Gröbner bases and standard monomial theory

Gröbner Bases and Standard Monomial Theory

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Abstract. For rings of polynomials on varieties corresponding to minuscule weight representations of Lie groups, we show how the standard monomial theory of Seshadri et al. can be used to compute Gröbner bases. This generalizes results of Sturmfels and White in which straightening has been interpreted as a normal form computation with respect to a Gröbner basis.

keywords: Lie groups, minuscule weights, straightening, Gröbner bases

1. Introduction

The Grassmann varieties have been studied in various algorithmic ways. For instance, rectangular Young tableaux have proved to be useful for describing monomials in the ring of polynomial functions on a Grassmannian. The associated straightening procedure is an effective way of expressing such a monomial as a linear combination of certain monomials, called standard monomials. Moreover, the standard monomials form a vector space basis of the ring of polynomial functions on the Grassmannian.

The theory of Gröbner bases deals with the effective study of rings of polynomial functions on algebraic varieties in much greater generality. It entails the existence of a vector space basis consisting of special monomials, which – by fortunate coincidence? – are also called standard!

One would expect from the very nature of the Grassmannians that at least certain Gröbner bases for the ring of polynomial functions on them would be obtainable from the much older procedure of straightening. This in fact is true. The correspondence between Gröbner bases and straightening has been discussed in [StWh1]. Here we show that this correspondence naturally extends to the standard monomial theory for minuscule weight varieties as developed by Seshadri (see [Sesh]). Thereby we answer part of the question raised in [StWh1]. The combinatorics we need to obtain the reduction ordering, an ingredient to the Gröbner basis, is that of the Bruhat order on cosets of the Weyl group with respect to parabolic subgroups. This is dealt with in §4. The main result is to be found in §5, at the end of which an explicit example, related to E_6 in a 27-dimensional representation, is worked out. An effective version of the main result is described in §6. There, it is shown how the Casimir operator can be used to find the reduced Gröbner basis. But first we survey the two main ingredients to this paper, Gröbner bases in §2, and straightening in §3.

2. Gröbner bases

This section contains the necessary background material from Gröbner basis theory. See [Buch] for further details and references.

Let $\mathbf{F}$ be a field. Let $\mathbf{F}[x]$ be the ring of polynomials in the variables $x = (x_1, \ldots, x_n)$. A *monomial* is an expression of the form $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$; it is frequently abbreviated to x^α, where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ is the *exponent* of the monomial. We can identify $\mathbb{N}^n$ with the monoid M of all monomials of $\mathbf{F}[x]$ via the map $\alpha \mapsto x^\alpha$. Let I be an ideal of $\mathbf{F}[x]$. Then the quotient $\mathcal{A} = \mathbf{F}[x]/I$ is a finitely generated commutative $\mathbf{F}$-algebra. A key procedure in many computational problems regarding $\mathcal{A}$ consists of producing a unique representative in $\mathbf{F}[x]$ of $f + I \in \mathcal{A}$ when given a polynomial $f \in \mathbf{F}[x]$.

To describe this procedure, we first order the variables $x_1, \ldots, x_n$ in $\mathbf{F}[x]$ as follows: $x_1 < x_2 < \ldots < x_n$. Then we extend it to a linear ordering $<$ on the set M of all monomials of $\mathbf{F}[x]$ by setting

$$x_1^{\alpha_1} \cdots x_n^{\alpha_n} < x_1^{\beta_1} \cdots x_n^{\beta_n}$$

if and only if either $\sum_{i=1}^n \alpha_i < \sum_{i=1}^n \beta_i$ or $\sum_{i=1}^n \alpha_i = \sum_{i=1}^n \beta_i$ and there is $j \in \{1, \ldots, n\}$ with $\beta_k = \alpha_k$ for all $k \in \{1, \ldots, j-1\}$ and $\beta_j > \alpha_j$. This ordering $<$ on M is called the *total degree lexicographic order*. When viewed as an order on the exponents, it has the following property:

$$\left.\begin{array}{l} \gamma \geq 0, \quad \text{and} \\ \text{if } \alpha < \beta \text{ then } \alpha + \gamma < \beta + \gamma \end{array}\right\} \text{ for all exponents } \alpha, \beta, \gamma \in \mathbb{N}^n. \qquad (1)$$

An ordering $<$ on M with this property is called a *reduction ordering*. An important feature of a reduction ordering is that it is *Noetherian*, that is, every strictly descending chain is finite. For nonzero $f \in \mathbf{F}[x]$, we denote by lead(f) the *leading monomial* of f, that is the highest monomial which occurs in f (with nonzero coefficient).

Suppose that $F = \{f_1, \ldots, f_m\}$ is a finite set of polynomials in $\mathbf{F}[x]$ generating the ideal I (such a finite set exists as $\mathbf{F}[x]$ is Noetherian). If $g \in \mathbf{F}[x]$ is given, then $h \in \mathbf{F}[x]$ is called a *reduct of g with respect to F* if lead$(h) <$ lead(g) and there is an $f \in F$ such that, for some $c \in \mathbf{F}$ and $u \in M$, we have $h = g - c\,u\,f$. (Note that then lead(g) is divisible by lead(f) with quotient u.) If we continue with h instead of g and form successive reducts with respect to F until no further reduction is possible, then we arrive at a polynomial which is called a *normal form of g with respect to F*. The normal form need not be unique. Nevertheless, we shall write $NormalForm_F(g)$ to denote a normal form of g with respect to F. Each reduct h of g represents the same element $h + (F) = g + (F)$ of $\mathcal{A}$, and so $NormalForm_F(g) \in g + I$.

Define the S-polynomial of two polynomials $f_1, f_2 \in \mathbf{F}[x]$ as follows:

$$S(f_1, f_2) = c_2 u_1 f_1 - c_1 u_2 f_2, \tag{2}$$

where, for $i = 1, 2$, c_i is the coefficient of the term in f_i containing lead(f_i) and $u_i$lead(f_i) is the least common multiple of lead(f_1) and lead(f_2).

Let F be a finite subset of $\mathbf{F}[x]$. For $I = (F)$, denote by lead(I) the $\mathbf{F}$-linear subspace of $\mathbf{F}[x]$ spanned by all leading monomials of polynomials in I. It is an ideal of $\mathbf{F}[x]$. For $m \in M$ define m to be *non-standard* if it belongs to lead(I), and *standard* otherwise. If J is the $\mathbf{F}$-linear span of all standard monomials, then

$$\mathbf{F}[x] = I \oplus J = \text{lead}(I) \oplus J. \tag{3}$$

2.1 Effective characterizations of Gröbner bases. *Let $<$ be a reduction ordering on the collection of monomials in the polynomial ring $\mathbf{F}[x]$. Then, for a fixed finite subset F of $\mathbf{F}[x]$ generating $I = (F)$, the following statements are equivalent.*

(i) *if $f, g \in F$, then $NormalForm_F(S(f, g)) = 0$ (for some computation of $NormalForm$);*

(ii) *for each $f \in I$, we have $NormalForm_F(f) = 0$ (for any computation of $NormalForm$);*

(iii) lead(I) *is spanned by* $\{\text{lead}(f) \mid f \in F\}$ *as an ideal, or, equivalently, by* $\{m\,\text{lead}(f) \mid f \in F, m \in M\}$ *as a vector space;*

(iv) *for each $f \in \mathbf{F}[x]$, we have $NormalForm_F(f) \in J$.*

If F satisfies these properties, it is called a *Gröbner basis*. An ideal I may have many Gröbner bases. Given one, say F, we first normalize each $f \in F$ so that the coefficients of the terms containing lead(f) are 1. We say that f is *monic* if it is normalized in this way. F is a *reduced Gröbner basis* for the ideal $I = (F)$ if the normal form of any $f \in F$ with respect to $F \setminus \{f\}$ is f and each $f \in F$ is monic. It is not hard to show that I has a *unique* reduced Gröbner basis. Some caution is in order though: the uniqueness only holds for a fixed ordering $<$ (because the ideal lead(I) may change if the ordering is changed).

