On some Bessel-function integrals arising in a telecommunication problem

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1. Introduction

The present note deals with the evaluation of the following Bessel-function integrals:

$$(1.1)\qquad \int_0^\infty \frac{u^m J_0^2(u)}{(u^2-u_1^2)^2}\,du\ ,\qquad \int_0^\infty \frac{u^m J_0^2(u)}{(u^2-u_1^2)(u^2-u_2^2)}\,du\ ,\qquad m = 2,3\ ,$$

where u_1, u_2 are the first and second zeros of the Bessel function $J_0(u)$, i.e., $u_1 = 2.4048$, $u_2 = 5.5201$ to four decimal places;

$$(1.2)\qquad ⨍_0^\infty \frac{u^m J_0(u)J_0(ur)}{(u^2-v_1^2)^2}\,du\ ,\qquad ⨍_0^\infty \frac{u^m J_0(u)J_0(ur)}{(u^2-v_1^2)(u^2-v_2^2)}\,du\ ,\qquad m = 1,3\ ,$$

$$(1.3)\qquad ⨍_0^\infty \frac{u^m J_1(u)J_0(ur)}{(u^2-v_1^2)^2}\,du\ ,\qquad ⨍_0^\infty \frac{u^m J_1(u)J_0(ur)}{(u^2-v_1^2)(u^2-v_2^2)}\,du\ ,\qquad m = 2,4\ ,$$

where $0 < r < 1$ and v_1, v_2 are the first and second zeros of the function $f(u) = J_0(u)Y_0(ur) - J_0(ur)Y_0(u)$. The notation $⨍$ in (1.2) and (1.3) denotes that the Cauchy principal value of the integral is to be taken. The integrals above were encountered by Mr. S. Worm (Eindhoven University of Technology, Department of Electrical Engineering, Group ET) in his research on satellite antennas. The integrals (1.1) with $J_0^2(u)$ replaced by $J_1^2(u)$ and u_1, u_2 being zeros of $J_1(u)$, were studied before by Dörr [1]. Dörr's integrals came up in the mathematical analysis of elastically supported, thick circular plates.

The integrals (1.1) are special cases of the general integral

$$(1.4)\qquad I_m(a,b) = ⨍_0^\infty \frac{x^m J_0^2(x)}{(x^2-a^2)(x^2-b^2)}\,dx$$

where a and b are real. Likewise, we introduce the integrals

$$K_m(a,b;r) = \int_0^\infty \frac{x^m J_0(x)J_0(rx)}{(x^2-a^2)(x^2-b^2)}\,dx\,, \tag{1.5}$$

$$L_m(a,b;r) = \int_0^\infty \frac{x^m J_1(x)J_0(rx)}{(x^2-a^2)(x^2-b^2)}\,dx\,, \tag{1.6}$$

with real a and b, and $0 \le r \le 1$, which contain the integrals (1.2) and (1.3) as special cases. In section 2, $I_m(a,b)$ is expressed in terms of the Hilbert transforms $H\{J_0^2(x)\}$ and $H\{J_0^2(x)\ \mathrm{sgn}(x)\}$. Likewise, $K_m(a,b;r)$ with $m = 1,3$, and $L_m(a,b;r)$ with $m = 2,4$, can be expressed in terms of the Hilbert transforms $H\{J_0(x)J_0(rx)\ \mathrm{sgn}(x)\}$ and $H\{|x|J_1(x)J_0(rx)\}$, respectively. These Hilbert transforms are evaluated in section 3. Final results for $I_m(a,b)$ when $m = 0,1,2,3$; for $K_m(a,b;r)$ when $m = 1,3$; for $L_m(a,b;r)$ when $m = 2,4$; and for Worm's integrals (1.1)-(1.3) are presented in section 4. It is found that I_1 and I_3 are expressible in terms of Bessel functions J and Y; I_0 and I_2 can be expressed in terms of a generalized hypergeometric function of the type ${}_2F_3$. Section 5 deals with some extensions involving the Hilbert transforms $H\{J_n^2(x)\}$ and $H\{J_n^2(x)\ \mathrm{sgn}(x)\}$ where $n = 0,1,2,\ldots$. In the Appendix it is shown that the Hilbert transforms $H\{J_0^2(x)\}$ and $H\{J_0^2(x)\ \mathrm{sgn}(x)\}$ can be expressed as integrals of the complete elliptic integral K; a table of such integrals was compiled by Glasser [7].

2. Reduction of I_m, K_m, L_m to Hilbert transforms

Starting from the partial fraction decomposition

$$\frac{1}{(x^2-a^2)(x^2-b^2)} = \frac{1}{a^2-b^2}\left[\frac{1}{x^2-a^2} - \frac{1}{x^2-b^2}\right] \tag{2.1}$$

we have

$$I_0(a,b) = \frac{1}{a^2-b^2}\left[\int_0^\infty \frac{J_0^2(x)}{x^2-a^2}\,dx - \int_0^\infty \frac{J_0^2(x)}{x^2-b^2}\,dx\right] = \tag{2.2}$$

$$= \frac{1}{2(a^2-b^2)}\left[\frac{1}{a}\int_0^\infty J_0^2(x)\left(\frac{1}{x-a}-\frac{1}{x+a}\right)dx - \frac{1}{b}\int_0^\infty J_0^2(x)\left(\frac{1}{x-b}-\frac{1}{x+b}\right)dx\right]$$

$$= \frac{1}{2a(a^2-b^2)}\int_{-\infty}^{\infty}\frac{J_0^2(x)}{x-a}\,dx - \frac{1}{2b(a^2-b^2)}\int_{-\infty}^{\infty}\frac{J_0^2(x)}{x-b}\,dx .$$

In the same manner we establish

$$(2.3)\qquad I_1(a,b) = \frac{1}{2(a^2-b^2)}\int_{-\infty}^{\infty}\frac{J_0^2(x)\,\mathrm{sgn}(x)}{x-a}\,dx - \frac{1}{2(a^2-b^2)}\int_{-\infty}^{\infty}\frac{J_0^2(x)\,\mathrm{sgn}(x)}{x-b}\,dx ,$$

$$(2.4)\qquad I_2(a,b) = \frac{a}{2(a^2-b^2)}\int_{-\infty}^{\infty}\frac{J_0^2(x)}{x-a}\,dx - \frac{b}{2(a^2-b^2)}\int_{-\infty}^{\infty}\frac{J_0^2(x)}{x-b}\,dx ,$$

$$(2.5)\qquad I_3(a,b) = \frac{a^2}{2(a^2-b^2)}\int_{-\infty}^{\infty}\frac{J_0^2(x)\,\mathrm{sgn}(x)}{x-a}\,dx - \frac{b^2}{2(a^2-b^2)}\int_{-\infty}^{\infty}\frac{J_0^2(x)\,\mathrm{sgn}(x)}{x-b}\,dx .$$

In the above results it is understood that $a \neq b$. Thus for $m = 0,1,2,3$, $I_m(a,b)$ has been expressed in terms of the two Hilbert transforms

$$(2.6)\qquad \mathcal{H}\{J_0^2(x)\} = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{J_0^2(x)}{x-y}\,dx ,$$

$$(2.7)\qquad \mathcal{H}\{J_0^2(x)\,\mathrm{sgn}(x)\} = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{J_0^2(x)\,\mathrm{sgn}(x)}{x-y}\,dx ,$$

where y is real.

