Regular finite planar maps with equal edges

EINDHOVEN UNIVERSITY OF TECHNOLOGY

Department of Mathematics and Computing Science

Memorandum 1982-12

July 1982

Regular finite planar maps with equal edges

by

A. Blokhuis

University of Technology

Department of Mathematics and Computing Science

PO Box 513, Eindhoven

The Netherlands

REGULAR FINITE PLANAR MAPS WITH EQUAL EDGES

by

A. Blokhuis

Abstract.

There doesn't exist a finite planar map with all edges having the same length, and each vertex on exactly 5 edges.

Introduction.

At the 1981-meeting for discrete geometry in Oberwolfach, H. Harborth posed the following problem: Is it possible to put a finite set of match-sticks in the plane such that in each endpoint a constant number k of matches meet, and no two match-sticks overlap? Also if possible, what is the minimum number of match-sticks in such a configuration. He proceeded to give minimal examples for k = 2 (fig.1) and k = 3 (fig.2) and a non minimal example for k = 4 (fig.3).

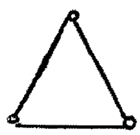


fig.1

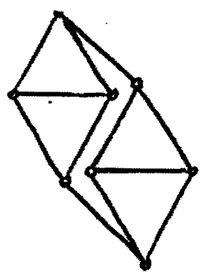


fig.2

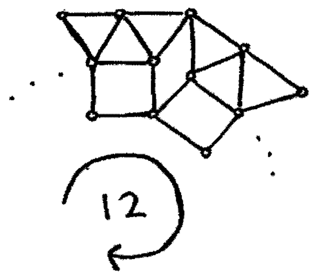
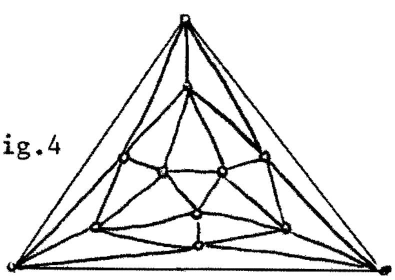


fig.3

For $k \geq 6$ there exist no finite regular planar map of valency k by consequence of Euler's theorem: $V - E + F = 2$ where V = the number of vertices, E = the number of edges, F = the number of faces.

For k = 5 there do exist finite regular maps, the smallest one is the graph of the icosahedron (fig.4), but it is not possible to draw it in such a way that all edges have the same length,

fig.4



We will show that this is true for all finite planar graphs that are regular of degree 5.

Theorem. No finite planar map with straight edges of equal length exists that is a regular of degree 5.

Proof.

Let V denote the number of vertices, E the number of edges and F the number of faces of a planar map. We then have Euler's relation:

$$V - E + F = 2 . \qquad (1)$$

If, furthermore each point is on 5 edges then

$$5V = 2E . \qquad (2)$$

Write F_i for the number of faces with i sides, then

$$F = F_3 + F_4 + \dots \qquad (3)$$

and

$$2E = 3F_3 + 4F_4 + \dots . \qquad (4)$$

We may combine (2), (3) and (4) to get

$$F_3 - 2F_4 - 5F_5 - 8F_6 - \dots = 20 . \qquad (5)$$

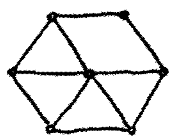
For any vertex v we define

$$f_i(v) = \#\ \text{i-gonal faces containing } v ,$$
$$f(v) = \frac{f_3(v)}{3} - \frac{2f_4(v)}{4} - \frac{5f_5(v)}{5} - \dots . \qquad (6)$$

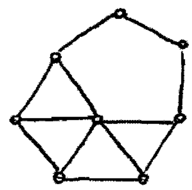
From (5) and (6) and $\sum_{v\in V} \frac{f_i(v)}{i} = F_i$ we obtain

$$\sum_v f(v) = 20 . \qquad (7)$$

From now on, we assume that the edges in the map all have the same length. A point is then surrounded by at most 4 triangles, and the only possibilities for a point v, making a positive contribution to $\sum_v f(v)$ are 4 triangles + a tetragon, or a pentagon:



$f(v) = \frac{4}{3}$



$f(v) = \frac{1}{3}$

We will show that the positive contribution is killed by the surrounding points, yielding $\sum_v f(v) \le 0$ clearly a contradiction.

First we define a modified map: we add the diagonal in diamonds as in figure 5: thus producing 2 equilateral triangles. The effect upon $\sum_v f(v)$ is as follows:

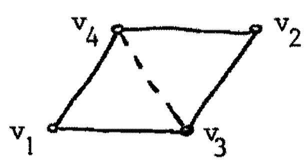


fig.5

$f(v_1)$ and $f(v_2)$ are increased by $\frac{1}{3}$, $f(v_3)$ and $f(v_4)$ are increased by $\frac{1}{2} + \frac{2}{3}$; therefore each added diagonal produces an increment of 4:

$$\sum_v f(v) = 20 + 4 \times (\#\text{ added diagonals}) . \qquad (8)$$

After the addition of extra diagonals points are produced of valency 6 and maybe 7, say v_6 of valency 6, v_7 of valency 7 and we have the relation

$$2 \times (\#\text{ added diagonals}) = v_6 + 2v_7 \qquad (9)$$

together with (8) this gives:

$$\sum_{v\in V} f(v) - 2v_6 - 4v_7 = 20 . \tag{10}$$

The contribution of points with valency 6 or 7 to the left hand side of this relation is negative, this shows we may limit us to the study of points that are in a pentagon, since all other points do not make a positive contribution. Let $\mathcal{P}$ denote the set of pentagonal faces, and $\cup\mathcal{P}$ the set of points contained in a pentagonal face.

Let $\tilde{f}(v) = f(v) - 2(d(v) - 5)$ where $d(v)$ is the degree of v, we then rewrite relation (10) as

$$\sum_{v\in V} \tilde{f}(v) = 20$$

or, separating pentagonal points and non-pentagonal points:

$$\sum_{v\in V\setminus\cup\mathcal{P}} \tilde{f}(v) + \sum_{v\in\cup\mathcal{P}} \tilde{f}(v) = 20 . \tag{11}$$

Since $\tilde{f}(v) \le 0$ for $v \in V\setminus\cup\mathcal{P}$ we will now investigate

$$\sum_{v\in\cup\mathcal{P}} \tilde{f}(v) .$$

Write:

$$\sum_{v\in\cup\mathcal{P}} \tilde{f}(v) = \sum_{P\in\mathcal{P}} \sum_{v\in P} \frac{\tilde{f}(v)}{f_5(v)} .$$

We will finish the proof by showing that

$$\sum_{v\in P} \frac{\tilde{f}(v)}{f_5(v)} \le 0$$

for all possible pentagons P.

Now the only way for $v \in P$ to have $\frac{\tilde{f}(v)}{f_5(v)} > 0$ is that v is surrounded by 4 triangles and a pentagon, in which case $\tilde{f}(v) = \frac{1}{3}$.

In all other cases $\frac{\tilde{f}(v)}{f_5(v)} \leq -\frac{1}{2}$.

A pentagon making a positive contribution must therefore have four vertices of the first kind, this is clearly impossible, so we are finished.