

# The matrices of the differential operators $\frac{d}{dx}$ and $x\frac{d}{dx}$ with respect to orthonormal bases of Jacobi polynomials

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THE MATRICES OF THE DIFFERENTIAL OPERATORS  $\frac{d}{dx}$  and  $x \frac{d}{dx}$   
WITH RESPECT TO ORTHONORMAL BASES OF JACOBI POLYNOMIALS

by

S.J.L. van Eijndhoven

Eindhoven University of Technology

Department of Mathematics and Computing Science

PO Box 513, 5600 MB Eindhoven

The Netherlands

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Abstract

From the recurrence relations of the Jacobi polynomials we compute the matrix entries of the differential operators  $\frac{d}{dx}$  and  $x \frac{d}{dx}$  with respect to the corresponding orthonormal bases of normalized Jacobi polynomials.

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Some notations

In this paper we consider the Hilbert spaces

$$X_{\alpha, \beta} = L_2([-1, 1], (1-x)^\alpha (1+x)^\beta dx)$$

and the positive self-adjoint operators  $A_{\alpha, \beta}$  in  $X_{\alpha, \beta}$

$$A_{\alpha, \beta} = -(1-x^2) \frac{d^2}{dx^2} - ((\beta - \alpha) - (\alpha + \beta + 2)x) \frac{d}{dx}$$

where we take  $\alpha$  and  $\beta$  larger than  $-1$ . The operator  $A_{\alpha, \beta}$  has a discrete spectrum  $\{n(n + \alpha + \beta + 1) \mid n \in \mathbb{N} \cup \{0\}\}$ . Its normalized eigenvectors are the normalized Jacobi polynomials  $R_n^{(\alpha, \beta)}$

$$R_n^{(\alpha, \beta)} = \left[ \frac{\alpha + \beta + 2n + 1}{2^{\alpha + \beta + 1}} \frac{\Gamma(n+1) \Gamma(n + \alpha + \beta + 1)}{\Gamma(n + \alpha + 1) \Gamma(n + \beta + 1)} \right]^{\frac{1}{2}} P_n^{(\alpha, \beta)}$$

where

$$P_n^{(\alpha, \beta)}(x) = \frac{(-1)^n}{n! 2^n} \frac{1}{(1-x)^\alpha (1+x)^\beta} \left( \frac{d}{dx} \right)^n [(1-x)^{\alpha+n} (1+x)^{\beta+n}]$$

(cf. [2], p. 209).

In our study of distribution spaces based on Jacobi polynomials, cf. [1], we needed an estimation of the matrix entries  $(\mathcal{D}R_n^{(\alpha, \beta)}, R_k^{(\alpha, \beta)})_{\alpha, \beta}$  and  $((x\mathcal{D})R_n^{\alpha, \beta}, R_k^{(\alpha, \beta)})_{\alpha, \beta}$  where  $(\cdot, \cdot)_{\alpha, \beta}$  denotes the inner product in the Hilbert space  $X_{\alpha, \beta}$  and where  $\mathcal{D}$  denotes the differential operator  $\frac{d}{dx}$ .

Exact expressions for these matrix entries are not known. In this note we present such expressions. Also we give estimates for the considered matrix entries.

Results

In [2], p. 213, the following relations can be found

$$(1) \quad \mathcal{D}_P^{(\alpha, \beta)} P_n^{(\alpha, \beta)} = \frac{1}{2}(n+\alpha+\beta+1)P_{n-1}^{(\alpha+1, \beta+1)}, \quad n = 1, 2, \dots$$

We express the polynomials  $P_{n-1}^{(\alpha+1, \beta+1)}$  as finite combinations of the polynomials  $P_k^{(\alpha, \beta)}$ ,  $k = 0, 1, 2, \dots, n-1$ . So we write

$$P_{n-1}^{(\alpha+1, \beta+1)} = \sum_{k=0}^{n-1} \gamma_{n-1, k}^{(\alpha, \beta)} P_k^{(\alpha, \beta)}.$$

