i(m_1, \dots, m_i) = \frac{1}{(2\pi j)^i} \oint_{C_1} \dots \oint_{C_i} H_i(z_1, \dots, z_i) z_1^{m_1-1} \dots z_i^{m_i-1} dz_1 \dots dz_i . \quad (56)$$

Finding the output  $\{y(m)\}$  of such a system at a given input  $\{x(m)\}$  goes, along a similar four step procedure as given in Section IV, as follows

- 1) Calculate the  $z$ -transform  $X(z)$  of the input sequence  $\{x(m)\}$ :
- 2) Construct the set of  $n$  functions

$$Y_i(z_1, \dots, z_i) = H_i(z_1, \dots, z_i) \prod_{j=1}^i X(z_j) \quad i = 1, \dots, n \quad (57)$$

of the  $n$ th order system.

- 3) Calculate the inverse transforms  $\{y_i(m_1, \dots, m_i)\}$  of (57).
- 4) The output  $\{y(m)\}$  is found by means of

$$y(m) = \sum_{i=1}^n y_i(m, \dots, m). \quad (58)$$

Here we meet the same difficulty as in the continuous case, namely the need for extensive tables for multi-dimensional  $z$ -transforms as needed in step 3) of the procedure. But again, since we are only interested in  $y_i(m_1, \dots, m_i) = y_i(m, \dots, m)$ , the association of variables is possible, as will be shown in Section IX, and thus the difficulty mentioned above can be avoided.

### VIII. Properties of the system functions

In Section VI it has been shown that a sufficient condition for the stability of a system is that the impulse response function of any order is absolutely summable. Moreover, it has been shown, that it is always allowed to assume that the impulse responses are symmetrical in all variables. From (55) it follows that the system functions  $H_i(z_1, \dots, z_i)$  have the same property. If a system function is not symmetrical it can always be written in a symmetrical form as follows

$$[H_i(z_1, \dots, z_i)]_{symm} = \frac{1}{i!} \left[ \sum_j H_i^j(\dots) \right], \quad (59)$$

where the summation over  $j$  extends over all possible permutations of the  $i$  variables  $z_1, \dots, z_i$ . For the case  $i=3$  this means

$$\begin{aligned}
 [H_3(z_1, z_2, z_3)]_{\text{symm.}} &= \frac{1}{6} [H_3(z_1, z_2, z_3) + H_3(z_2, z_3, z_1) + H_3(z_3, z_2, z_1) + \\
 &\quad + H_3(z_3, z_1, z_2) + H_3(z_2, z_1, z_3) + H_3(z_1, z_3, z_2)] \quad .
 \end{aligned}
 \tag{60}$$

The impulse response functions can be symmetrized in the same way. For the sake of simpler notation the unsymmetrical form is often preferred.

Assuming that the sufficient conditions for stability are satisfied, we shall derive analytic properties of the system functions. Let us consider a second order system, then we have

$$\begin{aligned}
 |H_2(z_1, z_2)| &= \left| \sum_{m_1} \sum_{m_2} h_2(m_1, m_2) z_1^{-m_1} z_2^{-m_2} \right| \leq \\
 &\leq \sum_{m_1} \sum_{m_2} |h_2(m_1, m_2)| \cdot |z_1^{-m_1}| \cdot |z_2^{-m_2}| \quad .
 \end{aligned}
 \tag{61}$$

Because of (34), and if causality is assumed, the system function  $H_2(z_1, z_2)$  is analytic outside and on the unit circles, i.e.

$$|z_1|, |z_2| \geq 1 \quad .
 \tag{62}$$

Of course this property is also valid for higher dimensions, so that in (56) the unit circle can serve as contour for all integrations.

#### IX. Association of variables

As indicated in Section VII the use of tables of multidimensional  $z$ -transforms is avoided by a technique named 'association of variables' [3], [5], [7] and [9]. This technique reduces the multidimensional  $z$ -transform to a one-dimensional  $z$ -transform, which is transformed back to the time domain by means of a one-dimensional  $z$ -transform table or the integral (54) with  $n=1$ . Back transforming the  $n$ -dimensional  $z$ -transform  $Y_n(z_1, \dots, z_n)$  results in a function  $y_n(m_1, \dots, m_n)$  of  $n$  variables in the time domain. For time-invariant systems one is only interested in the function of one time variable  $y_n(m, \dots, m)$ ,

where  $m_1 = m_2 = \dots = m_n = m$ . Consider the two-dimensional transform given by (41), evaluated for  $m_1 = m_2 = m$

$$y_2(m) = y_2(m,m) = \frac{1}{(2\pi j)^2} \oint_{C_1} \oint_{C_2} Y_2(z_1, z_2) z_1^{m-1} z_2^{m-1} dz_1 dz_2 \quad (63)$$

Let

$$\begin{aligned} z_1 &= zu^{-1} \\ z_2 &= u \\ dz_1 dz_2 &= u^{-1} du dz \end{aligned} \quad (64)$$

Substituting (64) into (63) yields

$$\begin{aligned} y_2(m) &= \frac{1}{(2\pi j)^2} \oint_{C_1} \oint_{C_2} Y_2(zu^{-1}, u) z^{m-1} u^{-1} du dz = \\ &= \frac{1}{2\pi j} \oint_{C_1} \left[ \frac{1}{2\pi j} \oint_{C_2} Y_2(zu^{-1}, u) u^{-1} du \right] z^{m-1} dz \end{aligned} \quad (65)$$

The integral between the brackets reduces the two-dimensional  $z$ -transform to a one-dimensional  $z$ -transform, whereas the contour integral along  $C_1$  represents in fact the one-dimensional inverse transformation of the  $z$ -transform in the brackets. It is the operation between the brackets that is called the 'association of variables' and reads for the two-dimensional case

$$\Gamma_2[Y_2(z_1, z_2)] \triangleq \frac{1}{2\pi j} \oint_{C_2} Y_2(zu^{-1}, u) u^{-1} du \quad (66)$$

where  $\Gamma_2[Y_2(z_1, z_2)]$  is the formal notation of the operation which reduces the two-dimensional  $z$ -transform to a one-dimensional  $z$ -transform. For systems satisfying the sufficient conditions for stability and excited by bounded and causal input sequences, the function  $Y_2(z_1, z_2)$  has no singularities outside  $C_1$  (see Section VIII). Suppose that  $Y_2(z_1, z_2)$  has a singularity at  $z_1 = a$ , then  $|a| < 1$ . Consider the function  $Y_2(zu^{-1}, u)$  and let us have a look

at the point  $zu^{-1} = \alpha$ . Because  $|a| < 1$  it follows that  $|u| > |z|$ , which means that the function  $Y(zu^{-1}, u)$  has no singularities inside  $C_2$  due to singularities inside  $C_1$  arising from the term  $zu^{-1}$  [7].

In the n-dimensional case the operations (63)-(65) are performed successively  $n-1$  times. This yields the general association formula ([7] and [9])

$$\Gamma_n[Y_n(z_1, \dots, z_n)] \stackrel{\Delta}{=} \frac{1}{(2\pi j)^{n-1}} \oint_{C_{n-1}} \dots \oint_{C_2} Y_n(zu_1^{-1} u_2^{-1} \dots u_{n-1}^{-1}, u_1, u_2, \dots, u_{n-1}) \cdot u_1^{-1} u_2^{-1} \dots u_{n-1}^{-1} du_1 du_2 \dots du_{n-1} \quad (67)$$

At integration along  $C_i$  it should be kept in mind that there are no singularities within  $C_i$  arising from singularities inside  $C_{i-1}$ . In (67) the association is being given in a certain order, but the association can be performed in any arbitrary order. The  $(n-2)$ -fold integral (67) is readily evaluated by repeated application of the residue theorem.

Example

To illustrate the method, given in this section, we consider the system as depicted in Figure 5 and ask for the second order impulse response sequence, the z-transform of which is given by (52). From (52) we find

$$H_2(zu^{-1}, u) = \frac{z}{z-e^{-\beta}} \frac{zu^{-1}}{zu^{-1}-e^{-\alpha}} \frac{u}{u-e^{-\alpha}} \quad (68)$$

and

$$\begin{aligned} \Gamma_2[H_2(z_1, z_2)] &= \frac{1}{2\pi j} \oint_{C_2} \frac{z}{z-e^{-\beta}} \frac{z}{zu^{-1}-e^{-\alpha}} \frac{1}{u-e^{-\alpha}} u^{-1} du = \\ &= \frac{z}{z-e^{-\beta}} \operatorname{Res.}_{u=e^{-\alpha}} \left\{ \frac{z}{zu^{-1}-e^{-\alpha}} \frac{1}{u-e^{-\alpha}} u^{-1} \right\} = \\ &= \frac{z^2}{(z-e^{-\beta})(z-e^{-2\alpha})} \quad (69) \end{aligned}$$



By means of the inverse one-dimensional z-transform integral, this expression is back transformed to the time domain. This gives

$$\begin{aligned}
 h_2(m) &= \frac{1}{2\pi j} \oint_{C_1} \frac{z^2}{(z-e^{-\beta})(z-e^{-2\alpha})} z^{m-1} dz = \\
 &= \text{Res.}_{z=e^{-\beta}} \left\{ \frac{z^{m+1}}{(z-e^{-\beta})(z-e^{-2\alpha})} \right\} + \\
 &+ \text{Res.}_{z=e^{-2\alpha}} \left\{ \frac{z^{m+1}}{(z-e^{-\beta})(z-e^{-2\alpha})} \right\} = \frac{e^{-2\alpha m}}{1-e^{-2\alpha+\beta}} + \frac{e^{-\beta m}}{1-e^{-2\alpha+\beta}} \quad (70)
 \end{aligned}$$

An important rule connected with the association of variables, is given by

$$\begin{aligned}
 \Gamma_n[G(z_1, z_2, \dots, z_n) Y_n(z_1, z_2, \dots, z_n)] &= \\
 = G(z) \Gamma_n[Y_n(z_1, z_2, \dots, z_n)] \quad , \quad (71)
 \end{aligned}$$

which is easy to verify by means of (67).