Likewise, we express $K_m(a,b;r)$ with $m = 1,3$, and $L_m(a,b;r)$ with $m = 2,4$, in terms of the Hilbert transforms $\mathcal{H}\{J_0(x)J_0(rx)\,\mathrm{sgn}(x)\}$ and $\mathcal{H}\{|x|J_1(x)J_0(rx)\}$, respectively, viz.,

$$(2.8)\qquad K_1(a,b;r) = \frac{1}{2(a^2-b^2)}\int_{-\infty}^{\infty}\frac{J_0(x)J_0(rx)\,\mathrm{sgn}(x)}{x-a}\,dx$$

$$- \frac{1}{2(a^2-b^2)}\int_{-\infty}^{\infty}\frac{J_0(x)J_0(rx)\,\mathrm{sgn}(x)}{x-b}\,dx ,$$

(2.9)
$$K_3(a,b;r) = \frac{a^2}{2(a^2-b^2)} \int_{-\infty}^{\infty} \frac{J_0(x)J_0(rx)\,\mathrm{sgn}(x)}{x-a}\,dx - \frac{b^2}{2(a^2-b^2)} \int_{-\infty}^{\infty} \frac{J_0(x)J_0(rx)\,\mathrm{sgn}(x)}{x-b}\,dx ,$$

(2.10)
$$L_2(a,b;r) = \frac{1}{2(a^2-b^2)} \int_{-\infty}^{\infty} \frac{|x|J_1(x)J_0(rx)}{x-a}\,dx - \frac{1}{2(a^2-b^2)} \int_{-\infty}^{\infty} \frac{|x|J_1(x)J_0(rx)}{x-b}\,dx ,$$

(2.11)
$$L_4(a,b;r) = \frac{a^2}{2(a^2-b^2)} \int_{-\infty}^{\infty} \frac{|x|J_1(x)J_0(rx)}{x-a}\,dx - \frac{b^2}{2(a^2-b^2)} \int_{-\infty}^{\infty} \frac{|x|J_1(x)J_0(rx)}{x-b}\,dx .$$

The pertaining Hilbert transforms are determined in the next section.

3. Evaluation of Hilbert transforms

Consider first the Hilbert transform $\mathcal{H}\{J_0(px)J_0(qx)\,\mathrm{sgn}(x)\}$ where $0 \le p \le q$, $q \ne 0$. We start from the contour integral

(3.1)
$$\int_C \frac{J_0(pz)H_0^{(1)}(qz)}{z-y}\,dz = 0$$

where the contour C consists of the real axis with a semi-circular indentation $|z-y| = \delta$ above y, and a closing semi-circle $|z| = R \to \infty$ in the upper half-plane. It is understood that $\arg z = \pi$ along the negative real axis. From the asymptotic behavior of $J_0(pz)$ and the Hankel function $H_0^{(1)}(qz)$ it is easily found that the contribution of the semi-circle $|z| = R$ vanishes as $R \to \infty$. The contribution of the semi-circle $|z-y| = \delta$ tends to $-\pi i\,J_0(py)H_0^{(1)}(qy)$ as $\delta \to 0$. Thus by taking limits in (3.1) as $R \to \infty$ and $\delta \to 0$, we find

$$(3.2) \qquad \int_{-\infty}^{\infty} \frac{J_0(px)H_0^{(1)}(qx)}{x-y}\,dx = \pi i\; J_0(py)H_0^{(1)}(qy) \;.$$

We now take real and imaginary parts of (3.2). Remember that for $x > 0$,

$$(3.3) \qquad \begin{aligned} H_0^{(1)}(x) &= J_0(x) + iY_0(x) \;, \\ H_0^{(1)}(xe^{i\pi}) &= H_0^{(1)}(x) - 2J_0(x) = -J_0(x) + iY_0(x) \;, \end{aligned}$$

cf. Watson [2, eq. 3.62(5)]. Thus we find

$$(3.4) \qquad \begin{aligned} H\{J_0(px)J_0(qx)\,\mathrm{sgn}(x)\} &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{J_0(px)J_0(qx)\,\mathrm{sgn}(x)}{x-y}\,dx \\ &= -J_0(py)Y_0(q|y|) \;, \end{aligned}$$

$$(3.5) \qquad H\{J_0(px)Y_0(q|x|)\} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{J_0(px)Y_0(q|x|)}{x-y}\,dx = J_0(py)J_0(qy)\,\mathrm{sgn}(y) \;,$$

valid for $0 \le p \le q$, $q \neq 0$. The present results are in accordance with the well-known reciprocity relation (cf. [3, form. 15.1(1),(2)])

$$(3.6) \qquad H\{f(x)\} = g(y) \iff H\{g(x)\} = -f(y) \;.$$

The result (3.4) can also be derived from Watson [2, eq. 13.53(4)], viz.,

$$(3.7) \qquad \begin{aligned} \int_0^{\infty} \frac{xJ_0(px)J_0(qx)}{x^2 - r^2}\,dx &= \frac{1}{2}\int_{-\infty}^{\infty} \frac{J_0(px)J_0(qx)\,\mathrm{sgn}(x)}{x-r}\,dx \\ &= \tfrac{1}{2}\pi i\, J_0(pr)H_0^{(1)}(qr) \;, \end{aligned}$$

valid for $\mathrm{Im}\, r > 0$. Taking the limit $r \to y$ real, we have according to Plemelj's formulae [4]

$$(3.8) \qquad \begin{aligned} \tfrac{1}{2}\pi i\, J_0(py)J_0(qy)\,\mathrm{sgn}(y) + \frac{1}{2}\int_{-\infty}^{\infty} \frac{J_0(px)J_0(qx)\,\mathrm{sgn}(x)}{x-y}\,dx \\ = \tfrac{1}{2}\pi i\, J_0(py)H_0^{(1)}(qy) \end{aligned}$$

from which the result (3.4) is obvious.

Starting from (3.4) we readily find the Hilbert transforms

$$(3.9)\qquad H\{J_0(x)J_0(rx)\,\mathrm{sgn}(x)\} = \begin{cases} -J_0(ry)Y_0(|y|)\ , & 0 \le r \le 1\ , \\ -J_0(y)Y_0(r|y|)\ , & r \ge 1\ , \end{cases}$$

$$(3.10)\qquad H\{J_0^2(x)\,\mathrm{sgn}(x)\} = -\,J_0(y)Y_0(|y|)\ ,$$

as needed in the evaluation of the integrals I_1, I_3, K_1, K_3 of section 2. It is remarked that reciprocal Hilbert transforms may be established by use of (3.6). The same remark applies to all further Hilbert transforms evaluated in this section.