In order to compute the coefficients  $\gamma_{n, k}^{(\alpha, \beta)}$ ,  $n = 0, 1, 2, \dots$ ,  $k = 0, 1, 2, \dots, n$ , we use the following relations which can be derived from [2], p. 213

$$(2.i) \quad P_\ell^{(\alpha+1, \beta+1)} = c_\ell P_\ell^{(\alpha, \beta+1)} + d_\ell P_{\ell-1}^{(\alpha+1, \beta+1)}$$

$$(2.ii) \quad P_\ell^{(\alpha, \beta+1)} = a_\ell P_\ell^{(\alpha, \beta)} + b_\ell P_{\ell-1}^{(\alpha, \beta+1)}$$

where

$$a_\ell = \frac{2\ell + \alpha + \beta + 1}{\ell + \alpha + \beta + 1}, \quad c_\ell = \frac{2\ell + \alpha + \beta + 2}{\ell + \alpha + \beta + 2},$$

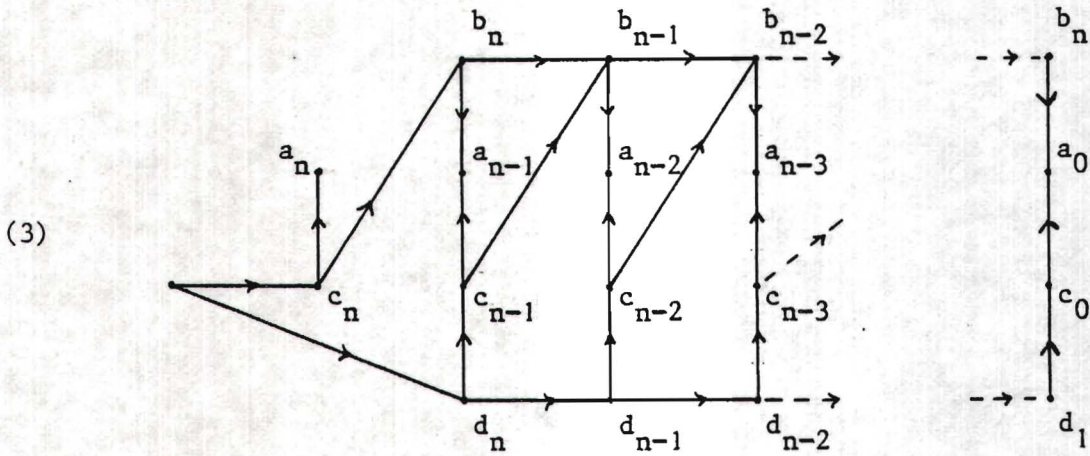
$$b_\ell = -\frac{\ell + \alpha}{\ell + \alpha + \beta + 1}, \quad d_\ell = \frac{\ell + \beta + 1}{\ell + \alpha + \beta + 2}.$$

So starting from  $P_\ell^{(\alpha+1, \beta+1)}$  we get  $c_\ell P_\ell^{(\alpha, \beta+1)} + d_\ell P_{\ell-1}^{(\alpha+1, \beta+1)}$  by (2.i)

and then by (2.ii)  $P_\ell^{(\alpha, \beta+1)} = a_\ell P_\ell^{(\alpha, \beta)} + b_\ell P_{\ell-1}^{(\alpha, \beta+1)}$ , and also by (2.i)

$$P_{\ell-1}^{(\alpha+1, \beta+1)} = c_{\ell-1} P_{\ell-1}^{(\alpha, \beta+1)} + d_{\ell-1} P_{\ell-2}^{(\alpha+1, \beta+1)}, \text{ etc.}$$

The sketched process terminates, because  $\forall_{p,q > -1} : P_0^{(p,q)} \equiv 1$ . It can be described by the following directed graph.