X. Cascading and inversion of nonlinear systems

Describing the overall system function of a cascade connection of linear systems is very easy. Linear system theory learns that one only needs to multiply the system functions of the constituting subsystems. For nonlinear systems matters are more complicated because of the interaction between the system functions of different order of the subsystems, as will be seen in the sequel. We shall calculate the first, second and third order terms, which are the most important ones for practical applications; of course, terms of higher order are derived in a similar way. Consider the cascade connection of Figure 6.

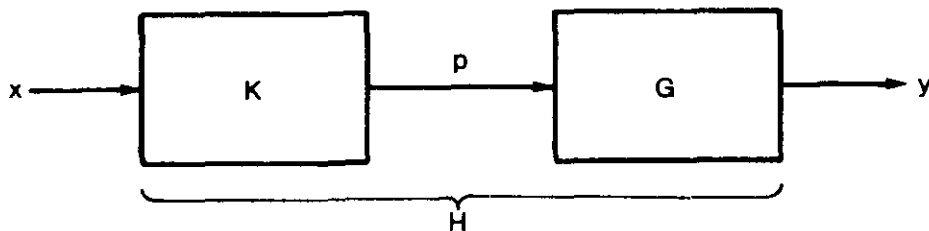


Figure 6 : The cascade connection of two nonlinear systems.

The input output relation for the subsystem  $K$  is denoted in the  $z$ -domain as follows

$$P(z) = K_1(z)X(z) + \Gamma_2[K_2(z_1, z_2)X(z_1)X(z_2)] + \\ + \Gamma_3[K_3(z_1, z_2, z_3)X(z_1)X(z_2)X(z_3)] \quad , \quad (72)$$

whereas this relation for the subsystem  $G$  is given by

$$Y(z) = G_1(z)P(z) + \Gamma_2[G_2(z_1, z_2)P(z_1)P(z_2)] + \\ + \Gamma_3[G_3(z_1, z_2, z_3)P(z_1)P(z_2)P(z_3)] \quad . \quad (73)$$

Substituting (72) into (73) gives us the input output relation of the cascade system  $H$

$$Y(z) = G_1(z)K_1(z)X(z) + G_1(z)\Gamma_2[K_2(z_1, z_2)X(z_1)X(z_2)] + \\ + G_1(z)\Gamma_3[K_3(z_1, z_2, z_3)X(z_1)X(z_2)X(z_3)] + \\ + \Gamma_2[G_2(z_1, z_2)\{K_1(z_1)X(z_1) + \Gamma_2[K_2(z_1, z_2)X(z_1)X(z_2)]_{z_1} + \dots\} \cdot \\ \cdot \{K_1(z_2)X(z_2) + \Gamma_2[K_2(z_1, z_2)X(z_1)X(z_2)]_{z_2} + \dots\}] + \\ + \Gamma_3[G_3(z_1, z_2, z_3)\{K_1(z_1)X(z_1) + \dots\} \cdot \{K_1(z_2)X(z_2) + \dots\} \cdot \\ \cdot \{K_1(z_3)X(z_3) + \dots\}] + \dots \quad , \quad (74)$$

where  $\Gamma_2[\dots]_{z_1}$  means that the one-dimensional  $z$ -transform, resulting from the reduction by means of the association of variables, has to be evaluated at  $z=z_1$ ;  $\Gamma_2[\dots]_{z_2}$  means evaluating at  $z=z_2$ . From (74) it can directly be seen that

$$H_1(z) = G_1(z)K_1(z) \quad . \quad (75)$$

Collecting the second order terms yields

$$\begin{aligned}
 & G_1(z)\Gamma_2[K_2(z_1, z_2)X(z_1)X(z_2)] + \Gamma_2[G_2(z_1, z_2)K_1(z_1)X(z_1)K_1(z_2)X(z_2)] = \\
 & = \Gamma_2[G_1(z_1, z_2)K_2(z_1, z_2)X(z_1)X(z_2) + G_2(z_1, z_2)K_1(z_1)K_1(z_2) \\
 & \quad \cdot X(z_1)X(z_2)] \quad . \quad (76)
 \end{aligned}$$

In this latter equality use is made of rule (71) and it follows

$$H_2(z_1, z_2) = G_1(z_1, z_2)K_2(z_1, z_2) + G_2(z_1, z_2)K_1(z_1)K_1(z_2) \quad . \quad (77)$$

To find the third order term  $H_3(z_1, z_2, z_3)$  we collect the third order terms of (74)

$$\begin{aligned}
 & \Gamma_3[H(z_1, z_2, z_3)X(z_1)X(z_2)X(z_3)] = \\
 & = G_1(z)\Gamma_3[K_3(z_1, z_2, z_3)X(z_1)X(z_2)X(z_3)] + \\
 & \quad + \Gamma_2[G_2(z_1, z_2)K_1(z_1)X(z_1)\Gamma_2[K_2(z_1, z_2)X(z_1)X(z_2)]_{z_2}] + \\
 & \quad + \Gamma_2[G_2(z_1, z_2)K_1(z_2)X(z_2)\Gamma_2[K_2(z_1, z_2)X(z_1)X(z_2)]_{z_1}] + \\
 & \quad + \Gamma_3[G_3(z_1, z_2, z_3)K_1(z_1)K_1(z_2)K_1(z_3)X(z_1)X(z_2)X(z_3)] \quad . \quad (78)
 \end{aligned}$$

Applying (71) to the first term of this expression yields  $\Gamma_3[G_1(z_1, z_2, z_3) \cdot K_3(z_1, z_2, z_3)X(z_1)X(z_2)X(z_3)]$ . Using the symmetry properties as discussed in Section VIII, the second and third term are shown to be equal. After addition of these terms it follows  $2\Gamma_2[G_2(z_1, z_2)K_1(z_1)X(z_1)\Gamma_2[K_2(z_1, z_2) \cdot X(z_1)X(z_2)]_{z_2}]$ . This functional consists of two operators that reduce a two-dimensional  $z$ -transform to a one-dimensional  $z$ -transform. By means of the definition integral (67) for the association of variables it is shown that this term equals  $\Gamma_3[G_2(z_1, z_3, z_2)K_2(z_1, z_3)K_1(z_2)X(z_1)X(z_2)X(z_3)]$ . Thus

for the system  $H$  the third order system function becomes

$$H_3(z_1, z_2, z_3) = G_1(z_1 z_2 z_3) K_3(z_1, z_2, z_3) + \\ + 2G_2(z_1 z_3, z_2) K_2(z_1, z_3) K_1(z_2) + G_3(z_1, z_2, z_3) K_1(z_1) K_1(z_2) K_1(z_3). \quad (79)$$

It can be seen that the first order term of the cascaded system depends only on the first order terms of the constituting subsystems. The second order term of the overall system depends on the first and second order terms of the subsystems and the third order term of  $H$  results from the first, second and third order terms of  $G$  and  $K$ . It appears to be a general rule that an  $n$ th order term of the cascade connection contains only terms of order  $n$  and lower of the constituting subsystems [10]. As an interesting exercise it is left to the reader to derive the system functions of the second order system of Figure 5 from the constituting first order systems and the squarer.

Once the formulae for cascading are derived, the inversion can be treated as a special case thereof. If the action of a system is, in a symbolic way, denoted by  $K$  and the action of a cascade connection by (see Figure 6)

$$H = GK, \quad (80)$$

then the inverse of a system  $K$  is denoted by  $K^{-1}$  and defined as

$$K^{-1}K = 1. \quad (81)$$

So the formulae for inversion are found by taking (75) equal to unity and (77) and (79) equal to zero and by solving the resulting equations for  $G$ .

This leads to the inverse operators  $G=K^{-1}$  as follows

$$G_1(z) = \frac{1}{K_1(z)}, \\ G_2(z_1, z_2) = \frac{-K_2(z_1, z_2)}{K_1(z_1)K_1(z_2)K_1(z_1 z_2)},$$

$$G_3(z_1, z_2, z_3) = \frac{-K_3(z_1, z_2, z_3)K_1(z_1 z_3) + 2K_2(z_1, z_3)K_2(z_1 z_3, z_2)}{K_1(z_1)K_1(z_2)K_1(z_3)K_1(z_1 z_3)K_1(z_1 z_2 z_3)}. \quad (82)$$

We want to emphasize that none of these inverse system functions exists if the linear term  $K_1(z)$  equals zero. It has been shown [10] and it is readily verified that the pre-inverses of a system are identical to the post-inverses.

