# Optimization of hydro energy power plants

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## OPTIMIZATION OF HYDRO ENERGY POWER PLANTS

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# Optimization of Hydro Energy Power Plants

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## Information about this Report

In 1989, from September 22 till September 29, Prof. H.J. Wacker from the Johannes Kepler Universität (Linz, Austria) gave a course on Optimization at the University of Technology Eindhoven (The Netherlands). This course was given by him to an audience of ECMI-students and faculty staff members.

After having finished his lectures Prof. Wacker left some problems to be solved behind him in Eindhoven. One of those problems was given to three students as a problem to be solved for the "Modelling Colloquium".

This "Modelling Colloquium" is an important part of the ECMI-education in Eindhoven. All the Dutch ECMI-students have to play an active part in this course. They are splitted into small groups, two to four persons, and must try to solve a (mathematical) modelling problem. They work on this problem for about three to four months, and finish it by writing a report, in English, about it. As the name of the course suggests emphasis is placed on the modelling aspects of the problem at hand rather than on the mathematical techniques to solve it.

This report is the result of the work done by the three students on the problem from Prof. Wacker. This was a problem about the optimization of hydro energy power plants. In his lectures Prof. Wacker presented a rather difficult theorem which was used in order to obtain a solution for the optimization problem. However, we do not use this theorem. Instead of using this theorem, we tried to find an easier way to get a solution. We chose this option because we wanted to make new models of a power plant. In this way we were able to train our modelling skills.

Our solution to the problem is the subject of this report.

We hope you enjoy reading this report.

# 1. Introduction to the Problem

## § 1.1 General Introduction

In Austria the energy production by means of hydro energy power plants is, in contrast to most European countries, a very important economic factor. In 1983, for instance, hydro energy constituted 68% of the total energy production.

There are two types of hydro energy power plants: river power plants and storage power plants. The total energy production of river power plants is more than twice as large as that of storage power plants. However, storage power plants are more important. The reason for this is that storage power plants allow the producer to produce the most energy when the demands are the highest.

Storage power plants can be classified according to their "characteristic" periods. A characteristic period of a storage power plant is the time-interval in which the full reservoir can be emptied and refilled again. For small reservoirs this time-interval is at least some hours, and for large reservoirs at most one year.

Another important feature of a storage power plant is whether or not it is part of a system of (serially) connected storage power plants. Of course the operation of such systems is much more difficult to optimize than only one storage power plant.

In this report we try to maximize the profit of two storage power plants over one week. These storage power plants are situated near the Schwarzenegger lake in Austria. The two power plants operate differently with respect to the discharges. For one reservoir, the older one, the discharges can only change at some fixed time-points during the day. For the other power plant, the modern power plant, the discharge can vary continuously during the day.

The profit depends on the tariff, which is time-dependent (e.g. different tariffs for day and night), on the discharge and on the height of water in the reservoir. During a period of maximal tariff we would like to have both maximal discharge and maximal height of water in the reservoirs. This is obviously impossible. So, we have to find some sort of balance. It is important to realize that the reservoirs and the outlets of the reservoirs have limited capacity. For example, the characteristic period of the reservoirs under consideration is about one day.

## § 1.2 The Problem

As already mentioned in the preceding section, in this report we consider a system with two power plants (See figure 1.1). Because these power plants are rather small (the characteristic periods are about one day), and because the power plants are connected to the comparatively large Schwarzenegger lake, which guarantees a constant influx of water in the reservoirs, we can consider the two power plants to be independent. This means that we can optimize the two power plant independently. The resulting two optimal controls together form the optimal control for the whole system.

We try to maximize the profits of the given power plants over a week. The profit is the integral of the product of the tariff, which is a time-dependent block-function, and the momental power production. This momental power production is a function of the head of the water reservoir and the discharge. By means of the discharge the producer can control the momental power production.

The capacity of the reservoirs and the capacity of the outlet lead to some constraints on the total amount of available water and on the discharge respectively. Moreover for one reservoir, the old one, only discharges of a special type are possible. Since for this power plant it is only possible to switch from one constant discharge to another one at fixed time-points.

With these constraints, taking the profit as object-function, together with integral formula to describe the water flow, we have a nice way to consider the actual problem as an optimal control problem.

The following numerical data are given to us by the producer of the hydro energy in the two power plants. These data are summarized again in Chapter 4.

<u>The old power plant:</u>

- The volume of the reservoir V must satisfy $V_{min} \leq V \leq V_{max}$.
  Where $V_{min} = 500000 \text{ m}^3$ from Saturday 6 am till Sunday 12 pm.
  $V_{min} = 50000 \text{ m}^3$ the rest of the week, and $V_{max} = 750000 \text{ m}^3$.
- The constant influx Z is given by $Z = 10 \text{ m}^3/\text{s}$.
- The maximal height $H_{max} = 165$ m.
- The discharge Q must satisfy $Q_{min} \leq Q \leq Q_{max}$.
  Where $Q_{min} = 0 \text{ m}^3/\text{s}$ and $Q_{max} = 30 \text{ m}^3/\text{s}$.
- Because this is an old power plant, which is difficult to operate, the operators decided to change the discharge at fixed time-points. They

choose these fixed time-points to be the moments at which the tariff switches.

For the modern power plant we have the same numerical data.

The modern power plant:

- The volume of the reservoir V must satisfy $V_{min} \leq V \leq V_{max}$.
  Where $V_{min} = 500000\ m^3$ from Saturday 6 am till Sunday 12 pm,
  $V_{min} = 50000\ m^3$ the rest of the week, and $V_{max} = 750000\ m^3$.
- The constant influx $Z = 10\ m^3/s$.
- The maximal height $H_{max} = 165$ m.
- The discharge Q must satisfy $Q_{min} \leq Q \leq Q_{max}$.
  Where $Q_{min} = 0\ m^3/s$, and $Q_{max} = 30\ m^3/s$.
- With this modern power plant it is possible to change the discharges at any moment.

(N.B. We use the same notation for the old and for the modern power plant. Because the power plants are optimized independently, we think that this will not confuse the reader.)

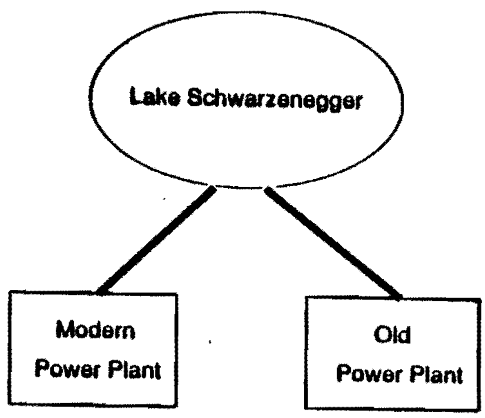


Figure 1.1

## § 1.3 The Results

The optimal discharge structures are drawn in Figure 5.1 for the old power plant, and in Figure 5.2 for the modern power plant. These optimal discharge functions will be discussed in Chapter 5. In general, we can say that the optimal discharge functions have a bang-bang like structure. This is proven in Chapter 3.

# 2. The Problem

## § 2.1 Mathematical Formulation of the Problem

In this section we define some constants, parameters, variables and functions. Some of them have been defined implicitly in the preceding section. Here, we also state which relations and constraints between them must be satisfied. We end by giving the mathematical formulation of our optimal control problem.