The graph (3) shows the following:

- $c_\ell$  is multiplied by  $a_\ell$  or  $b_\ell$
- $d_\ell$  is multiplied by  $d_{\ell-1}$  or  $c_{\ell-1}$
- $b_\ell$  is multiplied by  $b_{\ell-1}$  or  $a_{\ell-1}$
- every factor ends with some  $a_q$ .

The above examinations yield the following result:

$$P_n^{(\alpha+1, \beta+1)} = \sum_{p=0}^n \left( \sum_{k=0}^p d_n \dots d_{n-k+1} c_{n-k} b_{n-k} \dots b_{n-p+1} a_{n-p} \right) P_{n-p}^{(\alpha, \beta)}$$

with the convention  $d_n d_{n+1} = 1$  and  $b_{n-p} b_{n-p+1} = 1$ ; equivalently

$$(4) \quad P_n^{(\alpha+1, \beta+1)} = \sum_{\ell=0}^n \left( \sum_{k=0}^{n-\ell} d_n \dots d_{n-k+1} c_{n-k} b_{n-k} \dots b_{\ell+1} a_\ell \right) P_\ell^{(\alpha, \beta)}$$

Thus we find that

$$\gamma_{n, \ell}^{(\alpha, \beta)} = \sum_{k=0}^{n-\ell} (d_n \dots d_{n-k+1} c_{n-k} b_{n-k} \dots b_{\ell+1} a_\ell)$$

A simple calculation yields

$$(5) \quad \gamma_{n,\ell}^{(\alpha,\beta)} = (-1)^\ell \frac{\Gamma(\ell+\alpha+\beta+1)\Gamma(n+\beta+2)}{\Gamma(n+\alpha+\beta+3)\Gamma(\ell+\alpha+1)} (2\ell+\alpha+\beta+1) \cdot \\ \cdot \sum_{k=\ell}^n (-1)^k (2k+\alpha+\beta+2) \frac{\Gamma(k+\alpha+1)}{\Gamma(k+\beta+2)}.$$

Let  $O_k^{(\alpha,\beta)}$  denote the  $X_{\alpha,\beta}$ -normalization factor for the Jacobi polynomials,

$$(6) \quad O_k^{(\alpha,\beta)} = \left( \frac{2k + \alpha + \beta + 1}{2^{\alpha+\beta+1}} \frac{\Gamma(k+1)\Gamma(k+\alpha+\beta+1)}{\Gamma(k+\alpha+1)\Gamma(k+\beta+1)} \right)^{\frac{1}{2}}.$$

Then we obtain for the matrix of  $\mathcal{D}$  with respect to the orthonormal basis

$(R_n^{(\alpha,\beta)})_{n=0}^\infty$  of  $X_{\alpha,\beta}$

$$(7) \quad (\mathcal{D}R_n^{(\alpha,\beta)}, R_\ell^{(\alpha,\beta)})_{X_{\alpha,\beta}} = \begin{cases} 0 & \text{if } \ell \geq n, \ell, n \in \mathbb{N} \cup \{0\} \\ \frac{1}{2} \frac{O_n^{(\alpha,\beta)}}{O_\ell^{(\alpha,\beta)}} \gamma_{n-1,\ell}^{(\alpha,\beta)} (n+\alpha+\beta+1) & \text{if } \ell = 0, 1, \dots, n-1, n \in \mathbb{N}. \end{cases}$$

With similar methods we next compute the matrix of the differential operator  $x\mathcal{D}$  with respect to  $(R_n^{(\alpha,\beta)})_{n=0}^\infty$ . From [2], p.213, we obtain the following identities:

$$(1-x)P_n^{(\alpha,\beta)}(x) = \frac{2(n+\alpha)}{2n+\alpha+\beta+1} P_n^{(\alpha-1,\beta)}(x) - \frac{2(n+1)}{2n+\alpha+\beta+1} P_{n+1}^{(\alpha-1,\beta)}(x)$$

and

$$(1+x)P_n^{(\alpha,\beta)}(x) = \frac{2(n+\beta)}{2n+\alpha+\beta+1} P_n^{(\alpha,\beta-1)}(x) + \frac{2(n+1)}{2n+\alpha+\beta+1} P_{n-1}^{(\alpha,\beta-1)}(x).$$

