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Abstract

From the recurrence relations of the Jacobi polynomials we compute the matrix entries of the differential operators $\frac{d}{dx}$ and $x\frac{d}{dx}$ with respect to the corresponding orthonormal bases of normalized Jacobi polynomials.

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Some notations

In this paper we consider the Hilbert spaces

$$X_{\alpha,\beta} = L_2([-1,1], (1-x)^{\alpha}(1+x)^{\beta}\,dx)$$

and the positive self-adjoint operators $A_{\alpha,\beta}$ in $X_{\alpha,\beta}$

$$A_{\alpha,\beta} = -(1-x^2)\frac{d^2}{dx^2} - ((\beta-\alpha) - (\alpha+\beta+2)x)\frac{d}{dx}$$

where we take α and β larger than -1. The operator $A_{\alpha,\beta}$ has a discrete spectrum $\{n(n+\alpha+\beta+1) \mid n \in \mathbb{N} \cup \{0\}\}$. Its normalized eigenvectors are the normalized Jacobi polynomials $R_n^{(\alpha,\beta)}$

$$R_n^{(\alpha,\beta)} = \left[\frac{\alpha+\beta+2n+1}{2^{\alpha+\beta+1}}\ \frac{\Gamma(n+1)\,\Gamma(n+\alpha+\beta+1)}{\Gamma(n+\alpha+1)\,\Gamma(n+\beta+1)}\right]^{\frac{1}{2}} P_n^{(\alpha,\beta)}$$

where

$$P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{n!\,2^n}\ \frac{1}{(1-x)^{\alpha}(1+x)^{\beta}} \left(\frac{d}{dx}\right)^n [(1-x)^{\alpha+n}(1+x)^{\beta+n}]$$

(cf. [2], p. 209).

In our study of distribution spaces based on Jacobi polynomials, cf. [1], we needed an estimation of the matrix entries $(\mathcal{D}R_n^{(\alpha,\beta)}, R_k^{(\alpha,\beta)})_{\alpha,\beta}$ and $((x\mathcal{D})R_n^{\alpha,\beta}, R_k^{(\alpha,\beta)})_{\alpha,\beta}$ where $(\cdot,\cdot)_{\alpha,\beta}$ denotes the inner product in the Hilbert space $X_{\alpha,\beta}$ and where $\mathcal{D}$ denotes the differential operator $\frac{d}{dx}$. Exact expressions for these matrix entries are not known. In this note we present such expressions. Also we give estimates for the considered matrix entries.

Results

In [2], p. 213, the following relations can be found

(1) $$\mathcal{D}P_n^{(\alpha,\beta)} = \tfrac{1}{2}(n+\alpha+\beta+1)P_{n-1}^{(\alpha+1,\beta+1)}, \qquad n = 1,2,\ldots .$$

We express the polynomials $P_{n-1}^{(\alpha+1,\beta+1)}$ as finite combinations of the polynomials $P_k^{(\alpha,\beta)}$, $k = 0,1,2,\ldots,n-1$. So we write

$$P_{n-1}^{(\alpha+1,\beta+1)} = \sum_{k=0}^{n-1} \gamma_{n-1,k}^{(\alpha,\beta)} P_k^{(\alpha,\beta)} .$$

In order to compute the coefficients $\gamma_{n,k}^{(\alpha,\beta)}$, $n = 0,1,2,\ldots$, $k = 0,1,2,\ldots,n$, we use the following relations which can be derived from [2], p. 213

(2.i) $$P_\ell^{(\alpha+1,\beta+1)} = c_\ell P_\ell^{(\alpha,\beta+1)} + d_\ell P_{\ell-1}^{(\alpha+1,\beta+1)}$$