We define :

(*) $t_0$ is the time at which we start to optimize.
$t_N$ is the time at which we end.

(*) $V_0$ is the volume of water in the reservoir at time $t_0$.
$V_N$ is the minimal volume of water in the reservoir at time $t_N$.

(*) $V_{min}$ and $V_{max}$ are the minimal and maximal volume of water respectively in the reservoir under consideration.

(*) $Q_{min}$ and $Q_{max}$ are the minimal and maximal discharge respectively of this reservoir.

We also define the following functions :

(*) $Q(t)$ is the discharge from the reservoir at time t. We are able to control this function. So, in mathematical terms we can take this as a control variable.

(*) $Z(t)$ is the known influx at time t.

(*) $V_Q(t)$ is the volume of water in the reservoir at time t if the discharge is given by the function Q.

(*) $A(t)$ is the tariff function. This function can be described by :
$A(t) = A_i$ for $t \in (t_{i-1}, t_i]$ $\quad ( i = 1,..., N )$
for certain moments $t_0, t_1, t_2,..., t_N$. The $A_i$'s are constants.

With these functions we define :

(*) The height H of water in the reservoir is a function of $V_Q(t)$.
$H(t) = h(V_Q(t))$ , with h strictly increasing.

A typical example, which actually describes the modern reservoir, is

$$h(V) = 160 + \sqrt{V / 30000}\ .$$

(*) The power production P at time t is a function of $Q(t)$ and $V_Q(t)$.
$P(t) = p(Q(t),V_Q(t))$. We will use $p(Q,V) = Qf(V)$ , with f strictly

increasing. In Section 3.1 and Section 3.2 we take $f(V) = ch(V)$ with c a constant number.

(*) The spillage S is a function of $V_Q(t)$.

$S(t) = s(V_Q(t))$.

We have the following relations and constraints :

(1) $V_Q(t) = V_0 + \int_{t_0}^{t} (Z(\tau) - Q(\tau) - S(\tau))d\tau$

(2) $V_Q(0) = V_0$

(3) $V_Q(t_N) \geq V_N$

(4) $V_{min} \leq V_Q(t) \leq V_{max}$ ( $t \in [t_0,t_N]$ )

(5) $Q_{min} \leq Q(t) \leq Q_{max}$ ( $t \in [t_0,t_N]$ )

Our optimal control problem can be stated in the following way:
Maximize the quantity $\int_{t_0}^{t_N} A(t)p(Q(t),V_Q(t))dt$ (= the profit) with respect to all discharges such that (1), (2), (3), (4) and (5) are satisfied.

## § 2.2 A Dynamic Programming Approach

### § 2.2.1 Introduction

An approximation of the solution of the mathematical problem defined in Section 2.1 can be found by solving a corresponding discrete problem. For a suitable discretization of time, the mathematical model sketched in Section 2.1 can be reduced to a discrete problem, in the sense that integrals are replaced by sums. The solution of this discrete problem is an approximation of the solution of the continuous problem, and can be handled with the aid of dynamic programming. The general method to do this is sketched in this section. In Section 3 it is applied to some special situations.

### § 2.2.2 Formulation of the Discrete Problem

The general idea is the following. Subdivide the period $[t_0,t_N]$ in M subperiods. At the beginning of each subperiod i the decision must be taken of what type the discharge must be in this subperiod . So, we also have M decision time-points.

More formally we have :

$\tau_0, \ldots, \tau_M$ : decision time-points,

$V_i$ : volume at time-point $\tau_i$ (= state variable),

$Q_i$ : discharge function for the period i (= control function).

In order to apply the principle of dynamic programming, the subdivision must be chosen in such a way that the tariff is constant on each period.

The volume $V_{i+1}$ depends on the volume $V_i$, and on the discharge function $Q_i$ defined on $[\tau_{i-1},\tau_i]$. The discharges $Q_i$ are taken out of some function space $\mathcal{F}_i$.

More formally we define transition functions $f_i$ by $f_i(V_{i-1},Q_i) =: V_i$. The function $f_i$ describes which state we obtain at time-point $\tau_i$ if we start with state $V_{i-1}$ at time-point $\tau_{i-1}$, and use a discharge $Q_i$ on $[\tau_{i-1},\tau_i]$. Our goal is to maximize the profit $G_i$ over the whole period $[t_0,t_N]$. Therefore, we define a profit function $G_i$ for each interval i.

In the problem under consideration we have :

$$f_i(V_{i-1},Q_i) = V_{i-1} + \int_{\tau_{i-1}}^{\tau_i} (Z(\tau) - Q(\tau) - S(\tau))d\tau \qquad \text{for all } i \in \{1,\ldots,M\},$$

and

$$G_i(V_{i-1},Q_i) = \int_{\tau_{i-1}}^{\tau_i} A_i p(Q(\tau),V_Q(\tau))\, d\tau \qquad \text{for all } i \in \{1,\ldots,M\},$$

, where $A_i$ = tariff of period i (see Section 2.1).

We now arrive at the following discretized problem:

Maximize $\sum_{i=1}^{M} G_i(V_{i-1},Q_i)$ such that $Q_i \in \mathcal{F}_i$, and $V_i = f_i(V_{i-1},Q_i)$ (for all i $\in \{1,\ldots,M\}$), here $\mathcal{F}_i$ is the decision function space for period i. It is at least to be taken such that $V_{Q_i}(t) \in [V_{min},V_{max}]$, and $Q_i(t) \in [Q_{min},Q_{max}]$ for all $t \in [t_{i-1},t_i]$.

To solve this optimal control problem we can use the principle of dynamic programming (Ref. 1). For this we define the following functions:

$Q_{i,opt}$: optimal decision function for the decision at time-point $t_{M-i+1}$,

$\mathcal{P}_i$: total value of returns over the last i periods when optimal decisions are taken.

The functions $Q_{i,opt}$ and $\mathcal{P}_i$ are determined recursively (Ref. 1):

$$\mathcal{P}_1(V_{M-1}) = \max_{Q_M \in \mathcal{F}_M} G_M(V_{M-1},Q_M),$$

$$\mathcal{P}_2(V_{M-2}) = \max_{Q_{M-1} \in \mathcal{F}_{m-1}} \{G_{M-1}(V_{m-2},Q_{M-1}) + f_1(S_{M-1}(V_{M-2},Q_{M-1}))\}$$

etc. until

$$\mathcal{P}_M(V_0) = \max_{Q_1 \in \mathcal{F}_1} \{G_1(V_0,Q_1) + f_{M-1}(S_1(V_0,Q_1))\}.$$

## 3. Some Models

In this chapter, we give some models to describe the hydro energy power plants. Because in former times the strategy was to keep the reservoirs full all the time, there used to be no need to measure the spillage. The general opinion of the operators is however, that spillage is negligible. For this reason, we concentrate on models without spillage. In fact, only for these models we have numerical results. However, in Section 3.3 and Section 3.4 we do describe two models which include spillage. These models might be useful if spillage turns out to be not negligible.

Note that if spillage is constant we can still use the models without spillage. In this case one can say that the net influx is the constant influx minus the constant spillage. Then one can use the net influx instead of the influx in the calculations.

As we already indicated in the previous chapter, we use a dynamic programming approach in order to obtain an approximation for the optimal solution.

### § 3.1 A Model for the Old Power Plant

We start with a model for the old power plant. In this model there is no spillage, and the influx Z is constant in time. Furthermore time is made discrete by splitting each tariff period into smaller subperiods. During these subperiods the discharge Q is constant.