Adding these relations, we obtain the following formula

$$xP_n^{(\alpha, \beta)}(x) = \frac{1}{2n + \alpha + \beta + 1} \left[ - (n+\alpha)P_n^{(\alpha-1, \beta)}(x) + (n+1)P_{n-1}^{(\alpha-1, \beta)}(x) + \right. \\ \left. + (n+\beta)P_n^{(\alpha, \beta-1)}(x) + (n+1)P_{n+1}^{(\alpha, \beta-1)}(x) \right].$$

Thus it follows that

$$(8) \quad (xD)P_{n+1}^{(\alpha, \beta)}(x) = \frac{1}{2} (n+\alpha+\beta+2) xP_n^{(\alpha+1, \beta+1)}(x) = \\ = \frac{1}{2} \frac{n + \alpha + \beta + 2}{2n + \alpha + \beta + 3} \left[ - (n+\alpha+1)P_n^{(\alpha, \beta+1)}(x) + (n+1)P_{n+1}^{(\alpha, \beta+1)}(x) + \right. \\ \left. + (n+\beta+1)P_n^{(\alpha+1, \beta)}(x) + (n+1)P_{n+1}^{(\alpha+1, \beta)}(x) \right].$$

With the relations (2.i) and (2.ii) we get

$$P_k^{(\alpha+1, \beta)} = \sum_{\ell=0}^k \tilde{d}_k \dots \tilde{d}_{\ell+1} \tilde{c}_\ell P_\ell^{(\alpha, \beta)}$$

and

$$P_k^{(\alpha, \beta+1)} = \sum_{\ell=0}^k \tilde{b}_k \dots \tilde{b}_{\ell+1} \tilde{a}_\ell P_\ell^{(\alpha, \beta)}$$

where

$$\tilde{a}_j = \frac{2j + \alpha + \beta + 1}{j + \alpha + \beta + 1}, \quad \tilde{b}_j = - \frac{j + \alpha}{j + \alpha + \beta + 1},$$

$$\tilde{c}_j = \frac{2j + \alpha + \beta + 1}{j + \alpha + \beta + 1}, \quad \tilde{d}_j = \frac{j + \beta}{j + \alpha + \beta + 1}.$$

Finally, substituting the above values in (8) we get for the  $\ell$ -th coefficient,  $0 \leq \ell \leq n$ , in the expression of  $(xD)P_{n+1}^{(\alpha, \beta)}$