(2.ii) $$P_\ell^{(\alpha,\beta+1)} = a_\ell P_\ell^{(\alpha,\beta)} + b_\ell P_{\ell-1}^{(\alpha,\beta+1)}$$

where

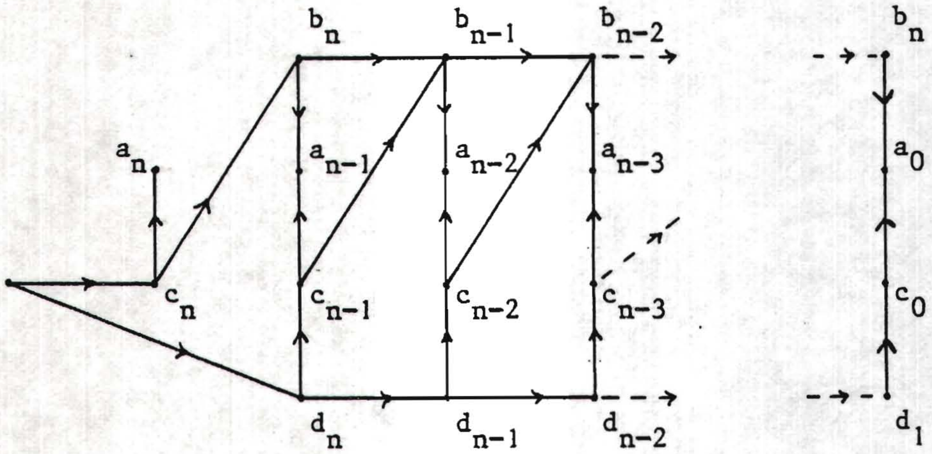
$$a_\ell = \frac{2\ell+\alpha+\beta+1}{\ell+\alpha+\beta+1}, \qquad c_\ell = \frac{2\ell+\alpha+\beta+2}{\ell+\alpha+\beta+2},$$

$$b_\ell = -\frac{\ell+\alpha}{\ell+\alpha+\beta+1}, \qquad d_\ell = \frac{\ell+\beta+1}{\ell+\alpha+\beta+2}.$$

So starting from $P_\ell^{(\alpha+1,\beta+1)}$ we get $c_\ell P_\ell^{(\alpha,\beta+1)} + d_\ell P_{\ell-1}^{(\alpha+1,\beta+1)}$ by (2.i) and then by (2.ii) $P_\ell^{(\alpha,\beta+1)} = a_\ell P_\ell^{(\alpha,\beta)} + b_\ell P_{\ell-1}^{(\alpha,\beta+1)}$, and also by (2.i) $P_{\ell-1}^{(\alpha+1,\beta+1)} = c_{\ell-1} P_{\ell-1}^{(\alpha,\beta+1)} + d_{\ell-1} P_{\ell-2}^{(\alpha+1,\beta+1)}$, etc.

The sketched process terminates, because $\forall_{p,q>-1}: P_0^{(p,q)} \equiv 1$. It can be described by the following directed graph.

(3)



The graph (3) shows the following:

c_ℓ is multiplied by a_ℓ or b_ℓ

d_ℓ is multiplied by $d_{\ell-1}$ or $c_{\ell-1}$

b_ℓ is multiplied by $b_{\ell-1}$ or $a_{\ell-1}$

every factor ends with some a_q.

The above examinations yield the following result:

$$P_n^{(\alpha+1,\beta+1)} = \sum_{p=0}^{n} \left(\sum_{k=0}^{p} d_n \cdots d_{n-k+1}\, c_{n-k}\, b_{n-k} \cdots b_{n-p+1}\, a_{n-p} \right) P_{n-p}^{(\alpha,\beta)}$$

with the convention $d_n d_{n+1} = 1$ and $b_{n-p} b_{n-p+1} = 1$; equivalently

$$(4) \qquad P_n^{(\alpha+1,\beta+1)} = \sum_{\ell=0}^{n} \left(\sum_{k=0}^{n-\ell} d_n \cdots d_{n-k+1}\, c_{n-k}\, b_{n-k} \cdots b_{\ell+1}\, a_\ell \right) P_\ell^{(\alpha,\beta)} \; .$$

Thus we find that

$$\gamma_{n,\ell}^{(\alpha,\beta)} = \sum_{k=0}^{n-\ell} (d_n \cdots d_{n-k+1}\, c_{n-k}\, b_{n-k} \cdots b_{\ell+1}\, a_\ell) \; .$$