Next we differentiate (3.4) with respect to p or q, thus leading to

$$(3.11)\qquad H\{|x|J_1(px)J_0(qx)\} = -\,y\,J_1(py)Y_0(q|y|)\ ,$$

$$(3.12)\qquad H\{|x|J_0(px)J_1(qx)\} = -\,|y|\,J_0(py)Y_1(q|y|)\ ,$$

valid for $0 \le p < q$. As a special case we have

$$(3.13)\qquad H\{|x|J_1(x)J_0(rx)\} = \begin{cases} -\,|y|\,J_0(ry)Y_1(|y|)\ , & 0 \le r < 1\ , \\ -\,y\,J_1(y)Y_0(r|y|)\ , & r > 1\ ; \end{cases}$$

this result is needed in the evaluation of the integrals L_2, L_4 of section 2. Notice that (3.11), (3.12) can be rewritten as

$$(3.14)\qquad H\{|x|J_1(px)J_0(qx)\} = \frac{1}{\pi}\int_{-\infty}^{\infty} J_1(px)J_0(qx)\,\mathrm{sgn}(x)\,dx + y\,H\{J_1(px)J_0(qx)\,\mathrm{sgn}(x)\}\ ,$$

$$(3.15)\qquad H\{|x|J_0(px)J_1(qx)\} = \frac{1}{\pi}\int_{-\infty}^{\infty} J_0(px)J_1(qx)\,\mathrm{sgn}(x)\,dx + y\,H\{J_0(px)J_1(qx)\,\mathrm{sgn}(x)\}\ .$$

Then, by use of the auxiliary integral [2, eq. 13.42(9)]

$$(3.16)\qquad \int_0^{\infty} J_0(at)J_1(bt)\,dt = \begin{cases} 0\ , & b < a\ , \\ 1/(2b)\ , & b = a\ , \\ 1/b\ , & b > a\ , \end{cases}$$

we establish two additional Hilbert transforms

$$(3.17)\qquad \mathcal{H}\{J_1(px)J_0(qx)\,\mathrm{sgn}(x)\} = -J_1(py)Y_0(q|y|)\ ,$$

$$(3.18)\qquad \mathcal{H}\{J_0(px)J_1(qx)\,\mathrm{sgn}(x)\} = -\frac{2}{\pi qy} - J_0(py)Y_1(q|y|)\,\mathrm{sgn}(y)\ ,$$

valid for $0 \le p \le q$. These transforms become identical when $p = q$, because of the Wronskian relation [2, eq. 3.63(12)]

$$(3.19)\qquad J_0(qy)Y_1(q|y|) - J_1(qy)Y_0(q|y|)\,\mathrm{sgn}(y) = -\frac{2}{\pi qy}\,\mathrm{sgn}(y)\ .$$

Setting $p = q$ in (3.14) and (3.15), we may proceed backwards to obtain

$$(3.21)\qquad \mathcal{H}\{|x|J_1(qx)J_0(qx)\} = \frac{1}{\pi q} - yJ_1(qy)Y_0(q|y|) = -\frac{1}{\pi q} - |y|J_0(qy)Y_1(q|y|)\ ,$$

or equivalently

$$(3.22)\qquad \mathcal{H}\{|x|J_1(qx)J_0(qx)\} = -\tfrac{1}{2}yJ_1(qy)Y_0(q|y|) - \tfrac{1}{2}|y|J_0(qy)Y_1(q|y|)\ .$$

This result may be interpreted as the average of the limits of the transforms (3.11) and (3.12) when $p \to q$. In the same manner we may determine the Hilbert transform (3.13) when $r = 1$, viz.,

$$(3.23)\qquad \mathcal{H}\{|x|J_1(x)J_0(x)\} = -\tfrac{1}{2}yJ_1(y)Y_0(|y|) - \tfrac{1}{2}|y|J_0(y)Y_1(|y|)\ .$$

Consider next the Hilbert transform $\mathcal{H}\{J_0^2(x)\}$, as needed in the evaluation of the integrals I_0, I_2 of section 2. Referring to Luke [5, eqs. 13.4.6(12), (13)] we have

$$(3.24)\qquad \mathcal{H}\{J_0^2(x)\} = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{J_0^2(x)}{x-y}\,dx = -\frac{2}{\pi}\int_0^{\pi/2}\mathbf{H}_0(2y\cos\theta)\,d\theta$$

where $\mathbf{H}_0$ denotes Struve's function of order zero. The derivation of (3.24) goes back to Dörr [1] who proceeded as follows. Replace $J_0^2(x)$ by the integral representation [2, eq. 2.6(1)]

$$(3.25)\qquad J_0^2(x) = J_0^2(|x|) = \frac{2}{\pi}\int_0^{\pi/2} J_0(2|x|\cos\theta)\,d\theta$$

and interchange the order of integration, thus leading to

$$(3.26)\qquad \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{J_0^2(x)}{x-y}\,dx = \frac{2}{\pi^2}\int_0^{\pi/2} d\theta \int_{-\infty}^{\infty}\frac{J_0(2|x|\cos\theta)}{x-y}\,dx\ .$$

Here the inner integral may be found from [3, form. 15.3(13)], viz.,

$$(3.27)\qquad \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{J_0(2|x|\cos\theta)}{x-y}\,dx = -\,\mathrm{sgn}(y)H_0(2|y|\cos\theta) = -\,H_0(2y\cos\theta)$$

since H_0 is an odd function of its argument. As a check, the result (3.27) has also been derived from Watson [2, eq. 13.51(7)]. By inserting (3.27) into (3.26), the result (3.24) is precisely recovered.

Starting from the series representation of H_0 (cf. [2, eq. 10.4(2)])

$$(3.28)\qquad H_0(z) = \sum_{n=0}^{\infty}\frac{(-1)^n(\frac{1}{2}z)^{2n+1}}{\Gamma(n+\frac{3}{2})\Gamma(n+\frac{3}{2})}\ ,$$

we have through a term-by-term integration

$$(3.29)\qquad \int_0^{\pi/2} H_0(2y\cos\theta)\,d\theta = \sum_{n=0}^{\infty}\frac{(-1)^n\,y^{2n+1}}{\Gamma(n+\frac{3}{2})\Gamma(n+\frac{3}{2})}\int_0^{\pi/2}(\cos\theta)^{2n+1}\,d\theta$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n\,y^{2n+1}}{\Gamma(n+\frac{3}{2})\Gamma(n+\frac{3}{2})}\,\frac{1}{2}\,\frac{n!\,\Gamma(\frac{1}{2})}{\Gamma(n+\frac{3}{2})} = \frac{\Gamma(\frac{1}{2})y}{2\Gamma^3(\frac{3}{2})}\sum_{n=0}^{\infty}\frac{(1)_n(1)_n}{(\frac{3}{2})_n(\frac{3}{2})_n(\frac{3}{2})_n}\,\frac{(-y^2)^n}{n!}$$

where we used the notation (Pochhammer's symbol)

$$(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1)\ ,\quad n = 1,2,3,\ldots\ ;\quad (\alpha)_0 = 1\ .$$

The final result in (3.29) is immediately recognized as a generalized hypergeometric function of the type ${}_2F_3$; for the general definition of ${}_pF_q$ see [6, Sec. 4.1]. Thus we find

$$(3.30)\qquad H\{J_0^2(x)\} = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{J_0^2(x)}{x-y}\,dx = -\frac{2}{\pi}\int_0^{\pi/2} H_0(2y\cos\theta)\,d\theta$$

$$= -\frac{8y}{\pi^2}\,{}_2F_3\left(\begin{matrix}1\ ,\ 1 & ;\ -y^2\\ \frac{3}{2}\ ,\ \frac{3}{2}\ ,\ \frac{3}{2} & \end{matrix}\right)\ .$$

We have tried to simplify the ${}_2F_3$-function by expressing it as a product of series ${}_pF_q$ with smaller parameters p and q. However, a search through the list in [6, Sec. 4.3] was not successful.