#### XI. Measurement of the Volterra Kernels

In the preceding sections it was assumed that the multidimensional impulse response functions, or Volterra kernels,  $\{h_i(m_1, \dots, m_i)\}$  were known. One can imagine situations where these functions are unknown. In these cases one would like to have a measuring procedure to find the kernels. Eykhoff [11] and Alper [12] gave a method based on excitation of the system by a stochastic process and the kernels follow from correlation measurements.

Because of the theoretical and practical difficulties of this procedure we shall discuss here a method based on the direct measurement of responses at deterministic input signals. The method is developed by Schetzen [13] for nonlinear continuous systems, but can, in a complete similar way, be applied to nonlinear discrete systems as is shown in the sequel.

The response of a system at an excitation by an input sequence  $\{x(m)\}$  is denoted by  $\{y(m)_x\}$ . Let us consider a power system of the second order. Suppose that the system is excited by the sum of two input sequences  $\{x_1(m)\}$  and  $\{x_2(m)\}$ . Then it follows from (17)

$$\begin{aligned} y_2(m)_{x_1+x_2} &= \sum_i \sum_j h_2(i, j) [x_1(m-i) + x_2(m-i)] [x_1(m-j) + x_2(m-j)] \\ &= \sum_i \sum_j h_2(i, j) x_1(m-i) x_1(m-j) + \sum_i \sum_j h_2(i, j) x_2(m-i) x_2(m-j) + \\ &\quad + 2 \sum_i \sum_j h_2(i, j) x_1(m-i) x_2(m-j), \end{aligned} \quad (83)$$

where in the last term use has been made of the fact that  $h_2(i, j)$  is symmetric in  $i$  and  $j$ . Equation (83) is rewritten as

$$y_2^{(m)}(x_1 + x_2) = y_2^{(m)}(x_1) + y_2^{(m)}(x_2) + 2y_2^{(m)}(x_1 x_2), \quad (84)$$

where

$$y_2^{(m)}(x_1 x_2) \triangleq \sum_i \sum_j h_2(i, j) x_1^{(m-i)} x_2^{(m-j)}. \quad (85)$$

For  $\{x_1(m)\}$  and  $\{x_2(m)\}$  we take respectively the unit impulse sequences  $\delta_{m0}$  and  $\delta_{mn}$ , where  $\delta_{mn}$  is the Kronecker delta function

$$\delta_{mn} \triangleq \begin{cases} 1 & \text{for } m = n \\ 0 & \text{for } m \neq n \end{cases}. \quad (86)$$

From (85) it follows

$$y_2^{(m)}(\delta_{m0} \delta_{mn}) = h_2(m, m-n). \quad (87)$$

Substituting (87) into (84) yields

$$h_2(m, m-n) = \frac{1}{2} \{ y_2^{(m)}(\delta_{m0} + \delta_{mn}) - y_2^{(m)}(\delta_{m0}) - y_2^{(m)}(\delta_{mn}) \}. \quad (88)$$

Now the measuring procedure is clear and proceeds into the following steps.

- 1) Excite the system by the sequence  $\delta_{m0} + \delta_{mn}$  and measure the response.
- 2) Excite the system by the sequence  $\delta_{m0}$  and measure the response.
- 3) Calculate from the result of step 2) the response at the excitation by  $\delta_{mn}$ . Because the system is assumed to be time-invariant this is only a shift over an  $n$  units delay.
- 4) Subtract the responses of steps 2) and 3) from the response of step 1).
- 5) The second order kernel  $h_2(m, m-n)$  is found after dividing the result of step 4) by two.

Of course the procedure has to be repeated for all relevant values of  $n$  to

determine the second order kernel over the whole range of integers  $i$  and  $j$ . The method as given, results into  $h_2(m, m-n)$ , i.e. the values of  $h_2(i, j)$  that are on the straight line  $j=i-n$  in the  $\{i, j\}$ -plane. For the full dimensioning of the kernel it suffices to determine the values on the diagonal and the upper left or lower right half of the  $\{i, j\}$ -plane. This because of the symmetry of  $h_2(i, j)$ .

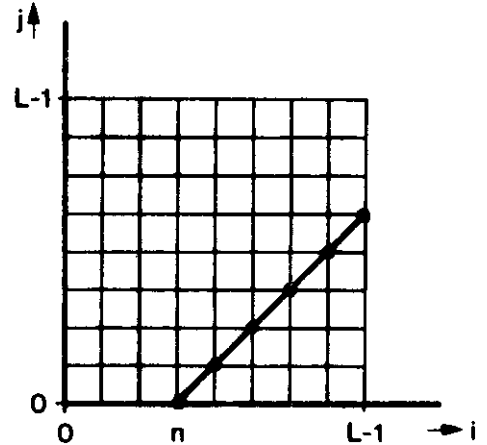


Figure 7 : The range of  $h(i, j)$  in the  $\{i, j\}$ -plane and the points on the line  $j=i-n$ .

In Figure 7 the range of  $i$  and  $j$  of a system with finite duration of the two-dimensional impulse response of length  $L$  is depicted, together with the points on the line  $j=i-n$ .

If we are dealing with a second order polynomial system, the procedure as described above will still result into the second order kernel  $h_2(i, j)$ . This can readily be seen, since from (20) it follows

$$y_1^{(m)}(x_1+x_2) - y_1^{(m)}(x_1) - y_1^{(m)}(x_2) = 0 \quad (89)$$

The first order kernel of a second order polynomial system is obtained by establishing the impulse response of such a system being  $h_1(m) + h_2(m, m)$ ; once  $h_2(i, j)$  is known,  $h_1(i)$  is found by subtracting  $h_2(m, m)$  from the impulse response.

The given method is also valid for higher order systems. To obtain  $y_n^{(m)}(x_1 \dots x_n)$  use is made of the identity

$$\begin{aligned}
 n!x_1 \dots x_n &= (x_1 + \dots + x_n)^n - [(x_1 + \dots + x_{n-1})^n + \dots] + \\
 &+ [(x_1 + \dots + x_{n-2})^n + \dots] + (-1)^{n-1} [x_1^n + \dots + x_n^n] \quad (90)
 \end{aligned}$$

The first term of this expression consists of the sum of all  $x$ 's raised to the power  $n$ . In the second term, each term is the sum of  $(n-1)$  different  $x$ 's raised to the power  $n$ . There are  $\binom{n}{n-1}$  terms of this type. Each term in the third term of (90) is the sum of  $(n-2)$  different  $x$ 's raised to the power  $n$ . There are  $\binom{n}{n-2}$  such terms, etc. for all terms in (90). In such a way the function  $h_n(i_1, \dots, i_n)$  is found, since the contribution of kernels of order less than  $n$  is zero. This follows from (23) and

$$\begin{aligned}
 (x_1 + \dots + x_n)^i - [(x_1 + \dots + x_{n-1})^i + \dots] + \dots + (-1)^{n-1} [x_1^n + \dots + x_n^n] &= 0 \\
 \text{for } i = 1, \dots, n-1. & \quad (91)
 \end{aligned}$$

The validity of (91) is not shown here, but is found in [13]. A system of order  $n$  is fully determined by starting with the measurement of  $h_n(i_1, \dots, i_n)$ . Then the measurement can proceed for  $h_{n-1}(i_1, \dots, i_{n-1})$  by subtracting the contribution due to  $h_n$ , etc. In this way all kernels are successively measured, starting with the kernel of the highest order.

### XII Synthesis

The synthesis of circuits is a problem that is more difficult to solve than the analysis. This statement holds especially for the most general case. From the analysis it is most times possible to derive synthesis methods for specific cases. In this section we shall deal with two such cases. The methods will be outlined for second order power systems and since the generalization to higher order is quite straightforward, this is being left to the reader.

For the first case we repeat Equation (49) here



$$H_2(z_1, z_2) = B(z_1 z_2) A(z_1) A(z_2) . \quad (92)$$

From Figure 5 it follows that a system with this kind of transfer function can be represented as two linear systems, separated by a squaring device as depicted in Figure 8.

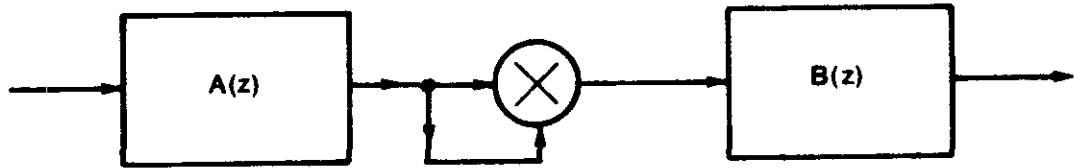


Figure 8 : Synthesis of a second order system, whose system function is decomposed as  $B(z_1 z_2) A(z_1) A(z_2)$ .

From this it follows that a second order function that can be decomposed like (92), is synthesized as given by Figure 8, namely a linear system  $A(z)$  followed by a squarer and this followed by a second linear system  $B(z)$ .