Now, we can formulate this model as a dynamic programming problem as follows.

1. The time is discretized into $M \geq N$ periods $\tau_0 < \tau_1 < ... < \tau_M$, where $\tau_0 = t_0$, $\tau_M = t_N$ and $\{t_0,t_1,...,t_N\} \subseteq \{\tau_0,\tau_1,...,\tau_M\}$.
2. For ease of notation we introduce new tariff constants $B_1,B_2,...,B_M$, such

that for each time t, the tariff at time t will be $B_i$, when $\tau_{i-1} \leq t < \tau_i$.
3. The state at time-point i, $V_i$, should be between $V_{max}$ and $V_{min}$.
4. The transition function $f_i(V_i,Q_i) = V_i + (Z - Q_i)(\tau_{i+1} - \tau_i)$.
5. The allowed decision set at decision time-point i, $S_i(V_i)$, should allow as decision functions only those $Q_i$ that would not let the reservoir overflow or underflow.

Therefore $S_i(V_i) = \{ Q_i \in [Q_{min},Q_{max}] \,|\, V_{min} \leq f(V_i,Q_i) \leq V_{max} \}$.

6. The profit earned during period i $G_i(V_i,Q_i)$ is given by:

$$G_i(V_i,Q_i) = \int_{\tau_{i-}}^{\tau_i} cQ_i h(V_i + (Z - Q_i)(t - \tau_{i-1}))dt.$$

With these definitions the problem can be stated as :

$$\max \{ \sum_{i=0}^{M-1} G_i(V_i,Q_i) \mid V_{i+1} = f_i(V_i,Q_i),\ Q_i \in S_i(V_i)\ (i = 0,\dots,M-1),\ Q_M = V_N \}.$$

## § 3.2 A Model for the Modern Power Plant

The previous model has two disadvantages if we use this as a model for the modern power plant.

The first disadvantage is that only an approximation of the optimal control function can be given.

The second disadvantage is that the total period has to be partitioned into very small periods in order to reach a certain precision. In this section we prove that the optimal control function has a bang-bang like structure, but the pieces where it is constant still can form any partition of the total period. A reasonably good approximation of the optimal control function can be obtained only if the total period is partitioned into sufficiently small periods. This will increase computing time.

The model for the modern power plant, which is described below, has only one disadvantage namely, still only an approximation of the optimal control function can be given. It is not necessary however, to partition the total period into very small subperiods in order to obtain precise results.

This model for the modern power plant is obtained by using two theorems. These two theorems describe the structure of the optimal control functions if the volumes of the water at the decision time-points are known. In the model for the modern power plant we use these optimal parts of the control function instead of constant parts.

The state at the end of a tariff period depends on the volume of the reservoir at the beginning of that period and on the *total* discharge of water in that period. Since there is no spillage , the state does not depend on the way in which water is discharged. However, the way in which the water is discharged does influence the profit earned in that period, since the height from which the water falls down is important. This height should be as great as possible, in order to maximize the profit during that period, given the initial and final volume. The maximization of the height can be accomplished by waiting as long as possible before discharging the water, thus filling the reservoir. If $V_{max}$ is reached, the discharge Q is taken equal to the influx Z, thus keeping $V = V_{max}$. This is the idea behind the optimal strategy proven in the next theorem.

**Theorem 3.1** Let A(t) be constant during $t \in [a,b)$ and $V(a) = V_a$. Let the objective be to reach a final volume $V(b) = V_b$, then the optimal strategy to reach this final volume is as follows:

$Q_{opt}(t) = Q_{min}$ if $t \in [a,\tau_a)$

$Q_{opt}(t) = Z(t)$ if $t \in [\tau_a,\tau_b)$

$Q_{opt}(t) = Q_{max}$ if $t \in [\tau_b, b)$

With $\tau_a$ such that

$$V_a + \int_a^{\tau_a}(Z(t) - Q_{min})dt = V_{max},$$

and $\tau_b$ such that

$$V_{max} + \int_{\tau_b}^{b}(Z(t) - Q_{max})dt = V_b$$

Unless $\tau_a \leq \tau_b$, in which case $\tau_a = \tau_b$ are such that

$$V_a + \int_a^{\tau_a}(Z(t) - Q_{min})dt + \int_{\tau_a}^{b}(Z(t) - Q_{max})dt = V_b.$$

**Proof:** The proof is rather straightforward. Let Q be an arbitrarily control function. Then

$$V_{Q_{opt}}(t) = V_a + \int_a^t(Z - Q_{min})dt \geq V_a + \int_a^t(Z - Q)dt = V_Q(t) \quad \text{if } t \in [a,\tau_a)$$

$$V_{Q_{opt}}(t) = V_{max} \geq V_Q(t) \quad \text{if } t \in [\tau_a,\tau_b)$$

$$V_{Q_{opt}}(t) = V_b - \int_t^b(Z - Q_{max})dt \geq V_b + \int_t^b(Z - Q)dt = V_Q(t) \quad \text{if } t \in [\tau_b,b)$$

Thus $V_{Q_{opt}} \geq V_Q$ for all strategies Q. The profit for a strategy Q is

$$G_Q = \int_a^b cQ(t)h(V_Q(t))dt$$

$$= \int_a^b c(Q(t) - Z + Z)h(V_Q(t))dt$$

$$= \int_a^b cZh(V_Q(t))dt - \int_a^b c\dot{V}_Q(t)h(V_Q(t))dt$$

$$= \int_a^b cZh(V_Q(t))dt - cH(V_b) + cH(V_a),$$

in which H is the primitive function of h. Thus

$$G_{Q_{opt}} = \int_a^b cZh(V_{Q_{opt}}(t))dt + cH(V_a) - cH(V_b)$$

$$\geq \int_a^b cZh(V_Q(t))dt + cH(V_a) - cH(V_b)$$
$$= G_Q,$$

which concludes the proof. ∎

This theorem only says something about the optimal strategy between tariff switches. However, an extension to the whole period is easily made by the following theorem.

**Theorem 3.2** If an optimal strategy exists, then an optimal strategy exists with the optimal form described in theorem 3.1 during each period of constant tariff.

**Proof:** Let Q be an optimal strategy of any form. Then between two tariff switches $t_{i-1}$ and $t_i$ the strategy has a certain form. Replacing this form by the optimal form described in theorem 3.1, such that $V(t_{i-1})$ and $V(t_i)$ do not change, can only improve the profit during this period, while the other profits remain the same. Since this argument holds for all i, the theorem is proven. ∎

We can formulate the model for the modern power plant as a dynamic programming problem as follows:

1. The state at time-point $\tau_i$, $V_i$, should be between $V_{max}$ and $V_{min}$.
2. The decision variable at time-point $\tau_i$, $X_i$, indicates the volume you want to have at the end of a period. This volume should be attainable so the allowed decision set should take this into consideration.
3. The allowed decision set at time-point $\tau_i$ is given by

$$\mathcal{F}_i(V_i) = \{ X_i \in [V_{min},V_{max}] \mid \begin{array}{l} X_i \geq V_i + (Z - Q_{max})(t_{i+1} - t_i) \\ X_i \leq V_i + (Z - Q_{min})(t_{i+1} - t_i) \end{array} \}.$$

4. The transition function $f_i(V_i,X_i) = X_i$.
5. The profit during period i is given by:

$$G_i(V_i,X_i) = \int_{t_{i-1}}^{t_i} cQ_{i,opt}h(V_i + (Z - Q_{i,opt})(t - t_{i-1}))dt,$$

where $Q_{i,opt}$ is a strategy of the form described in Theorem 3.1, and such that $V_{Qopt}(t_{i-1}) = V_i$ and $V_{Qopt}(t_i) = X_i$.