In algebraic geometry, the quotient ring $\mathcal{A} = \mathbf{F}[x]/I$ is the ring of all polynomial functions on the subset of $\mathbf{F}^n$ consisting of all common zeros of the polynomials in I. If I is a homogeneous ideal, the zero set can be interpreted as a subset of the projective space $\mathcal{P}\mathbf{F}^n$ on $\mathbf{F}^n$. The projective zero set X of I is then called a *projective variety* and the polynomial ring $\mathcal{A}$, denoted by $\mathbf{F}[X]$, will then be viewed as a graded ring. In the next section, we shall consider a special kind of projective variety, the Grassmann variety.

3. Straightening

We begin by reviewing some basic facts on Grassmann varieties. Viewing an $n \times d$-matrix as an array of d vectors $v_1, \ldots, v_d$ of length n we obtain the mapping $\phi : \mathcal{M}_{n\times d} \longrightarrow \bigwedge^d \mathbf{F}^n$ sending $(v_1, \ldots, v_d)$ to $v_1 \wedge \ldots \wedge v_d$. The image under ϕ of the subset of $\mathcal{M}_{n\times d}$ consisting of all matrices of rank d in the projective space $\mathcal{P}(\bigwedge^d(\mathbf{F}^n))$ is called the *Grassmann variety* $G(n,d)$. Its embedding in $\mathcal{P}(\bigwedge^d(\mathbf{F}^n))$ is called the Plücker embedding. The projective points of this variety correspond bijectively to the d-dimensional subspaces of $\mathbf{F}^n$: the projective point (= 1-dimensional subspace) corresponding to $v_1 \wedge \ldots \wedge v_d \in \bigwedge^d \mathbf{F}^n$ does not depend on the choice of basis $v_1, \ldots, v_d$ in the d-dimensional subspace of $\mathbf{F}^n$ that it spans.

We shall now justify the use of the word variety by exhibiting a set of homogeneous polynomials $F_{n,d}$ whose projective zero set coincides with $G(n,d)$. For each $1 \le i_1, \ldots, i_d \le n$ define the *bracket* $[i_1, \ldots, i_d]$ to be the coordinate function $x_{i_1} \wedge \ldots \wedge x_{i_d}$, where $x_1, \ldots, x_n$ are the standard coordinate functions on $\mathbf{F}^n$. (Note, here we use the standard identification of $\bigwedge^d(\mathbf{F}^n)^*$ with $(\bigwedge^d \mathbf{F}^n)^*$ by means of the pairing determined by $(\mu_1 \wedge \ldots \wedge \mu_d, \nu_1 \wedge \ldots \wedge \nu_d) = \det\,(\mu_i(\nu_j))_{1\le i,j\le d}$ for $\mu_i \in (\mathbf{F}^n)^*$ and $\nu_i \in \mathbf{F}^n$.) Using the map ϕ the bracket $[i_1, \ldots, i_d]$ can be interpreted as the determinant of the $d \times d$-matrix formed by the rows $i_1, \ldots, i_d$ of the $n \times d$-matrix $(v_1, \ldots, v_d)$. Thus $\mathbf{F}[x]$, where $x = ([i_1, \ldots, i_d])_{1 \le i_1 < \ldots < i_d \le n}$, is the polynomial ring of the vector space $\bigwedge^d(\mathbf{F}^n)$. Usually the bracket $[i_1, \ldots, i_d]$ is defined for an arbitrary sequence $i_1, \ldots, i_d$ of elements of $\{1, \ldots, n\}$ by use of the conventions $[i_1, \ldots, i_\ell, i_{\ell+1}, \ldots, i_d] = -[i_1, \ldots, i_{\ell+1}, i_\ell, \ldots, i_d]$ and $[i_1, \ldots, i_{m-1}, i_m, i_m, i_{m+1} \ldots, i_{d-1}] = 0$ for all ℓ, m. We shall say that the bracket $[i_1, \ldots, i_d]$ is *normalized* if it satisfies $i_1 < \ldots < i_d$.

It is customary to write the *bracket monomial* $[i_{11}, \ldots, i_{1d}][i_{21}, \ldots, i_{2d}] \cdots [i_{\ell 1}, \ldots, i_{\ell d}]$ as a tableau

$$T = \begin{bmatrix} i_{11} & \ldots & i_{1d} \\ i_{21} & \ldots & i_{2d} \\ \vdots & \vdots & \vdots \\ i_{\ell 1} & \ldots & i_{\ell d} \end{bmatrix} \tag{4}$$

of length ℓ. Since rows may be interchanged at will, and since a transposition of entries from the same row results in a sign change of the corresponding term, we can always normalize a tableau so that it either vanishes or has the following properties:

- each row corresponds to a normalized bracket, and
- the rows are weakly increasing (with respect to the lexicographical ordering).

We shall refer to a tableau satisfying these two conditions as a *normalized tableau*. Furthermore, we shall say that the t-th column of T is weakly increasing if

$$i_{1t} \leq i_{2t} \leq \cdots \leq i_{\ell t}.$$

If X is a set, we write Sym$X$ for the group of all permutations of X. For each pair of brackets $[i_1, \ldots, i_d]$ and $[j_1, \ldots, j_d]$, and each $\ell \in \{1, \ldots, d\}$, consider the following polynomial

$$\sum_{\sigma} \operatorname{sign}(\sigma) \begin{bmatrix} i_1 & \ldots & i_{\ell-1} & \sigma(i_\ell) & \sigma(i_{\ell+1}) & \ldots & \sigma(i_d) \\ \sigma(j_1) & \ldots & \sigma(j_{\ell-1}) & \sigma(j_\ell) & j_{\ell+1} & \ldots & j_d \end{bmatrix} \tag{5}$$

where the sum runs over a system of coset representatives in $\text{Sym}\{i_\ell, \ldots, i_d, j_1, \ldots, j_\ell\}$ with respect to $\text{Sym}\{i_\ell, \ldots, i_d\} \times \text{Sym}\{j_1, \ldots, j_\ell\}$. For some choices, e.g., $j_1 \in \{i_1, \ldots, i_d\}$ and $\ell = 1$, the corresponding polynomial vanishes identically. Otherwise, (5) is a quadratic polynomial and vanishes on $G(n,d)$. Let $F_{n,d}$ denote the set of all nonzero quadratic polynomials occurring in (5). Its members, when equated to zero, are (special instances of) the well known *Plücker relations*.