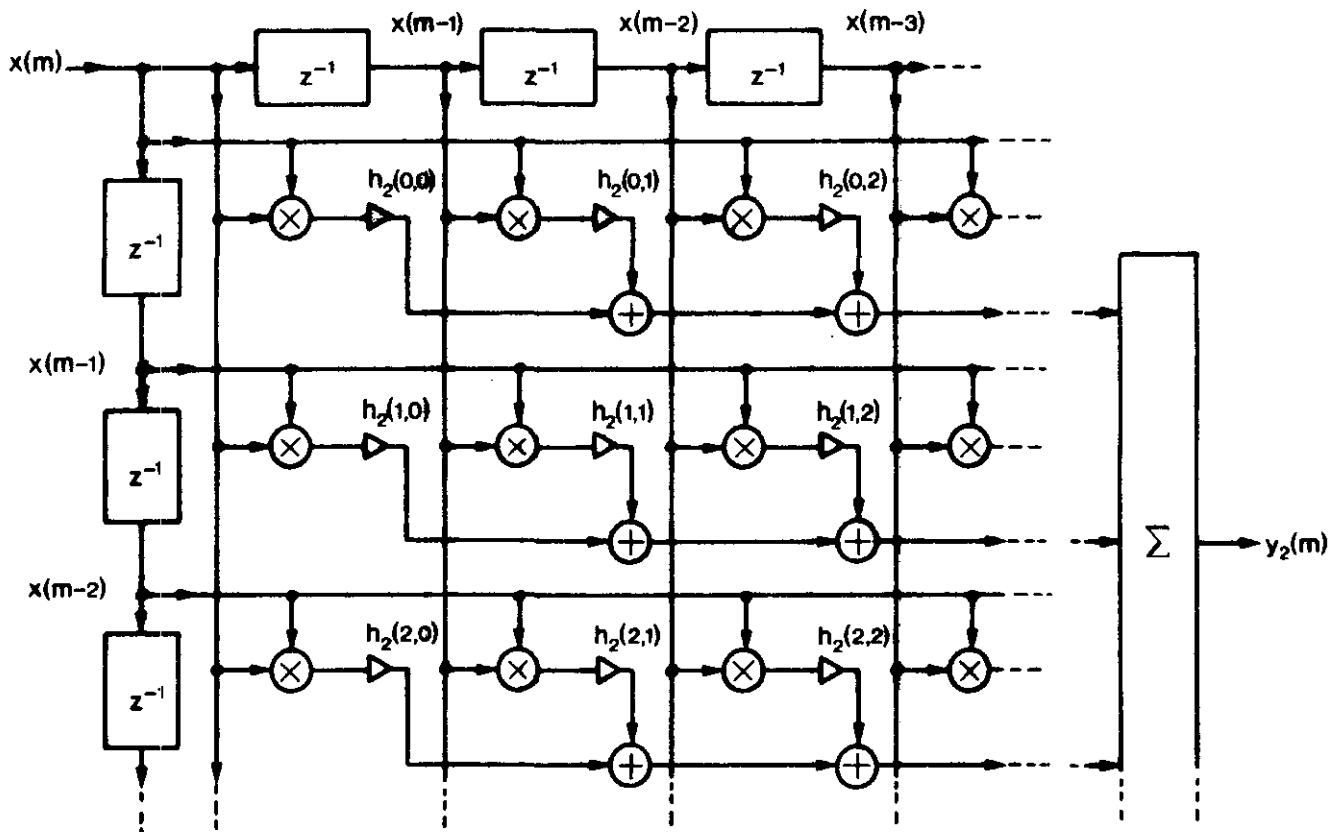


Figure 9 : Synthesis of a second order kernel.

The second method is more general; its only limitation being that it leads to realizations with a finite impulse response (FIR). To show the method, Equation (17) is repeated here

$$y_2(m) = \sum_i \sum_j h_2(i,j)x(m-i)x(m-j) \quad . \quad (93)$$

It is easy to verify that the circuit of Figure 9 realizes this function. The boxes with the label ' $z^{-1}$ ', represent delay devices, which give one unit delay.

The scheme is further simplified by using the symmetry property of  $h_2(i,j)$ . The lower left or upper right part of the weighting coefficients and multipliers is then omitted. The diagonal elements remain unchanged, but the other weighting coefficients get twice the value of the scheme of

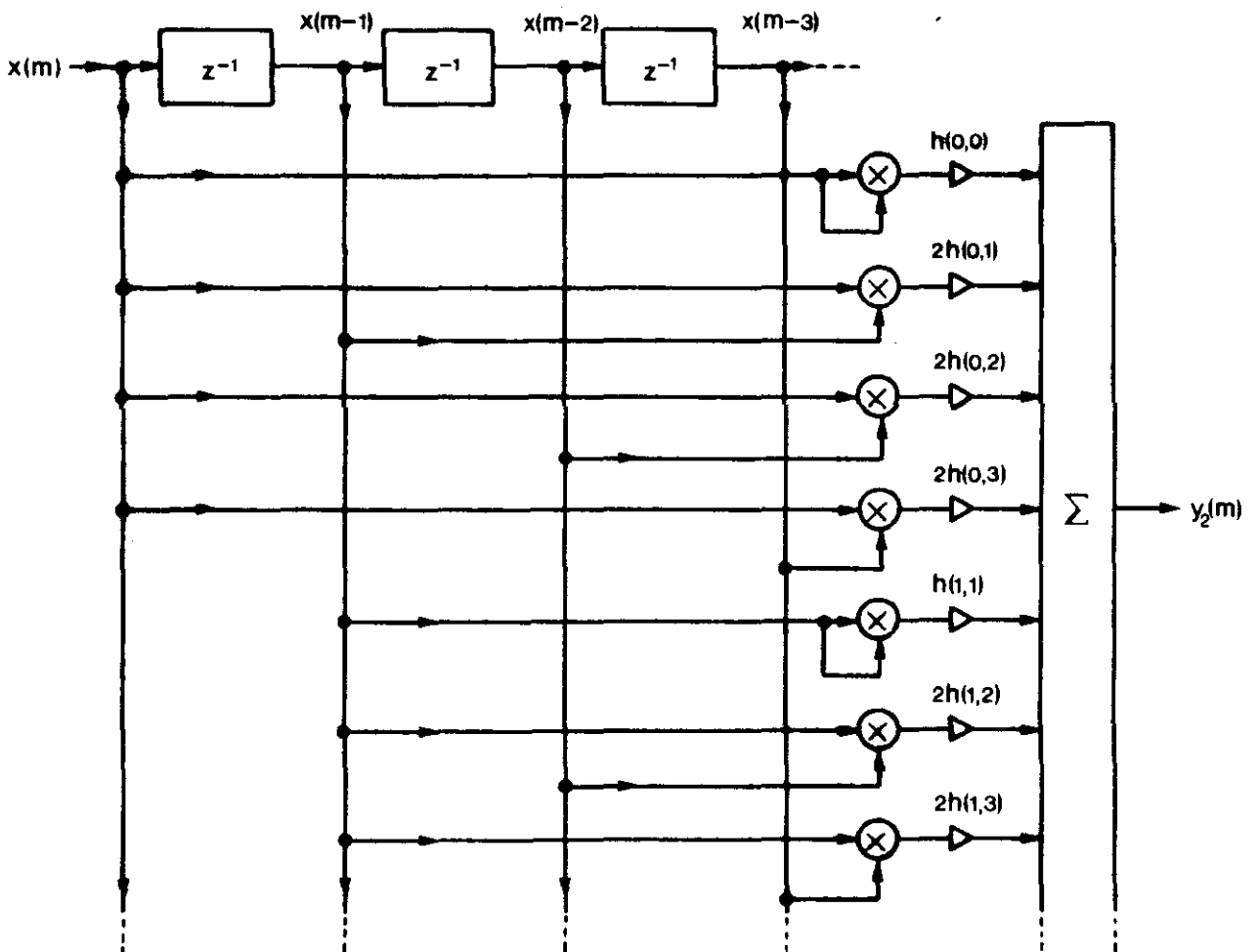


Figure 10 : The simplified realization scheme of a second order kernel.

Figure 8. In fact, one delay line suffices, because the information of the second delay line is also present in the first one. We then get the scheme of Figure 10.

The limitation of finite impulse response is rather theoretical than practical.

In Section VI it was shown that for stable systems

$$\lim_{i,j \rightarrow \infty} h(i,j) = 0 \quad . \quad (94)$$

So in practical situations the system is approximated by a FIR system of a satisfactory length.

### XIII. The equalization of a nonlinear digital communication channel

An important application of the theory given in this report is the equalization of nonlinear digital communication channels. Such channels can be considered as discrete-time (sampled data) systems. We implemented a system as given in Fig. 11.

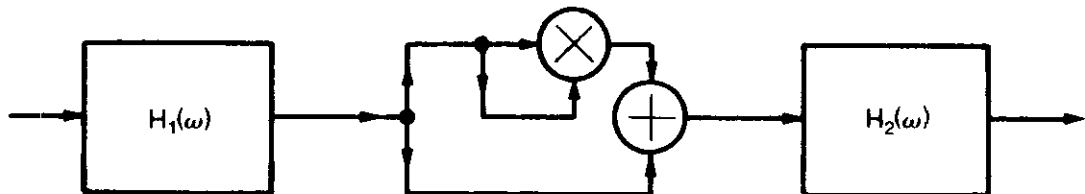


Figure 11: The simulated communication channel.