A simple calculation yields

$$(5)\qquad \gamma_{n,\ell}^{(\alpha,\beta)} = (-1)^{\ell}\,\frac{\Gamma(\ell+\alpha+\beta+1)\,\Gamma(n+\beta+2)}{\Gamma(n+\alpha+\beta+3)\,\Gamma(\ell+\alpha+1)}\,(2\ell+\alpha+\beta+1)\;\cdot$$

$$\cdot\;\sum_{k=\ell}^{n}(-1)^{k}(2k+\alpha+\beta+2)\,\frac{\Gamma(k+\alpha+1)}{\Gamma(k+\beta+2)}\,.$$

Let $o_k^{(\alpha,\beta)}$ denote the $X_{\alpha,\beta}$-normalization factor for the Jacobi polynomials,

$$(6)\qquad o_k^{(\alpha,\beta)} = \left(\frac{2k+\alpha+\beta+1}{2^{\alpha+\beta+1}}\,\frac{\Gamma(k+1)\,\Gamma(k+\alpha+\beta+1)}{\Gamma(k+\alpha+1)\,\Gamma(k+\beta+1)}\right)^{\frac{1}{2}}.$$

Then we obtain for the matrix of $\mathcal{D}$ with respect to the orthonormal basis $(R_n^{(\alpha,\beta)})_{n=0}^{\infty}$ of $X_{\alpha,\beta}$

$$(7)\qquad (\mathcal{D}R_n^{(\alpha,\beta)},R_\ell^{(\alpha,\beta)})_{X_{\alpha,\beta}} = \begin{cases} 0 & \text{if } \ell \ge n,\ \ell,n \in \mathbb{N}\cup\{0\} \\[2ex] \frac{1}{2}\,\dfrac{o_n^{(\alpha,\beta)}}{o_\ell^{(\alpha,\beta)}}\,\gamma_{n-1,\ell}^{(\alpha,\beta)}(n+\alpha+\beta+1) & \\ & \text{if } \ell = 0,1,\dots,n-1,\ n\in\mathbb{N}. \end{cases}$$

With similar methods we next compute the matrix of the differential operator $x\mathcal{D}$ with respect to $(R_n^{(\alpha,\beta)})_{n=0}^{\infty}$. From [2], p.213, we obtain the following identities:

$$(1-x)P_n^{(\alpha,\beta)}(x) = \frac{2(n+\alpha)}{2n+\alpha+\beta+1}\,P_n^{(\alpha-1,\beta)}(x) - \frac{2(n+1)}{2n+\alpha+\beta+1}\,P_{n+1}^{(\alpha-1,\beta)}(x)$$

and

$$(1+x)P_n^{(\alpha,\beta)}(x) = \frac{2(n+\beta)}{2n+\alpha+\beta+1}\,P_n^{(\alpha,\beta-1)}(x) + \frac{2(n+1)}{2n+\alpha+\beta+1}\,P_{n-1}^{(\alpha,\beta-1)}(x)\,.$$

Adding these relations, we obtain the following formula

$$xP_n^{(\alpha,\beta)}(x) = \frac{1}{2n+\alpha+\beta+1}\Big[-(n+\alpha)P_n^{(\alpha-1,\beta)}(x) + (n+1)P_{n-1}^{(\alpha-1,\beta)}(x) + $$

$$+ (n+\beta)P_n^{(\alpha,\beta-1)}(x) + (n+1)P_{n+1}^{(\alpha,\beta-1)}(x)\Big] .$$

Thus it follows that

$$(8) \qquad (x\mathcal{D})P_{n+1}^{(\alpha,\beta)}(x) = \tfrac{1}{2}(n+\alpha+\beta+2)\, xP_n^{(\alpha+1,\beta+1)}(x) =$$

$$= \tfrac{1}{2}\,\frac{n+\alpha+\beta+2}{2n+\alpha+\beta+3}\Big[-(n+\alpha+1)P_n^{(\alpha,\beta+1)}(x) + (n+1)P_{n+1}^{(\alpha,\beta+1)}(x) +$$

$$+ (n+\beta+1)P_n^{(\alpha+1,\beta)}(x) + (n+1)P_{n+1}^{(\alpha+1,\beta)}(x)\Big] .$$