Example The set $F_{n,1}$ is empty, whereas $F_{n,2}$ consists of all $[i,j][k,\ell] + [i,k][\ell,j] + [i,\ell][j,k]$ for i,j,k,ℓ distinct integers in $\{1, \ldots, n\}$.

3.1 The second fundamental theorem of invariant theory. *The Grassmann variety $G(n,d)$ is the zero set of the ideal $(F_{n,d})$. Conversely, any polynomial vanishing on $G(n,d)$ belongs to $(F_{n,d})$.*

PROOF. See [ACGH]. QED

Set $R = \mathbf{F}[([i_1, \ldots, i_d])_{1 \leq i_1 < \ldots < i_d \leq n}]$. We introduce the total degree lexicographical ordering $<$ on M, the set of monomials of R, which is determined by ordering the variables so that $[i_1, \ldots, i_d] < [j_1, \ldots, j_d]$ if, for some ℓ we have $i_s = j_s$ $(1 \leq s < \ell)$ and $i_\ell < j_\ell$. The theory of §2 applied to $F_{n,d}$ now gives:

3.2 Proposition. *A normalized tableau is a standard monomial (with respect to the total degree lexicographic ordering $<$) if and only if all of its columns are non-decreasing.*

PROOF. Let T be a tableau and suppose it is normalized but non-standard. Then there are two consecutive rows in T:

$$\begin{bmatrix} i_1 & \ldots & i_d \\ j_1 & \ldots & j_d \end{bmatrix}$$

and a column $\ell > 1$ such that $i_r \leq j_r$ for all $r < \ell$ and $i_\ell > j_\ell$. The Plücker relation (5) shows that T can be replaced with a linear combination of tableaux each of which is smaller than T with respect to $<$ on M. For, if σ

is a nontrivial coset representative as in (5), then $\sigma(i_m) \in \{j_1, \ldots, j_s\}$ for at least one $m \in \{\ell, \ldots, d\}$, yielding

$$[i_1 \quad \ldots \quad i_{\ell-1} \quad i_\ell \quad \ldots \quad i_d] >$$

$$[i_1 \quad \ldots \quad i_{\ell-1} \quad \sigma(i_\ell) \quad \ldots \quad \sigma(i_m) \quad \ldots \quad \sigma(i_d)]$$

as $\sigma(i_m) \leq j_\ell < i_\ell$. Thus, by use of (5), every nonstandard quadratic bracket monomial can be written as a sum of two smaller quadratic bracket monomials. Consequently, every nonstandard tableau may be *straightened*, that is, by use of $F_{n,d}$, it may be rewritten as a linear combination of standard tableaux modulo $(F_{n,d})$. Since this operation is nothing but the linear projection onto the **F**-span of the standard monomials with kernel $(F_{n,d})$, straightening of a monomial m produces the normal form $NormalForm_{F_{n,d}}(m)$. QED

Observe that, in general, the Plücker relations do not give a reduced basis. But, for $d = 2$, each polynomial in $F_{n,2}$ has a single non-standard term and each nonsquare quadratic monomial occurs in precisely one member of $F_{n,2}$, from which it is easy to see that we do obtain a reduced Gröbner basis. Putting all this together, we obtain the result covered by Sturmfels and White in [StWh1]:

3.3 Corollary. *The set $F_{n,d}$ is a Gröbner basis for the ideal with quotient ring* $\mathbf{F}[G(n,d)]$. *If $d = 2$, it is even a reduced Gröbner basis.*

4. The Bruhat order

Consider the set $P_{n,d}$ whose elements are the normalized brackets and supply it with the partial ordering $\prec$ in which $[i_1, \ldots, i_d] \preceq [j_1, \ldots, j_d]$ if $i_s \leq j_s$ for all $s \in \{1, \ldots, d\}$. Then $<$ is a refinement of $\prec$, and the tableau $[i_1, \ldots, i_d] \cdot [j_1, \ldots, j_d]$ is nonstandard if and only if the two brackets involved are incomparable (i.e., they are related by neither $\prec$ nor $\succ$).

Example The Hasse diagram of $(P_{6,2}, \prec)$ is given in Figure 1.

$$\begin{array}{ccccccccc}
56 & & & & & & & & \\
\downarrow & & & & & & & & \\
46 & \longrightarrow & 45 & & & & & & \\
\downarrow & & \downarrow & & & & & & \\
36 & \longrightarrow & 35 & \longrightarrow & 34 & & & & \\
\downarrow & & \downarrow & & \downarrow & & & & \\
26 & \longrightarrow & 25 & \longrightarrow & 24 & \longrightarrow & 23 & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
16 & \longrightarrow & 15 & \longrightarrow & 14 & \longrightarrow & 13 & \longrightarrow & 12
\end{array}$$

Figure 1. Hasse diagram of $(P_{6,2}, \prec)$.

The node ij represents the bracket $[i, j]$ and the arrow $b \longrightarrow a$ connects the two nodes a and b related by $a \prec b$ in the poset $(P_{6,2}, \prec)$ for which there is no node $c \in P_{6,2}$ such that $a \prec c \prec b$. In other words, b *covers* a. A straightforward check shows that there are 15 incomparable pairs in the Hasse diagram, each corresponding to a unique non-standard, square-free quadratic monomial in lead $F_{6,2}$.

The poset $(P_{n,d}, \prec)$ can be obtained from the theory of Coxeter groups. We first recall the definition of a Coxeter group. See [Bour], [Hum] or [Coh] for introductions.

A *Coxeter matrix* of rank n is an $n \times n$ matrix $M = (m_{i,j})_{1 \leq i,j \leq n}$ with $m_{i,i} = 1$ and $m_{i,j} = m_{j,i} > 1$ (possibly ∞) for all $i, j = 1, \ldots, n$. The *Coxeter group* associated with the Coxeter matrix M is the group generated by elements ρ_i $(i = 1, \ldots, n)$ subject to the relations

$$(\rho_i \rho_j)^{m_{i,j}} = 1.$$

It is denoted by $W(M)$ or just W. Furthermore, we set $I = \{1, \ldots, n\}$ and $R = \{\rho_i \mid i \in I\}$. The pair (W, R) is called the *Coxeter system* of type M. The number n is called the *rank* of the system (or group).

If $n = 1$ then $W = \{1\} \cup R \cong \mathbf{Z}/(2)$, the group of order 2. If $n = 2$, then $W \cong \langle r, s \mid r^2 = s^2 = (r\,s)^{m_{1\,2}} = 1 \rangle$, the dihedral group of order $2m_{1\,2}$.

It is common practice to provide a pictorial presentation of M by means of the labeled graph (I, M) with vertex set I, no loops, and (undirected) edges $\{i,j\}$, labeled $m_{i,j}$, if the latter number exceeds 2. Also, if $m_{i,j}=3$, the label may be omitted.