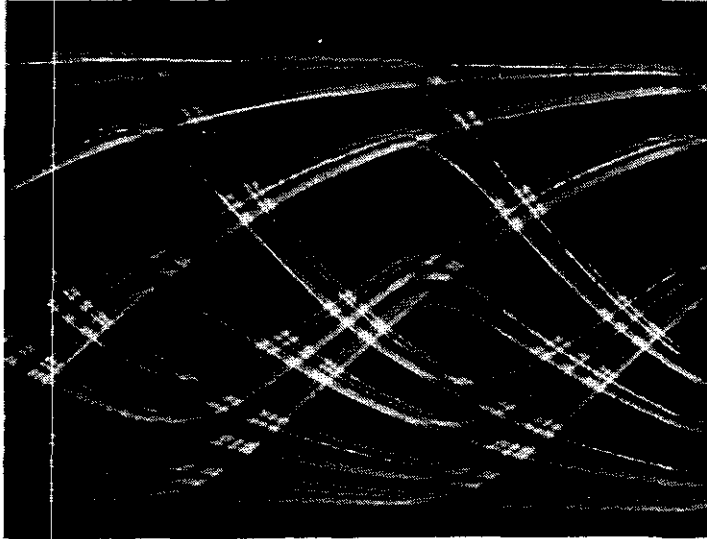
The input of this system is driven by a binary random sequence of rectangular pulses of duration 80  $\mu$ sec at a rate of 12.5 kb/s. and the output is synchronously sampled. Because the linear transfer functions  $H_1(\omega)$  and  $H_2(\omega)$  are not well defined the time-discrete transfer functions of the overall system are measured by means of the method given in

Section XI. The results are

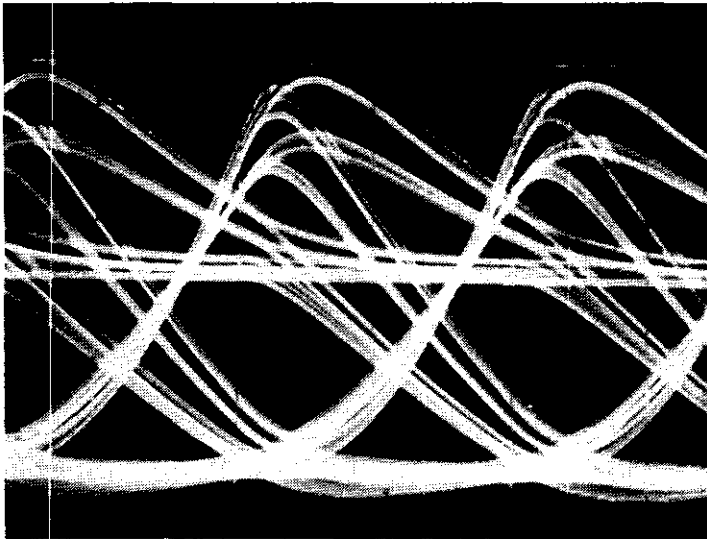
$$\begin{aligned}
 G_1(z) &= 0 \cdot z^0 + 0.168z^{-1} + 0.12z^{-2} + 0.048z^{-3} + 0.018z^{-4} + 0.008z^{-5}, \\
 G_2(z_1, z_2) &= 0.0448z_1^{-1}z_2^{-1} + 0.0264z_1^{-2}z_2^{-2} + 0.01z_1^{-3}z_2^{-3} + 0.004z_1^{-4}z_2^{-4} \\
 &\quad + 0.0016z_1^{-5}z_2^{-5} + 0.0156z_1^{-2}z_2^{-1} + 0.0058z_1^{-3}z_2^{-2} \\
 &\quad + 0.0018z_1^{-4}z_2^{-3} + 0.0008z_1^{-5}z_2^{-4} + 0.0156z_1^{-1}z_2^{-2} \\
 &\quad + 0.0058z_1^{-2}z_2^{-3} + 0.0018z_1^{-3}z_2^{-4} + 0.0008z_1^{-4}z_2^{-5}. \quad (95)
 \end{aligned}$$

Due to the dispersion of the system the several information carrying pulses overlap at the output of the system. This phenomenon is called intersymbol interference. Removing this disturbance is called equalization and is reached by cascading the system with another system such that the cascade connection has a transfer function equal to unity. In fact this means that we have to look for the inverse system as has been treated in Section X. For the system functions given by (95) the inverses  $K_1(z)$ ,  $K_2(z_1, z_2)$  and  $K_3(z_1, z_2, z_3)$  according to (82) are calculated. The inverse system has been realized by means of a four stage shift register, a set of "and"-gates forming the multipliers and a resistor matrix, forming the weighting coefficients. This network is used as a pre-inverse system. The quality of the equalization is shown in Figure 12, where the eye pattern [14] of the equalized system is given together with the unequalized one. As an intermediate result the eye pattern of the system with the linear equalization is also shown.

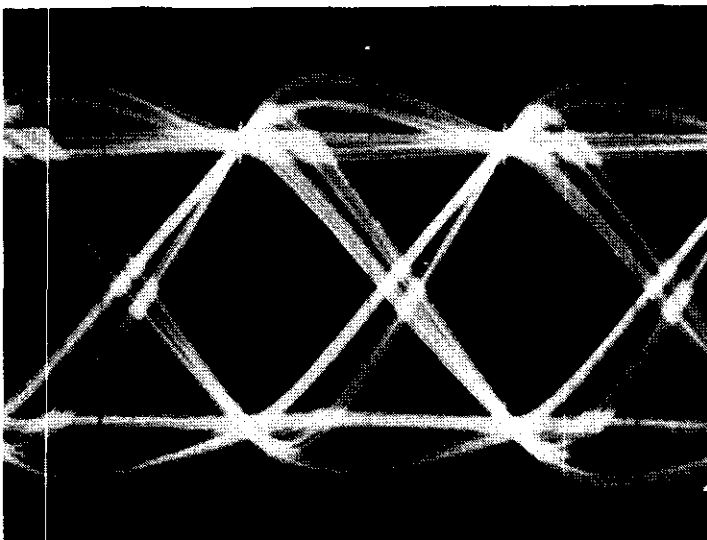
It is seen that the nonlinear functions  $K_2(z_1, z_2)$  and  $K_3(z_1, z_2, z_3)$  give a substantial contribution to the equalization. For a detailed description of the system and the equalization we refer to [15].



a - The eye pattern of the unequalized system.



b - The eye pattern in the case of linear equalization.



c - The eye pattern of the equalized system.

Figure 12

XIV. Concluding remarks

It has been shown that many aspects of nonlinear continuous systems can be treated in a similar way for nonlinear discrete-time systems. Not all questions have been answered and the treatment is by no means complete; for instance, neither the problems of stochastic input signals have been dealt with nor has the question of the convergence of the Volterra series been considered.

The purpose of this report is to gather the introductory material on nonlinear discrete-time systems as far as it is indispensable for the equalization of nonlinear digital communication channels. In [16] an extensive bibliography is found, which can serve as a guideline for further reading.

From the reported experiments it follows that the theory as treated in this report is sufficiently and serves quite well to be able to equalize a nonlinear communication channel.

References

1. Papoulis, A.,  
The Fourier integral and its applications,  
New York: McGraw-Hill, 1962.
2. Rabiner, L.R. and Gold, B.,  
Theory and application of digital signal processing,  
Englewood Cliffs, New Jersey: Prentice Hall Inc., 1975.
3. George, D.A.,  
Continuous nonlinear systems,  
Technical Report no.335, RLE, MIT, Cambridge, Mass., 1959.
4. Butterweck, H.J.,  
Frequenzabhängige nichtlineare Übertragungssysteme,  
Archiv der Elektrischen Übertragung, Vol.21, no.5, 1967, pp. 239 - 254.
5. Reddy, D.C. and Jagan, N.C.,  
Multidimensional transforms : new technique for the association of  
variables,  
Electronics Letters, Vol.7, no.10, May 20 1971, pp. 278 - 279.
6. Alper, P.,  
Higher-dimensional z-transforms and non-linear discrete systems,  
Revue A, Vol. VI, no.4, 1964, pp. 199 - 212.
7. Barker, H.A. and Ambati S.,  
Nonlinear sampled-data system analysis by multidimensional z-transforms,  
Proc. I.E.E., Vol. 119, no.9, September 1972, pp. 1407 - 1413.

8. Whittaker, E.T. and Watson, G.N.,  
A course of modern analysis,  
Cambridge : Cambridge University Press, 1969.
  
9. Jagan, N.C. and Reddy, D.C.,  
Evaluation of response of nonlinear discrete systems using multidimensional  
z-transforms,  
Proc. I.E.E., Vol. 119, no.10, October 1972, pp. 1521 - 1525.
  
10. Schetzen, M.,  
Theory of p th-order inverses of nonlinear systems,  
I.E.E.E. Trans. on Circuits and Systems, Vol. CAS-23, no.5, May 1976,  
pp. 285 - 291.
  
11. Eykhoff, P.,  
Some fundamental aspects of process-parameter estimation,  
I.E.E.E. Trans. on Automatic Control, Vol. AC-8, October 1963, pp. 347 -  
357.
  
12. Alper, P.,  
A consideration of the discrete Volterra series,  
I.E.E.E. Trans. on Automatic Control, Vol. AC-10, July 1965, pp. 322 - 327.
  
13. Schetzen, M.,  
Measurement of the kernels of a non-linear system of finite order,  
International Journal of Control, Vol.1, no.3, 1965, pp. 251 - 263.



14. Lucky, R.W., Salz, J. and Weldon, E.J.,  
Principles of data communication,  
New York: McGraw-Hill, 1968.
  
15. van Vught, F.J.,  
Optimale ontvangers voor datareeksen die verzonden zijn over niet-  
lineaire kanalen (in Dutch),  
Master thesis Eindhoven University of Technology, Department of  
Electrical Engineering, August 1978.
  
