Isosceles point sets in $R%5Ed$

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ISOSCELES POINT SETS IN R^d

by Aart Blokhuis

Abstract.

Let X be a set of points in R^d such that the triangle determined by an arbitrary triple from X is isosceles, then $\operatorname{card}(X) \leq \frac{1}{2}(d+1)(d+2)$.

Introduction.

An isosceles set X in R^d is a set of points such that any triple among them determines an isosceles triangle. Isosceles sets where introduced by Kelly, the problem goes back to Erdős [1] .
Troughout the article X will denote an isosceles set in R^d, $X = \{x_1, x_2, \ldots, x_v\}$ and we assume

$$\operatorname{aff}(X) = \left\{ \sum_{i=1}^{v} a_i x_i \,\middle|\, \sum a_i = 1 \right\} = R^d .$$

For any subset $X_1 \subset X$, $\dim(X_1)$ denotes the dimension of $\operatorname{aff}(X_1)$.
By $A(X)$ we mean the set of distances between points of X.
For $a \in A(X)$ we denote by X_a the graph with point set X and edges the pairs of points at distance a. X is called decomposable if it is possible to partition X in sets X_1 and X_2 with $|X_2| > 1$ such that a point of X_1 has the same distance to all points of X_2 . (This distance may be different for several points of X_1 though.)
Finally if $\operatorname{card}(A(X)) = 2$; X is called a two-distance set.

The structure of isosceles sets.

Lemma 1. If X is decomposable, and (X_1, X_2) is a decomposition for X, then $\dim(X_1) + \dim(X_2) \leq \dim(X)$.

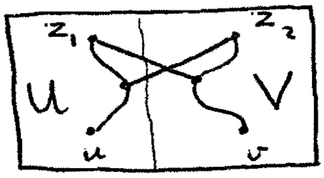
Proof. Let P be the orthogonal projection on $\operatorname{Aff}(X_2)$. Then for any $x_1 \in X_1$, Px_1 is the center of a sphere in $\operatorname{Aff}(X_2)$ containing X_2 . Since X_2 spans $\operatorname{Aff}(X_2)$ P maps X_1 onto a single point. Therefore the flats $\operatorname{Aff}(X_1)$ and $\operatorname{Aff}(X_2)$ are orthogonal and the result follows.

<u>Lemma</u> 2. If X is indecomposable then it is a two-distance set.

The proof is split into three parts, first we examine the case that there is some distance a for which X_a is disconnected. Then we look at the case where there is some a for which X_a has diameter larger then two. And finally we consider the case that X_a has diameter two for each $a \in A(X)$.

Case 1. Suppose there is an $a \in A(X)$ such that X_a is disconnected, then X is decomposable, for let X_2 be a component of X_a having more then 1 point. From the isosceles property it now follows that any point not in X_2 has the same distance to all points in X_2.

Case 2. Now suppose X_a is connected for all $a \in A(X)$ and let b be a distance such that there are two points, u and v at distance 3 in X_b. Let a be the euclidean distance between u and v. We claim that X is a two-distance set. Let U be the set of points in X that are closer to u than to v in the graph X_b and let $V = X \setminus U$. For any z in U there is a (u,z) path entirely in U so by the isosceles property v and z have distance a. Similarly u has (Euclidean) distance a to any point in V. Now take $z_1 \in U$ and $z_2 \in V$ and let P_1 be a shortest (z_1,u) path and P_2 a shortes (z_2,v) path. If z_1 is adjacent to z_2 in X_b they have distance b which is okay. If z_1 is not adjacent to any point on P_2 then they have distance a by the isosceles property, similarly if z_2 is not adjacent to any point of P_1. Now if both points do have a neighbour on the other path it is clear from the picture that the following holds :



$$d_b(v,z_1) \leq d_b(v,z_2) \leq d_b(u,z_2) \leq d_b(u,z_1),$$

contradiction. Now for any further distance c the graph cannot be connected since U and V are only joined by distances a and b. Therefore X is a two-distance set.