As a check we shall now re-derive (3.30) in two alternative ways.

In the first procedure we start from

$$(3.31)\qquad \int_{-\infty}^{\infty} \frac{J_0^2(x)}{x-z}\,dx = \int_{-\infty}^{\infty} J_0^2(x)\,dx\,(\pm i)\int_0^{\infty} e^{\mp is(x-z)}\,ds$$

$$= \pm i \int_0^{\infty} e^{\pm isz}\,ds \int_{-\infty}^{\infty} J_0^2(x)\,e^{\mp isx}\,dx$$

$$= \pm 2i \int_0^{\infty} e^{\pm isz}\,ds \int_0^{\infty} J_0^2(x)\cos(sx)\,dx\,, \qquad \operatorname{Im} z \gtrless 0\,.$$

Let $z \to y$ real, then by addition of the two results in (3.31) we obtain

$$(3.32)\qquad ⨍_{-\infty}^{\infty} \frac{J_0^2(x)}{x-y}\,dx = -2\int_0^{\infty} \sin(sy)\,ds \int_0^{\infty} J_0^2(x)\cos(sx)\,dx\,.$$

Thus the Hilbert transform of $J_0^2(x)$ has been expressed as a successive Fourier cosine and Fourier sine transform. From [3, form. 1.12(21)] we quote

$$(3.33)\qquad \int_0^{\infty} J_0^2(x)\cos(sx)\,dx = \begin{cases} \tfrac{1}{2}P_{-\frac{1}{2}}(\tfrac{1}{2}s^2-1)\,, & 0 < s < 2\,,\\ 0\,, & 2 < s < \infty\,. \end{cases}$$

By means of [6, eq. 3.4(6)] the Legendre function $P_{-\frac{1}{2}}$ can be expressed in terms of a hypergeometric function F, viz.,

$$(3.34)\qquad P_{-\frac{1}{2}}(\tfrac{1}{2}s^2-1) = F(\tfrac{1}{2},\tfrac{1}{2};1;1-\tfrac{1}{4}s^2)\,.$$

Inserting (3.33) and (3.34) into (3.32) we find

$$(3.35)\qquad ⨍_{-\infty}^{\infty} \frac{J_0^2(x)}{x-y}\,dx = -\int_0^{2} F(\tfrac{1}{2},\tfrac{1}{2};1;1-\tfrac{1}{4}s^2)\sin(sy)\,ds$$

$$= -2\int_0^{1} F(\tfrac{1}{2},\tfrac{1}{2};1;1-s^2)\sin(2sy)\,ds\,.$$

The latter integral is evaluated through series-expansion of sin(2sy) and term-by-term integration, yielding

$$(3.36)\qquad -2\int_0^1 F(\tfrac{1}{2},\tfrac{1}{2};1;1-s^2)\sin(2sy)\,ds$$

$$= -2\sum_{n=0}^{\infty}\frac{(-1)^n(2y)^{2n+1}}{(2n+1)!}\int_0^1 F(\tfrac{1}{2},\tfrac{1}{2};1;1-s^2)\,s^{2n+1}\,ds$$

$$= -\sum_{n=0}^{\infty}\frac{(-1)^n(2y)^{2n+1}}{(2n+1)!}\int_0^1 F(\tfrac{1}{2},\tfrac{1}{2};1;t)(1-t)^n\,dt$$

$$= -\sum_{n=0}^{\infty}\frac{(-1)^n(2y)^{2n+1}}{(2n+1)!}\frac{\Gamma(1)\Gamma(n+1)}{\Gamma(n+2)}F(\tfrac{1}{2},\tfrac{1}{2};n+2;1)$$

$$= -\sum_{n=0}^{\infty}\frac{(-1)^n(2y)^{2n+1}}{(2n+1)!}\frac{\Gamma(1)\Gamma(n+1)}{\Gamma(n+2)}\frac{\Gamma(n+2)\Gamma(n+1)}{\Gamma(n+\frac{3}{2})\Gamma(n+\frac{3}{2})}$$

where we used [6, eqs. 2.4(2), 2.8(46)]. The final series in (3.36) can be rewritten as

$$(3.37)\qquad -\sum_{n=0}^{\infty}\frac{(-1)^n(2y)^{2n+1}}{2^{2n}\,n!(\frac{3}{2})_n}\frac{n!\,n!}{\Gamma^2(\frac{3}{2})(\frac{3}{2})_n(\frac{3}{2})_n} = -\frac{8y}{\pi}\,{}_2F_3\left(\begin{matrix}1\,,\,1\\ \frac{3}{2}\,,\,\frac{3}{2}\,,\,\frac{3}{2}\end{matrix};-y^2\right)$$

in accordance with (3.30).

Our second approach uses Mellin transforms. From [3, form. 6.8(33)] we quote

$$(3.38)\qquad \mathcal{M}\{J_0^2(x)\} = \int_0^{\infty} J_0^2(x)\,x^{s-1}\,dx = \frac{2^{s-1}\,\Gamma(1-s)\Gamma(\frac{1}{2}s)}{\Gamma^3(1-\frac{1}{2}s)}\,,\qquad 0 < \operatorname{Re} s < 1\,.$$

Then by means of the inversion formula for Mellin transforms [3, form. 6.1(1)] we arrive at the integral representation

$$(3.39)\qquad J_0^2(x) = J_0^2(|x|) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{2^{s-1}\,\Gamma(1-s)\Gamma(\frac{1}{2}s)}{\Gamma^3(1-\frac{1}{2}s)}\,|x|^{-s}\,ds\,,\qquad 0 < c < 1\,.$$

According to [3, form. 15.2(29)] we have

$$(3.40)\qquad \mathcal{H}\{|x|^{-s}\} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{|x|^{-s}}{x-y}\, dx = -\tan(\tfrac{1}{2}s\pi)\,\mathrm{sgn}(y)\,|y|^{-s}\ , \quad 0 < \mathrm{Re}\ s < 1\ .$$

Combining (3.39) and (3.40) we find

$$(3.41)\qquad \mathcal{H}\{J_0^2(x)\} = -\frac{\mathrm{sgn}(y)}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{2^{s-1}\ \Gamma(1-s)\Gamma(\frac{1}{2}s)}{\Gamma^3(1-\frac{1}{2}s)} \tan(\tfrac{1}{2}s\pi)\,|y|^{-s}\, ds\ .$$