With these definitions we find the following problem:

$$\max_{X_0,\ldots,X_{N-1}} \{ \sum_{i=0}^{N-1} G_i(V_i,X_i) \mid V_{i+1} = f_i(V_i,X_i),\ X_i \in \mathcal{F}_i(V_i) \quad \text{for } i = 0,\ldots,N-1 \}$$

This is a problem which can be solved with the aid of dynamic programming.

## § 3.3 A General Model with Spillage

In this section we return to the most general model as it is presented in Chapter 2. In contrast to the two preceding sections, in this section and in the following section we assume that the spillage is of importance. It turns out that for the old power plant it is possible to find the optimal solution through the use of dynamic programming, but for the modern power plant this is in general not possible. However, we can always calculate an approximation of the optimal solution. Moreover, in case of certain special spillage functions we can find the optimal solution for the modern power plant. This is done in Section 3.4.

**Formulation of the Discrete Problem.** Again we introduce M decision time-points. In order to apply dynamic programming, we assume Q to be constant during each subperiod. It is clear that the decision function set $\mathcal{F}_i$ is compact for all i, thus dynamic programming can be applied. It is easy to formulate the discrete problem. See for instance Section 3.1.

## § 3.4 Models with Special Spillage Functions

In Section 3.2 the solution of the initial problem is obtained by first determining the optimal structure in subperiods with a constant tariff. Then the optimal solution for the whole period is found by applying the principle of dynamic programming. In this section a similar procedure is applied to solve a problem in which spillage is included. As in Section 3.2 we assume that the discharge can be changed any moment. So, these models are only suited for the modern power plant.

**The optimal structure of the solution.** Consider a time interval [a,b] in which the tariff is constant. We assume that the influx Z is constant on [a,b]. We try to find the optimal structure on [a,b] given the volume at t = a, $V_a$, and the volume at t = b, $V_b$. In the deduction of the optimal structure the following function plays a crucial role:

$I(V) = [Z - S(V)]h(V) \qquad V \in [V_{min}, V_{max}].$

We can present three conditions for the function I which assure the

existence of an optimal structure.
(1) I has a maximum on $[V_{min},V_{max}]$ at $V = V^*$.
(2) I is strictly increasing on $[V_{min},V^*]$.
(3) I is strictly decreasing on $[V^*,V_{max}]$.
If the conditions (1),(2), and (3) are fulfilled then an optimal structure exists. This optimal structure is given in Theorem 3.3.

**Theorem 3.3** Let $t_a$ and $t_b$ be time-points in [a,b] such that $a \leq t_a \leq t_b \leq b$, and $V(t_a) = V^* = V(t_b)$. Then the optimal control function $Q_{opt}$ is given by:

$Q_{opt} = Q_{max}$ on $[a,t_a]$ if $V_a > V^*$
$Q_{opt} = Q_{min}$ on $[a,t_a]$ if $V_a < V^*$
$Q_{opt} = Z - S(V^*)$ on $(t_a,t_b)$
$Q_{opt} = Q_{min}$ on $[t_b,b]$ if $V_b > V^*$
$Q_{opt} = Q_{max}$ on $[t_b,b]$ if $V_b < V^*$.

If $V_a = V^*$ then $t_a$ will be equal to a, and if $V_b = V^*$ then $t_b$ will be equal to b.

In Figure 3.1 and Figure 3.2 two examples of the optimal structure are presented. In these figures the volume V is chosen instead of the control function Q in order to clarify the optimal structure. The relation between Q and V is given by: $\dot{V} = Z - Q - S$.

**Proof of Theorem 3.3:** Let $Q_{opt}$ be the control function on [a,b] as in the theorem, and let $V_{opt}$ be the corresponding volume function. The goal is to maximize the profit with respect to the control function Q. The profit is given by:

$$\int_a^b cQ(t)h(V(t))dt$$ , where c is a constant including tariff etc.

We substitute $Q = Z - S - \dot{V}$ in the integral above, and find the problem:

$$\max_Q \left\{ \int_a^b c(Z - S)h(V)dt - \int_a^b c\dot{V}h(V)dt \right\}$$ , which is equivalent to

$$\max_Q \int_a^b c(Z - S)h(V)dt - \max_Q \int_a^b ch(V)dV \qquad (*)$$

Since $\int_a^b ch(V)dV$ is independent of Q, we only have to maximize the first part of (*). Furthermore, because C is a constant (*) is equivalent to maximizing $\int_a^b I(V)dt$ with respect to Q. (**).

In order to have $Q_{opt}$ as the optimal control function and $V_{opt}$ as the corresponding optimal volume function, the following equation must be valid:

$$\int_a^b I(V_{opt})dt - \int_a^b I(V_{opt} + V')dt \geq 0$$

, where $V_{opt} + V'$ is a feasible volume function on [a,b]. In order to prove this relation we subdivide [a,b] into three subintervals, namely: $[a,t_a]$, $(t_a,t_b)$, and $[t_b,b]$.

<u>Subinterval $[a,t_a]$:</u>

On this interval we have the following equations. These equations follow immediately from the definitions of $Q_{opt}$ and $V_{opt}$.

$V_{opt}(t) \leq V_{opt}(t) + V'(t)$ if $V_a > V^*$

$V_{opt}(t) \geq V_{opt}(t) + V'(t)$ if $V_a < V^*$

$V_{opt}(t) = V^*$ if $V_a = V^*$

From the conditions (1),(2), and (3) on page 11 follows:

$V_{opt}(t) \leq V_{opt}(t) + V'(t) \Rightarrow I(V_{opt}(t)) \geq I(V_{opt}(t) + V'(t))$ if $V_{opt}(t) > V^*$

$V_{opt}(t) \geq V_{opt}(t) + V'(t) \Rightarrow I(V_{opt}(t)) \geq I(V_{opt}(t) + V'(t))$ if $V_{opt}(t) \leq V^*$

$V_{opt}(t) = V^* \Rightarrow I(V_{opt}(t)) \geq I(V_{opt}(t) + V'(t))$

So, for the interval $[a,t_a]$ we can conclude that

$$\int_a^{t_a} I(V_{opt}(t))dt \geq \int_a^{t_a} I(V_{opt}(t) + V'(t))dt. \qquad \text{(i)}$$

<u>Subinterval $(t_a,t_b)$:</u>

On this interval we have $V_{opt}= V^*$, and thus we have

$I(V_{opt}(t)) \geq I(V_{opt}(t) + V'(t))$.