With the relations (2.i) and (2.ii) we get

$$P_k^{(\alpha+1,\beta)} = \sum_{\ell=0}^{k} \tilde{d}_k \cdots \tilde{d}_{\ell+1}\, \tilde{c}_\ell\, P_\ell^{(\alpha,\beta)}$$

and

$$P_k^{(\alpha,\beta+1)} = \sum_{\ell=0}^{k} \tilde{b}_k \cdots \tilde{b}_{\ell+1}\, \tilde{a}_\ell\, P_\ell^{(\alpha,\beta)}$$

where

$$\tilde{a}_j = \frac{2j+\alpha+\beta+1}{j+\alpha+\beta+1}, \qquad \tilde{b}_j = -\frac{j+\alpha}{j+\alpha+\beta+1},$$

$$\tilde{c}_j = \frac{2j+\alpha+\beta+1}{j+\alpha+\beta+1}, \qquad \tilde{d}_j = \frac{j+\beta}{j+\alpha+\beta+1} .$$

Finally, substituting the above values in (8) we get for the ℓ-th coefficient, $0 \le \ell \le n$, in the expression of $(x\mathcal{D})P_{n+1}^{(\alpha,\beta)}$

$$\frac{1}{2}\,\frac{n+\alpha+\beta+2}{2n+\alpha+\beta+3}\Bigg[(-1)^{n-\ell-1}\,(n+\alpha+1)\,\frac{\Gamma(n+\alpha+1)}{\Gamma(n+\alpha+\beta+2)}\,\frac{\Gamma(\ell+\alpha+\beta+1)}{\Gamma(\ell+\alpha+1)}\,(2\ell+\alpha+\beta+1)\,+$$

$$+\,(-1)^{n-\ell+1}(n+1)\,\frac{\Gamma(n+\alpha+2)}{\Gamma(n+\alpha+\beta+3)}\,\frac{\Gamma(\ell+\alpha+\beta+1)}{\Gamma(\ell+\alpha+1)}\,(2\ell+\alpha+\beta+1)\,+$$

$$+\,(n+\beta+1)\,\frac{\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+2)}\,\frac{\Gamma(\ell+\alpha+\beta+1)}{\Gamma(\ell+\beta+1)}\,(2\ell+\alpha+\beta+1)\,+$$

$$+\,(n+1)\,\frac{\Gamma(n+\beta+2)}{\Gamma(n+\alpha+\beta+3)}\,\frac{\Gamma(\ell+\alpha+\beta+1)}{\Gamma(\ell+\beta+1)}\,(2\ell+\alpha+\beta+1)\Bigg]=$$

$$=\frac{1}{2}\,\frac{n+\alpha+\beta+2}{2n+\alpha+\beta+3}\,(2\ell+\alpha+\beta+1)\,\frac{\Gamma(\ell+\alpha+\beta+1)}{\Gamma(n+\alpha+\beta+2)}\,\Big(1+\frac{n-1}{n+\alpha+\beta+2}\Big)\,\cdot$$

$$\cdot\,\Bigg[(-1)^{n-\ell+1}\,\frac{\Gamma(n+\alpha+2)}{\Gamma(\ell+\alpha+2)}+\frac{\Gamma(n+\beta+2)}{\Gamma(\ell+\beta+1)}\Bigg]=$$

$$=\frac{1}{2}(2\ell+\alpha+\beta+1)\,\frac{\Gamma(\ell+\alpha+\beta+1)}{\Gamma(n+\alpha+\beta+2)}\Bigg[(-1)^{n-\ell+1}\,\frac{\Gamma(n+\alpha+2)}{\Gamma(\ell+\alpha+2)}+\frac{\Gamma(n+\beta+2)}{\Gamma(\ell+\beta+1)}\Bigg]\,.$$

The (n-1)-th coefficient in (8) is given by

$$\frac{1}{2}\,\frac{n+\alpha+\beta+2}{2n+\alpha+\beta+3}\,\big((n+1)\tilde{a}_{n+1}+(n+1)\tilde{c}_{n+1}\big)=n+1\,.$$

Remark. If $\alpha=\beta=\lambda-\frac{1}{2}$, then the polynomials $P_n^{(\lambda-\frac{1}{2},\lambda-\frac{1}{2})}$ lead to the so-called Gegenbauer polynomials

$$C_n^{(\lambda)}(x)=\frac{(2\lambda)_n}{(\lambda+\frac{1}{2})_n}\,P_n^{(\lambda-\frac{1}{2},\lambda-\frac{1}{2})}(x)\,.$$