The latter integral can be evaluated by closing the integration path by an infinite semi-circle to the left. The integrand has simple poles at $s = -2n-1$, $n = 0,1,2,\ldots$, inside the contour. By means of the residue theorem we obtain

$$(3.42)\qquad \mathcal{H}\{J_0^2(x)\} = -\,\mathrm{sgn}(y) \sum_{n=0}^{\infty} \frac{2^{-2n-2}\ \Gamma(2n+2)\Gamma(-n-\frac{1}{2})}{\Gamma^3(n+\frac{3}{2})} (-\tfrac{2}{\pi})\,|y|^{2n+1}$$

$$= \frac{2}{\pi}\, y \sum_{n=0}^{\infty} \frac{2^{-2n-2}(2n+1)!}{\Gamma^3(n+\frac{3}{2})} \frac{\pi(-1)^{n+1}}{\Gamma(n+\frac{3}{2})}\, y^{2n}$$

$$= -\frac{y}{2\Gamma^4(\frac{3}{2})} \sum_{n=0}^{\infty} \frac{n!}{(\frac{3}{2})_n(\frac{3}{2})_n(\frac{3}{2})_n} (-y^2)^n$$

$$= -\frac{8y}{\pi^2}\ {}_2F_3\left(\begin{matrix}1\ ,\ 1 & ; -y^2\\ \frac{3}{2}, \frac{3}{2}, \frac{3}{2} & \end{matrix}\right)$$

in accordance with (3.30).

Yet another (formal) derivation of (3.30) uses the theory of Meijer's G-function. From [6, eq. 5.6(56)] we have

$$(3.43)\qquad J_0^2(x) = \pi^{-\frac{1}{2}}\ G_{13}^{11}\left(x^2\ \middle|\ \begin{matrix}\frac{1}{2}\\ 0,0,0\end{matrix}\right)\ .$$

The Hilbert transform of this G-function is given in [3, form. 15.3(61)], viz.,

$$(3.44)\qquad \mathcal{H}\{J_0^2(x)\} = \pi^{-\frac{1}{2}}\ \mathrm{sgn}(y)\ G_{35}^{22}\left(y^2\ \middle|\ \begin{matrix}\frac{1}{2},\frac{1}{2},1\\ \frac{1}{2},0,0,0,1\end{matrix}\right)\ .$$

The result in [3, form. 15.3(61)] is stated under the condition $p+q < 2(m+n)$ for the original G^{mn}_{pq}. Strictly speaking this condition is not fulfilled for G^{11}_{13}. Using the series-definition [6, eq. 5.3(5)] of the G-function, we find from (3.44)

$$(3.45)\qquad H\{J_0^2(x)\} = \pi^{-\frac{1}{2}}\,\mathrm{sgn}(y)\,\frac{\Gamma(-\frac{1}{2})\Gamma(1)\Gamma(1)}{\Gamma(\frac{3}{2})\Gamma(\frac{3}{2})\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})}\,(y^2)^{\frac{1}{2}}\;{}_3F_4\left(\begin{matrix}1\,,\,1\,,\,\frac{1}{2} & ;\,-y^2\\ \frac{3}{2}\,,\,\frac{3}{2}\,,\,\frac{3}{2}\,,\,\frac{1}{2} & \end{matrix}\right)$$

$$= -\frac{8y}{\pi^2}\;{}_2F_3\left(\begin{matrix}1\,,\,1 & ;\,-y^2\\ \frac{3}{2}\,,\,\frac{3}{2}\,,\,\frac{3}{2} & \end{matrix}\right);$$

so in spite of the condition $p+q < 2(m+n)$ being violated, the correct result is recovered.

4. Final results for I_m, K_m, L_m and Worm's integrals

By means of the Hilbert transforms $H\{J_0^2(x)\,\mathrm{sgn}(x)\}$ and $H\{J_0^2(x)\}$ as given in (3.10) and (3.30), we now determine the integral $I_m(a,b)$ as defined by (1.4), for $m = 0,1,2,3$. The results presented below pertain to $I_m(a,b)$ when $a \neq b$. Then $I_m(a,a)$ is found by taking limits as $b \to a$.

Case $m = 0$. From (2.2) we derive

$$(4.1)\qquad I_0(a,b) = -\frac{4}{\pi(a^2-b^2)}\left[{}_2F_3\left(\begin{matrix}1\,,\,1 & ;\,-a^2\\ \frac{3}{2}\,,\,\frac{3}{2}\,,\,\frac{3}{2} & \end{matrix}\right) - {}_2F_3\left(\begin{matrix}1\,,\,1 & ;\,-b^2\\ \frac{3}{2}\,,\,\frac{3}{2}\,,\,\frac{3}{2} & \end{matrix}\right)\right],\quad a \neq b,$$

then

$$(4.2)\qquad I_0(a,a) = \lim_{b\to a} I_0(a,b) = -\frac{2}{\pi a}\frac{d}{da}\;{}_2F_3\left(\begin{matrix}1\,,\,1 & ;\,-a^2\\ \frac{3}{2}\,,\,\frac{3}{2}\,,\,\frac{3}{2} & \end{matrix}\right)$$

$$= \frac{32}{27\pi}\;{}_2F_3\left(\begin{matrix}2\,,\,2 & ;\,-a^2\\ \frac{5}{2}\,,\,\frac{5}{2}\,,\,\frac{5}{2} & \end{matrix}\right).$$

Case m = 1. From (2.3) we find

$$(4.3)\qquad I_1(a,b) = -\frac{\pi}{2(a^2-b^2)}\left[J_0(a)Y_0(|a|) - J_0(b)Y_0(|b|)\right], \quad a \neq b,$$

then

$$(4.4)\qquad I_1(a,a) = -\frac{\pi}{4a}\frac{d}{da}\left[J_0(a)Y_0(|a|)\right]$$

$$= \frac{\pi}{4a}\left[J_0(a)Y_1(|a|)\,\mathrm{sgn}(a) + J_1(a)Y_0(|a|)\right].$$

Case m = 2. From (2.4) we derive

$$(4.5)\qquad I_2(a,b) = -\frac{4}{\pi(a^2-b^2)}\left[a^2\,{}_2F_3\left(\begin{matrix}1, 1 & ; -a^2\\ \frac{3}{2}, \frac{3}{2}, \frac{3}{2} & \end{matrix}\right) - b^2\,{}_2F_3\left(\begin{matrix}1, 1 & ; -b^2\\ \frac{3}{2}, \frac{3}{2}, \frac{3}{2} & \end{matrix}\right)\right], \quad a \neq b,$$

then

$$(4.6)\qquad I_2(a,a) = -\frac{2}{\pi a}\frac{d}{da}\left[a^2\,{}_2F_3\left(\begin{matrix}1, 1 & ; -a^2\\ \frac{3}{2}, \frac{3}{2}, \frac{3}{2} & \end{matrix}\right)\right] = -\frac{4}{\pi}\,{}_2F_3\left(\begin{matrix}1, 2 & ; -a^2\\ \frac{3}{2}, \frac{3}{2}, \frac{3}{2} & \end{matrix}\right).$$

Case m = 3. From (2.5) we find

$$(4.7)\qquad I_3(a,b) = -\frac{\pi}{2(a^2-b^2)}\left[a^2\,J_0(a)Y_0(|a|) - b^2\,J_0(b)Y_0(|b|)\right], \quad a \neq b,$$

then

$$(4.8)\qquad I_3(a,a) = -\frac{\pi}{4a}\frac{d}{da}\left[a^2\,J_0(a)Y_0(|a|)\right]$$

$$= \frac{\pi}{4}\left[|a|J_0(a)Y_1(|a|) + aJ_1(a)Y_0(|a|) - 2J_0(a)Y_0(|a|)\right].$$