$$\begin{aligned}
 & \frac{1}{2} \frac{n + \alpha + \beta + 2}{2n + \alpha + \beta + 3} \left[ (-1)^{n-\ell-1} (n+\alpha+1) \frac{\Gamma(n+\alpha+1)}{\Gamma(n+\alpha+\beta+2)} \frac{\Gamma(\ell+\alpha+\beta+1)}{\Gamma(\ell+\alpha+1)} (2\ell+\alpha+\beta+1) + \right. \\
 & \quad + (-1)^{n-\ell+1} (n+1) \frac{\Gamma(n+\alpha+2)}{\Gamma(n+\alpha+\beta+3)} \frac{\Gamma(\ell+\alpha+\beta+1)}{\Gamma(\ell+\alpha+1)} (2\ell+\alpha+\beta+1) + \\
 & \quad + (n+\beta+1) \frac{\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+2)} \frac{\Gamma(\ell+\alpha+\beta+1)}{\Gamma(\ell+\beta+1)} (2\ell+\alpha+\beta+1) + \\
 & \quad \left. + (n+1) \frac{\Gamma(n+\beta+2)}{\Gamma(n+\alpha+\beta+3)} \frac{\Gamma(\ell+\alpha+\beta+1)}{\Gamma(\ell+\beta+1)} (2\ell+\alpha+\beta+1) \right] = \\
 & = \frac{1}{2} \frac{n + \alpha + \beta + 2}{2n + \alpha + \beta + 3} (2\ell+\alpha+\beta+1) \frac{\Gamma(\ell+\alpha+\beta+1)}{\Gamma(n+\alpha+\beta+2)} \left( 1 + \frac{n-1}{n + \alpha + \beta + 2} \right) \cdot \\
 & \quad \cdot \left[ (-1)^{n-\ell+1} \frac{\Gamma(n+\alpha+2)}{\Gamma(\ell+\alpha+2)} + \frac{\Gamma(n+\beta+2)}{\Gamma(\ell+\beta+1)} \right] = \\
 & = \frac{1}{2} (2\ell+\alpha+\beta+1) \frac{\Gamma(\ell+\alpha+\beta+1)}{\Gamma(n+\alpha+\beta+2)} \left[ (-1)^{n-\ell+1} \frac{\Gamma(n+\alpha+2)}{\Gamma(\ell+\alpha+2)} + \frac{\Gamma(n+\beta+2)}{\Gamma(\ell+\beta+1)} \right].
 \end{aligned}$$

The  $(n-1)$ -th coefficient in (8) is given by

$$\frac{1}{2} \frac{n + \alpha + \beta + 2}{2n + \alpha + \beta + 3} \left( (n+1) \tilde{a}_{n+1} + (n+1) \tilde{c}_{n+1} \right) = n + 1.$$

Remark. If  $\alpha = \beta = \lambda - \frac{1}{2}$ , then the polynomials  $P_n^{(\lambda-\frac{1}{2}, \lambda-\frac{1}{2})}$  lead to the so-called Gegenbauer polynomials

$$C_n^{(\lambda)}(x) = \frac{(2\lambda)_n}{(\lambda+\frac{1}{2})_n} P_n^{(\lambda-\frac{1}{2}, \lambda-\frac{1}{2})}(x).$$

From the above computation we obtain

$$(xD) C_{2n}^{(\lambda)} = \sum_{k=0}^{n-1} (4k+2\lambda) C_{2k}^{(\lambda)} + 2n C_{2n}^{(\lambda)}$$

$$(xD) C_{2n+1}^{(\lambda)} = \sum_{k=0}^{n-1} (4k+2\lambda-2) C_{2k+1}^{(\lambda)} + (2n+1) C_{2n+1}^{(\lambda)}.$$

This result corresponds with the well-known formula

$$(\mathbf{x}\mathcal{D})(C_n^{(\lambda)} - C_{n-2}^{(\lambda)}) = nC_n^{(\lambda)} + (n-2+2\lambda)C_{n-2}^{(\lambda)},$$

cf. [2], p. 221.

Now for  $0 \leq \ell < n+1$  we put

$$(9) \quad \theta_{n+1, \ell}^{(\alpha, \beta)} = \frac{1}{2}(2\ell + \alpha + \beta + 1) \frac{\Gamma(\ell + \alpha + \beta + 1)}{\Gamma(n + \alpha + \beta + 2)} \left( (-1)^{n-\ell+1} \frac{\Gamma(n + \alpha + 2)}{\Gamma(\ell + \alpha + 1)} + \frac{\Gamma(n + \beta + 2)}{\Gamma(\ell + \beta + 1)} \right).$$