Let I be an index set and W any group generated by a set $R=\{\rho_i \mid i\in I\}$. The free monoid on the alphabet I with unit (usually denoted by ϵ) is denoted by I^* and $\rho : I^* \to W(R)$ stands for the monoid morphism determined by $\rho(i)=\rho_i$ $(i\in I)$. There is a natural notion of length for an element of I^*; the length of the empty element is 0, the length of an element of the alphabet I equals 1, and so on. A typical element of I^* will be written as $\mathbf{i}$ and its length as $\ell(\mathbf{i})$. Thus, if $\ell(\mathbf{i})=q$, there are $i_j\in I$ $(1\le j\le q)$ such that $\mathbf{i}=i_1\cdots i_q$. The *length* of an element $w\in W$, denoted by $\ell(w)$, or $\ell_R(w)$ if more precision is required, is $\min\{\ell(\mathbf{i}) \mid \rho(\mathbf{i})=w\}$. For each element $\mathbf{i}=i_1\cdots i_q\in I^*$ with $\rho(\mathbf{i})=w$, we call the product $\rho(i_1)\cdots\rho(i_q)$ an *expression* of w. If $q=\ell(w)$, the expression is called *reduced.*

Let (W,R) be a Coxeter system with $R=\{\rho_1,\ldots,\rho_n\}$ of cardinality n. We shall write $\prec$ for the relation on W defined by $x\prec w$ if there is a reduced expression $s_1\cdots s_q$ $(s_j\in R)$ for w such that $x=s_{i_1}\cdots s_{i_m}$ (where $1\le i_1<i_2<\ldots<i_m\le q$). It is well known (cf. [Bour]) that $\succ$ defines an ordering on W; it is called the *Bruhat order.* Clearly, 1 is the smallest element of W; in case W is finite, the group W has a unique longest element w_0 which is the largest element with respect to $\succ$.

Let I be a subset of R. The subgroup $\langle I\rangle$ of W, also denoted by W_I, is the subgroup generated by I. It is again a Coxeter group. In particular, $W_\emptyset=\{1\}$ and $W_R=W$. Set

$$D_I=\{w\in W \mid \ell(ws)>\ell(w) \text{ for all } s\in I\}.$$

The set D_I is a natural system of W_I-coset representatives. As W-sets $D_I\cong W/W_I$.

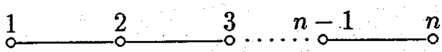
4.1 Proposition. *Let (W,R) be a Coxeter system, and let I be a subset of R. The map $D_I\to W/W_I$ sending $w\in D_I$ to wW_I is a bijection. Each $w\in W$ has a reduced expression $w=dv$ with $d\in D_I$, $v\in W_I$; in particular, w is the unique shortest element of wW_I.* QED

PROOF. See [Bour]. QED

The order $\succ$ on W induces a partial order, also denoted by $\succ$, on W/W_I. It can be characterized by $xW_I\succ yW_I$ if and only if $x\succ y$ for $x,y\in D_I$ and has the following useful property ([Deodh] Cor. 3.5]):

if $x,y\in D_I$ and $x\succ y$ then there is a chain $x=x_0\succ x_1\succ\ldots\succ x_t=y\in W$ of elements of D_I such that $\ell(x_{i-1})=\ell(x_i)-1$ for all $i\in\{1,\ldots,t\}$.

Example The symmetric group Sym_{n+1} on $n+1$ letters is the Coxeter group $W(A_n)$ where

$A_n=$ 

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The evident morphism $W \to \mathrm{Sym}_{n+1}$ sending ρ_i to $(i, i+1)$ for each $i \in I$ is an isomorphism. Set $W = W(A_{n-1}) = \mathrm{Sym}_n$, where we identify ρ_i with the transposition $(i, i+1)$. It is easy to check that the longest element is the permutation $w_0 = (1,n)(2,n-1)(3,n-2)\cdots$ which is the mapping $[[n, n-1, \ldots, 1]]$ sending $j \in \{1, \ldots, n\}$ to $n+1-j$. (The double square bracket notation for a sequence of length n indicates the permutation sending $i \in \{1, \ldots, n\}$ to the i-th element of the sequence.)

We now look at the coset space W/W_J, where $J = \{\rho_1, \ldots, \rho_{d-1}, \rho_{d+1}, \ldots, \rho_{n-1}\}$. It has $\binom{n}{d}$ elements. The subgroup W_J is isomorphic to $\mathrm{Sym}_d \times \mathrm{Sym}_{n-d}$, and there is a bijection between $D_{\emptyset,J}$ and the set of normalized tableaux $t = [i_1, \ldots, i_d]$ with $1 \le i_1 < \ldots < i_d \le n$. The element in $D_{\emptyset,J}$ corresponding to t is the permutation

$$w_t = (\rho_{i_1-1} \cdots \rho_1)(\rho_{i_2-1} \cdots \rho_2) \cdots (\rho_{i_d-1} \cdots \rho_d), \tag{6}$$

that is, the permutation $(i_1, \ldots, 1)(i_2, \ldots, 2) \cdots (i_d, \ldots, d)$, or, written in yet another notation, $[[i_1, \ldots, i_d, i_{d+1}, \ldots, i_n]]$ with $i_{d+1} < i_{d+2} < \cdots < i_n$ and $i_\ell \in \{1, \ldots, n\} \setminus \{i_1, \ldots, i_d\}$ for $\ell = d+1, \ldots, n$. For example, if $[i,j] = [2,5]$ then

$$w_{2,5} = [[2,5,1,3,4]] = (1,2)(4,5)(3,4)(2,3) = \rho_1 \rho_4 \rho_3 \rho_2.$$

From (6) we see that if t and t' are normalized tableaux with $t' = [i_1, \ldots, i_{j-1}, i_j - 1, i_{j+1}, \ldots, i_d]$ and $t = [i_1, \ldots, i_{j-1}, i_j, i_{j+1}, \ldots, i_d]$, then t' is covered by t. This follows because $w_{t'}$ is obtained from w_t by removing ρ_{i_j-1}.

4.2 Proposition. *If* $W = W(A_{n-1}) = \mathrm{Sym}_n$ *and* $J = \{\rho_1, \ldots, \rho_{d-1}, \rho_{d+1}, \ldots, \rho_{n-1}\}$, *the Bruhat poset* $(D_J, \prec)$ *is isomorphic to the poset* $(P_{n,k}, \prec)$; *the isomorphism is given by*

$$w\langle J\rangle \mapsto w[[i_1, \ldots, i_d]] \ (w \in D_J), \quad \text{and its inverse by} \quad t \mapsto w_t \ (t \in P_{n,k}).$$

5. The highest weight orbit

Let $\mathbf{F}$ be a field of characteristic 0. Consider a simple split algebraic group G defined over $\mathbf{F}$ and fix a Tits system (B, N, W, R) in G, where B is a Borel subgroup of G (cf. [Bourb]). Denote by T the maximal torus $B \cap N$ of G in B. Then $N = N_G(T)$ is the normalizer in G of T and $W = N/T$ is a Weyl group with Coxeter system (W, R). For example, if G is the special linear group $SL(n, \mathbb{C})$ consisting of all $n \times n$ matrices with determinant 1, then B can be taken to be the subgroup of all upper triangular matrices in G, and the subgroup N can be taken to be the subgroup of all monomial matrices (one nonzero entry in each row and each column). The group W, which arises as the quotient of N by the diagonal subgroup $T = B \cap N$ is then isomorphic to Sym_n, and the Coxeter system (W, R) is of type A_{n-1}.