And so for this interval too, we can conclude that

$$\int_{t_a}^{t_b} I(V_{opt}(t))dt \geq \int_{t_a}^{t_b} I(V_{opt}(t) + V'(t))dt. \qquad \text{(ii)}$$

<u>Subinterval $[t_b,b]$:</u>

In the same way as for subinterval $[a,t_a]$ we can derive the following equations:

$V_{opt}(t) \leq V_{opt}(t) + V'(t)$ if $V_b > V^*$

$V_{opt}(t) \geq V_{opt}(t) + V'(t)$ if $V_b < V^*$

$V_{opt}(t) = V^*$ if $V_b = V^*$

In the same way as for subinterval we can prove the following equation:

$$\int_{t_b}^{b} I(V_{opt}(t))dt \geq \int_{t_b}^{b} I(V_{opt}(t) + V'(t))dt. \qquad \text{(iii)}$$

Taking the sum of (i), (ii), and (iii) concludes the proof. ■

To get an idea which kind of spillage functions fulfill the conditions (1), (2), and (3) on page 11, we examine the derivative of I(V) with respect to V.

$$\frac{dI(V)}{dV} = (Z - S(V))\frac{dh}{dV} - \frac{dS}{dV}h(V)$$

The conditions (2) and (3) give the following conditions for the spillage function S:

(2') $(Z - S(V))\frac{dh}{dV} > \frac{dS}{dV}h(V)$ if $V < V^*$

(3') $(Z - S(V))\frac{dh}{dV} < \frac{dS}{dV}h(V)$ if $V > V^*$

Where we assume for the moment that a maximum for the function I exists. Conditions (2') and (3') tell us that in cases where the spillage function has sharp increases there does not exist an optimal structure. An example of such a function is a step function as is shown in Figure 3.3. However, if the spillage function is sufficiently smooth then the conditions (2') and (3') are fulfilled. For example, it is easy to prove that conditions (2') and (3') are fulfilled for convex spillage functions.

**The optimal solution.** In the same way as in Section 3.2 an extension to the whole period can be made.

**Theorem 3.4** If an optimal solution for the whole period $[t_0,t_N]$ exists, then this optimal solution has the optimal structure as is defined in Theorem 3.3 during subperiods with constant tariff.

**Proof:** Let Q be any feasible control function on the whole interval $[t_0,t_N]$. Then the function Q has a certain form between two tariff switches $t_{i-1}$ and $t_i$. Replacing this form by the optimal form described in Theorem 3.3, such that $V(t_{i-1})$ and $V(t_i)$ do not change, can only improve the profit during this period, while the other profits remain the same. Since this argument holds for all i, the theorem is proven. ■

From Theorem 3.4 we can conclude that to find the optimal solution for the period $[t_0,t_N]$, only the optimal volumes $V(t_i)$ have to be determined.

This can be done by formulating a suitable dynamic programming problem.

1. The state at time-point $\tau_i$, $V_i$, should be between $V_{max}$ and $V_{min}$.
2. The allowed decision set at time-point $\tau_i$ is given by:

$$\mathcal{F}_i(V_i) = \{ X_i \in \mathcal{G}_i \mid V_{X_i}(t) \in [V_{min},V_{max}] \text{ for all } t \in [t_{i-1},t_i], \text{ and } V_{X_i}(t_{i-1}) = V_{i-1} \}$$

Where $\mathcal{G}_i$ is the set of functions on the interval $[t_{i-1},t_i]$ having the structure given by Theorem 3.3.

3. The transition function $f_i(V_i,Q_i) = V_{i+1}$.
4. The profit made during period i is given by:

$$G_i(V_i,Q_i) = \int_{t_{i-1}}^{t_i} CQ_{i,opt}h(V_i + (Z - Q_{i,opt})(t - t_{i-1}))dt,$$

where $Q_{i,opt}$ is a strategy of the form described in Theorem 3.1, and such that $V_{Q_{opt}}(t_{i-1}) = V_i$ and $V_{Q_{opt}}(t_i) = V_{i+1}$.

With these definitions we find the following problem.

$$\max_{Q_1,\ldots,Q_N} \{ \sum_{i=1}^{N} G_i(V_i,Q_i) \mid V_i = f_i(V_{i-1},Q_i),\ Q_i \in \mathcal{F}_i(V_i) \quad \text{for } i = 1,\ldots,N \}.$$

This is again a problem which can be solved by dynamic programming.

# 4. The Numerical Data

## § 4.1 The Two Power Plants

Both power plants are situated near the Schwarzenegger lake in Austria. Because the reservoirs of these power plants are small, and the Schwarzenegger lake is large in comparison to them, the operators of the power plants can decide for themselves how much energy they will produce. This can be done for each power plant separately. The amount of energy the power plants can produce is only determined by the technical constraints of the maximal amount of discharge.

Because of the touristic nature of the district, it was decided by the local council that the reservoirs should be relatively full during the weekends and the holiday season.

This project was initiated because the control of one power plant was modernized, and the tariff structure was altered. Because of this change of the tariff structure the strategy of the old power plant needed to be changed also. The former strategy was to keep the reservoirs full all the time.

Together with the modernization an experiment would be held in which a day is divided in four different tariff periods.(In fact, this experiment was the main reason for the modernization.) These periods correspond to four periods during the day in which a significant difference in energy usage is measured. The first period is from 6 am till 6 pm, the normal working hours, during which the most energy is consumed. The second period is only from 6 pm till 8 pm, dinner hours, during which the energy consumption drops considerably. The energy consumption increases to a fairly high rate again from 8 pm till 12 pm. This increase is due to the fun-fair, which is open during the same hours. Finally, the last period is from 12 pm till 6 am, during which of course the least energy is consumed. The tariff from 6 am till 6 pm in the weekends is only half of the normal tariff, since nobody works during the weekends.

Our aim was to calculate the strategies for a week in the winter period. These strategies should be an improvement of the current strategy, which was to keep the lake full at all times. If possible the strategies should even optimize the profit.

The numerical data concerning the tariffs are given by

- There are 4 tariff periods in a day. With $\tau$ in hours, and $\tau = 0$ corresponding to Wednesday 0 am, we get the following time-points:

$\tau_{4i} = 24i + 6$ 6 am,

$\tau_{4i+1} = 24i + 18$ 6 pm,

$\tau_{4i+2} = 24i + 20$ 8 pm,

$\tau_{4i+3} = 24i + 24$ 12 pm, $i = 0,..., 6$,

and the final time-point $\tau_{28} = 154$.

N.B. we start on Wednesday 6 am, because both intuition and calculation have shown that it is optimal for the reservoirs to be full at this time-point. This is because 6 am is the start of the period with the highest tariff, and because on Wednesday the weekend is still far away.

- The tariffs during these periods are

$A_{4i+1} = 0.8$ ATS/kWh if $i \in \{0,1,2,5,6\}$,

$A_{4i+1} = 0.4$ ATS/kWh if $i \in \{3,4\}$,

$A_{4i+2} = 0.4$ ATS/kWh,

$A_{4i+3} = 0.6$ ATS/kWh,

$A_{4i+4} = 0.3$ ATS/kWh, with $i = 0,...,6$.

## § 4.2 The Old Power Plant

The numerical data for the old power plant are given below.

- The volume V of the water of the reservoir must satisfy the following constraint: $V_{i,min} \leq V \leq V_{i,max}$.

$V_{i,min}$ and $V_{i,max}$ are given by

$V_{i,max} = 750000\ m^3$ $i = 1,...,28$,

$V_{i,min} = 50000\ m^3$ $i = 1,...,12$,

$V_{i,min} = 500000\ m^3$ $i = 13,...,19$,

$V_{i,min} = 50000\ m^3$ $i = 20,...,28$.