Worm's integrals (1.1) correspond to the special cases $I_m(u_1,u_1)$, $I_m(u_1,u_2)$ where m = 2,3. The results for m = 3 simplify because of $J_0(u_1) = J_0(u_2) = 0$; furthermore, we employ the Wronskian relation [2, eq. 3.63(12)]

$$(4.9)\qquad J_0(u_1)Y_1(u_1) - J_1(u_1)Y_0(u_1) = - J_1(u_1)Y_0(u_1) = - \frac{2}{\pi u_1} .$$

In this manner we find for Worm's integrals:

$$(4.10)\qquad I_2(u_1,u_1) = - \frac{4}{\pi}\, {}_2F_3\left(\begin{matrix} 1\,,\,2 & ; -u_1^2 \\ \frac{3}{2},\frac{3}{2},\frac{3}{2} & \end{matrix}\right) ,$$

$$(4.11)\qquad I_2(u_1,u_2) = - \frac{4}{\pi(u_1^2 - u_2^2)}\left[u_1^2\, {}_2F_3\left(\begin{matrix} 1\,,\,1 & ; -u_1^2 \\ \frac{3}{2},\frac{3}{2},\frac{3}{2} & \end{matrix}\right) - u_2^2\, {}_2F_3\left(\begin{matrix} 1\,,\,1 & ; -u_2^2 \\ \frac{3}{2},\frac{3}{2},\frac{3}{2} & \end{matrix}\right)\right] ,$$

$$(4.12)\qquad I_3(u_1,u_1) = \frac{1}{2} , \quad I_3(u_1,u_2) = 0 .$$

It is remarked that the present results hold for any zero u_1 or any pair of zeros u_1,u_2 of $J_0(u)$.

The ${}_2F_3$-series in (4.10) and (4.11) were numerically evaluated by mr. A. Baayens & dr.ir. J.K.M. Jansen, thus leading to the numerical values

$$(4.13)\qquad I_2(u_1,u_1) = 0.17983191 , \quad I_2(u_1,u_2) = - 0.010883441$$

where u_1, u_2 stand for the first and second zeros of $J_0(u)$.

Consider next the integrals $K_m(a,b;r)$, $m = 1,3$, and $L_m(a,b;r)$, $m = 2,4$, as defined by (1.5) and (1.6). In section 2 these integrals were shown to be expressible in terms of the Hilbert transforms $H\{J_0(x)J_0(rx)\,\mathrm{sgn}(x)\}$ and $H\{|x|J_1(x)J_0(rx)\}$. The latter transforms were evaluated in section 3, see (3.9) and (3.13). The results presented below pertain to $K_m(a,b;r)$, $L_m(a,b;r)$ when $a \neq b$. Then $K_m(a,a;r)$, $L_m(a,a;r)$ are found by taking limits as $b \to a$. Throughout it is understood that $0 \le r < 1$, although the case $r = 1$ might be handled as well by use of (3.23) instead of (3.13).

<u>Case of K_1</u>. From (2.8) we derive

$$(4.14)\qquad K_1(a,b;r) = - \frac{\pi}{2(a^2 - b^2)}\,[J_0(ar)Y_0(|a|) - J_0(br)Y_0(|b|)] , \quad a \neq b ,$$

then

$$(4.15)\qquad K_1(a,a;r) = \lim_{b\to a} K_1(a,b;r) = -\frac{\pi}{4a}\frac{d}{da}[J_0(ar)Y_0(|a|)]$$

$$= \frac{\pi}{4a}[J_0(ar)Y_1(|a|)\,\mathrm{sgn}(a) + rJ_1(ar)Y_0(|a|)] .$$

Case of K_3. From (2.9) we find

$$(4.16)\qquad K_3(a,b;r) = -\frac{\pi}{2(a^2-b^2)}[a^2J_0(ar)Y_0(|a|) - b^2J_0(br)Y_0(|b|)] ,\quad a \neq b ,$$

then

$$(4.17)\qquad K_3(a,a;r) = -\frac{\pi}{4a}\frac{d}{da}[a^2\,J_0(ar)Y_0(|a|)]$$

$$= \frac{\pi}{4}[|a|J_0(ar)Y_1(|a|) + ar\,J_1(ar)Y_0(|a|) - 2J_0(ar)Y_0(|a|)] .$$

Case of L_2. From (2.10) we derive

$$(4.18)\qquad L_2(a,b;r) = -\frac{\pi}{2(a^2-b^2)}[|a|J_0(ar)Y_1(|a|) - |b|J_0(br)Y_1(|b|)] ,\quad a \neq b ,$$

then

$$(4.19)\qquad L_2(a,a;r) = -\frac{\pi}{4a}\frac{d}{da}[|a|J_0(ar)Y_1(|a|)]$$

$$= \frac{\pi}{4}[-J_0(ar)Y_0(|a|) + rJ_1(ar)Y_1(|a|)\,\mathrm{sgn}(a)] .$$

Case of L_4. From (2.11) we find

$$(4.20)\qquad L_4(a,b;r) = -\frac{\pi}{2(a^2-b^2)}[|a|^3J_0(ar)Y_1(|a|) - |b|^3\,J_0(br)Y_1(|b|)] ,\ a \neq b ,$$

then

$$(4.21)\qquad L_4(a,a;r) = -\frac{\pi}{4a}\frac{d}{da}[|a|^3\,J_0(ar)Y_1(|a|)]$$

$$= \frac{\pi}{4}[-a^2J_0(ar)Y_0(|a|) + a^2rJ_1(ar)Y_1(|a|)\,\mathrm{sgn}(a) - 2|a|J_0(ar)Y_1(|a|)] .$$

Worm's integrals (1.2) and (1.3) correspond to the special cases $K_m(v_1,v_1;r)$, $K_m(v_1,v_2;r)$ and $L_m(v_1,v_1;r)$, $L_m(v_1,v_2;r)$, respectively. Hence, explicit results for Worm's integrals can be obtained from (4.14)-(4.21) by substitution of $a = v_1$, $b = v_2$. No further simplification of these results can be achieved, as it is only known that v_1, v_2 are zeros of the function $f(u) = J_0(u)Y_0(ur) - J_0(ur)Y_0(u)$.