If $\mathbf{F} = \mathbb{C}$, the Bruhat order is closely related to the topological structure of G (viewed as a Lie group). For each $I \subset R$, the subgroup $P_I = B\langle I\rangle B$ has the property that G/P_I inherits the structure of a variety. The subset BwP_I/P_I for $w \in W$ is a cell in G/P_I, whose closure is $\{bxP_I \mid x \preceq w, b \in B\}$. We shall exploit the Bruhat order in a slightly different way.

Suppose λ is a dominant weight. Then $\lambda \in \mathbb{N}^n$, where n is the Lie rank of G (i.e., the dimension of T), and the highest weight module $V(\lambda)$ has a highest weight vector v_λ (unique up to scalar multiples). The stabilizer of the projective point $\langle v_\lambda\rangle$ is the parabolic subgroup $P = B\langle I\rangle B$ containing B of type I, where $\langle I\rangle = W_\lambda$ (= the stabilizer in W of the weight λ). Thus the projective variety G/P embeds in the projective space $\mathcal{P}V(\lambda)$ with image $G\langle v_\lambda\rangle$. If V is any G-module, then, as a T-module, it has a basis of eigenvectors whose corresponding projective points are permuted by W. We shall call such a basis of V a *T-frame*.

We shall be primarily interested in the easiest cases, namely those where λ is *minuscule*, that is, $V(\lambda)$ has a T-frame consisting of a single W-orbit. (See [Proc] or [Hill] for many equivalent definitions.) Fix such a T-frame $(v_\mu)_{\mu\in W\lambda}$ in $V(\lambda)$. Accordingly, for $\mu \in W\lambda$ we denote by x_μ the element of the dual space $V(\lambda)^*$ that is dual to v_μ with respect to the selected basis of $V(\lambda)$. Write $x_\mu \prec x_\nu$ whenever $w \prec v$ for $w, v \in W$ with $\mu = w\lambda$ and $\nu = v\lambda$. Thus we transport the poset structure from W/W_λ, where W_λ is the stabilizer of λ, to $\{x_\mu\}_{\mu\in W\lambda}$. Since $W_\lambda = W_I$ for some subset I of R, we can apply Proposition 4.1. By the way, we shall write λ^* for the dominant weight satisfying $V(\lambda^*) \cong V(\lambda)^*$.

For $G = SL(n, \mathbb{C})$ the representation $V(\lambda)$ is the one obtained from the standard n-dimensional one by taking the symmetrized power (or plethysm) of the standard representation V with respect to the partition $(\lambda_1 + \cdots + \lambda_{n-1}, \lambda_2 + \cdots + \lambda_{n-1}, \ldots, \lambda_{n-1})$. Thus $V((0,0,1,0,0))$ and $V((3,0,0,0,0))$ are the third exterior and the third symmetric power of V, respectively. In particular, the minuscule weights of $SL(n, \mathbb{C})$ are the fundamental ones, i.e., those of the form $V((0, \ldots, 0, 1, 0, \ldots, 0))$, the T-frames of which are the Plücker coordinates.

The straightening phenomena in §2 have the following generalizations:

5.1 Theorem. *Let F be a field of characteristic 0. Suppose G is a simple split algebraic group over F with Tits system (B, N, W, R), and set $T = B \cap N$. Let λ be a dominant weight, and let $P = BW_\lambda B$ be the corresponding parabolic subgroup of G.*

(i) (cf. [Brion], [Lich]) *The G-module $S^2V(\lambda)$ contains the highest weight module $V(2\lambda^*)$ with multiplicity 1 and has a G-invariant complement M. The ideal I in $\mathbf{F}[V(\lambda)]$ of the highest weight orbit $G\langle v_\lambda \rangle \cong G/P$ of G in $\mathcal{P}V(\lambda)$ is generated by the polynomial quadratic maps forming a T-frame of M.*

(ii) (cf. [Sesh]) *If λ is minuscule, then $\{x_\alpha x_\beta \mid \alpha, \beta \in W\lambda, \ \ \alpha \geq \beta\}$ is a basis of a complement of M in $S^2V(\lambda)$.*

(iii) *If λ is minuscule, then there is a set $F_{G,\lambda}$ of polynomials $f_{\phi,\tau}$ indexed by the incomparable (unordered) pairs $\{\phi, \tau\}$ from the poset $(W/W_J, \prec)$ in such a way that, for each indexing pair $\{\phi, \tau\}$, the polynomial $f_{\phi,\tau}$ has shape*

$$x_\phi x_\tau - \sum_{\substack{\alpha,\beta \in W\lambda \\ x_\alpha \succeq x_\beta}} c_{\alpha,\beta} x_\alpha x_\beta \qquad (7)$$

for certain coefficients $c_{\alpha,\beta}$.

(iv) (cf. [SeSh]) *In (7), any pair α, β with $c_{\alpha,\beta} \neq 0$ satisfies*

$$x_\alpha \succeq x_\phi, \quad x_\alpha \succeq x_\tau, \quad \text{and} \quad x_\beta \preceq x_\phi, \quad x_\beta \preceq x_\tau.$$

Consequently, $F_{G,\lambda}$ is a reduced Gröbner basis (with respect to any total degree lexicographical ordering $<$ extending $\prec$ on the variables) for the projective variety G/P embedded in $\mathcal{P}V(\lambda)$.

Proof. For (i) and (ii), we refer to the cited papers.

(iii) The standard monomials form a complement J_2 of M in $S^2V(\lambda)$. But the non-standard monomials span a complementary space I_2 to J_2. Thus, modulo J_2, the vector spaces M and I_2 are isomorphic. This means that there is a map $M \to I_2$ and a basis

$$\{f_{\phi,\tau} \mid \phi, \tau \in W\lambda, \ \phi, \tau \text{ incomparable}\}$$

of M such that each $f_{\phi,\tau}$ is the sum of $x_\phi x_\tau \in I_2$ and a linear combination of standard monomials.

(iv) The first statement is due to [Sesh], see also [LaSh]. It implies that, for $f \in I_2$, we have $NormalForm_F(f) \in J_2$, where $F = F_{G,\lambda}$. Together with (i), this readily implies that, for any $f \in \mathbf{F}[x]$, we have $NormalForm_F(f) \in J$, the $\mathbf{F}$-linear span of all monomials in which no product $x_\phi x_\tau$ with incomparable ϕ, $\tau \in W\lambda$ occurs. Hence F is a Gröbner basis. Since removal of an element g from F would make the non-standard monomial lead(g) reduced with respect to $F \setminus \{g\}$, the Gröbner basis is reduced. QED

To give an impression of the scope of the theorem, we list all miniscule weights of simple Lie groups. We use the labeling of Dynkin diagrams of [Bour]. If ω_j is the miniscule weight in the case of type Y_n, we write $Y_{n,j}$.

$$A_{n,j},\ B_{n,n},\ C_{n,1},\ D_{n,1}, D_{n,n-1},\ D_{n,n},\ E_{6,1},\ E_{6,6},\ E_{7,7}.$$

We finish this section by giving examples of Gröbner bases in some of these cases.

Example $A_{3,2}$. Here $V(\lambda)$ has dimension 6, so $S^2V(\lambda)$ has dimension 21. As $V(2\lambda^*)$ has dimension 20, the Gröbner basis consists of a single relation. This is the well known one, given in the $G(4,2)$ example of §3.