Then the matrix of the operator  $(\mathbf{x}\mathcal{D})$  with respect to  $(R_n^{(\alpha, \beta)})_{n=0}^{\infty}$  is given by

$$(10) \quad ((\mathbf{x}\mathcal{D})R_n^{(\alpha, \beta)}, R_\ell^{(\alpha, \beta)}) = \begin{cases} 0 & \text{if } \ell > n, \quad n \in \mathbb{N} \cup \{0\} \\ n & \text{if } \ell = n, \quad n \in \mathbb{N} \cup \{0\} \\ \theta_{n, \ell}^{(\alpha, \beta)} \frac{O_n^{(\alpha, \beta)}}{O_\ell^{(\alpha, \beta)}} & \text{if } 0 \leq \ell < n \text{ and } n \in \mathbb{N} \end{cases}$$

Above we have computed the explicit values of the matrix elements of the operators  $\mathcal{D}$  and  $(\mathbf{x}\mathcal{D})$  with respect to each orthonormal basis  $(R_n^{(\alpha, \beta)})_{n=0}^{\infty}$ .

The next step is the derivation of sufficiently sharp upper bounds for these values. Therefore we need the following result.

(11) Lemma

Let  $c, d > 0$ . Then there exists a positive constant  $K_{c, d} > 0$  such that for all  $m \in \mathbb{N}$

$$\frac{\Gamma(m+c)}{\Gamma(m+d)} \leq K_{c, d} m^{c-d}.$$

Proof

From [3] we take the following inequality:

$$\forall m \in \mathbb{N} \quad \forall s, 0 \leq s \leq 1: m^{1-s} \leq \frac{\Gamma(m+1)}{\Gamma(m+s)} \leq (m+1)^{1-s}.$$

We proceed as follows. Let  $m \in \mathbb{N}$ . Then

$$\frac{\Gamma(m+c)}{\Gamma(m+d)} = \frac{\Gamma(m+c)}{\Gamma(m+1)} \frac{\Gamma(m+1)}{\Gamma(m+d)} .$$

Moreover we have

$$\begin{aligned} \frac{\Gamma(m+c)}{\Gamma(m+1)} &= (m+c-1) \dots (m+c-[c]) \frac{\Gamma(m+c-[c])}{\Gamma(m+1)} \leq \\ &\leq (m+c-1)^{[c]} m^{c-[c]-1} \end{aligned}$$

and, also

$$\begin{aligned} \frac{\Gamma(m+1)}{\Gamma(m+d)} &= \frac{1}{(m+d-1) \dots (m+d-[d])} \frac{\Gamma(m+1)}{\Gamma(m+d-[d])} \leq \\ &\leq \left( \frac{1}{m+d-[d]} \right)^{[d]} (m+1)^{1-d+[d]} . \end{aligned}$$

Since

$$\begin{aligned} \frac{(m+c-1)^{[c]}}{(m+d-[d])^{[d]}} &= m^{[c]-[d]} \frac{\left(1 + \frac{c-1}{m}\right)^{[c]}}{\left(1 + \frac{d-[d]}{m}\right)^{[d]}} \leq \\ &\leq (c)^{[c]} m^{[c]-[d]} \end{aligned}$$

we finally get

$$\begin{aligned} \frac{\Gamma(m+c)}{\Gamma(m+d)} &\leq (c)^{[c]} m^{[c]-[d]} m^{c-[c]-1} (m+1)^{1-d+[d]} \leq \\ &\leq (c)^{[c]} 2^{1-[d]+d} m^{c-d} . \end{aligned}$$

□

The previous lemma gives rise to the following estimates

$$(12.i) \quad |O_k^{(\alpha, \beta)}| = \left( \frac{2k + \alpha + \beta + 1}{2^{\alpha+\beta+1}} \frac{\Gamma(k+1)\Gamma(k+\alpha+\beta+1)}{\Gamma(k+\alpha+1)\Gamma(k+\beta+1)} \right)^{\frac{1}{2}} =$$