5. Evaluation of $H\{J_n^2(x)\,\mathrm{sgn}(x)\}$ and $H\{J_n^2(x)\}$

The results (3.10), (3.30) for $H\{J_0^2(x)\,\mathrm{sgn}(x)\}$ and $H\{J_0^2(x)\}$ can easily be extended to the Hilbert transforms $H\{J_n^2(x)\,\mathrm{sgn}(x)\}$ and $H\{J_n^2(x)\}$ where $n = 0,1,2,\ldots$. The derivation runs along the same lines as in section 3. Thus we find as an extension of (3.10),

$$(5.1)\qquad H\{J_n^2(x)\,\mathrm{sgn}(x)\} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{J_n^2(x)\,\mathrm{sgn}(x)}{x-y}\,dx = -\,J_n(|y|)Y_n(|y|)\ ,$$

with the reciprocal transform

$$(5.2)\qquad H\{J_n(|x|)Y_n(|x|)\} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{J_n(|x|)Y_n(|x|)}{x-y}\,dx = J_n^2(y)\,\mathrm{sgn}(y)\ .$$

The extension of (3.30) is found to be

$$(5.3)\qquad H\{J_n^2(x)\} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{J_n^2(x)}{x-y}\,dx = \frac{8y}{\pi^2(4n^2-1)}\ {}_2F_3\left(\begin{matrix}1\,,\ 1 \\ \frac{3}{2}\,,\ \frac{3}{2}+n\,,\ \frac{3}{2}-n\end{matrix}\ ;\ -y^2\right)\ .$$

To derive the latter result we start from the integral representation [2, eq. 2.6(3)]

$$(5.4)\qquad J_n^2(x) = J_n^2(|x|) = \frac{2}{\pi}\,(-1)^n \int_0^{\pi/2} J_0(2|x|\cos\theta)\cos(2n\theta)\,d\theta\ ,$$

then

$$(5.5) \qquad \mathcal{H}\{J_n^2(x)\} = \frac{2}{\pi^2}(-1)^n \int_0^{\pi/2} \cos(2n\theta)\,d\theta \; ⨍_{-\infty}^{\infty} \frac{J_0(2|x|\cos\theta)}{x-y}\,dx$$

$$= -\frac{2}{\pi}(-1)^n \int_0^{\pi/2} \cos(2n\theta) H_0(2y\cos\theta)\,d\theta$$

by means of (3.27). The latter integral is evaluated by term-by-term integration of the series expansion (3.28) for H_0, employing the integral [6, eq. 1.5.1(30)]

$$(5.6) \qquad \int_0^{\pi/2} \cos(2n\theta)(\cos\theta)^{2\ell+1}\,d\theta = \frac{\pi}{2^{2\ell+2}} \frac{\Gamma(2\ell+2)}{\Gamma(\frac{3}{2}+\ell+n)\Gamma(\frac{3}{2}+\ell-n)}$$

as an auxiliary result. As a check the Hilbert transform (5.3) has also been derived by each of the alternative approaches of section 3.

Appendix

The Legendre function $P_{-\frac{1}{2}}$ occurring in (3.33) is expressible in terms of a complete elliptic integral K of the first kind [6, p.174], viz.,

$$(A1) \qquad P_{-\frac{1}{2}}(\cos\theta) = \frac{2}{\pi} K(\sin\tfrac{1}{2}\theta) \; .$$

Starting from (3.32) and (3.33), we make the substitution $\frac{1}{2}s^2 - 1 = \cos\theta$, $s = 2\cos\frac{1}{2}\theta$, leading to

$$(A2) \qquad ⨍_{-\infty}^{\infty} \frac{J_0^2(x)}{x-y}\,dx = -\int_0^2 P_{-\frac{1}{2}}(\tfrac{1}{2}s^2-1)\sin(sy)\,ds$$

$$= -\int_0^{\pi} P_{-\frac{1}{2}}(\cos\theta)\,\sin(2y\cos\tfrac{1}{2}\theta)\,\sin\tfrac{1}{2}\theta\,d\theta$$

$$= -\frac{2}{\pi}\int_0^{\pi} K(\sin\tfrac{1}{2}\theta)\,\sin(2y\cos\tfrac{1}{2}\theta)\,\sin\tfrac{1}{2}\theta\,d\theta \; .$$

Setting $u = \sin\frac{1}{2}\theta$, we find by means of (3.30), (A3) $$\int_0^1 K(u)\sin(2y(1-u^2)^{\frac{1}{2}})\,\frac{u}{(1-u^2)^{\frac{1}{2}}}\,du = 2y\;{}_2F_3\left(\begin{matrix}1\,,\,1 & ;\,-y^2\\ \frac{3}{2},\frac{3}{2},\frac{3}{2} & \end{matrix}\right).$$

The present result is not contained in Glasser's table [7] of integrals of the complete elliptic integral K. By expansion of (A3) in a power-series in y, we obtain

(A4) $$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}(2y)^{2n+1}\int_0^1 K(u)(1-u^2)^n u\,du$$

$$= 2y\sum_{n=0}^{\infty}\frac{(1)_n(1)_n}{(\frac{3}{2})_n(\frac{3}{2})_n(\frac{3}{2})_n}\,\frac{(-y^2)^n}{n!}\,.$$

By equating corresponding powers of y, we are led to

(A5) $$\int_0^1 K(u)(1-u^2)^n u\,du = \frac{n!\,n!}{(\frac{3}{2})_n(\frac{3}{2})_n} = \frac{\pi}{4}\left[\frac{\Gamma(n+1)}{\Gamma(n+\frac{3}{2})}\right]^2.$$

The latter result can also be obtained as a special case of [7, form. I(23)]. A direct derivation of (A5) proceeds by replacing K(u) by its hypergeometric-series representation followed by a term-by-term integration. The result (A5) holds generally for $n > -1$, where n is not necessarily an integer.
A further result is found by differentiation of (A3) with respect to y:

(A6) $$\int_0^1 K(u)\cos(2y(1-u^2)^{\frac{1}{2}})\,u\,du = {}_2F_3\left(\begin{matrix}1\,,\,1 & ;\,-y^2\\ \frac{1}{2},\frac{3}{2},\frac{3}{2} & \end{matrix}\right).$$

This result does not appear in Glasser's list [7] either.
Next we shall evaluate the Hilbert transform $H\{J_0^2(x)\,\mathrm{sgn}(x)\}$ by the same procedure. Similar to (3.31) we have

(A7) $$\int_{-\infty}^{\infty}\frac{J_0^2(x)\,\mathrm{sgn}(x)}{x-z}\,dx = \int_{-\infty}^{\infty}J_0^2(x)\,\mathrm{sgn}(x)\,dx\,(\pm i)\int_0^{\infty}e^{\mp is(x-z)}\,ds$$

$$= \pm i\int_0^{\infty}e^{\pm isz}\,ds\int_{-\infty}^{\infty}J_0^2(x)\,\mathrm{sgn}(x)\,e^{\mp isx}\,dx =$$

$$= 2 \int_0^\infty e^{\pm isz}\, ds \int_0^\infty J_0^2(x)\ \sin(sx)\, dx\ , \quad \mathrm{Im}\ z \gtrless 0\ .$$

Let $z \to y$ real, then by addition of the two results in (A7) we obtain

$$\text{(A8)} \qquad \fint_{-\infty}^{\infty} \frac{J_0^2(x)\,\mathrm{sgn}(x)}{x-y}\, dx = 2 \int_0^\infty \cos(sy)\, ds \int_0^\infty J_0^2(x)\ \sin(sx)\, dx\ .$$