Case 3. Finally we suppose that X_a is connected for every distance a and has diameter 2. Now suppose there are three distances, call them a, b and c. We will construct an infinite subset of X thus obtaining a contradiction.

Let z be an arbitrary point in X and a_1 a point at distance a from z. In X there is a point b_1 having distance b to both z and a_1 for the diameter of X_b is 2. Similarly we can find a point c_1 having distance c to both z and b_1. Since c_1a_1 is part of the triangle $c_1a_1b_1$, c_1a_1 is either c or b, but since it is also a side of the triangle c_1a_1z it is either a or c, and therefore it has to be c. Now let a_2 be a point at distance a from both c_1 and z and define $b_2, c_2, a_3, \ldots$ in the way indicated above, we will show that at each step at the construction of the infinite set the last constructed point has the same distance to all previous constructed points. Suppose the last point we added was a_k, we assume that our induction assumption holds for all points preceeding a_k, i.e. if d_j is a point of the sequence, where $d = a, b$ or c and $j < k$ then d_j has distance d to all points preceeding d_j. By definition a_k has distance a to z and c_{k-1}. Comparing the triangles za_kb_j and $c_{k-1}a_kb_j$ we see that a_kb_j is a. Similarly, comparing the triangles za_kc_j and $b_{j+1}a_kc_j$ (where $j+1 < k$) we conclude that a_kc_j is a. Finally the triangles $b_{k-1}a_ka_j$ and $c_{k-1}a_ka_j$ force a_ka_j to be a. Since every point has a different distance to it's immediate successor and it's predecessors all points we obtain in this way are new, therefore we constructed an infinite subset of X, a contradiction. Therefore X is two-distance set.

<u>Theorem.</u> Let X be an isosceles set in R^d, then

$$\operatorname{card}(X) \leq \tfrac{1}{2}(d+1)(d+2) \quad ,$$

equality implies that X is a two-distance set, or a spherical two-distance set together with the center.

Proof. The proof is by induction. If d=1 then 3 is the maximal cardinality and X is a spherical set together with it's center. For d=2 Kelly [1] proved that the maximum is 6 realized only by the centered regular pentagon.

Now let $d > 2$. If X is a two-distance set then we have the requiered inequality (see [3]). Now suppose X is decomposable, (X_1, X_2) being a decomposition.

Case 1. $\dim X_1 \neq 0$. Since $|X_2| > 1$ we have $0 < \dim X_1 < d$. Let $d_1 = \dim X_1$,then by induction we have:

$$|X| \leq \tfrac{1}{2}(d_1+1)(d_1+2) + \tfrac{1}{2}(d-d_1+1)(d-d_1+2) < \tfrac{1}{2}(d+1)(d+2) .$$

Case 2. $\dim X_1 = 0$. In this case X_1 is a single point and therefore X_2 is spherical. If X_2 is not a two-distance set it is decomposable say $X_2 = (X_2', X_2'')$. But now $(X_1 \cup X_2', X_2'')$ is a decomposition of X as in Case 1. This finishes the proof.

Final Remarks.

Cases 2 and 3 in the proof of Lemma 2. can be considered as the proof of the following pure graph-theoretic theorem:

Let K_n (the complete graph on n vertices) be edge-colored with k colors, such that (i) every triangle has at most two colors, and (ii) for each color, the induced graph on that color is connected. Then $k=2$.

I wonder what the implications and possible generalizations of a theorem like this are in graph-theory.

References.

[1] L.M. Kelly , "Elementary Problems and Solutions, Isosceles n-points", Amer. Math. Monthly, 54 (1947), 227-229 .

[2] D.G. Larman, C.A. Rogers and J.J. Seidel, "On Two-Distance Sets in Euclidean Space. Bull. London Math. Soc., 9 (1977) 261-267 .

[3] A. Blokhuis, A new upperbound for the cardinality of 2-distance sets in Euclidean space. THE memorandum 1981-04 .