16. Barrett, J.F.,  
Bibliography on Volterra series Hermite functional expansions and  
related subjects,  
TH-Report 77-E-71, Eindhoven University of Technology, Netherlands,  
March 1977.

**EINDHOVEN UNIVERSITY OF TECHNOLOGY  
THE NETHERLANDS  
DEPARTMENT OF ELECTRICAL ENGINEERING**

**Reports:**

- 1) **Dijk, J., M. Jeuken and E.J. Maanders**  
AN ANTENNA FOR A SATELLITE COMMUNICATION GROUND STATION  
(PROVISIONAL ELECTRICAL DESIGN).  
TH-Report 68-E-01. 1968. ISBN 90-6144-001-7
- 2) **Veefkind, A., J.H. Blom and L.H.Th. Rietjens**  
THEORETICAL AND EXPERIMENTAL INVESTIGATION OF A NON-EQUILIBRIUM  
PLASMA IN A MHD CHANNEL. Submitted to the Symposium on Magnetohydrodynamic  
Electrical Power Generation, Warsaw, Poland, 24-30 July, 1968.  
TH-Report 68-E-02. 1968. ISBN 90-6144-002-5
- 3) **Boom, A.J.W. van den and J.H.A.M. Melis**  
A COMPARISON OF SOME PROCESS PARAMETER ESTIMATING SCHEMES.  
TH-Report 68-E-03. 1968. ISBN 90-6144-003-3
- 4) **Eykhoff, P., P.J.M. Ophay, J. Severs and J.O.M. Oome**  
AN ELECTROLYTIC TANK FOR INSTRUCTIONAL PURPOSES REPRESENTING THE  
COMPLEX-FREQUENCY PLANE.  
TH-Report 68-E-02. 1968. ISBN 90-6144-004-1
- 5) **Vermij, L. and J.E. Daalder**  
ENERGY BALANCE OF FUSING SILVER WIRES SURROUNDED BY AIR.  
TH-Report 68-E-05. 1968. ISBN 90-6144-005-X
- 6) **Houben, J.W.M.A. and P. Masee**  
MHD POWER CONVERSION EMPLOYING LIQUID METALS.  
TH-Report 69-E-06. 1969. ISBN 90-6144-006-8
- 7) **Heuvel, W.M.C. van den and W.F.J. Kersten**  
VOLTAGE MEASUREMENT IN CURRENT ZERO INVESTIGATIONS.  
TH-Report 69-E-07. 1969. ISBN 90-6144-007-6
- 8) **Vermij, L.**  
SELECTED BIBLIOGRAPHY OF FUSES.  
TH-Report 69-E-08. 1969. ISBN 90-6144-008-4
- 9) **Westenberg, J.Z.**  
SOME IDENTIFICATION SCHEMES FOR NON-LINEAR NOISY PROCESSES.  
TH-Report 69-E-09. 1969. ISBN 90-6144-009-2
- 10) **Koop, H.E.M., J. Dijk and E.J. Maanders**  
ON CONICAL HORN ANTENNAS.  
TH-Report 70-E-10. 1970. ISBN 90-6144-010-6
- 11) **Veefkind, A.**  
NON-EQUILIBRIUM PHENOMENA IN A DISC-SHAPED MAGNETOHYDRODYNAMIC  
GENERATOR.  
TH-Report 70-E-11. 1970. ISBN 90-6144-011-4
- 12) **Jansen, J.K.M., M.E.J. Jeuken and C.W. Lamrechtse**  
THE SCALAR FEED.  
TH-Report 70-E-12. 1969. ISBN 90-6144-012-2
- 13) **Teuling, D.J.A.**  
ELECTRONIC IMAGE MOTION COMPENSATION IN A PORTABLE TELEVISION CAMERA.  
TH-Report 70-E-13. 1970. ISBN 90-6144-013-0

**EINDHOVEN UNIVERSITY OF TECHNOLOGY  
THE NETHERLANDS  
DEPARTMENT OF ELECTRICAL ENGINEERING**

**Reports:**

- 14) **Lorencin, M.**  
AUTOMATIC METEOR REFLECTIONS RECORDING EQUIPMENT.  
TH-Report 70-E-14. 1970. ISBN 90-6144-014-9
- 15) **Smets, A.S.**  
THE INSTRUMENTAL VARIABLE METHOD AND RELATED IDENTIFICATION SCHEMES.  
TH-Report 70-E-15. 1970. ISBN 90-6144-015-7
- 16) **White, Jr., R.C.**  
A SURVEY OF RANDOM METHODS FOR PARAMETER OPTIMIZATION.  
TH-Report 70-E-16. 1971. ISBN 90-6144-016-5
- 17) **Talmon, J.L.**  
APPROXIMATED GAUSS-MARKOV ESTIMATORS AND RELATED SCHEMES.  
TH-Report 71-E-17. 1971. ISBN 90-6144-017-3
- 18) **Kalásek, V.**  
MEASUREMENT OF TIME CONSTANTS ON CASCADE D.C. ARC IN NITROGEN.  
TH-Report 71-E-18. 1971. ISBN 90-6144-018-1
- 19) **Hosselet, L.M.L.F.**  
OZONBILDUNG MITTELS ELEKTRISCHER ENTLADUNGEN.  
TH-Report 71-E-19. 1971. ISBN 90-6144-019-X
- 20) **Arts, M.G.J.**  
ON THE INSTANTANEOUS MEASUREMENT OF BLOODFLOW BY ULTRASONIC MEANS.  
TH-Report 71-E-20. 1971. ISBN 90-6144-020-3
- 21) **Roer, Th.G. van de**  
NON-ISO THERMAL ANALYSIS OF CARRIER WAVES IN A SEMICONDUCTOR.  
TH-Report 71-E-21. 1971. ISBN 90-6144-021-1
- 22) **Jeulen, P.J., C. Huber and C.E. Mulders**  
SENSING INERTIAL ROTATION WITH TUNING FORKS.  
TH-Report 71-E-22. 1971. ISBN 90-6144-022-X
- 23) **Dijk, J., J.M. Berendís and E.J. Maanders**  
APERTURE BLOCKAGE IN DUAL REFLECTOR ANTENNA SYSTEMS - A REVIEW.  
TH-Report 71-E-23. 1971. ISBN 90-6144-023-8
- 24) **Kregting, J. and R.C. White, Jr.**  
ADAPTIVE RANDOM SEARCH.  
TH-Report 71-E-24. 1971. ISBN 90-6144-024-6
- 25) **Damen, A.A.H. and H.A.L. Piceni**  
THE MULTIPLE DIPOLE MODEL OF THE VENTRICULAR DEPolarISATION.  
TH-Report 71-E-25. 1971. ISBN 90-6144-025-4
- 26) **Bremmer, H.**  
A MATHEMATICAL THEORY CONNECTING SCATTERING AND DIFFRACTION  
PHENOMENA, INCLUDING BRAGG-TYPE INTERFERENCES.  
TH-Report 71-E-26. 1971. ISBN 90-6144-026-2
- 27) **Bokhoven, W.M.G. van**  
METHODS AND ASPECTS OF ACTIVE RC-FILTERS SYNTHESIS.  
TH-Report 71-E-27. 1970. ISBN 90-6144-027-0
- 28) **Boeschoten, F.**  
TWO FLUIDS MODEL REEXAMINED FOR A COLLISIONLESS PLASMA IN THE  
STATIONARY STATE.  
TH-Report 72-E-28. 1972. ISBN 90-6144-028-9