- The height of the water in the reservoir is given by

$$h(V) = 160 + \sqrt{V/30000} \ .$$

- The discharge Q satisfies: $Q_{min} \leq Q \leq Q_{max}$. Where $Q_{min} = 0\ m^3/s$, and $Q_{max} = 30\ m^3/sec$.

- The constant influx is $Z = 10\ m^3/s$.

## § 4.3 The Modern Power Plant

The numerical data for the modern power plant are given in this section.

- The volume of the water in the reservoir must satisfy the following constraint: $V_{i,min} \leq V \leq V_{i,max}$.

$V_{i,min}$ and $V_{i,max}$ are given by:

$V_{i,max} = 750000\ m^3 \qquad i = 1,...,28,$

$V_{i,min} = 50000\ m^3 \qquad i = 1,...,12,$

$V_{i,min} = 500000\ m^3 \qquad i = 13,...,19,$

$V_{i,min} = 50000\ m^3 \qquad i = 20,...,28.$

The height of the water in the reservoir is given by:

$$h(V) = 160 + \sqrt{V/30000} \ .$$

- The discharge Q satisfies: $Q_{min} \leq Q \leq Q_{max}$. Where $Q_{min} = 0\ m^3/s$, and $Q_{max} = 30\ m^3/s$.

- The constant influx $Z = 10\ m^3/s$.

# 5. The Results

In this section we will give the results for the optimal strategies of the two power plants connected to the Schwarzenegger lake. Since these two power plants have no spillage, they can be modelled by the models from Section 3.1 respectively Section 3.2. The optimal strategies calculated by dynamic programming can be seen in Figure 5.1 respectively Figure 5.2.

## § 5.1 The Results for the Old Power Plant

The (almost) optimal strategy for the old power plant is summarized in Table 5.1. In this table the new discharges at switching points, and the volumes at these time-points, are shown. So, for instance, on Friday 6 am the new discharge will be 25.8 $m^3/s$, and the volume at that moment is 750000 $m^3$.

In Table 5.3 this strategy is, partially, summarized again, but this time at the switching points of the modern power plant. This is done in order to make a comparison between the two strategies somewhat easier.
(N.B. In the table-heading we call the switching time-points of the modern power plant "the new switching points".)

The optimal strategy is also displayed in Figure 5.1.

## § 5.2 The Results for the Modern Power Plant

The (almost) optimal strategy for the modern power plant is summarized in Table 5.2. This table is similar to Table 5.1. Here, however, the new discharges and volumes at the (new) switching points are given.

In Table 5.4 this strategy is, partially, summarized again, but this time at the possible switching points of the old power plant. This table can be used again to compare the two strategies.
(N.B. In the table-heading we call the possible switching time-points of the old power plant, which are the tariff switching points, "the switching points".)

The optimal strategy is also displayed in Figure 5.2.

**Table 5.1:** The optimal strategy for the old power plant.

| Day | Time | New Discharge | Volume |
|---|---|---|---|
| Wednesday | 6:00 am | 20 $m^3/s$ | 750000 $m^3$ |
| Wednesday | 6:00 pm | 0 $m^3/s$ | 318000 $m^3$ |
| Wednesday | 8:00 pm | 0 $m^3/s$ | 390000 $m^3$ |
| Thursday | 0:00 am | 0 $m^3/s$ | 534000 $m^3$ |
| Thursday | 6:00 am | 20 $m^3/s$ | 750000 $m^3$ |
| Thursday | 6:00 pm | 0 $m^3/s$ | 318000 $m^3$ |
| Thursday | 8:00 pm | 0 $m^3/s$ | 390000 $m^3$ |
| Friday | 0:00 am | 0 $m^3/s$ | 534000 $m^3$ |
| Friday | 6:00 am | 25.8 $m^3/s$ | 750000 $m^3$ |
| Friday | 6:00 pm | 0 $m^3/s$ | 68000 $m^3$ |
| Friday | 8:00 pm | 0 $m^3/s$ | 140000 $m^3$ |
| Saturday | 0:00 am | 0 $m^3/s$ | 284000 $m^3$ |
| Saturday | 6:00 am | 4.2 $m^3/s$ | 500000 $m^3$ |
| Saturday | 6:00 pm | 10 $m^3/s$ | 750000 $m^3$ |
| Saturday | 8:00 pm | 27.4 $m^3/s$ | 750000 $m^3$ |
| Sunday | 0:00 am | 0 $m^3/s$ | 500000 $m^3$ |
| Sunday | 6:00 am | 9.2 $m^3/s$ | 716000 $m^3$ |
| Sunday | 6:00 pm | 10 $m^3/s$ | 750000 $m^3$ |
| Sunday | 8:00 pm | 25 $m^3/s$ | 750000 $m^3$ |
| Monday | 0:00 am | 0 $m^3/s$ | 534000 $m^3$ |
| Monday | 6:00 am | 20 $m^3/s$ | 750000 $m^3$ |
| Monday | 6:00 pm | 0 $m^3/s$ | 318000 $m^3$ |
| Monday | 8:00 pm | 0 $m^3/s$ | 390000 $m^3$ |
| Tuesday | 0:00 am | 0 $m^3/s$ | 534000 $m^3$ |
| Tuesday | 6:00 am | 20 $m^3/s$ | 750000 $m^3$ |
| Tuesday | 6:00 pm | 0 $m^3/s$ | 318000 $m^3$ |
| Tuesday | 8:00 pm | 0 $m^3/s$ | 390000 $m^3$ |
| Wednesday | 0:00 pm | 0 $m^3/s$ | 750000 $m^3$ |

**Table 5.2**: The optimal strategy for the modern power plant.

| Day | Time | New Discharge | Volume |
|---|---|---|---|
| Wednesday | 6:00 am | 10 $m^3/s$ | 750000 $m^3$ |
| Wednesday | 0:00 pm | 30 $m^3/s$ | 750000 $m^3$ |
| Wednesday | 6:00 pm | 0 $m^3/s$ | 318000 $m^3$ |
| Thursday | 6:00 am | 10 $m^3/s$ | 750000 $m^3$ |
| Thursday | 0:00 pm | 30 $m^3/s$ | 750000 $m^3$ |
| Thursday | 6:00 pm | 0 $m^3/s$ | 318000 $m^3$ |
| Friday | 6:00 am | 10 $m^3/s$ | 750000 $m^3$ |
| Friday | 8:32 am | 30 $m^3/s$ | 750000 $m^3$ |
| Friday | 6:00 pm | 0 $m^3/s$ | 68000 $m^3$ |
| Saturday | 0:57 pm | 10 $m^3/s$ | 750000 $m^3$ |
| Saturday | 8:32 pm | 30 $m^3/s$ | 750000 $m^3$ |
| Sunday | 0:00 am | 0 $m^3/s$ | 500000 $m^3$ |
| Sunday | 6:57 am | 10 $m^3/s$ | 750000 $m^3$ |
| Sunday | 9:00 pm | 30 $m^3/s$ | 750000 $m^3$ |
| Monday | 0:00 am | 0 $m^3/s$ | 534000 $m^3$ |
| Monday | 6:00 am | 10 $m^3/s$ | 750000 $m^3$ |
| Monday | 0:00 pm | 30 $m^3/s$ | 750000 $m^3$ |
| Monday | 6:00 pm | 0 $m^3/s$ | 318000 $m^3$ |
| Tuesday | 6:00 am | 10 $m^3/s$ | 750000 $m^3$ |
| Tuesday | 0:00 pm | 30 $m^3/s$ | 750000 $m^3$ |
| Tuesday | 6:00 pm | 0 $m^3/s$ | 318000 $m^3$ |
| Wednesday | 6:00 am | 10 $m^3/s$ | 750000 $m^3$ |