$$\begin{aligned}
 &= \left( \frac{(2k+\alpha+\beta+1)(k+\alpha+1)(k+\beta+1)\Gamma(k+2)\Gamma(k+\alpha+\beta+3)}{2^{\alpha+\beta+1}(k+\alpha+\beta+1)(k+\alpha+\beta+2)(k+1)\Gamma(k+\alpha+2)\Gamma(k+\beta+2)} \right)^{\frac{1}{2}} \leq \\
 &\leq \left( \sup_{k \in \mathbb{N} \cup \{0\}} \left\{ \frac{(2k+\alpha+\beta+1)(k+\alpha+1)(k+\beta+1)}{2^{\alpha+\beta+1}(k+\alpha+\beta+1)(k+\alpha+\beta+2)(k+1)} \right\} K_{1, \alpha+1} K_{\alpha+\beta+2, \beta+1} (k+1) \right)^{\frac{1}{2}} \\
 &=: C_{\alpha, \beta} (k+1)^{\frac{1}{2}}
 \end{aligned}$$

$$\begin{aligned}
 (12.ii) \quad |O_k^{(\alpha, \beta)}|^{-1} &= \left( \frac{2^{\alpha+\beta+1}(k+\alpha+\beta+1)(k+\alpha+\beta+2)(k+1)}{(2k+\alpha+\beta+1)(k+\alpha+1)(k+\beta+1)} \frac{\Gamma(k+\alpha+2)\Gamma(k+\beta+2)}{\Gamma(k+2)\Gamma(k+\alpha+\beta+3)} \right)^{\frac{1}{2}} \leq \\
 &\leq D_{\alpha, \beta} (k+1)^{-\frac{1}{2}}
 \end{aligned}$$

for some positive constant  $D_{\alpha, \beta}$ .

$$\begin{aligned}
 (12.iii) \quad |Y_{n, \ell}^{(\alpha, \beta)}| &\leq \left| \frac{\Gamma(n+\alpha+2)}{\Gamma(n+\alpha+\beta+3)} \frac{\Gamma(\ell+\alpha+\beta+1)}{\Gamma(\ell+\alpha+1)} (2\ell+\alpha+\beta+1) \right| \cdot \\
 &\quad \cdot \left( \sum_{k=\ell}^n (2k+\alpha+\beta+2) \frac{\Gamma(k+\alpha+1)}{\Gamma(k+\beta+2)} \right) = \\
 &= \frac{\Gamma(n+\beta+2)}{\Gamma(n+\alpha+\beta+3)} \frac{\Gamma(\ell+\alpha+\beta+3)}{\Gamma(\ell+\alpha+2)} \left| \frac{2\ell+\alpha+\beta+1}{(\ell+\alpha+\beta+1)} \frac{(\ell+\alpha+1)}{(\ell+\alpha+\beta+2)} \right| \cdot \\
 &\quad \cdot \left( \sum_{k=\ell}^n \frac{2k+\alpha+\beta+2}{k+\alpha+1} \frac{\Gamma(k+\alpha+2)}{\Gamma(k+\beta+2)} \right) \leq \\
 &\leq \left\{ \sup_{\ell \in \mathbb{N} \cup \{0\}} \left| \frac{(2\ell+\alpha+\beta+1)(\ell+\alpha+1)}{(\ell+\alpha+\beta+1)(\ell+\alpha+\beta+2)} \right| \right\} K_{\beta+1, \alpha+\beta+1} K_{\alpha+\beta+1, \alpha+1} \cdot \\
 &\quad \cdot (n+1)^{-(\alpha+1)} (\ell+1)^{\beta+1} \left\{ \sup_{k \in \mathbb{N} \cup \{0\}} \left( \frac{2k+\alpha+\beta+2}{k+\alpha+1} \right) \right\} \cdot \\
 &\quad \cdot \sum_{k=\ell}^n K_{\alpha+1, \beta+1} (k+1)^{\alpha-\beta} = \\
 &=: E_{\alpha, \beta} \sum_{k=\ell}^n \left( \frac{k+1}{n+1} \right)^{\alpha+1} \left( \frac{\ell+1}{k+1} \right)^{\beta+1} < E_{\alpha, \beta} (n-\ell+1)
 \end{aligned}$$