Example $C_{n,1}$. We consider Sp_{2n} in its natural $2n$-dimensional representation. Every nonzero vector is in the high weight vector orbit, so the set of quadratic equations is the empty set. Correspondingly, the Bruhat poset is a chain (of length $2n$). Thus the Gröbner basis is empty.

Example $D_{n,1}$. Another easy example is furnished by the standard representation of the orthogonal group in $2n$ dimensions of maximal Witt index (the split group of type D_n). Here the Hasse diagram looks like

$$\begin{array}{ccccccccc} -1 & & & & & & & & \\ \downarrow & & & & & & & & \\ -2 & & & & & & & & \\ \vdots & & & & & & & & \\ 1-n & \longrightarrow & -n & & & & & & \\ \downarrow & & \downarrow & & & & & & \\ n & \longrightarrow & n-1 & \longrightarrow & \dots & \longrightarrow & 2 & \longrightarrow & 1 \end{array}$$

Figure 2. Hasse diagram of $D_{n,1}$.

This is in accordance with the well-known fact that the high weight orbit is a quadric in the natural representation space. Thus, the Gröbner basis is nothing but the quadric.

Example $E_{6,1}$. Let $\mathbf{K}$ denote the 27-dimensional vector space over the field $\mathbf{F}$ consisting of all ordered triples $x = [x^{(1)}, x^{(2)}, x^{(3)}]$ of 3×3-matrices $x^{(i)}$

$(1 \le i \le 3)$ (addition and scalar multiplication are entrywise.) The vector space $\mathbf{K}$ is supplied with the symmetric cubic form $D : \mathbf{K} \to \mathbf{F}$ given by :

$$D(x) = \det x^{(1)} + \det x^{(2)} + \det x^{(3)} - \operatorname{trace}\ x^{(1)}x^{(2)}x^{(3)} \quad (x \in \mathbf{K}) \quad (8)$$

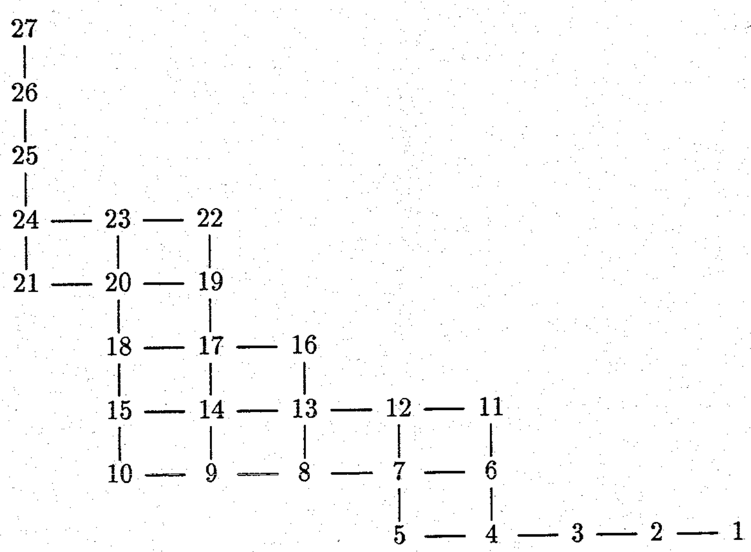
The group $G = GL(\mathbf{K})_D$ of (invertible) linear transformations g of $\mathbf{K}$ such that $D(g(x)) = D(x)$ for all $x \in \mathbf{K}$ is the simply connected Lie group G of type E_6. Moreover, $\mathbf{K}$ is a highest weight representation space of G with highest weight ω_1.

Next, we describe the ordering of the variables which we exploit. We shall write $\mathbf{F}[\mathbf{K}] = \mathbf{F}[x]$ with $x = (x_i)_{1 \le i \le 27}$ such that x_i is the coordinate function attached to the standard basis element labeled i the following scheme:

$$\left[\begin{pmatrix} 6 & 4 & 11 \\ 7 & 5 & 12 \\ 25 & 26 & 27 \end{pmatrix}, \begin{pmatrix} 14 & 13 & 15 \\ 17 & 16 & 18 \\ 9 & 8 & 10 \end{pmatrix}, \begin{pmatrix} 23 & 20 & 2 \\ 24 & 21 & 3 \\ 22 & 19 & 1 \end{pmatrix}\right].$$

Thus, for example $x_6 = x^{(1)}_{1,1}$.

The Bruhat ordering of W/W_J, where $J = \{2,3,4,5,6\}$ is depicted in the Hasse diagram of Figure 3. It is refined to a total ordering on the variables by putting $1 < x_1 < x_2 < \ldots < x_{27}$.



igure 3. Hasse diagram of $(W/W_J, \prec)$, where $W = W(E_6)$ and $J = \{2,3,4,5,6\}$.

There are 27 incomparable pairs in the poset $(W/W_{\{2,3,4,5,6\}}, \prec)$. On the other hand, the variety G/P is well known (cf. [CoCo]) to be the zero set of the 27 partial derivatives $\partial_{x_i} D(x)$ $(i \in \{1, \ldots, 27\})$ of the cubic form D of (8). These defining equations are

$$\begin{aligned}
&x_{10}x_{19} - x_{25}x_4 + x_{26}x_6 + x_9x_{20} + x_8x_{21},\\
&x_{15}x_{22} - x_5x_{27} + x_{12}x_{26} + x_{14}x_{23} + x_{13}x_{24},\\
&x_{22}x_{20} - x_{19}x_{23} - x_{25}x_{13} - x_{26}x_{16} - x_{27}x_8,\\
&x_{11}x_9 - x_{21}x_1 + x_3x_{19} + x_6x_{14} + x_4x_{17},\\
&x_9x_{13} - x_8x_{14} - x_{22}x_4 - x_{19}x_5 - x_1x_{26},\\
&x_{16}x_{10} - x_{18}x_8 - x_{23}x_6 - x_{20}x_7 - x_2x_{25},\\
&x_{11}x_5 - x_4x_{12} + x_{14}x_2 + x_{13}x_3 + x_{15}x_1,\\
&x_{18}x_{22} - x_{12}x_{25} + x_7x_{27} + x_{17}x_{23} + x_{16}x_{24},\\
&x_{11}x_{10} - x_{20}x_3 + x_2x_{21} + x_6x_{15} + x_4x_{18},\\
&x_{12}x_9 - x_3x_{22} + x_{24}x_1 + x_7x_{14} + x_5x_{17},\\
&x_{15}x_{16} - x_{13}x_{18} + x_{23}x_{11} + x_{20}x_{12} + x_2x_{27},\\
&x_{17}x_{10} - x_{18}x_9 + x_{24}x_6 + x_{21}x_7 + x_3x_{25},\\
&x_{11}x_7 - x_6x_{12} - x_{17}x_2 - x_{16}x_3 - x_{18}x_1,\\
&x_{10}x_{22} - x_7x_{26} + x_5x_{25} + x_9x_{23} + x_8x_{24},\\
&x_{12}x_{10} - x_2x_{24} + x_{23}x_3 + x_7x_{15} + x_5x_{18},\\
&x_{21}x_{22} - x_{24}x_{19} + x_{25}x_{14} + x_{26}x_{17} + x_{27}x_9,\\
&x_{15}x_{17} - x_{14}x_{18} - x_{24}x_{11} - x_{21}x_{12} - x_3x_{27},\\
&x_{16}x_9 - x_{17}x_8 + x_{22}x_6 + x_{19}x_7 + x_1x_{25},\\
&x_6x_5 - x_4x_7 - x_9x_2 - x_8x_3 - x_{10}x_1,\\
&x_{15}x_{19} - x_{26}x_{11} + x_{27}x_4 + x_{14}x_{20} + x_{13}x_{21},\\
&x_{23}x_{21} - x_{20}x_{24} - x_{25}x_{15} - x_{26}x_{18} - x_{27}x_{10},\\
&x_{11}x_8 - x_{19}x_2 + x_1x_{20} + x_6x_{13} + x_4x_{16},\\
&x_{14}x_{16} - x_{13}x_{17} - x_{22}x_{11} - x_{19}x_{12} - x_1x_{27},\\
&x_{10}x_{13} - x_8x_{15} + x_{23}x_4 + x_{20}x_5 + x_2x_{26},\\
&x_{18}x_{19} - x_{27}x_6 + x_{25}x_{11} + x_{17}x_{20} + x_{16}x_{21},\\
&x_{12}x_8 - x_1x_{23} + x_{22}x_2 + x_7x_{13} + x_5x_{16},\\
&x_{10}x_{14} - x_9x_{15} - x_{24}x_4 - x_{21}x_5 - x_3x_{26}.
\end{aligned}$$