**Table 5.3**: The strategy of the old power plant at the new switching points.

| Day | Time | New Discharge | Volume |
|---|---|---|---|
| Wednesday | 6:00 am | 20 $m^3/s$ | 750000 $m^3$ |
| Wednesday | 0:07 pm | 20 $m^3/s$ | 750000 $m^3$ |
| Wednesday | 6:00 pm | 0 $m^3/s$ | 325758 $m^3$ |
| Thursday | 6:00 am | 20 $m^3/s$ | 750000 $m^3$ |
| Thursday | 0:07 pm | 20 $m^3/s$ | 750000 $m^3$ |
| Thursday | 6:00 pm | 0 $m^3/s$ | 325758 $m^3$ |
| Friday | 6:00 am | 25.8 $m^3/s$ | 750000 $m^3$ |
| Friday | 8:40 am | 25.8 $m^3/s$ | 750000 $m^3$ |
| Friday | 6:00 pm | 0 $m^3/s$ | 78283 $m^3$ |
| Saturday | 0:48 pm | 4.2 $m^3/s$ | 750000 $m^3$ |
| Saturday | 8:32 pm | 27.4 $m^3/s$ | 750000 $m^3$ |
| Sunday | 0:48 am | 0 $m^3/s$ | 500000 $m^3$ |
| Sunday | 8:59 am | 8.4 $m^3/s$ | 750000 $m^3$ |
| Sunday | 9:01 pm | 25 $m^3/s$ | 750000 $m^3$ |
| Monday | 0:00 am | 0 $m^3/s$ | 535354 $m^3$ |
| Monday | 6:00 am | 20 $m^3/s$ | 750000 $m^3$ |
| Monday | 0:07 pm | 20 $m^3/s$ | 750000 $m^3$ |
| Monday | 6:00 pm | 0 $m^3/s$ | 325758 $m^3$ |
| Tuesday | 6:00 am | 20 $m^3/s$ | 750000 $m^3$ |
| Tuesday | 0:07 pm | 20 $m^3/s$ | 750000 $m^3$ |
| Tuesday | 6:00 pm | 0 $m^3/s$ | 325758 $m^3$ |
| Wednesday | 6:00 am | 20 $m^3/s$ | 750000 $m^3$ |

**Table 5.4**: The strategy of the modern power plant at the switching points.

| Day | Time | New Discharge | Volume |
|---|---|---|---|
| Wednesday | 6:00 am | 10 $m^3/s$ | 750000 $m^3$ |
| Wednesday | 6:00 pm | 0 $m^3/s$ | 318000 $m^3$ |
| Wednesday | 8:00 pm | 0 $m^3/s$ | 390000 $m^3$ |
| Thursday | 0:00 am | 0 $m^3/s$ | 534000 $m^3$ |
| Thursday | 6:00 am | 10 $m^3/s$ | 750000 $m^3$ |
| Thursday | 6:00 pm | 0 $m^3/s$ | 318000 $m^3$ |
| Thursday | 8:00 pm | 0 $m^3/s$ | 390000 $m^3$ |
| Friday | 0:00 am | 0 $m^3/s$ | 534000 $m^3$ |
| Friday | 6:00 am | 10 $m^3/s$ | 750000 $m^3$ |
| Friday | 6:00 pm | 0 $m^3/s$ | 68000 $m^3$ |
| Friday | 8:00 pm | 0 $m^3/s$ | 140000 $m^3$ |
| Saturday | 0:00 am | 0 $m^3/s$ | 284000 $m^3$ |
| Saturday | 6:00 am | 0 $m^3/s$ | 500000 $m^3$ |
| Saturday | 6:00 pm | 10 $m^3/s$ | 750000 $m^3$ |
| Saturday | 8:00 pm | 10 $m^3/s$ | 750000 $m^3$ |
| Sunday | 0:00 am | 30 $m^3/s$ | 500000 $m^3$ |
| Sunday | 6:00 am | 0 $m^3/s$ | 716000 $m^3$ |
| Sunday | 6:00 pm | 10 $m^3/s$ | 750000 $m^3$ |
| Sunday | 8:00 pm | 10 $m^3/s$ | 750000 $m^3$ |
| Monday | 0:00 am | 0 $m^3/s$ | 534000 $m^3$ |
| Monday | 6:00 am | 10 $m^3/s$ | 750000 $m^3$ |
| Monday | 6:00 pm | 0 $m^3/s$ | 318000 $m^3$ |
| Monday | 8:00 pm | 0 $m^3/s$ | 390000 $m^3$ |
| Tuesday | 0:00 am | 0 $m^3/s$ | 534000 $m^3$ |
| Tuesday | 6:00 am | 10 $m^3/s$ | 750000 $m^3$ |
| Tuesday | 6:00 pm | 0 $m^3/s$ | 318000 $m^3$ |
| Tuesday | 8:00 pm | 0 $m^3/s$ | 390000 $m^3$ |
| Wednesday | 0:00 am | 0 $m^3/s$ | 750000 $m^3$ |