The inner integral is obtainable from Watson [2, eq. 13.46(4),(5) with $\mu = \frac{1}{2}$] or from [3, form. 2.12(27)] (here the minus sign in front of the second result for $s > 2$ is incorrect and should be omitted):

$$\text{(A9)} \qquad \int_0^\infty J_0^2(x)\ \sin(sx)\, dx = \begin{cases} \frac{1}{2} P_{-\frac{1}{2}}(1 - \frac{1}{2}s^2)\ , & 0 < s < 2\ , \\ \frac{1}{\pi} Q_{-\frac{1}{2}}(\frac{1}{2}s^2 - 1)\ , & s > 2\ . \end{cases}$$

Both functions can be expressed in terms of the complete elliptic integral K according to (A1) and [6, eq. 3.13(8)],

$$\text{(A10)} \qquad Q_{-\frac{1}{2}}(\cosh \eta) = 2e^{-\eta/2}\ K(e^{-\eta})\ .$$

Thus combining (A8) and (A9), we make the substitutions $1 - \frac{1}{2}s^2 = \cos\theta$, $s = 2 \sin \frac{1}{2}\theta$, and $\frac{1}{2}s^2 - 1 = \cosh \eta$, $s = 2 \cosh \frac{1}{2}\eta$, leading to

$$\text{(A11)} \qquad \fint_{-\infty}^{\infty} \frac{J_0^2(x)\,\mathrm{sgn}(x)}{x-y}\, dx$$

$$= \int_0^2 P_{-\frac{1}{2}}(1 - \tfrac{1}{2}s^2)\ \cos(sy)\, ds + \frac{2}{\pi} \int_2^\infty Q_{-\frac{1}{2}}(\tfrac{1}{2}s^2 - 1)\ \cos(sy)\, ds$$

$$= \int_0^\pi P_{-\frac{1}{2}}(\cos\theta)\ \cos(2y \sin \tfrac{1}{2}\theta)\ \cos \tfrac{1}{2}\theta\ d\theta$$

$$+ \frac{2}{\pi} \int_0^\infty Q_{-\frac{1}{2}}(\cosh\eta)\ \cos(2y \cosh \tfrac{1}{2}\eta)\ \sinh \tfrac{1}{2}\eta\, d\eta =$$

$$= \frac{2}{\pi} \int_0^{\pi} K(\sin \tfrac{1}{2}\theta) \cos(2y \sin \tfrac{1}{2}\theta) \cos \tfrac{1}{2}\theta \, d\theta$$

$$+ \frac{4}{\pi} \int_0^{\infty} K(e^{-\eta}) \cos(2y \cosh \tfrac{1}{2}\eta) \sinh \tfrac{1}{2}\eta \, e^{-\eta/2} \, d\eta .$$

In the latter integrals we set $u = \sin \frac{1}{2}\theta$ and $u = e^{-\eta}$, respectively, then by means of (3.10) we find

(A12) $$\int_0^1 K(u) \cos(2yu)\,du + \frac{1}{2} \int_0^1 K(u) \cos\left(y \frac{1+u}{u^{\frac{1}{2}}}\right) \frac{1-u}{u} \, du = -\frac{\pi^2}{4} J_0(y) Y_0(|y|) .$$

Here the second integral can be simplified by applying Gauss' transformation [8, form. 164.02]:

(A13) $$K(u) = \tfrac{1}{2}[1 + (1-t^2)^{\frac{1}{2}}]K(t) \quad \text{where} \quad u = \frac{1 - (1-t^2)^{\frac{1}{2}}}{1 + (1-t^2)^{\frac{1}{2}}} .$$

By making the latter substitution, we have $t = 2u^{\frac{1}{2}}/(1+u)$ and

(A14) $$\frac{1}{2} \int_0^1 K(u) \cos\left(y \frac{1+u}{u^{\frac{1}{2}}}\right) \frac{1-u}{u} \, du$$

$$= \tfrac{1}{2} \int_0^1 \tfrac{1}{2}[1 + (1-t^2)^{\frac{1}{2}}]K(t) \cos\left(\frac{2y}{t}\right) \frac{2(1-t^2)^{\frac{1}{2}}}{1 - (1-t^2)^{\frac{1}{2}}} \frac{2t}{(1-t^2)^{\frac{1}{2}}[1 + (1-t^2)^{\frac{1}{2}}]^2} \, dt$$

$$= \int_0^1 K(t) \cos\left(\frac{2y}{t}\right) \frac{dt}{t} .$$

Inserting (A14) into (A12), we arrive at the elegant result

(A15) $$\int_0^1 K(u) \cos(2yu)\,du + \int_0^1 K(u) \cos\left(\frac{2y}{u}\right) \frac{du}{u} = -\frac{\pi^2}{4} J_0(y) Y_0(|y|) .$$

The present result can also be found in a more direct manner by use of Okui [9, form. 2.5(1)], yielding

$$\text{(A16)} \qquad \int_0^\infty J_0^2(x)\sin(sx)\,dx = \begin{cases} \frac{1}{\pi}K(\tfrac{1}{2}s), & 0 < s < 2, \\ \frac{2}{\pi s}K(\frac{2}{s}), & s > 2, \end{cases}$$

which can be shown to be equivalent to (A9), though it is of course much simpler. The result (A15) does not appear in Glasser's list [7]. It does not seem possible to separately evaluate the integrals in (A15). On the other hand, by rewriting (A15) as

$$\text{(A17)} \qquad \int_0^1 K(u)\cos(yu)\,du + \int_1^\infty K(\frac{1}{u})\cos(yu)\frac{du}{u} = -\frac{\pi^2}{4}J_0(\tfrac{1}{2}y)Y_0(\tfrac{1}{2}y), \quad y > 0,$$

it is easily recognized as the inverse of the Fourier transform [9, form. 2.7(1)]

$$\text{(A18)} \qquad \frac{2}{\pi}\int_0^\infty \left(-\frac{\pi^2}{4}\right)J_0(\tfrac{1}{2}y)\,Y_0(\tfrac{1}{2}y)\cos(yu)\,dy = \begin{cases} K(u), & 0 < u < 1, \\ \frac{1}{u}K(\frac{1}{u}), & u > 1. \end{cases}$$

In the same manner one may evaluate the Hilbert transform $\mathcal{H}\{J_0(x)Y_0(|x|)\}$ obtainable from (3.5). Similar to (3.31) and (3.32) we find

$$\text{(A19)} \qquad ⨍_{-\infty}^{\infty} \frac{J_0(x)Y_0(|x|)}{x-y}\,dx = -2\int_0^\infty \sin(sy)\,ds \int_0^\infty J_0(x)Y_0(x)\cos(sx)\,dx$$

$$= \pi J_0^2(y)\,\mathrm{sgn}(y).$$

Here the inner Fourier cosine transform can be obtained from (A18). Thus we are led to the following companion result of (A15),

$$\text{(A20)} \qquad \int_0^1 K(u)\sin(2yu)\,du + \int_0^1 K(u)\sin(\frac{2y}{u})\frac{du}{u} = \frac{\pi^2}{4}J_0^2(y)\,\mathrm{sgn}(y).$$

The result (A20) is again recognized as the inverse of the Fourier sine transform (A16) due to Okui [9].

As a final remark, Glasser's list [7] of integrals of K could almost trivially be extended by inverses of the Fourier transform results as compiled by Okui [9].

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