Each polynomial contains a unique monomial indexed by an incomparable pair. Since $S^2V(\omega_1) \cong V(2\omega_6) \oplus V(\omega_1)$, Theorem §5.1 yields that these 27 polynomials form a reduced Gröbner basis for the highest weight orbit of E_6 in $\mathcal{P}V(\omega_1)$.

6. The algorithm

The Gröbner bases in the examples of §5 have been obtained by ad hoc arguments. In order to present an effective version of Theorem 5.1, we need the Casimir operator.

Retain the notation of §5 for G, B, T, W and J. Let Φ denote the root system of G with respect to T, let $\{\alpha_1, \ldots, \alpha_n\}$ be a set of fundamental roots of T, defining B. The dual basis is the set of fundamental weights, $\{\omega_1, \ldots, \omega_n\}$. The numbering of the roots is as in [Bour]. Consider the Lie algebra L corresponding to G, with Chevalley basis $(X_\alpha)_{\alpha\in\Phi}$, $(H_i)_{1\leq i\leq n}$, where $H_i = H_{\alpha_i} = [X_{\alpha_i}, X_{-\alpha_i}]$. Let $Z_\alpha \in \mathbf{F}X_\alpha$ be a vector with $\kappa(X_\alpha, Z_\alpha) = 1$, where κ is the Killing form on L. In the Cartan subalgebra $\sum_i \mathbf{F}H_i$, take $\{K_i\}_i$ to be a dual basis of $\{H_i\}_i$ with respect to κ. Then

$$\Omega = \sum_{\alpha\in\Phi} X_\alpha Z_\alpha + \sum_{i=1}^{\ell} H_i K_i$$

is the *Casimir element* of L. It is a central element of the universal enveloping algebra of L. As such, Ω is an operator on each G-module V, in particular on SV, the ring of polynomial functions on V. The crucial property of the Casimir operator is that, on the G-module with highest weight μ, we have

$$\Omega|_{V(\mu)} \text{ is scalar multiplication by } (\mu + 2\delta, \mu),$$

where $\delta = \omega_1 + \cdots + \omega_n$ and $(\cdot,\cdot)$ is defined on weights by $(\mu,\nu) = \sum_{i=1}^n \mu(H_i)\nu(K_i)$.

6.1 Lemma. *Let G, B, λ, P be as in Theorem 5.1. If λ is minuscule, then, for every dominant weight μ occurring in $W\lambda + W\lambda$ distinct from 2λ,*

$$(\mu + 2\delta, \mu)(2\lambda + 2\delta, 2\lambda).$$

In particular, M as defined in Theorem 5.1 is generated by all $(\Omega - (\lambda + 2\delta, \lambda))\, v$ for v running through a T-frame of $S^2V(\lambda)$.

Proof. Using the standard inner product on the root space to identify the weights with vectors in the root space, the inner product $(\cdot,\cdot)$ on the weight space becomes a positive multiple of the standard inner product on the root space. Thus, $(\omega_i, \alpha_j) > 0$ if $i = j$ and equal to 0 otherwise. Set $\nu = 2\lambda - \mu$. By assumption $\mu \in W\lambda + W\lambda$, so there are $w_1, w_2 \in W$ such that $\mu = w_1\lambda + w_2\lambda$. Now $\lambda - w_i\lambda$ is a sum of positive roots for both $i \in \{1,2\}$ (cf. [Proc, Lemma 3.1]), so the same holds for $\nu = (\lambda - w_1\lambda) + (\lambda - w_2\lambda)$. Writing $\nu = \sum_{i=1}^n \nu_i\alpha_i$ with $\nu_i \in \mathbb{N}$, and $\lambda = \omega_k$ for some k (recall that λ is fundamental hence of this form), we obtain $(\nu, \lambda + \delta) = \nu_k(\omega_k, \alpha_k) + \sum_{i=1}^n \nu_i(\omega_i, \alpha_i)$. Hence

$$(\nu, \lambda + \delta) \geq 0, \qquad \text{with equality if and only if } \nu = 0. \tag{9}$$

Next note, that, since μ is dominant (that is, of the form $\sum_{i=1}^n \mu_i\omega_i$ with $\mu_i \geq 0$ for all i) we have $(\mu,\nu) = \sum_{i=1}^n \mu_i\nu_i(\omega_i,\alpha_i) \geq 0$. In view of (1), this yields

$$(2\lambda + 2\delta, 2\lambda) - (\mu + 2\delta, \mu) = 2(\nu, \lambda + \delta) + (\nu, \mu) \geq 0,$$

with a strict inequality whenever $\nu \neq 0$, that is, whenever $\mu \neq 2\lambda$. Hence the first statement.

For the second statement, observe that $\delta^* = \delta$ and $(\mu+2\delta,\mu) = (\mu^*+2\delta,\mu^*)$ for each dominant weight μ. QED

For computations, it is convenient to rewrite Ω. Let Φ^+ be the set of positive roots in Φ and put

$$C = \sum_{\alpha\in\Phi^+} (\alpha,\alpha)X_{-\alpha}X_\alpha. \tag{10}$$

Then, on a vector v in a weight space of some G-module with weight μ, we have

$$\Omega v = Cv + (\mu + 2\delta, \mu)v.$$

From this we obtain

$$\begin{aligned}\Omega(v\,w) &= C(v\,w) + (\mu+\nu+2\delta,\mu+\nu)v\,w \\ &= (\mu+\nu+2\delta,\mu+\nu)v\,w + \\ &\quad (Cv)\,w + v\,(Cw) + \sum_{\alpha\in\Phi^+}(\alpha,\alpha)\left((X_\alpha v)(X_{-\alpha}w) + (X_{-\alpha}v)(X_\alpha w)\right)\end{aligned}$$

for v a vector of weight μ^* and w a vector of weight ν^* in $S^2V(\lambda)$. Thus, the computation of the Casimir operator on $S^2V(\lambda)$ is brought back to a computation of X_α on monomials from a T-frame.