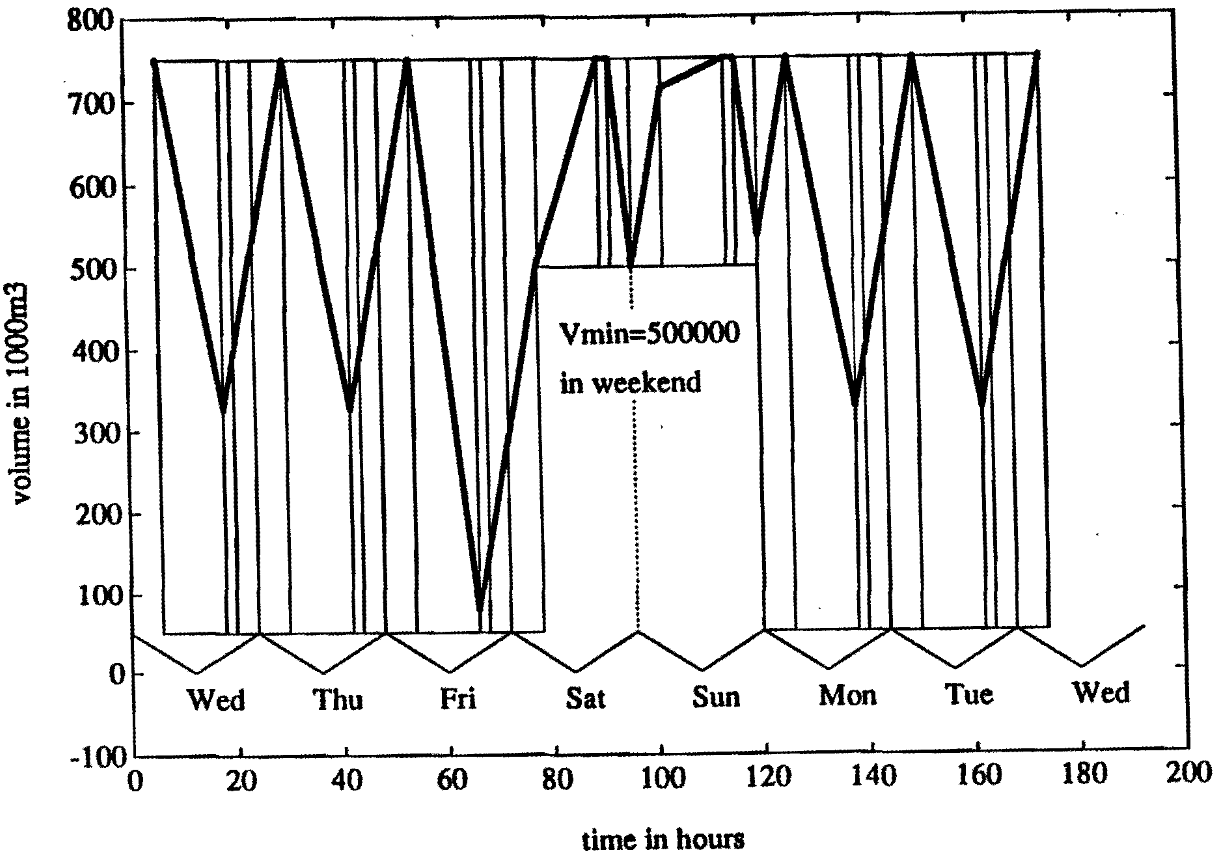


Figure 5.1. The optimal strategy for the old power plant.

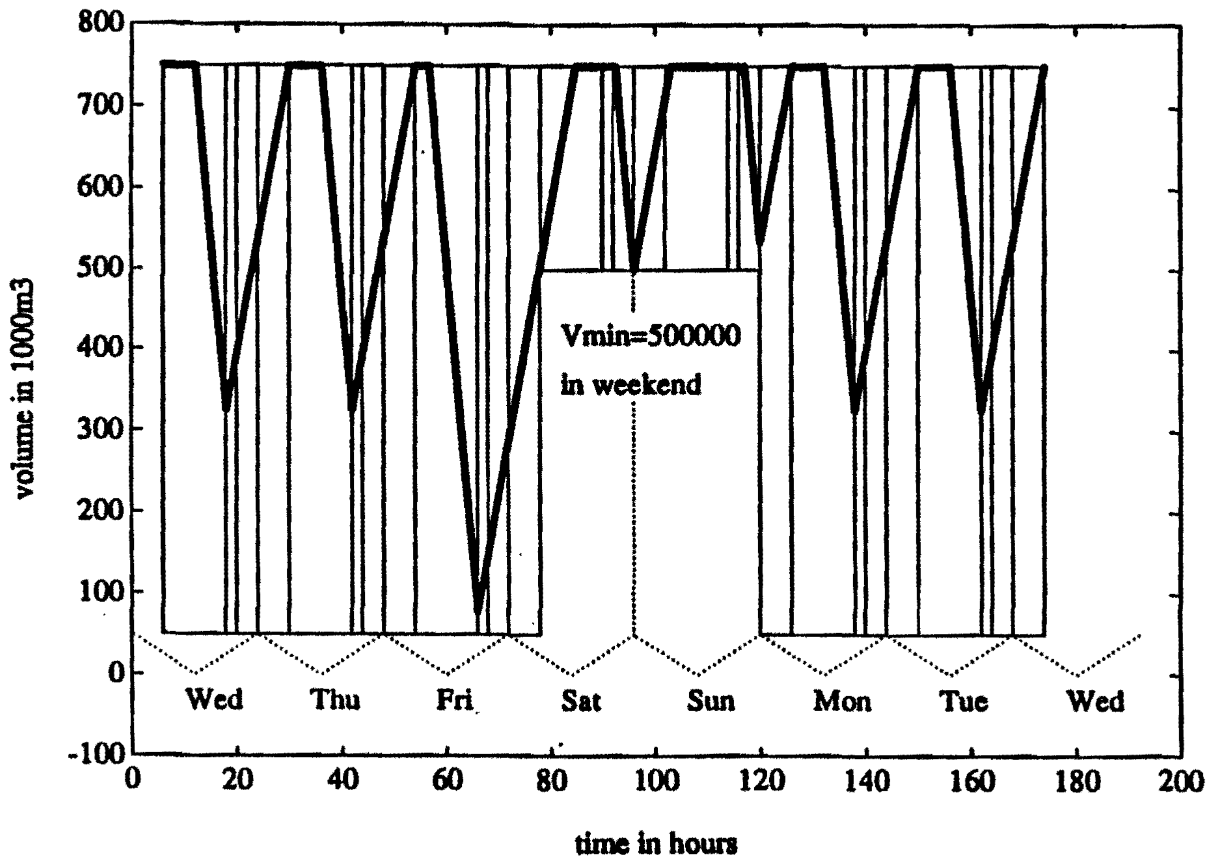


Figure 5.2. The optimal strategy for the modern power plant.

## § 5.3 A Comparison of the Results

We can compare these strategies with the old strategy of keeping the reservoir full all the time.

The new strategy for the old power plant earns a profit of 719,342 ATS in a week. The strategy for the new power plant earns 725,670 ATS in a week. The old strategy however, earned only 550,044 ATS in a week. We can conclude that the new strategies are considerably better than the old strategy. Hence, the strategies calculated with dynamic programming are major improvements over the old strategy. But the difference in profits between the two power plants are negligible.

## 6. Conclusions

In this report we demonstrate that dynamic programming, despite its simplicity, is an efficient way to approximate the optimal control of an hydro energy power plant. We were able to get a very good approximation of the optimal solution.

For the two power plants we were able to obtain a profit increase of about 30%. The difference in profit for the two power plants is however negligible.

In this report, we assume that there is no spillage of water. Stochastic influences are also neglected.

## References


1. *Applied Dynamic Programming*, R.E. Bellman and S.E. Dreyfus, 1962,
   Princeton University Press, Princeton, New Jersey.
2. *Dynamic Programming*, R.E. Bellman, 1957,
   Princeton University Press, Princeton, New Jersey.
3. *Mathematical Models based on Optimization Problems*,
   S.J.L. van Eijndhoven, 1990
   (Report based on a course given by Prof. H.J. Wacker.)

## Acknowledgement

First of all we would like to thank Prof. H.J Wacker for giving the course on optimization. We also want to thank him for introducing us to the problem, and inspiring us to try to solve it.

Secondly we would like to thank Dr.Ir. J.J.A.M. Brands, Dr.Ir. S.J.L. van Eijndhoven, and Dr.Ir. J.L. de Jong for their various contributions to this report. Without them this report would have been much thinner.

We hope you liked our work, and thank you for spending your time reading it.

To end this report we would like to quote a bear of little brain:

*"We've come to wish you a Very Happy Thursday," said Pooh, when he had gone in and out once or twice just to make sure that he could get out again.*

*"Why, what's going to happen on Thursday?" asked Rabbit, and when Pooh had explained, and Rabbit, whose life was made up of Important Things, said, "Oh, I thought you'd really come about something," they sat down for a little ... and by-and-by Pooh and Piglet went on again. The wind was behind them now, so they didn't have to shout.*

*"Rabbit's clever," said Pooh thoughtfully.*

*"Yes," said Piglet, "Rabbit's clever."*

*"And he has Brain."*

*"Yes," said Piglet, "Rabbit has Brain."*

*There was a long silence.*

*"I suppose," said Pooh, "that that's why he never understands anything."*