From the above computation we obtain

$$(xD)\,C_{2n}^{(\lambda)}=\sum_{k=0}^{n-1}(4k+2\lambda)\,C_{2k}^{(\lambda)}+2n\,C_{2n}^{(\lambda)}$$

$$(xD)\,C_{2n+1}^{(\lambda)}=\sum_{k=0}^{n-1}(4k+2\lambda-2)\,C_{2k+1}^{(\lambda)}+(2n+1)\,C_{2n+1}^{(\lambda)}\,.$$

This result corresponds with the well-known formula

$$(xD)(C_n^{(\lambda)}-C_{n-2}^{(\lambda)}) = n\,C_n^{(\lambda)} + (n-2+2\lambda)C_{n-2}^{(\lambda)} ,$$

cf. [2], p. 221.

Now for $0 \leq \ell < n+1$ we put

$$\text{(9)} \qquad \theta_{n+1,\ell}^{(\alpha,\beta)} = \tfrac{1}{2}(2\ell+\alpha+\beta+1)\,\frac{\Gamma(\ell+\alpha+\beta+1)}{\Gamma(n+\alpha+\beta+2)}\left((-1)^{n-\ell+1}\frac{\Gamma(n+\alpha+2)}{\Gamma(\ell+\alpha+1)} + \frac{\Gamma(n+\beta+2)}{\Gamma(\ell+\beta+1)}\right).$$

Then the matrix of the operator (xD) with respect to $(R_n^{(\alpha,\beta)})_{n=0}^{\infty}$ is given by

$$\text{(10)} \qquad ((xD)R_n^{(\alpha,\beta)}, R_\ell^{(\alpha,\beta)}) = \begin{cases} 0 & \text{if } \ell > n , \quad n \in \mathbb{N} \cup \{0\} \\ n & \text{if } \ell = n , \quad n \in \mathbb{N} \cup \{0\} \\ \theta_{n,\ell}^{(\alpha,\beta)} \dfrac{O_n^{(\alpha,\beta)}}{O_\ell^{(\alpha,\beta)}} & \text{if } 0 \leq \ell < n \text{ and } n \in \mathbb{N} \end{cases}$$

Above we have computed the explicit values of the matrix elements of the operators D and (xD) with respect to each orthonormal basis $(R_n^{(\alpha,\beta)})_{n=0}^{\infty}$. The next step is the derivation of sufficiently sharp upper bounds for these values. Therefore we need the following result.

(11) <u>Lemma</u>

Let $c,d > 0$. Then there exists a positive constant $K_{c,d} > 0$ such that for all $m \in \mathbb{N}$

$$\frac{\Gamma(m+c)}{\Gamma(m+d)} \leq K_{c,d}\, m^{c-d} .$$

<u>Proof</u>

From [3] we take the following inequality:

$$\forall_{m\in\mathbb{N}}\ \forall_{s,0\leq s\leq 1}:\ m^{1-s} \leq \frac{\Gamma(m+1)}{\Gamma(m+s)} \leq (m+1)^{1-s} .$$

We proceed as follows. Let $m \in \mathbb{N}$. Then

$$\frac{\Gamma(m+c)}{\Gamma(m+d)} = \frac{\Gamma(m+c)}{\Gamma(m+1)} \frac{\Gamma(m+1)}{\Gamma(m+d)} .$$

Moreover we have

$$\frac{\Gamma(m+c)}{\Gamma(m+1)} = (m+c-1) \ldots (m+c-[c]) \frac{\Gamma(m+c-[c])}{\Gamma(m+1)} \leq$$

$$\leq (m+c-1)^{[c]} m^{c-[c]-1}$$

and, also

$$\frac{\Gamma(m+1)}{\Gamma(m+d)} = \frac{1}{(m+d-1)\ldots(m+d-[d])} \frac{\Gamma(m+1)}{\Gamma(m+d-[d])} \leq$$