Now, by Lemma 6.1, the operator $\Omega - (2\lambda + \delta, 2\lambda)$ projects $S^2V(\lambda)$ onto M. Thus, we come to the following algorithm for finding the basis in each of the minuscule cases. The partitioning of the Gröbner basis by means of weights is a good tool to cut down the size of the computations.

MinusculeWeightStandardBase Algorithm *Given a minuscule weight λ of a simple Lie group G, compute a Gröbner basis of the ideal of polynomials vanishing on the highest weight orbit.*

$MWStandardBase(G, \lambda)$ =

$\Lambda := W\lambda$; $\Lambda_2 := \Lambda + \Lambda$;

$C := \sum_{\alpha \in \Phi^+} (\alpha, \alpha) X_{-\alpha} X_\alpha$; # the operator of (10)

for each $\mu \in \Lambda_2$ **do**

$S_\mu := \{x_\alpha x_\beta \mid \alpha, \beta \in \Lambda, \ \alpha + \beta = -\mu\}$;

a − sign as x_α has weight $-\alpha$

$E_\mu := \{(C - (2\lambda + 2\delta, 2\lambda) + (\mu + 2\delta, \mu))p \mid p \in S_\mu\}$;

$E_\mu := GaussElim(B_\mu)$;

to get the identity submatrix on incomparable pairs

$E := E \cup E_\mu$

od ;

return E.

Here, the procedure *GaussElim* sees to it that the principal minor of E_μ, viewed as a matrix with columns indexed by monomials, the non-standard ones coming first, becomes the identity (of size the number of incomparable pairs from $W\lambda$ whose sums equal μ) by means of elementary row operations.

In general, the computation of Ω (or, to be more precise, of X_α on a weight vector) requires knowledge of the structure constants of the Lie algebra L. Apart from these data, all necessary ingredients are implemented in LiE (cf. [LiE]).

Example In the case of A_n, the knowledge of the structure constants is hidden in the tableaux calculus. We have

$$\Phi^+ = \{\epsilon_i - \epsilon_j \mid 1 \le i < j \le n+1\}$$

$$X_\alpha = X_{ij} \text{ if } \alpha = \epsilon_i - \epsilon_j \in \Phi$$

$$X_{ij}[k_1, \ldots, k_t] = \begin{cases} [k_1, \ldots, k_{\ell-1}, i, k_{\ell+1} \ldots, k_t] & \text{if } j = k_\ell \text{ for a unique } \ell \in \{1, \ldots, t\} \\ 0 & \text{otherwise} \end{cases}$$

$$X_{ji} X_{ij}[k_1, \ldots, k_t] = \begin{cases} [k_1, \ldots, k_t] & \text{if } j = k_\ell \text{ for a unique } \ell \in \{1, \ldots, t\} \text{ and } i \text{ does not occur,} \\ 0 & \text{otherwise} \end{cases}$$

$$\kappa(\alpha, \beta) = 2(n+1)\alpha^\top \mathcal{C} \beta \quad (\alpha, \beta \in \Phi),$$

where $\mathcal{C}$ is the Cartan matrix of A_n

$$(\mu, \nu) = \mu^\top \mathcal{C}^{-1} \nu \quad (\mu, \nu \text{ weights})$$

We finish by illustrating the algorithm for $A_{5,3}$, focusing on the most difficult weight occurring in $S^2(V(\lambda))$, namely the zero weight. The computations have been done in LiE. In the lexicographic total ordering, there are 10 quadratic monomials of weight 0. They are

$$[136][245],[145][236],[146][235],[156][234],[126][345],$$
$$[123][456],[124][356],[125][346],[134][256],[135][246].$$

Their order of appearance is such that the non-standard ones come first. Applying $\Omega - (2\lambda+\delta, 2\lambda) = \Omega - 24$ to each of these monomials gives polynomials which are presented as rows of the following coefficient matrix. Here, the j-th entry of the i-th row represents the coefficient of the j-th monomial in the image of the i-th monomial under $\Omega - 24$.

$$2\cdot\begin{pmatrix}
-3 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1\\
1 & -3 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1\\
1 & 1 & -3 & 1 & -1 & -1 & -1 & -1 & 1 & 1\\
-1 & -1 & 1 & -3 & 1 & 1 & -1 & -1 & 1 & 1\\
1 & -1 & -1 & 1 & -3 & 1 & -1 & 1 & -1 & 1\\
-1 & 1 & -1 & 1 & 1 & -3 & 1 & -1 & -1 & 1\\
-1 & 1 & -1 & -1 & -1 & 1 & -3 & 1 & 1 & 1\\
1 & -1 & -1 & -1 & 1 & -1 & 1 & -3 & 1 & 1\\
-1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -3 & 1\\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -3
\end{pmatrix}.$$

The rank of this matrix is 5. Now we can apply classic Gauss elimination to obtain reduced Gröbner basis elements. The result is:

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -1\\
0 & 1 & 0 & 0 & 0 & 1 & -1 & 1 & 1 & -1\\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & -1\\
0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 & -1 & 0\\
0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 & 0 & 0
\end{pmatrix}$$

which tells us that the contribution E_0 to E is the following set of five quadratic polynomials:

$$E_0 = \left\{\begin{array}{l}
[136][245]+[123][456]+[134][256]-[135][246],\\
[145][236]+[123][456]-[124][356]+[125][346]+\\
\quad +[134][256]-[135][246],\\
[146][235]+[123][456]+[125][346]-[135][246],\\
[156][234]-[123][456]+[124][356]-[134][256],\\
[126][345]-[123][456]+[124][356]-[125][346]
\end{array}\right\}.$$

Together with the 30 Plücker relations of the form

$$[i\,j\,k]\,[i\,\ell\,m]+[i\,j\,\ell]\,[i\,m\,k]+[i\,j\,m]\,[i\,k\,\ell]$$

$$(\{i, j, k, \ell, m\} \text{ a 5-set in } \{1, 2, 3, 4, 5, 6\}),$$

the set E_0 yields the reduced Gröbner basis E (of size 35) of the Grassmannian $G(6,3)$ with respect to any lexicographic ordering $<$ in which $x_\alpha > x_\beta$ whenever $x_\alpha \succ x_\beta$.

7. Concluding remarks

Theorem 5.1(i) gives a clear indication that in the non-minuscule weights case, all hope for finding a Gröbner basis need not be lost. Also, standard monomial theory has been generalized to the case of quasi-minuscule weights (see [LaSh]). It is our intention to study how far the correspondence with Gröbner basis theory can be pushed.

As we have indicated above, the Plücker relations are in general different from the reduced Gröbner basis elements obtained by use of the Casimir operator. To derive the actual Plücker relations, we have reason to believe that Casimir operators relative to Levi subalgebras are the appropriate tool.

In [StWh2], it is indicated how the poset $(P_{n,d}, \succ)$ can be used to find a Stanley decomposition of the ring $\mathcal{A}$. By means of a lexicographical shelling of the poset (cf. [BjWa]), this construction can also be generalized to one for all minuscule weights.

8. References

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