**EINDHOVEN UNIVERSITY OF TECHNOLOGY  
THE NETHERLANDS  
DEPARTMENT OF ELECTRICAL ENGINEERING**

**Reports:**

- 29) **REPORT ON THE CLOSED CYCLE MHD SPECIALIST MEETING.** Working group of the joint ENEA/IAEA International MHD Liaison Group. Eindhoven, The Netherlands, September 20-22, 1971. Edited by **L.H.Th. Rietjens**. TH-Report 72-E-29. 1972. ISBN 90-6144-029-7
- 30) **Kessel, C.G.M. van and J.W.M.A. Houben**  
**LOSS MECHANISMS IN AN MHD GENERATOR.**  
TH-Report 72-E-30. 1972. ISBN 90-6144-030-0
- 31) **Veefkind, A.**  
**CONDUCTION GRIDS TO STABILIZE MHD GENERATOR PLASMAS AGAINST IONIZATION INSTABILITIES.**  
TH Report 72-E-31. 1972. ISBN 90-6144-031-9
- 32) **Daalder, J.E., and C.W.M. Vos**  
**DISTRIBUTION FUNCTIONS OF THE SPOT DIAMETER FOR SINGLE- AND MULTI-CATHODE DISCHARGES IN VACUUM.**  
TH-Report 73-E-32. 1973. ISBN 90-6144-032-7
- 33) **Daalder, J.E.**  
**JOULE HEATING AND DIAMETER OF THE CATHODE SPOT IN A VACUUM ARC.**  
TH-Report 73-E-33. 1973. ISBN 90-6144-033-5
- 34) **Huber, C.**  
**BEHAVIOUR OF THE SPINNING GYRO ROTOR.**  
TH-Report 73-E-34. 1973. ISBN 90-6144-034-3
- 35) **Bastian, C. et al.**  
**THE VACUUM ARC AS A FACILITY FOR RELEVANT EXPERIMENTS IN FUSION RESEARCH.** Annual Report 1972. EURATOM-T.H.E. Group 'Rotating Plasma'. TH-Report 73-E-35. 1973. ISBN 90-6144-035-1
- 36) **Blom, J.A.**  
**ANALYSIS OF PHYSIOLOGICAL SYSTEMS BY PARAMETER ESTIMATION TECHNIQUES.**  
TH-Report 73-E-36. 1973. ISBN 90-6144-036-X
- 37) **Cancelled**
- 38) **Andriessen, F.J., W. Boerman and I.F.E.M. Holtz**  
**CALCULATION OF RADIATION LOSSES IN CYLINDER SYMMETRIC HIGH PRESSURE DISCHARGES BY MEANS OF A DIGITAL COMPUTER.**  
TH-Report 73-E-38. 1973. ISBN 90-6144-038-6
- 39) **Dijk, J., C.T.W. van Diepenbeek, E.J. Maanders and L.F.G. Thurlings**  
**THE POLARIZATION LOSSES OF OFFSET ANTENNAS.**  
TH-Report 73-E-39. 1973. ISBN 90-6144-039-4
- 40) **Goes, W.P.**  
**SEPARATION OF SIGNALS DUE TO ARTERIAL AND VENOUS BLOOD FLOW IN THE DOPPLER SYSTEM THAT USES CONTINUOUS ULTRASOUND.**  
TH-Report 73-E-40. 1973. ISBN 90-6144-040-8
- 41) **Damen, A.A.H.**  
**A COMPARATIVE ANALYSIS OF SEVERAL MODELS OF THE VENTRICULAR DEPOLARIZATION; INTRODUCTION OF A STRING-MODEL.**  
TH-Report 73-E-41. 1973. ISBN 90-6144-041-6

EINDHOVEN UNIVERSITY OF TECHNOLOGY  
THE NETHERLANDS  
DEPARTMENT OF ELECTRICAL ENGINEERING

Reports:

- 42) Dijk, G.H.M. van  
THEORY OF GYRO WITH ROTATING GIMBAL AND FLEXURAL PIVOTS.  
TH-Report 73-E-42. 1973. ISBN 90-6144-042-4
- 43) Breimer, A.J.  
ON THE IDENTIFICATION OF CONTINUOUS LINEAR PROCESSES.  
TH-Report 74-E-43. 1974. ISBN 90-6144-043-2
- 44) Lier, M.C. van and R.H.J.M. Otten  
CAD OF MASKS AND WIRING.  
TH-Report 74-E-44. 1974. ISBN 90-6144-044-0
- 45) Bastian, C. et al.  
EXPERIMENTS WITH A LARGE SIZED HOLLOW CATHODE DISCHARGE FED WITH ARGON. Annual Report 1973. EURATOM-T.H.E. Group 'Rotating Plasma'.  
TH-Report 74-E-45. 1974. ISBN 90-6144-045-9
- 46) Roer, Th.G. van de  
ANALYTICAL SMALL-SIGNAL THEORY OF BARITT DIODES.  
TH-Report 74-E-46. 1974. ISBN 90-6144-046-7
- 47) Leliveld, W.H.  
THE DESIGN OF A MOCK CIRCULATION SYSTEM.  
TH-Report 74-E-47. 1974. ISBN 90-6144-047-5
- 48) Damen, A.A.H.  
SOME NOTES ON THE INVERSE PROBLEM IN ELECTRO CARDIOGRAPHY.  
TH-Report 74-E-48. 1974. ISBN 90-6144-048-3
- 49) Meeberg, L. van de  
A VITERBI DECODER.  
TH-Report 74-E-49. 1974. ISBN 90-6144-049-1
- 50) Poel, A.P.M. van der  
A COMPUTER SEARCH FOR GOOD CONVOLUTIONAL CODES.  
TH-Report 74-E-50. 1974. ISBN 90-6144-050-5
- 51) Sampic, G.  
THE BIT ERROR PROBABILITY AS A FUNCTION PATH REGISTER LENGTH IN THE VITERBI DECODER.  
TH-Report 74-E-51. 1974. ISBN 90-6144-051-3
- 52) Schalkwijk, J.P.M.  
CODING FOR A COMPUTER NETWORK.  
TH-Report 74-E-52. 1974. ISBN 90-6144-052-1
- 53) Stapper, M.  
MEASUREMENT OF THE INTENSITY OF PROGRESSIVE ULTRASONIC WAVES BY MEANS OF RAMAN-NATH DIFFRACTION.  
TH-Report 74-E-53. 1974. ISBN 90-6144-053-X
- 54) Schalkwijk, J.P.M. and A.J. Vinck  
SYNDROME DECODING OF CONVOLUTIONAL CODES.  
TH-Report 74-E-54. 1974. ISBN 90-6144-054-8
- 55) Yakimov, A.  
FLUCTUATIONS IN IMPATT-DIODE OSCILLATORS WITH LOW Q-FACTORS.  
TH-Report 74-E-55. 1974. ISBN 90-6144-055-6

EINDHOVEN UNIVERSITY OF TECHNOLOGY  
THE NETHERLANDS  
DEPARTMENT OF ELECTRICAL ENGINEERING

Reports:

- 56) **Plaats, J. van der**  
ANALYSIS OF THREE CONDUCTOR COAXIAL SYSTEMS. Computer-aided determination of the frequency characteristics and the impulse and step response of a two-port consisting of a system of three coaxial conductors terminating in lumped impedances.  
TH-Report 75-E-56. 1975. ISBN 90-6144-056-4
- 57) **Kalken, P.J.H. and C. Kooy**  
RAY-OPTICAL ANALYSIS OF A TWO DIMENSIONAL APERTURE RADIATION PROBLEM.  
TH-Report 75-E-57. 1975. ISBN 90-6144-057-2
- 58) **Schalkwijk, J.P.M., A.J. Vinck and L.J.A.E. Rust**  
ANALYSIS AND SIMULATION OF A SYNDROME DECODER FOR A CONSTRAINT LENGTH  $k = 5$ , RATE  $R = \frac{1}{2}$  BINARY CONVOLUTIONAL CODE.  
TH-Report 75-E-58. 1975. ISBN 90-6144-058-0.
- 59) **Boeschoten, F. et al.**  
EXPERIMENTS WITH A LARGE SIZED HOLLOW CATHODE DISCHARGE FED WITH ARGON, II. Annual Report 1974. EURATOM-T.H.E. Group 'Rotating Plasma'.  
TH-Report 75-E-59. 1975. ISBN 90-6144-059-9
- 60) **Maanders, E.J.**  
SOME ASPECTS OF GROUND STATION ANTENNAS FOR SATELLITE COMMUNICATION.  
TH-Report 75-E-60. 1975. ISBN 90-6144-060-2
- 61) **Mawira, A. and J. Dijk**  
DEPOLARIZATION BY RAIN: Some Related Thermal Emission Considerations.  
TH-Report 75-E-61. 1975. ISBN 90-6144-061-0
- 62) **Safak, M.**  
CALCULATION OF RADIATION PATTERNS OF REFLECTOR ANTENNAS BY HIGH-FREQUENCY ASYMPTOTIC TECHNIQUES.  
TH-Report 76-E-62. 1976. ISBN 90-6144-062-9
- 63) **Schalkwijk, J.P.M. and A.J. Vinck**  
SOFT DECISION SYNDROME DECODING.  
TH-Report 76-E-63. 1976. ISBN 90-6144-063-7
- 64) **Damen, A.A.H.**  
EPICARDIAL POTENTIALS DERIVED FROM SKIN POTENTIAL MEASUREMENTS.  
TH-Report 76-E-64. 1976. ISBN 90-6144-064-5
- 65) **Bakhuizen, A.J.C. and R. de Boer**  
ON THE CALCULATION OF PERMEANCES AND FORCES BETWEEN DOUBLY SLOTTED STRUCTURES.  
TH-Report 76-E-65. 1976. ISBN 90-6144-065-3
- 66) **Geutjes, A.J.**  
A NUMERICAL MODEL TO EVALUATE THE BEHAVIOUR OF A REGENERATIVE HEAT EXCHANGER AT HIGH TEMPERATURE.  
TH-Report 76-E-66. 1976. ISBN 90-6144-066-1
- 67) **Boeschoten, F. et al.**  
EXPERIMENTS WITH A LARGE SIZED HOLLOW CATHODE DISCHARGE, III; concluding work Jan. 1975 to June 1976 of the EURATOM-THE Group 'Rotating Plasma'.  
TH-Report 76-E-67. 1976. ISBN 90-6144-067-X
- 68) **Cancelled.**

EINDHOVEN UNIVERSITY OF TECHNOLOGY  
THE NETHERLANDS  
DEPARTMENT OF ELECTRICAL ENGINEERING

Reports:

- 69) Merck, W.F.H. and A.F.C. Sens  
THOMSON SCATTERING MEASUREMENTS ON A HOLLOW CATHODE DISCHARGE.  
TH-Report 76-E-69. 1976. ISBN 90-6144-069-6
- 70) Jongbloed, A.A.  
STATISTICAL REGRESSION AND DISPERSION RATIOS IN NONLINEAR SYSTEM  
IDENTIFICATION.  
TH-Report 77-E-70. 1977. ISBN 90-6144-070-X
- 71) Barrett, J.F.  
BIBLIOGRAPHY ON VOLTERRA SERIES HERMITE FUNCTIONAL EXPANSIONS AND  
RELATED SUBJECTS.  
TH-Report 77-E-71. 1977. ISBN 90-6144-071-8
- 72) Boeschoten, F. and R. Komen  
ON THE POSSIBILITY TO SEPARATE ISOTOPES BY MEANS OF A ROTATING PLASMA  
COLUMN: Isotope separation with a hollow cathode discharge.  
TH-Report 77-E-72. 1977. ISBN 90-6144-072-6
- 73) Schalkwijk, J.P.M., A.J. Vinck and K.A. Post  
SYNDROME DECODING OF BINARY RATE- $k/n$  CONVOLUTIONAL CODES.  
TH-Report 77-E-73. 1977. ISBN 90-6144-073-4
- 74) Dijk, J., E.J. Maanders and J.M.J. Oostvogels  
AN ANTENNA MOUNT FOR TRACKING GEOSTATIONARY SATELLITES.  
TH-Report 77-E-74. 1977. ISBN 90-6144-074-2
- 75) Vinck, A.J., J.G. van Wijk and A.J.P. de Paepe  
A NOTE ON THE FREE DISTANCE FOR CONVOLUTIONAL CODES.  
TH-Report 77-E-75. 1977. ISBN 90-6144-075-0
- 76) Daalder, J.E.  
RADIAL HEAT FLOW IN TWO COAXIAL CYLINDRICAL DISKS.  
TH-Report 77-E-76. 1977. ISBN 90-6144-076-9
- 77) Barrett, J.F.  
ON SYSTEMS DEFINED BY IMPLICIT ANALYTIC NONLINEAR FUNCTIONAL  
EQUATIONS.  
TH-Report 77-E-77. 1977. ISBN 90-6144-077-7
- 78) Jansen, J. and J.F. Barrett  
ON THE THEORY OF MAXIMUM LIKELIHOOD ESTIMATION OF STRUCTURAL  
RELATIONS. Part 1: One dimensional case.  
TH-Report 78-E-78. 1977. ISBN 90-6144-078-5
- 79) Borghi, C.A., A.F.C. Sens, A. Veefkind and L.H.Th. Rietjens  
EXPERIMENTAL INVESTIGATION ON THE DISCHARGE STRUCTURE IN A NOBLE  
GAS MHD GENERATOR.  
TH-Report 78-E-79. 1978. ISBN 90-6144-079-3
- 80) Bergmans, T.  
EQUALIZATION OF A COAXIAL CABLE FOR DIGITAL TRANSMISSION: Computer-  
optimized location of poles and zeros of a constant-resistance network to equalize a coaxial  
cable 1.2/4.4 for high-speed digital transmission (140 Mb/s).  
TH-Report 78-E-80. 1978. ISBN 90-6144-080-7

EINDHOVEN UNIVERSITY OF TECHNOLOGY  
THE NETHERLANDS  
DEPARTMENT OF ELECTRICAL ENGINEERING

Reports:

- 81) Kam, J.J. van der and A.A.H. Damen  
OBSERVABILITY OF ELECTRICAL HEART ACTIVITY STUDIES WITH THE SINGULAR  
VALUE DECOMPOSITION  
TH-Report 78-E-81. 1978. ISBN 90-6144-081-5
- 82) Jansen, J. and J.F. Barrett  
ON THE THEORY OF MAXIMUM LIKELIHOOD ESTIMATION OF STRUCTURAL  
RELATIONS. Part 2: Multi-dimensional case.  
TH-Report 78-E-82. 1978. ISBN 90-6144-082-3
- 83) Etten, W. van and E. de Jong  
OPTIMUM TAPPED DELAY LINES FOR THE EQUALIZATION OF MULTIPLE CHANNEL  
SYSTEMS.  
TH-Report 78-E-83. 1978. ISBN 90-6144-083-1
- 83) Vinck, A.J.  
MAXIMUM LIKELIHOOD SYNDROME DECODING OF LINEAR BLOCK CODES.  
TH-Report 78-E-84. 1978. ISBN 90-6144-084-X
- 85) Spruit, W.P.  
A DIGITAL LOW FREQUENCY SPECTRUM ANALYZER, USING A PROGRAMMABLE  
POCKET CALCULATOR.  
TH-Report 78-E-85. 1978. ISBN 90-6144-085-8
- 86) Beneken, J.E.W. et al  
TREND PREDICTION AS A BASIS FOR OPTIMAL THERAPY.  
TH-Report 78-E-86. 1978. ISBN 90-6144-086-6
- 87) Geus, C.A.M. and J. Dijk  
CALCULATION OF APERTURE AND FAR-FIELD DISTRIBUTION FROM MEASUREMENTS  
IN THE FRESNEL ZONE OF LARGE REFLECTOR ANTENNAS.  
TH-Report 78-E-87. 1978. ISBN 90-6144-087-4
- 88) Hajdasinski, A.K.  
THE GAUSS-MARKOV APPROXIMATED SCHEME FOR IDENTIFICATION OF  
MULTIVARIABLE DYNAMICAL SYSTEMS VIA THE REALIZATION THEORY.  
An Explicit Approach.  
TH-Report 78-E-88. 1978. ISBN 90-6144-088-2
- 89) Niederlinski, A.  
THE GLOBAL ERROR APPROACH TO THE CONVERGENCE OF CLOSED-LOOP  
IDENTIFICATION, SELF-TUNING REGULATORS AND SELF-TUNING PREDICTORS.  
TH-Report 78-E-89. 1978. ISBN 90-6144-089-0
- 90) Vinck, A.J. and A.J.P. de Paepe  
REDUCING THE NUMBER OF COMPUTATIONS IN STACK DECODING OF  
CONVOLUTIONAL CODES BY EXPLOITING SYMMETRIES OF THE ENCODER.  
TH-Report 78-E-90. 1978. ISBN 90-6144-090-4
- 91) Geutjes, A.J. and D.J. Kleyn  
A PARAMETRIC STUDY OF 1000 MWe COMBINED CLOSED CYCLE MHD/STEAM  
ELECTRICAL POWER GENERATING PLANTS.  
TH-Report 78-E-91. 1978. ISBN 90-6144-091-2
- 92) Masee, P.  
THE DISPERSION RELATION OF ELECTROTHERMAL WAVES IN A NONEQUILIBRIUM  
MHD PLASMA.  
TH-Report 78-E-92. 1978. ISBN 90-6144-092-0



EINDHOVEN UNIVERSITY OF TECHNOLOGY  
THE NETHERLANDS  
DEPARTMENT OF ELECTRICAL ENGINEERING

Reports:

- 93) Duin, C.A. van  
DIPOLE SCATTERING OF ELECTROMAGNETIC WAVES PROPAGATION THROUGH A RAIN MEDIUM. TH-Report 79-E-93. 1979. ISBN 90-6144-093-9
- 94) Kuijper, A.H. de and L.K.J. Vandamme  
CHARTS OF SPATIAL NOISE DISTRIBUTION IN PLANAR RESISTORS WITH FINITE CONTACTS. TH-Report 79-E-94. 1979. ISBN 90-6144-094-7
- 95) Hajdasinski, A.K. and A.A.H. Damen  
REALIZATION OF THE MARKOV PARAMETER SEQUENCES USING THE SINGULAR VALUE DECOMPOSITION OF THE HANKEL MATRIX. TH-Report 79-E-95. 1979. ISBN 90-6144-095-5
- 96) Stefanov, B.  
ELECTRON MOMENTUM TRANSFER CROSS-SECTION IN CESIUM AND RELATED CALCULATIONS OF THE LOCAL PARAMETERS OF Cs + Ar MHD PLASMAS. TH-Report 79-E-96. 1979. ISBN 90-6144-096-3
- 97) Worm, S.C.J.  
RADIATION PATTERNS OF CIRCULAR APERTURES WITH PRESCRIBED SIDELobe LEVELS. TH-Report 79-E-97. 1979. ISBN 90-6144-097-1
- 98) Kroezen, P.H.C.  
A SERIES REPRESENTATION METHOD FOR THE FAR FIELD OF AN OFFSET REFLECTOR ANTENNA. TH-Report 79-E-98. 1979. ISBN 90-6144-098-X
- 99) Koonen, A.M.J.  
ERROR PROBABILITY IN DIGITAL FIBER OPTIC COMMUNICATION SYSTEMS. TH-Report 79-E-99. 1979. ISBN 90-6144-099-8
- 100) Naidu, M.S.  
STUDIES ON THE DECAY OF SURFACE CHARGES ON DIELECTRICS. TH-Report 79-E-100. 1979. ISBN 90-6144-100-5
- 101) Verstappen, H.L.  
A SHAPED CYLINDRICAL DOUBLE-REFLECTOR SYSTEM FOR A BROADCAST-SATELLITE ANTENNA. TH-Report 79-E-101. 1979. ISBN 90-6144-101-3
- 102) Etten, W.C. van  
THE THEORY OF NONLINEAR DISCRETE-TIME SYSTEMS AND ITS APPLICATION TO THE EQUALIZATION OF NONLINEAR DIGITAL COMMUNICATION CHANNELS. TH-Report 79-E-102. 1979. ISBN 90-6144-102-1
- 103) Roer, Th.G. van de  
ANALYTICAL THEORY OF PUNCH-THROUGH DIODES. TH-Report 79-E-103. 1979. ISBN 90-6144-103-X