$$\leq \left(\frac{1}{m + d - [d]}\right)^{[d]} (m+1)^{1-d+[d]} .$$

Since

$$\frac{(m+c-1)^{[c]}}{(m+d- d)^{[d]}} = m^{[c]-[d]} \frac{\left(1 + \frac{c-1}{m}\right)^{[c]}}{\left(1 + \frac{d-[d]}{m}\right)^{[d]}} \leq$$

$$\leq (c)^{[c]} m^{[c]-[d]}$$

we finally get

$$\frac{\Gamma(m+c)}{\Gamma(m+d)} \leq (c)^{[c]} m^{[c]-[d]} m^{c-[c]-1} (m+1)^{1-d+[d]} \leq$$

$$\leq (c)^{[c]} 2^{1-[d]+d} m^{c-d} . \qquad \square$$

The previous lemma gives rise to the following estimates

$$(12.i) \qquad |0_k^{(\alpha,\beta)}| = \left(\frac{2k + \alpha + \beta + 1}{2^{\alpha+\beta+1}} \frac{\Gamma(k+1)\Gamma(k+\alpha+\beta+1)}{\Gamma(k+\alpha+1)\Gamma(k+\beta+1)}\right)^{\frac{1}{2}} =$$

$$= \left(\frac{(2k+\alpha+\beta+1)(k+\alpha+1)(k+\beta+1)\,\Gamma(k+2)\,\Gamma(k+\alpha+\beta+3)}{2^{\alpha+\beta+1}(k+\alpha+\beta+1)(k+\alpha+\beta+2)(k+1)\,\Gamma(k+\alpha+2)\,\Gamma(k+\beta+2)} \right)^{\frac{1}{2}} \leq$$

$$\leq \left(\sup_{k\in\mathbb{N}\cup\{0\}} \left\{ \frac{(2k+\alpha+\beta+1)(k+\alpha+1)(k+\beta+1)}{2^{\alpha+\beta+1}(k+\alpha+\beta+1)(k+\alpha+\beta+2)(k+1)} \right\} K_{1,\alpha+1}\, K_{\alpha+\beta+2,\beta+1}(k+1) \right)^{\frac{1}{2}}$$

$$=: C_{\alpha,\beta}(k+1)^{\frac{1}{2}} .$$

(12.ii) $$|O_k^{(\alpha,\beta)}|^{-1} = \left(\frac{2^{\alpha+\beta+1}(k+\alpha+\beta+1)(k+\alpha+\beta+2)(k+1)}{(2k+\alpha+\beta+1)(k+\alpha+1)(k+\beta+1)} \frac{\Gamma(k+\alpha+2)\,\Gamma(k+\beta+2)}{\Gamma(k+2)\,\Gamma(k+\alpha+\beta+3)} \right)^{\frac{1}{2}} \leq$$

$$\leq D_{\alpha,\beta}(k+1)^{-\frac{1}{2}}$$

for some positive constant $D_{\alpha,\beta}$.

(12.iii) $$|\gamma_{n,\ell}^{(\alpha,\beta)}| \leq \left| \frac{\Gamma(n+\alpha+2)}{\Gamma(n+\alpha+\beta+3)} \frac{\Gamma(\ell+\alpha+\beta+1)}{\Gamma(\ell+\alpha+1)} (2\ell+\alpha+\beta+1) \right| \cdot$$

$$\cdot \left(\sum_{k=\ell}^{n} (2k+\alpha+\beta+2) \frac{\Gamma(k+\alpha+1)}{\Gamma(k+\beta+2)} \right) =$$

$$= \frac{\Gamma(n+\beta+2)}{\Gamma(n+\alpha+\beta+3)} \frac{\Gamma(\ell+\alpha+\beta+3)}{\Gamma(\ell+\alpha+2)} \left| \frac{2\ell+\alpha+\beta+1}{(\ell+\alpha+\beta+1)} \frac{(\ell+\alpha+1)}{(\ell+\alpha+\beta+2)} \right| \cdot$$

$$\cdot \left(\sum_{k=\ell}^{n} \frac{2k+\alpha+\beta+2}{k+\alpha+1} \frac{\Gamma(k+\alpha+2)}{\Gamma(k+\beta+2)} \right) \leq$$