$$\begin{aligned}
 (12.iv) \quad |\theta_{n+1, \ell}^{(\alpha, \beta)}| &\leq \frac{1}{2}(n+\alpha+\beta+2) \left[ \frac{|2\ell+\alpha+\beta+1| |\ell+\alpha+1|}{|\ell+\alpha+\beta+1| |\ell+\alpha+\beta+2|} \frac{\Gamma(n+\alpha+2)\Gamma(\ell+\alpha+\beta+3)}{\Gamma(n+\alpha+\beta+3)\Gamma(\ell+\alpha+2)} + \right. \\
 &\quad \left. + \frac{|2\ell+\alpha+\beta+1| |\ell+\beta+1|}{|\ell+\alpha+\beta+1| |\ell+\alpha+\beta+2|} \frac{\Gamma(n+\beta+2)\Gamma(\ell+\alpha+\beta+3)}{\Gamma(n+\alpha+\beta+3)\Gamma(\ell+\beta+2)} \right] \leq \\
 &\leq \frac{1}{2}(n+\alpha+\beta+2) \left[ \sup_{\ell \in \mathbb{N} \cup \{0\}} \left( \frac{|2\ell+\alpha+\beta+1| |\ell+\alpha+1|}{|\ell+\alpha+\beta+1| |\ell+\alpha+\beta+2|} \right) \cdot \right. \\
 &\quad \cdot K_{\alpha+1, \alpha+\beta+2} K_{\alpha+\beta+2, \alpha+1} \left( \frac{\ell+1}{n+1} \right)^{\beta+1} + \\
 &\quad \left. + \left( \sup_{\ell \in \mathbb{N} \cup \{0\}} \frac{|2\ell+\alpha+\beta+1| |\ell+\beta+1|}{|\ell+\alpha+\beta+1| |\ell+\alpha+\beta+2|} \right) \cdot \right. \\
 &\quad \left. \cdot K_{\beta+1, \alpha+\beta+2} K_{\alpha+\beta+2, \beta+1} \left( \frac{\ell+1}{n+1} \right)^{\alpha+1} \right] \leq \\
 &\leq F_{\alpha, \beta}(n+1)
 \end{aligned}$$

for some well-chosen positive constant  $F_{\alpha, \beta}$ .

With the estimates (a.13.i-iv) we find

$$(13) \quad |(\mathcal{D}_n^{(\alpha, \beta)}, R_k^{(\alpha, \beta)})_{\alpha, \beta}| \leq \begin{cases} 0 & \text{if } k \geq n \\ G_{\alpha, \beta} \frac{n^{3/2}(n-k)}{(k+1)^{1/2}} & \text{if } 0 \leq k < n \end{cases}$$

Here  $G_{\alpha, \beta} > 0$  is a constant dependent on  $C_{\alpha, \beta}$ ,  $D_{\alpha, \beta}$  and  $E_{\alpha, \beta}$ . Also

$$(14) \quad |((x\mathcal{D})_n^{(\alpha, \beta)}, R_k^{(\alpha, \beta)})_{\alpha, \beta}| \leq \begin{cases} 0 & \text{if } k > n \\ n & \text{if } k = n \\ H_{\alpha, \beta}(n)^{3/2} (k+1)^{-1/2} & \text{if } 0 \leq k < n \end{cases}$$

References

- [1] Eindhoven, S.J.L. van, and J. de Graaf, On distribution spaces based on Jacobi polynomials. EUT Report 84-WSK-01, Eindhoven University of Technology, March 1984.
- [2] Magnus, W., F. Oberhettinger and R.P. Soni, Formulas and theorems for the special functions of mathematical physics. 3e Edition, Springer, Berlin, 1966.
- [3] Mitrinovic, D.S., Analytic inequalities. First edition, Springer, Berlin, 1970.