$$\leq \left\{ \sup_{\ell\in\mathbb{N}\cup\{0\}} \left| \frac{(2\ell+\alpha+\beta+1)(\ell+\alpha+1)}{(\ell+\alpha+\beta+1)(\ell+\alpha+\beta+2)} \right| \right\} K_{\beta+1,\alpha+\beta+1}\, K_{\alpha+\beta+1,\alpha+1} \cdot$$

$$\cdot (n+1)^{-(\alpha+1)}(\ell+1)^{\beta+1} \left\{ \sup_{k\in\mathbb{N}\cup\{0\}} \left(\frac{2k+\alpha+\beta+2}{k+\alpha+1} \right) \right\} \cdot$$

$$\cdot \sum_{k=\ell}^{n} K_{\alpha+1,\beta+1}(k+1)^{\alpha-\beta} =$$

$$=: E_{\alpha,\beta} \sum_{k=\ell}^{n} \left(\frac{k+1}{n+1} \right)^{\alpha+1} \left(\frac{\ell+1}{k+1} \right)^{\beta+1} < E_{\alpha,\beta}(n-\ell+1)$$

(12.iv) $$|\theta^{(\alpha,\beta)}_{n+1,\ell}| \le \tfrac{1}{2}(n+\alpha+\beta+2)\left[\frac{|2\ell+\alpha+\beta+1|\,|\ell+\alpha+1|}{|\ell+\alpha+\beta+1|\,|\ell+\alpha+\beta+2|}\,\frac{\Gamma(n+\alpha+2)\Gamma(\ell+\alpha+\beta+3)}{\Gamma(n+\alpha+\beta+3)\Gamma(\ell+\alpha+2)} + \right.$$

$$\left. + \frac{|2\ell+\alpha+\beta+1|\,|\ell+\beta+1|}{|\ell+\alpha+\beta+1|\,|\ell+\alpha+\beta+2|}\,\frac{\Gamma(n+\beta+2)\Gamma(\ell+\alpha+\beta+3)}{\Gamma(n+\alpha+\beta+3)\Gamma(\ell+\beta+2)}\right] \le$$

$$\le \tfrac{1}{2}(n+\alpha+\beta+2)\left[\sup_{\ell\in\mathbb{N}\cup\{0\}}\left(\frac{|2\ell+\alpha+\beta+1|\,|\ell+\alpha+1|}{|\ell+\alpha+\beta+1|\,|\ell+\alpha+\beta+2|}\right)\cdot\right.$$

$$\cdot K_{\alpha+1,\alpha+\beta+2}\,K_{\alpha+\beta+2,\alpha+1}\left(\frac{\ell+1}{n+1}\right)^{\beta+1} +$$

$$+\left(\sup_{\ell\in\mathbb{N}\cup\{0\}}\frac{|2\ell+\alpha+\beta+1|\,|\ell+\beta+1|}{|\ell+\alpha+\beta+1|\,|\ell+\alpha+\beta+2|}\right)\cdot$$

$$\left.\cdot K_{\beta+1,\alpha+\beta+2}\,K_{\alpha+\beta+2,\beta+1}\left(\frac{\ell+1}{n+1}\right)^{\alpha+1}\right] \le$$

$$\le F_{\alpha,\beta}(n+1)$$

for some well-chosen positive constant $F_{\alpha,\beta}$.

With the estimates (a.13.i-iv) we find

(13) $$|(\mathcal{D}R^{(\alpha,\beta)}_n, R^{(\alpha,\beta)}_k)_{\alpha,\beta}| \le \begin{cases} 0 & \text{if } k \ge n \\ G_{\alpha,\beta}\,\dfrac{n^{3/2}(n-k)}{(k+1)^{1/2}} & \text{if } 0 \le k < n \end{cases}$$

Here $G_{\alpha,\beta} > 0$ is a constant dependent on $C_{\alpha,\beta}$, $D_{\alpha,\beta}$ and $E_{\alpha,\beta}$. Also

(14) $$|((x\mathcal{D})R^{(\alpha,\beta)}_n, R^{(\alpha,\beta)}_k)_{\alpha,\beta}| \le \begin{cases} 0 & \text{if } k > n \\ n & \text{if } k = n \\ H_{\alpha,\beta}(n)^{3/2}(k+1)^{-1/2} & \text{if } 0 \le k < n \end{cases}$$

References

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